UNIT-4

UNIT-3

Centroid & Centre of gravity

Def :- Generally, the physical objects may be in one dim (1D), two dim (2D) (or) three dim (3D).
⇒ In case of 1D (or) 2D [i.e for a line (or) Area]
* we use centroid and ⇒ Incase of 3D [i.e for volume]
we use centre of gravity.

Centroid :- The point at which the total area of a plane surface is assumed to be concentrated is known as centroid.

Centre of gravity :- The point at which the total weight (volume)* of a body is assumed to be concentrated is known as centre of gravity

* **Centroid** : The summation of moments of area about the centroidal axes $(\bar{x}, \bar{y})$ is equal to zero
It is denoted as "C"

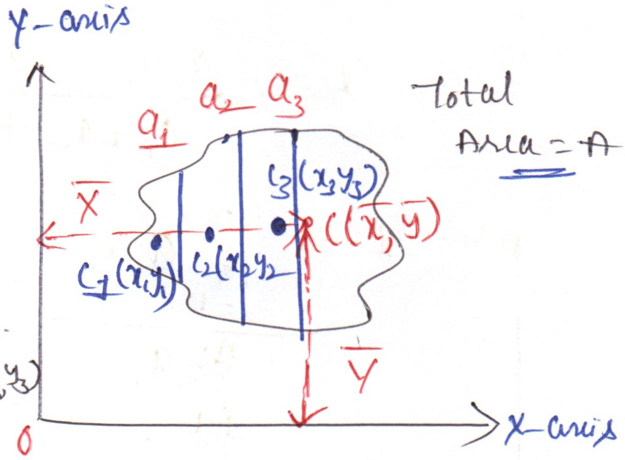
* **Centre of gravity** :- The summation of moments of volume about the centre of gravity axes $(\bar{x}, \bar{y}, \bar{z})$ is equal to zero → 'G' ←

Methods for the determination of Centroid & Centre of gravity

(A) **Determination of centroid** : we can determine the centroid of a given surface in two ways i.e method of moments and by the method of integration

1. Centroid by the method of moments :-

* Let us consider a plane surface of Area "A" with centroid 'C' $(\bar{x}, \bar{y})$ as shown in fig.



⇒ The moment of total Area from Y-axis $= A \times \bar{x}$ —— (1)

Now let us divide the total area into small areas a_1, a_2, a_3 --- with centroids $C_1(x_1, y_1)$, $C_2(x_2, y_2)$, $C_3(x_3, y_3)$ --- respectively

⇒ the moment of area a_1 from y-axis $= a_1 \times x_1$

a_2 from y-axis $= a_2 \times x_2$

a_3 from y axis $= a_3 \times x_3$

⇒ The summation of moments of all areas about y-axis $= a_1 x_1 + a_2 x_2 + a_3 x_3 + - - -$ —— (2)

$$\Rightarrow A \times \bar{x} = a_1 x_1 + a_2 x_2 + a_3 x_3 + - - .$$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{A}$$

$$\boxed{\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3}} \quad [\because A = a_1 + a_2 + a_3 + - -]$$

Similarly by taking the moments from x axis determine $\bar{y}$ as

$$\boxed{\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + - -}{a_1 + a_2 + a_3 + - -}}$$

Centroid by the method of integration:-

From the above derivation we know that

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}$$

i = no. of small area = large

Σ = replaced wit integration

$$\Rightarrow \bar{x} = \frac{\int x\, dA}{\int dA}$$

$\Rightarrow$ Simillarly

[* Material with uniform density]

$$\bar{y} = \frac{\int y \cdot dA}{dA}$$

(B) **Determination of Centre of gravity**: Similar to centroid we can also determine the C.O.G of body in two ways i.e., by the method of moments and by the method of integration.

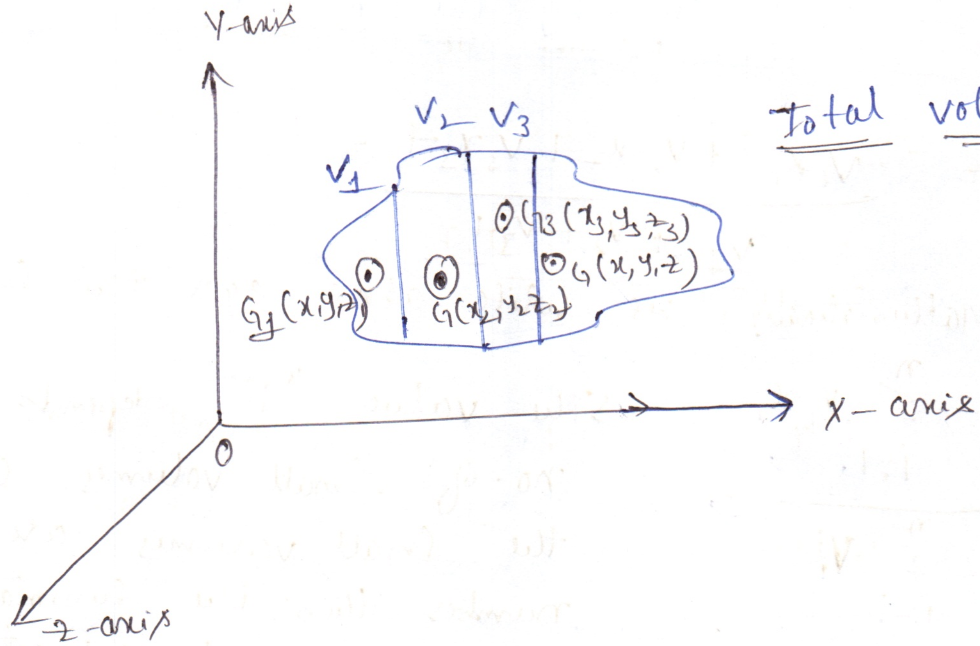
(1) **Centre of gravity by the method of moments:-**

