STATE SPAACE REPRESENTATION

State variable representation using physical variables

- The state variables are minimum number of variables which are associated with all the initial conditions of the system.
- As the sequence is not important and the state model of the system is not unique
- But for all the state models it is necessary that number of state variables are equal and minimal.
- The number n indicates the order of the system
- For the second order system minimum two variables are necessary and so on.

State variable representation using physical variables

- For electrical systems, the current through various indictors and the voltage across the various capacitors are selected to be the state variables.
- Then by any method of network analysis, the equations must be written in terms of the selected state variables, their derivatives and the inputs.
- The equations must be rearranged in the standard form so as to obtain required state model.
- *It is important that the equation for differentiation of one state variable should not involve the differentiation of any other state variable*.
- In the mechanical systems the displacements and velocities of energy storing elements such as spring and friction are selected as state variables.
- *In general, the physical variables associated with energy storing elements, which are responsible for initial conditions, are selected as the state variables of the given system.*

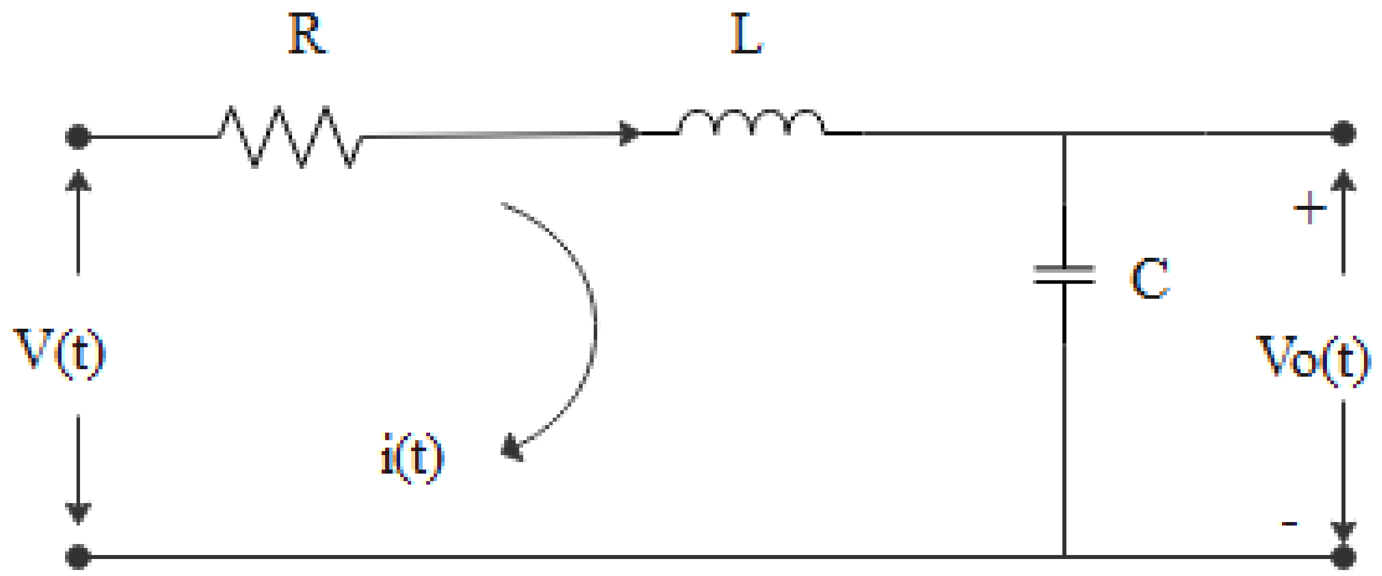
Advantages:

The advantages of using available physical variables as the state variables are,

1. The physical variables which are selected as the state variables are the physical quantities and can be measured.
2. As the state variables can be physically measured, the feedback may consists the information about state variables in addition to the output variables. Thus design with state feedback is possible.
3. Once the state equations are solved and solution is obtained, directly the behaviour of various physical variables with time is available.

But the important limitation of this method is that obtaining solution of such state equation with state variables as physical variables is very difficult and time consuming.

Q1. Obtain the state model of the given electrical network.



