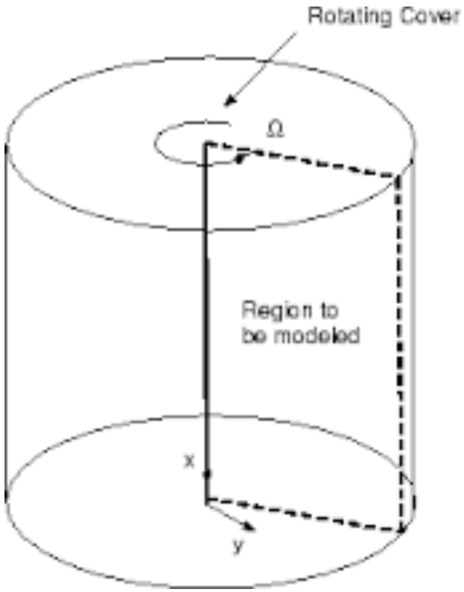
Lecture 17:Axisymmetric Problems

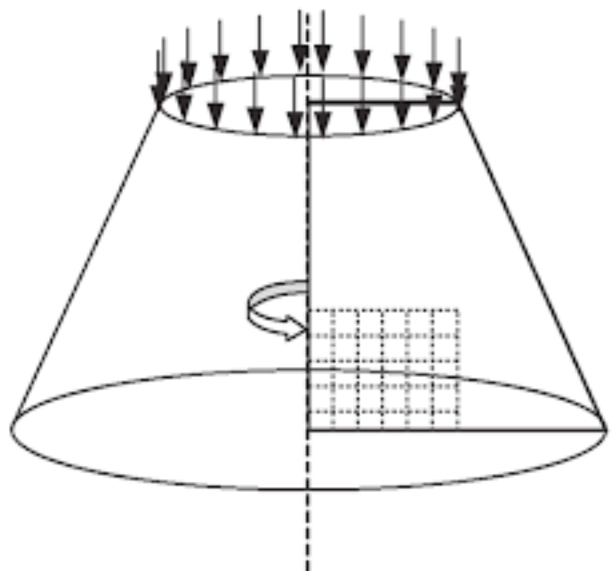
APL705 Finite Element Method

v18

Axisymmetric Solids

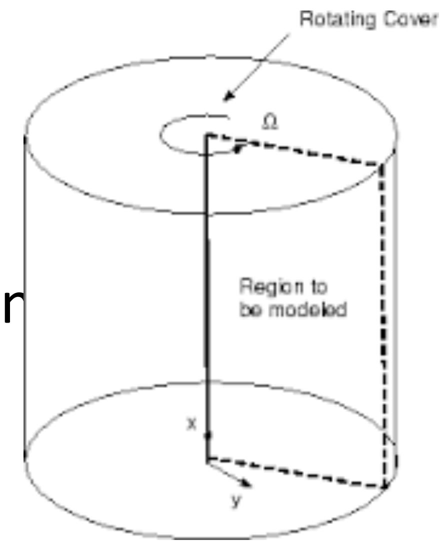
- The solids with an axis of symmetry or solids of revolution with axisymmetric loading (and support) are modeled as simple 2D elements. As seen from the figures below all stresses are independent of rotational angle. Eg. Flywheel, bearing, thick-walled pressure vessels





Condition for Axisymmetric Problems

1. The problem domain must have an axis of symmetry. It is customary to align this symmetry axis with the z-axis of the cylindrical (r,θ,z) coordinate system.
2. The boundary conditions are symmetric about the axis. Therefore all BCs are independent of θ
3. All loading conditions are symmetric about z hence these are also independent of circumferential direction θ
4. Also, the material properties must be homogeneous or symmetric. This condition always satisfied for isotropic materials
5. Lastly, the solutions are independent of θ