Let us Consider a body (with uniform density) with volume 'V' and with centre of gravity $G(\bar{x}, \bar{y}, \bar{z})$ as shown in fig.

The moment of total volume from YZ-plane

$$= V \times \bar{x} \quad \text{--- (1)}$$

Now let us Consider to divide the total volume into small volumes V_1, V_2, V_3 ---- with Centre of gravities $G_1(x_1, y_1, z_1)$, $G_2(x_2, y_2, z_2)$, $G_3(x_3, y_3, z_3)$ --- respectively



Total volume "V"

The moments of volume "V_1" from YZ-plane $= V_1 \times x_1$

The moment of volume "V_2" from YZ-plane $= V_2 \times x_2$

The moment of volume V_3 from YZ-plane $= V_3 \times x_3$

The Summation of moments of all volumes about

YZ-plane $= V_1 x_1 + V_2 x_2 + V_3 x_3 + \cdots$ (2)

we know that eq^n (1) & eq^n (2)

$$V \times \bar{x} = V_1 x_1 + V_2 x_2 + V_3 x_3 + \cdots$$

$$\bar{x} = \frac{V_1 x_1 + V_2 x_2 + V_3 x_3 + \cdots}{V}$$

$$\bar{x} = \frac{V_1 x_1 + V_2 x_2 + V_3 x_3 + \cdots}{V_1 + V_2 + V_3 + \cdots}$$

[Total volume $V = V_1 + V_2 + V_3 + \cdots$)

Simillarly by taking the moments from XZ-plane we can determine $\bar{y}$ as

$$\bar{y} = \frac{V_1 y_1 + V_2 y_2 + V_3 y_3 + \cdots}{V_1 + V_2 + V_3 + \cdots}$$

Simillarly by taking the moments from XY-plane we can determine $\bar{z}$ as

$$\bar{z} = \frac{V_1 z_1 + V_2 z_2 + V_3 z_3 + \cdots}{V_1 + V_2 + V_3}$$

Centre of gravity by the method of integration

$$\bar{x} = \frac{V_1 x_1 + V_2 x_2 + V_3 x_3 + \cdots}{V_1 + V_2 + V_3 + \cdots}$$

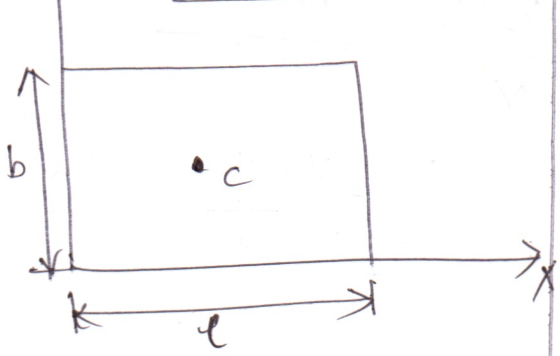
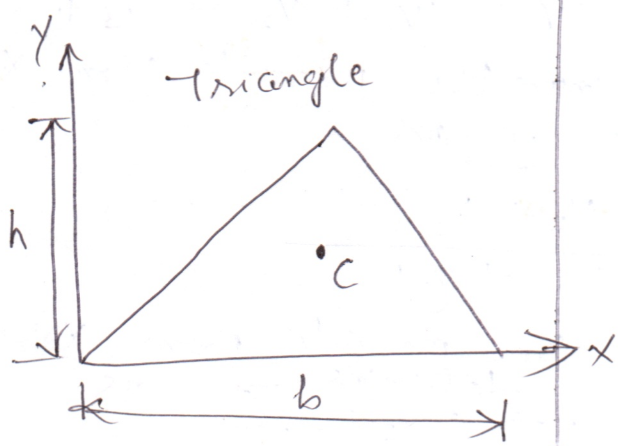
⇒ Mathematically we write above eqn the summation form