- **Solution:** There are two energy storing elements L and C. So the two state variables are current through indicator i(t) and the voltage across the capacitor i.e. V_o(t)

$$X_1(t) = \mathrm{i}\ (t) \text{ and } X_2(t) = V_0(t)$$

And $$\mathrm{U}\ (t) = V_i(t)$$

Applying the KVL to the loop,

$$V_i(t) = \mathrm{i}\ (t)\ \mathrm{R} + \mathrm{L}\ \frac{di(t)}{dt} + V_0(t)$$

Arrange it for $\frac{di(t)}{dt}$,

$$\therefore \qquad \frac{di(t)}{dt} = \frac{1}{L}V_i(t) - \frac{R}{L}i(t) - \frac{1}{L}V_o(t) \qquad but \quad \frac{di(t)}{dt} = \dot{X}_1(t)$$

$$i.e \quad \dot{X}_1(t) = -\frac{R}{L}X_1(t) - \frac{1}{L}X_2(t) + \frac{1}{L}U(t) \qquad \dots(1)$$

While $V_o(t)$ = Voltage across capacitor = $\frac{1}{C}\int i(t)dt$

$$\therefore \qquad \frac{dV_o(t)}{dt} = \frac{1}{C}i(t) \qquad but \qquad \frac{dV_o(t)}{dt} = \dot{X}_2(t)$$

i.e. $\qquad \dot{X}_2(t) = \frac{1}{C}X_1(t) \qquad \dots(2)$

The equations (1) and (2) give required state equation.

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U(t)$$

i.e. $\qquad \boldsymbol{\dot{X}(t) = A\,X(t) + B\,U(t)}$

While the output variable $Y(t) = V_o(t) = X_2(t)$

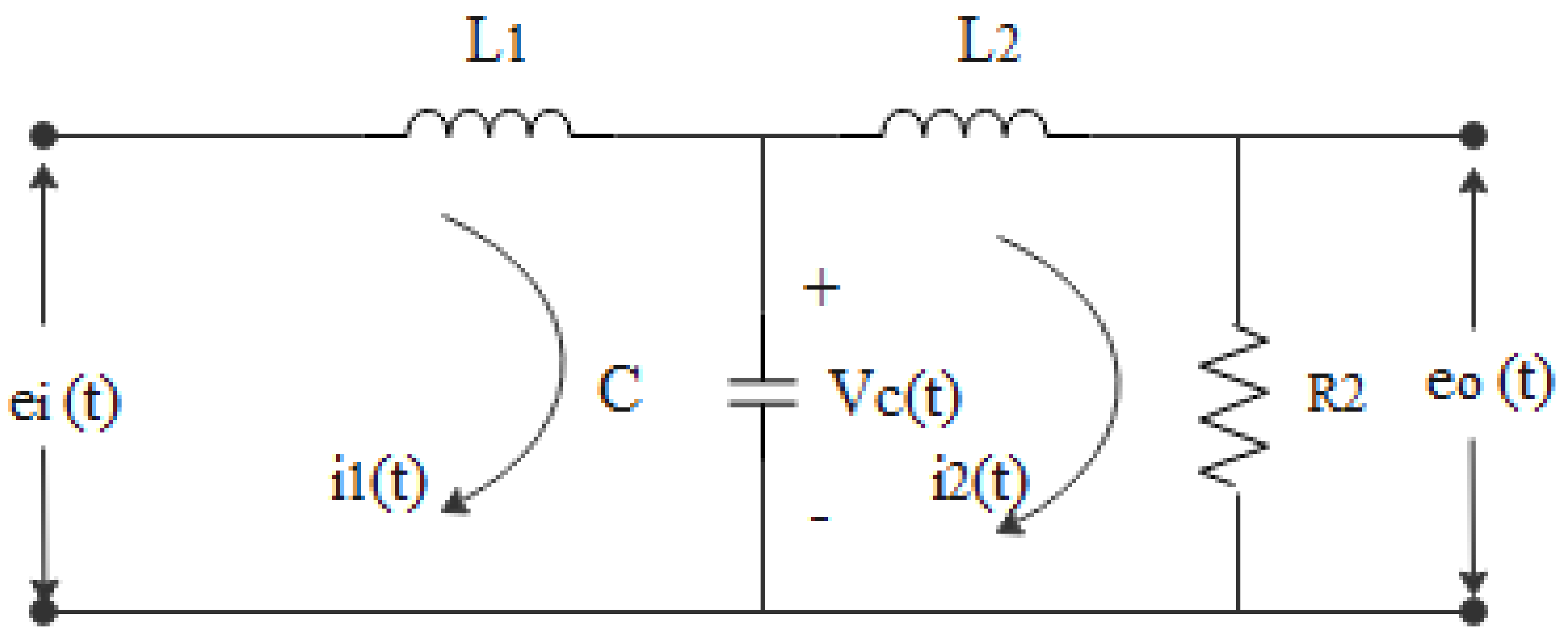
$$\therefore \qquad Y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} U(t)$$