Axisymmetric as Special 2D Problems

- Where there is symmetry in geometry and loads we can model such problems as special two-dimensional problems.
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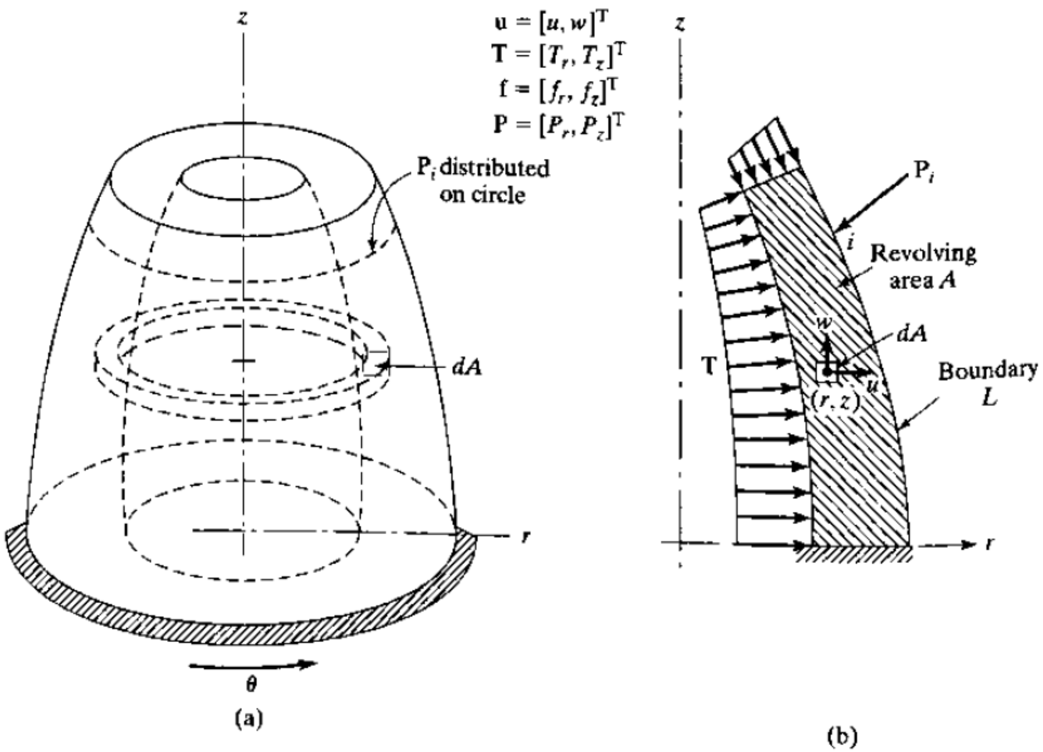
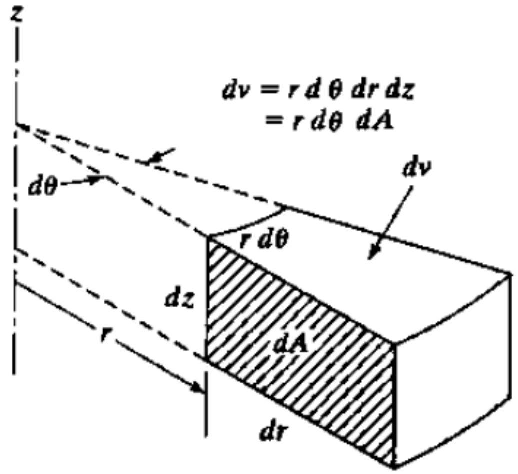


Axisymmetric Formulation

- Consider the elemental volume shown here. Now we write the potential energy in axisymmetric form
- Here we note that all integrals are independent of θ



$$\pi = \tfrac{1}{2}\int_0^{2\pi}\int_A \sigma^T \varepsilon r\, dA\, d\theta - \int_0^{2\pi}\int_A u^T f r\, dA\, d\theta - \int_0^{2\pi}\int_L u^T T r\, dl\, d\theta - \sum_i u_i^T P_i$$

$$\pi = 2\pi\left(\tfrac{1}{2}\int_A \sigma^T \varepsilon r dA - \int_A u^T f r dA - \int_L u^T T r dl\right) - \sum_i u_i^T P_i$$

- where

$$u = [u, w]^T; \qquad f = [f_r, f_z]^T; \qquad T = [T_r, T_z]^T$$

Stress Strain Relations

- We can write strains for this case as

$$\varepsilon = [\varepsilon_r, \varepsilon_z, \gamma_{rz}, \varepsilon_\theta]$$