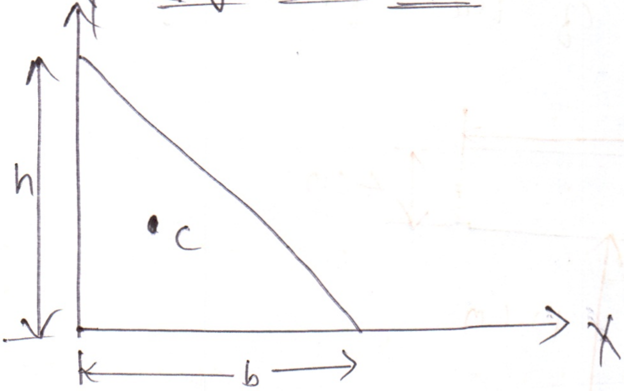
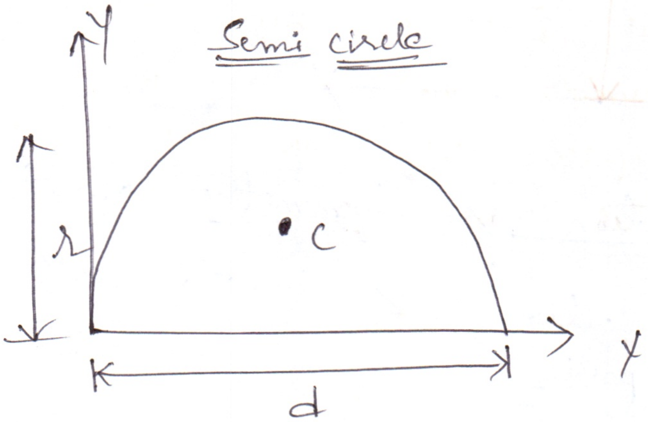
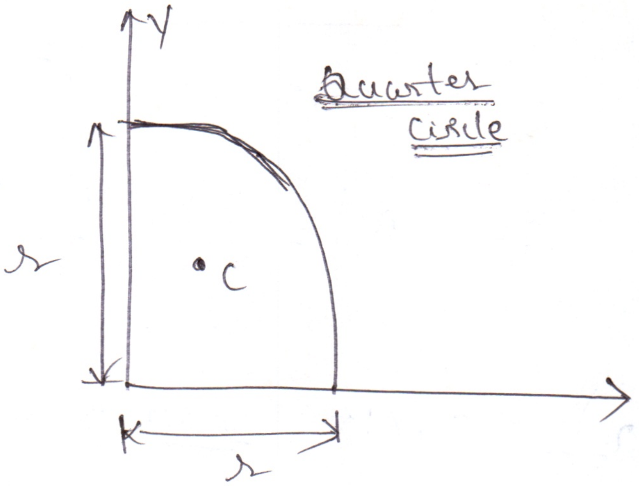
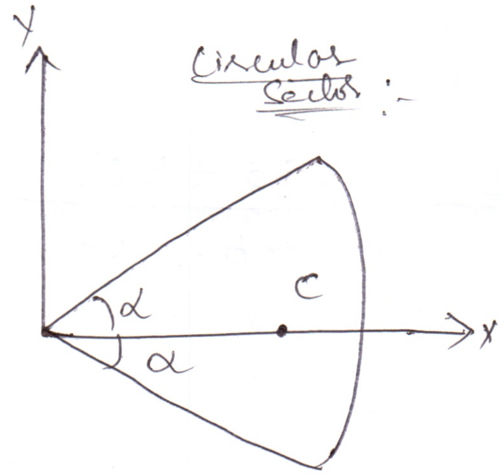
$$\bar{x} = \frac{\sum_{i=1}^{n} V_i x_i}{\sum_{i=1}^{n} V_i}$$

⇒ The value "i" depends on the no. of small volumes (n). If the small volumes are large in numbers then the summation can be replaced with the integration in the above eqn

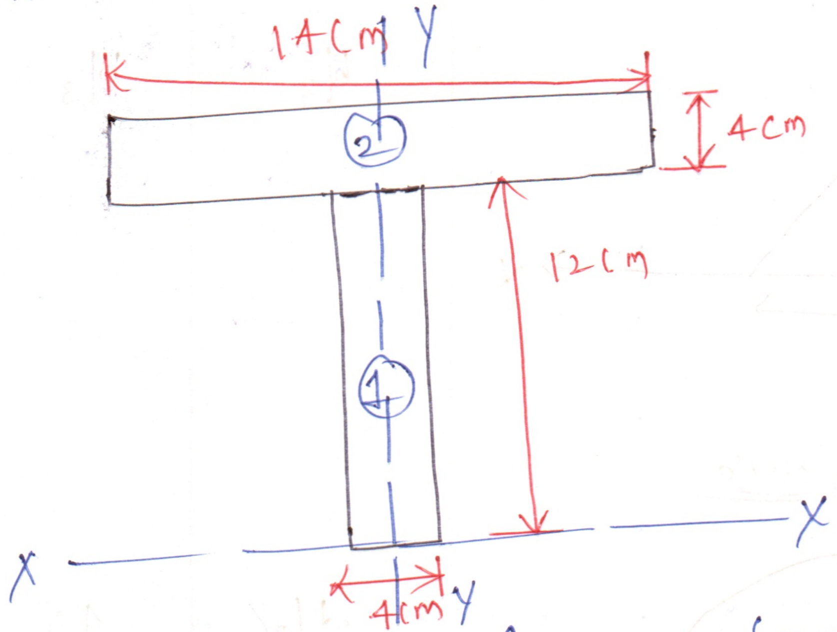
$$\Rightarrow \bar{x} = \frac{\int x\,dV}{\int dV} \quad \& \quad \bar{Y} = \frac{\int y\,dV}{\int dV} \quad \& \quad \bar{z} = \frac{\int z\,dV}{\int dV}$$