$$i.e \qquad Y(t) = C\,X(t) \quad and \quad D = [0]$$

This is required state model. As n = 2 it is second order system.

Note: The order of the state variable is not important. X_1(t) can be V_o(t) and X_2(t) can be i(t) due to which state model matrices get changed. Hence state model is not the unique property of the system.

Q2. Obtain the state model of the given electrical network in the standard form



- **Solution:**

$$U(t) = input = e_i(t)$$
$$Y(t) = output = e_o(t)$$

$state\ variables$: $X_1(t) = i_1(t), \quad X_2(t) = i_2(t), \quad X_3(t) = V_C(t)$

$writing\ equations$:

$$e_i(t) = L_1 \frac{di_1(t)}{dt} + V_C(t)$$

$i.e.$ $$\frac{di_1(t)}{dt} = \frac{1}{L_1} e_1(t) - \frac{1}{L_1} V_C(t)$$

$\therefore$ $$\boldsymbol{\dot{X}_1(t) = \frac{1}{L_1} U(t) - \frac{1}{L_1} X_3(t)} \qquad \dots.(1)$$

$Then \qquad V_C(t) = L_2 \frac{di_2(t)}{dt} + i(t)R_2$

$$\therefore \qquad \frac{di_2(t)}{dt} = \frac{1}{L_2} V_C(t) - \frac{R_2}{L_2} i_2(t)$$

$$\therefore \qquad \dot{X}_2(t) = \frac{1}{L_2} X_3(t) - \frac{R_2}{L_2} X_3(t) \qquad \ldots.(2)$$

$and \qquad C \frac{dV_C(t)}{dt} = i_1(t) - i_2(t) = Current\ through\ capacitor$

$$\therefore \qquad \frac{dV_C(t)}{dt} = \frac{1}{C} i_1(t) - \frac{1}{C} i_2(t)$$

$$\therefore \qquad \dot{X_3}(t) = \frac{1}{C} X_1(t) - \frac{1}{C} X_2(t) \qquad \ldots.(3)$$

$$\therefore \quad \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} U(t)$$

$i.e.$ $$\dot{\boldsymbol{X}}(\boldsymbol{t}) = \boldsymbol{A}\,\boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{B}\,\boldsymbol{U}(\boldsymbol{t})$$

And $$e_o(t) = i_2(t)\,R_2$$

$$\therefore \quad Y(t) = X_2(t)\,R_2$$

$$\therefore \quad Y(t) = [0 \quad R_2 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$i.e.$ $$\boldsymbol{Y}(\boldsymbol{t}) = \boldsymbol{C}\,\boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{D}\,\boldsymbol{U}(\boldsymbol{t})$$

Where D = 0. This is the required state model.

State variable representation using phase variables

- The phase variables are those variables which are obtained by assuming one of the system variable as a state variable and the other state variable are the derivatives of the selected system variable.
- Most of the time, the system variable used is the output variable which is used to select the state variable.
- Such set of phase variables is easily obtained if the differential equation of the system is known or the system transfer function is available.