- Using derivatives of displacements this can be written as

$$\varepsilon = \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]$$

- The stress vector for this case is

$$\sigma = [\sigma_r, \sigma_z, \tau_{rz}, \sigma_\theta]$$

- Now the stress-strain relation is given as $\sigma = D\varepsilon$ where D is

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & k & 0 & k \\ k & 1 & 0 & k \\ 0 & 0 & s & 0 \\ k & k & 0 & 1 \end{bmatrix}; k = \frac{\nu}{1-\nu}; s = \frac{1-2\nu}{2(1-\nu)}$$

Axisymmetric Galerkin Formulation

- The Galerkin formulation for an axisymmetric modeling can be given as

$$2\pi \int_A \sigma^T \varepsilon(\phi) r dA - (2\pi \int_A \phi^T f r dA + 2\pi \int_L \phi^T T r dl + \sum_i \phi_i^T P_i) = 0$$

- Here the strain vector is given as

$$\phi = [\phi_r, \phi_z]^T$$

$$\varepsilon(\phi) = \left[\frac{\partial \phi_r}{\partial r}, \frac{\partial \phi_z}{\partial z}, \frac{\partial \phi_r}{\partial z} + \frac{\partial \phi_z}{\partial r}, \frac{\phi_r}{r} \right]^T$$

Axisymmetric Modeling using Triangular Elements

- The area of axisymmetry is now divided into triangular finite elements. By revolving these elements we actually get the discretized solid object. Therefore though each element is a triangle, it actually represents a solid ring of triangular section about the axis of symmetry.
- Following the discussion on CST and replacing x, y coordinates with r,z we have a similar formulation now. Using 3 shape function N_1, N_2 and N_3 we write {u}=[N]{q} where

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$q = [q_1, q_2, q_3, q_4, q_5, q_6]^T$$

Axisymmetric - Isoparametric Formulation

- We can write the displacements in r and z directions as

$$u = (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5$$
$$v = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6$$

- The same functions are used to interpolate the r, z coordinates also in the isoparametric formulation

$$r = (r_1 - r_3)\xi + (r_2 - r_3)\eta + r_3$$
$$z = (z_1 - z_3)\xi + (z_2 - z_3)\eta + z_3$$

Calculating Strains

- The strain relation is converted to a matrix form

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial r}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix}$$

- Here the (2x2) matrix is the Jacobian of transformation, J
- $$J = \begin{bmatrix} \dfrac{\partial r}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial r}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \end{bmatrix}$$ On taking the derivatives we have

$$J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

Element Strains

- Using the definition of J we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = J \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix}$$

- The inverse transformation of the above is
- where J^{-1} is the inverse of J given by

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} r_{13} & -z_{13} \\ -r_{23} & z_{23} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{bmatrix} \qquad \begin{bmatrix} \dfrac{\partial w}{\partial r} \\ \dfrac{\partial w}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{bmatrix}$$

Strain Displacement Relation

- From strain-displacement relation

$$\varepsilon = \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial r} \\ \dfrac{u}{r} \end{bmatrix} = \begin{bmatrix} \left(z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5)\right) / \det J \\ \left(-r_{23}(q_2 - q_6) - r_{13}(q_4 - q_6)\right) / \det J \\ \left(-r_{23}(q_1 - q_5) - r_{13}(q_3 - q_5) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6)\right) / \det J \\ \left(N_1 q_1 + N_2 q_3 + N_1 q_5\right) / r \end{bmatrix}$$

- This will lead to {ε}=[B]{q}

Strain Displacement Relation

- The strain-displacement relation is $\varepsilon = Bq$
- Where B is the element strain-displacement matrix given as

$$B = \begin{bmatrix} z_{23}/\det J & 0 & z_{31}/\det J & 0 & z_{12}/\det J & 0 \\ 0 & r_{32}/\det J & 0 & r_{13}/\det J & 0 & r_{21}/\det J \\ r_{32}/\det J & z_{23}/\det J & r_{13}/\det J & z_{31}/\det J & r_{21}/\det J & z_{12}/\det J \\ N_1/r & 0 & N_2/r & 0 & N_3/r & 0 \end{bmatrix}$$