Centroids of some standard figures :-

S.NO	Name of shape	$\bar{x}$	$\bar{y}$	Area
1.	Rectangle 	$l/2$	$b/2$	lb
2.	Triangle 	$b/2$	$h/3$	$\frac{1}{2}bh$

3)	Right angle ∆le 	$b/3$	$h/3$	$\frac{1}{2}bh$
4)	Semi circle 	$d/2$	$\frac{4r}{3\pi}$	$\frac{\pi d^2}{8}$ (or) $\frac{\pi r^2}{2}$
6.	Quarter circle 	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi d^2}{16}$ (or) $\frac{\pi r^2}{4}$
7.	Circular Sector :- 	$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

(1) Determine the centroid of the following T-section



Sol: Here the given T-section is symmetry about y-axis so $\bar{X} = 0$

Now the given T-section is divided into two parts part-1 & part-2

part-1

$a_1 = lb = 4 \times 12 = 48\,cm^2$

$y_1 = \frac{b}{2} = \frac{12}{2} = 6\,cm$

$a_1 y_1 = 48 \times 6 = 288\,cm^3$

part-2

$a_2 = lb = 14 \times 4 = 56\,cm^2$

$y_2 = 12 + \frac{b}{2} = 12 + \frac{4}{2} = 14\,cm$

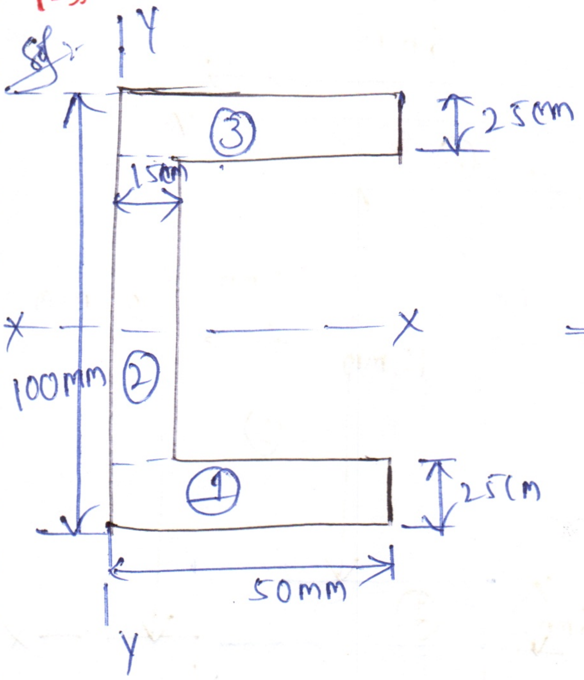
$a_2 y_2 = 56 \times 14 = 784\,cm^3$

we know $\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{288 + 784}{48 + 56}$

$\bar{y} = 10.3\,cm$

$C(\bar{x}, \bar{y}) = (0, 10.3)$

(2). Find the Centroid of following section



⇒ Here the given section is symmetry about x-axis

so $\boxed{\bar{y} = 0}$

⇒ Now the given 'c' section is divided into three parts as part-1 & part-2 & part 3

part-1	part-2	part-3
$a_1 = l_1 \times b_1 = 50 \times 25 = 1250\,mm^2$	⇒ $a_2 = l_2 b_2 = 15 \times 50 = 750\,mm^2$	⇒ $a_3 = l_3 b_3 = 50 \times 25 = 1250\,mm^2$
⇒ $x_1 = \frac{l_1}{2} = \frac{50}{2} = 25\,mm$	⇒ $x_2 = 7.5\,mm$	⇒ $x_3 = \frac{l_3}{2} = \frac{50}{2} = 25\,mm$
$a_1 x_1 = 31{,}250\,mm^3$	$a_2 x_2 = 5625\,mm^3$	$a_3 x_3 = 31{,}250\,mm^3$

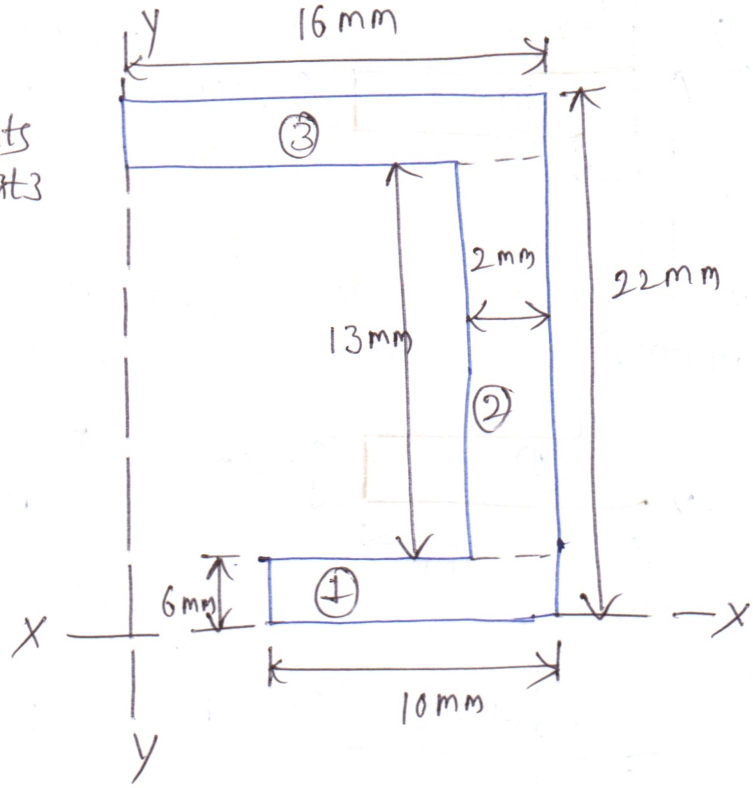
We know that

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} \Rightarrow \bar{x} = 20.96\,mm$$

$$c(\bar{x}, \bar{y}) = (20.96, 0)\,mm$$

③ Find the Centroid of the following Section

Sol: Now the given section is divided into three parts part 1 & part 2 & part 3



part 1

$a_1 = lb = 10 \times 6 = 60\,mm^2$

$x_1 = 6 + \frac{10}{2} = 11\,mm$

$y_1 = \frac{b}{2} = \frac{6}{2} = 3\,mm$

$a_1 x_1 = 60 \times 11 = 660\,mm^3$

$a_1 y_1 = 60 \times 3 = 18\,mm^3$

part - 2

$a_2 = lb = 2 \times 13 = 26\,mm^2$

$x_2 = 14 + \frac{2}{2} = 15\,mm$

$y_2 = 6 + \frac{b}{2} = 6 + \frac{13}{2}$

$= 12.5\,mm$

$a_2 x_2 = 26 \times 15 = 390\,mm^3$

$a_2 y_2 = 26 \times 12.5 = 325\,mm^3$

part 3

$a_3 = lb = 16 \times 3 = 48\,mm^2$

$x_3 = \frac{l}{2} = \frac{16}{2} = 8\,mm$

$y_3 = 6 + 13 + \frac{6}{2}$

$= 6 + 13 + \frac{3}{2} = 20.5\,mm$

$a_3 x_3 = 48 \times 8 = 384\,mm^3$

$a_3 y_3 = 48 \times 20.5 = 984\,mm^3$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{6(11) + 26(15) + 48(18)}{60 + 26 + 48}$$

$$\boxed{\bar{x} = 10.7\,mm}$$

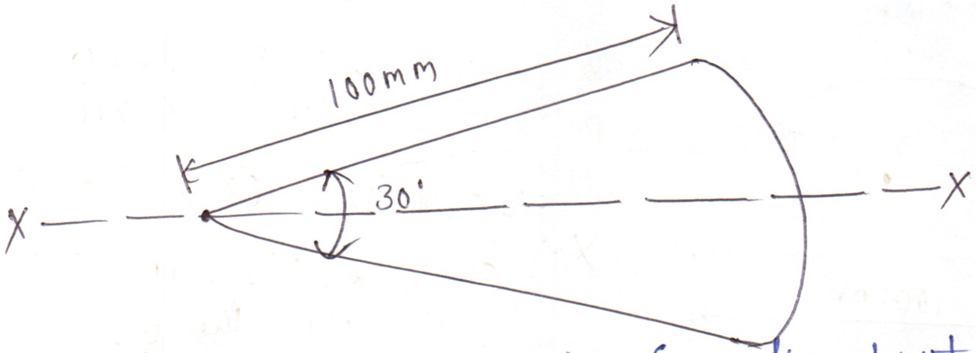
$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = 11.11\,mm$$

$\bar{y} = 11.11\,mm$

$c(\bar{x}, \bar{y}) = (10.7,\ 11.11)\,mm$

(14) A plane lamina of 100 mm radius is shown in figure. Find the centroid & Area

Sol:-



Sol: Since the given fig. is symmetry about x-axis $\bar{y} = 0$

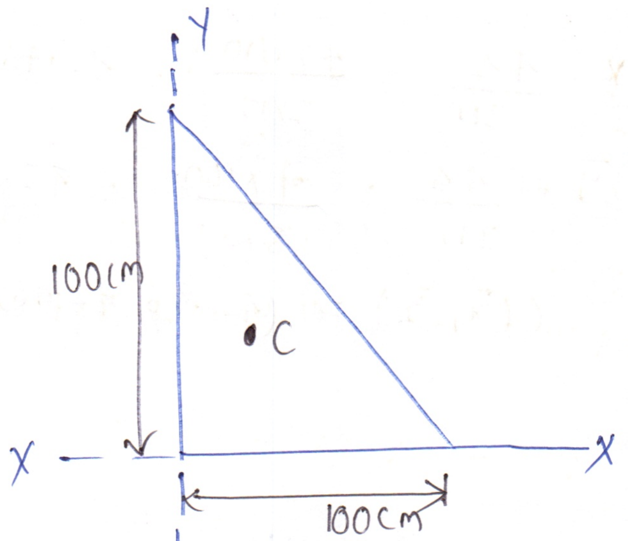
$$\bar{x} = \frac{2 r \sin\alpha}{3\alpha} = \frac{2 \times 100 \times \sin 15^\circ}{3 \times \frac{\pi}{12}} \quad \left[\because \alpha = \frac{30}{2} = 15^\circ \text{ in radians } 15^\circ = \frac{\pi}{12}\right]$$

$\bar{x} = 65.94$ mm $\Rightarrow$ $C(\bar{x}, \bar{y}) = (65.94, 0\text{ mm})$

Area $= \alpha r^2 = \left(\frac{\pi}{12}\right) \times 100^2 = 261.66\text{ mm}^2$

(15) Find the centroids of the following figures from the given X & Y axes.

(i)

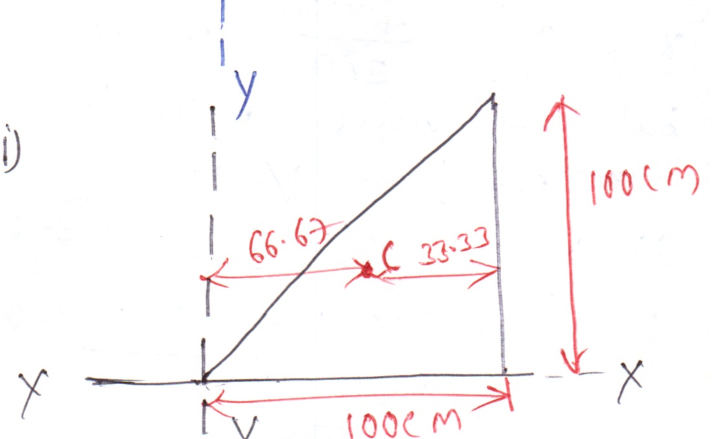


$\bar{x} = \frac{b}{3} = \frac{100}{3} = 33.33$ cm

$\bar{y} = \frac{h}{3} = \frac{100}{3} = 33.33$ cm

$C(\bar{x}, \bar{y}) = (33.33, 33.33)$ cm

(ii)



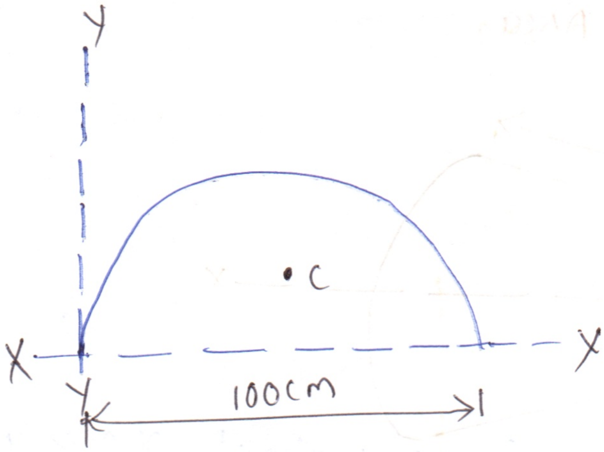
Actually, $\bar{x} = \frac{b}{3} = \frac{100}{3}$ $= 33.33$ but we have to find this value from Y-axis ($\because \bar{x}$ is distance from Y-axis)

$\Rightarrow \bar{x} = 100 - 33.33 = 66.67$ cm

$\bar{y} = \frac{h}{3} = \frac{100}{3} = 33.3$ [From ~~X~~-axis]

$C = (\bar{x}, \bar{y}) = (66.67, 33.33)$

(iii)



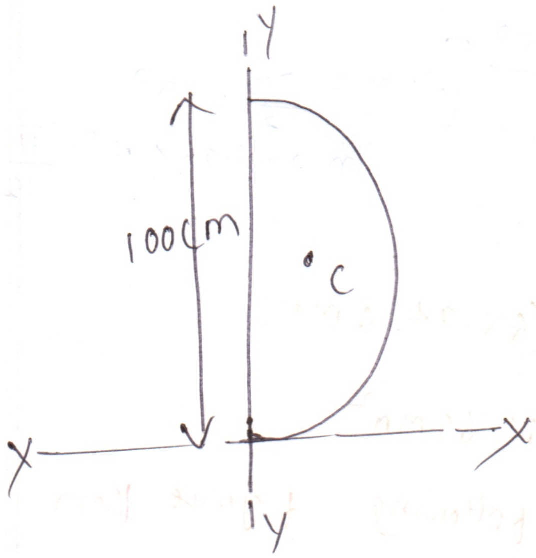
$\bar{x} = \frac{d}{2} = \frac{100}{2} = 50\,cm$

$\bar{y} = \frac{4r}{3\pi} = \frac{4\times 50}{3\times\pi} = 21.23\,cm$

$\Rightarrow C(\bar{x}, \bar{y}) = (50, 21.23\,cm)$

(Here $\bar{x}$, & $\bar{y}$ interchanging as the the fig is changed

(iv)

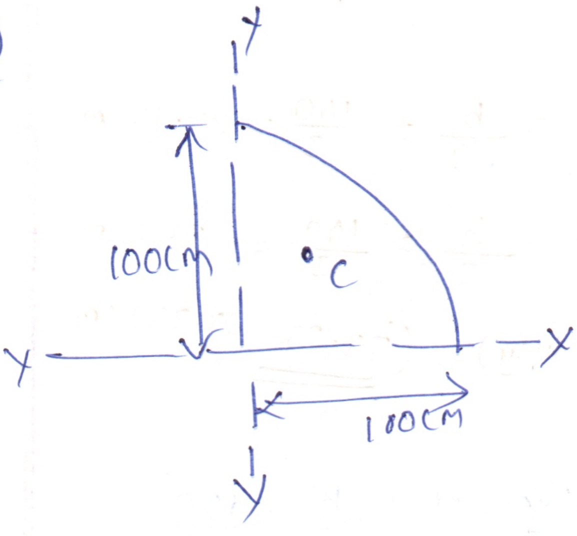


$\bar{x} = \frac{4r}{3\pi} = \frac{4\times 50}{3\times\pi} = 21.23\,cm$

$\bar{y} = \frac{d}{2} = \frac{100}{2} = 50\,cm$

$C(\bar{x}, \bar{y}) = (21.23, 50)\,cm$

(v)

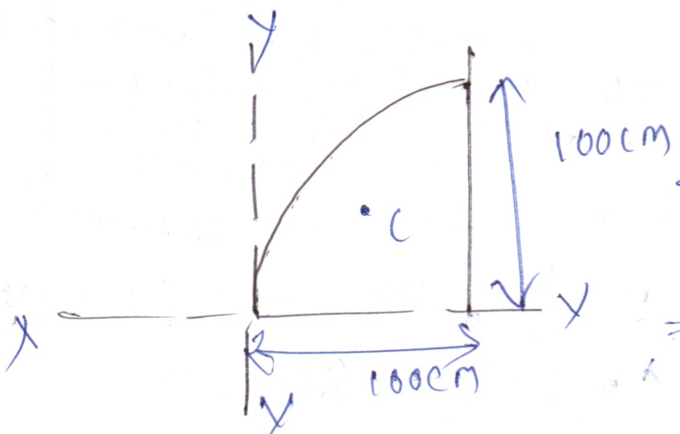


$\bar{x} = \frac{4r}{3\pi} = \frac{4\times 100}{3\times\pi} = 42.46\,cm$

$\bar{y} = \frac{4r}{3\pi} = \frac{4\times 100}{3\times\pi} = 42.46$

$C(\bar{x}, \bar{y}) = (42.46, 42.46)\,cm$

(vi)



$\bar{x} = \frac{4r}{3\pi} = \frac{4\times 100}{3\times\pi} = 42.46$

but this value must be determined from Y-axis

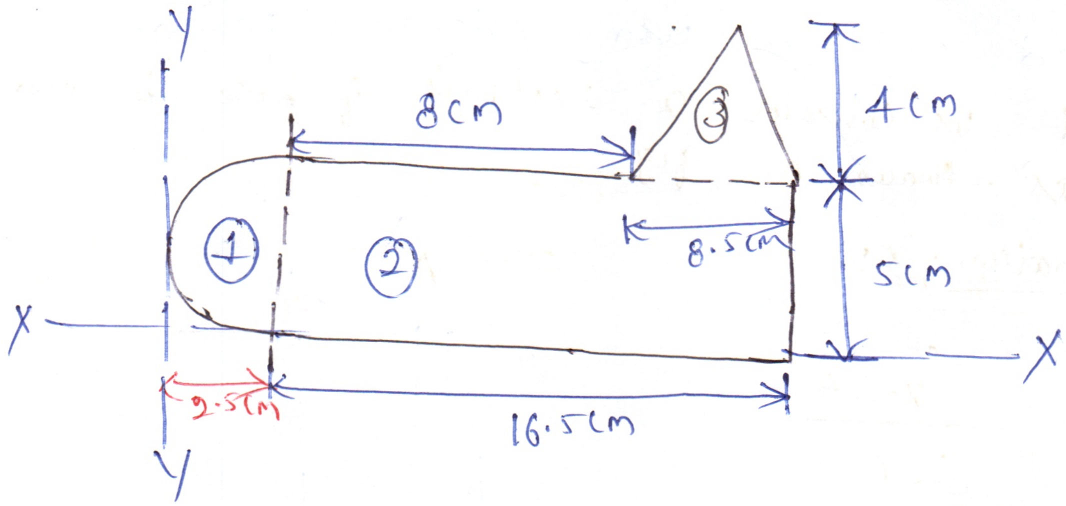
$\because$ $\bar{x}$ is distance from $\bar{y}$-axis

$\Rightarrow \bar{x} = 100 - 42.46 = 57.54\,cm$

$\bar{y} = \frac{4r}{3\pi} = 42.46$

* Centroid of Composite figures :-

1) Determine the centroid of following fig



Sol:

part 1

$r = 2.5 \, cm$

$a_1 = \frac{\pi r^2}{2} = \frac{(3.14)(2.5)^2}{2}$

$= 9.81 \, cm^2$

$\bar{x}_1 = r - \frac{4r}{3\pi} = 2.5 - \frac{4(2.5)}{3(3.14)}$

$= \underline{1.44 \, cm}$

$\bar{y}_1 = r = \underline{2.5 \, cm}$

$a_1 \bar{x}_1 = 14.12 \, cm^3$

$a_2 \bar{y}_1 = 24.52 \, cm^3$

part : 2

$a_2 = lb = 16.5 \times 5$

$= 82.5 \, cm^2$

$\bar{x}_2 = 2.5 + \frac{16.5}{2}$

$= 10.75 \, cm$

$\bar{y}_2 = \frac{b}{2} = \frac{5}{2} = 2.5 \, cm$

$a_2 \bar{x}_2 = 886.87 \, cm^3$

$a_2 \bar{y}_2 = 206.25 \, cm^3$

part : 3

$a_3 = \frac{1}{2} bh = \frac{1}{2} \times 8.5 \times 4$

$= 17 \, cm^2$

$\bar{x}_3 = 2.5 + 8 + \frac{b}{2}$

$= 2.5 + 8 + \frac{8.5}{2}$

$= 14.75 \, cm$

$\bar{y}_3 = 5 + \frac{h}{3}$

$= 5 + \frac{4}{3} = \underline{6.33 \, cm}$

$a_3 \bar{x}_3 = 250.75 \, cm^3$

$a_3 \bar{y}_3 = \underline{107.61 \, cm^3}$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{14.12 + 886.87 + 250.75}{9.81 + 82.5 + 17} = 10.5 \, cm$$

$\bar{x} = 10.5 \, cm$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{24.52 + 206.25 + 107.61}{9.81 + 82.5 + 17}$$

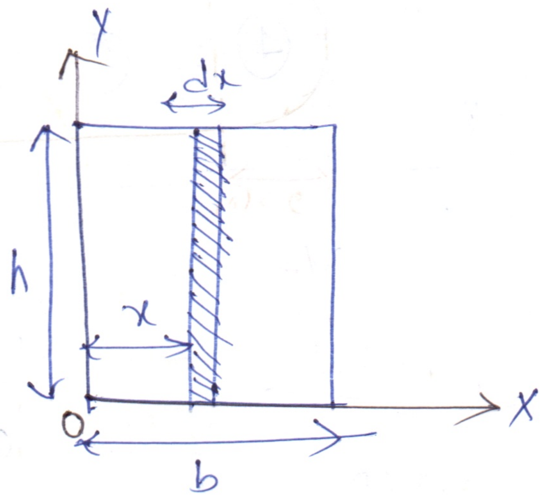
$= 3.09 \, cm$

$C(\bar{x}, \bar{y}) = (10.5, 3.09)$

Centroid By Integration:-

Eg:- (1) Determine the centroid of a rectangle of base 'b' and heigh "h" using integration method.

Sol:- Let us consider a rectangle of base "b" and height "h" as shown in fig



Determination: $\bar{x}$:-

$$\bar{x} = \frac{\int x \cdot dA}{\int dA}$$

From fig $dA = h \cdot dx$

$$A = \int_0^b dA$$

$$A = \int_0^b dA = \int_0^b h \cdot dx = h(x)_0^b \Rightarrow hb$$

$$\therefore \bar{x} = \frac{\int_0^b x \cdot dA}{\int_0^b dA} = \frac{\int_0^b x \cdot h\,dx}{bh} = \frac{h\int_0^b x \cdot dx}{bh}$$

$$h\left[\frac{x^2}{2}\right]$$