State model from differential equation

- Consider a linear continuous time system represented by n^{th}order differential equation as,
- $Y^n + a_{n-1}Y^{n-1} + a_{n-2}Y^{n-2} + \cdots + a_1\dot{Y} + a_0Y(t) = \quad b_0U + b_1\dot{U} + \cdots + b_{m-1}U^{m-1} + b_mU^m \qquad \ldots(1)$
- In the equation $y^n(t) = \frac{dy^n(t)}{dt^n} = n^{th}$ derivative of Y(t).
- For the time invariant system, the coefficients $a_{n-1}, a_{n-2}, \ldots a_0, \; b_0, b_1, \ldots . b_m$are constants
- For the system

  Y(t) = Output variable

  U(t) = Input variable
- $Y(0), Y(\dot{0}) \ldots \ldots Y(0)^{n-1}$ represents the initial condition of the system.

- Consider the simple case of the system in which derivatives of the control force U(t) are absent.
- Thus

$$\dot{U}(t) = \ddot{U}(t) = U^m(t) = 0$$

$$\therefore \quad Y^n + a_{n-1}Y^{n-1} + \cdots + a_1\dot{Y} + a_0Y(t) = b_0U(t) \qquad \ldots(2)$$

- Choice of state variable is generally output variable Y(t) itself, and other state variables are derivatives of the selected state variable Y(t).

$$\therefore \quad X_1(t) = Y(t)$$

$$\therefore \quad X_2(t) = \dot{Y}(t) = \dot{X_1}(t)$$

$$\therefore \quad X_3(t) = \ddot{Y}(t) = \ddot{X}_1(t) = \dot{X}_2(t)$$

- Thus the various state equations are

$$\dot{\boldsymbol{X}}_{\mathbf{1}}(\boldsymbol{t}) = \boldsymbol{X}_{\mathbf{2}}(\boldsymbol{t})$$

$$\dot{\boldsymbol{X}}_{\mathbf{2}}(\boldsymbol{t}) = \boldsymbol{X}_{\mathbf{3}}(\boldsymbol{t})$$

$$\vdots$$

$$\vdots$$

$$\dot{\boldsymbol{X}}_{\boldsymbol{n-1}}(\boldsymbol{t}) = \boldsymbol{X}_{\boldsymbol{n}}(\boldsymbol{t})$$

$$\dot{\boldsymbol{X}}_{\boldsymbol{n}}(\boldsymbol{t}) = ?$$

- Note that only n variables are to be defined to keep their number minimum. Thus $\dot{X}_{n-1}(t)$ gives n^{th}variable $X_n(t)$. But to complete the state model $\dot{X}_n(t)$ is necessary.

- Important $\dot{X}_n(t)$ is obtained by substituting the selected state variables in original differential equation (2).
- We have
- Y(t) =,$\dot{Y}(t)$= X_2 ,$\ddot{Y}(t)$ =X_3 …………………… $Y^{n-1}(t)$ = $X_n(t)$, $Y^n(t)$ = $\dot{X}_n(t)$

$$\therefore\ \dot{X}_n(t) +\ a_{n-1}X_n(t) +\ a_{n-2}X_{n-1}(t) + \cdots + a_1X_2(t) + a_0X_1(t) =\ b_0U(t)$$

$$\therefore\ \dot{X}_n(t)$$
$$= -\,a_0X_1(t) -\ a_1X_2(t) + \cdots - a_{n-2}X_{n-1}(t) + a_{n-1}X_n(t) +\ b_0U(t) \qquad \dots.(3)$$

- Hence all the equations now can be expressed in vector matrix form as

- $$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots. & 0 \\ 0 & 0 & 1 & \dots. & 0 \\ \vdots & & & & \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} U(t)$$

- i.e. $\dot{X} = A\,X(t) + B\,U(t)$

- Such set of state variables is called set of **phase variables.**
- The matrix A is called matrix in Phase variable form and it has following features,

Features of Phase Variable matrix

- Upper off-Diagonal i.e upper parallel row to the main principal diagonal contains all elements as 1.
- All other elements except last row are zeros.
- Last row consists of the negatives of the coefficients contained by the original differential equation.

- Such a form of matrix A is called **Bush form** or **Companion form.**
- Hence the method is also called **companion form realization**

The Output equation:

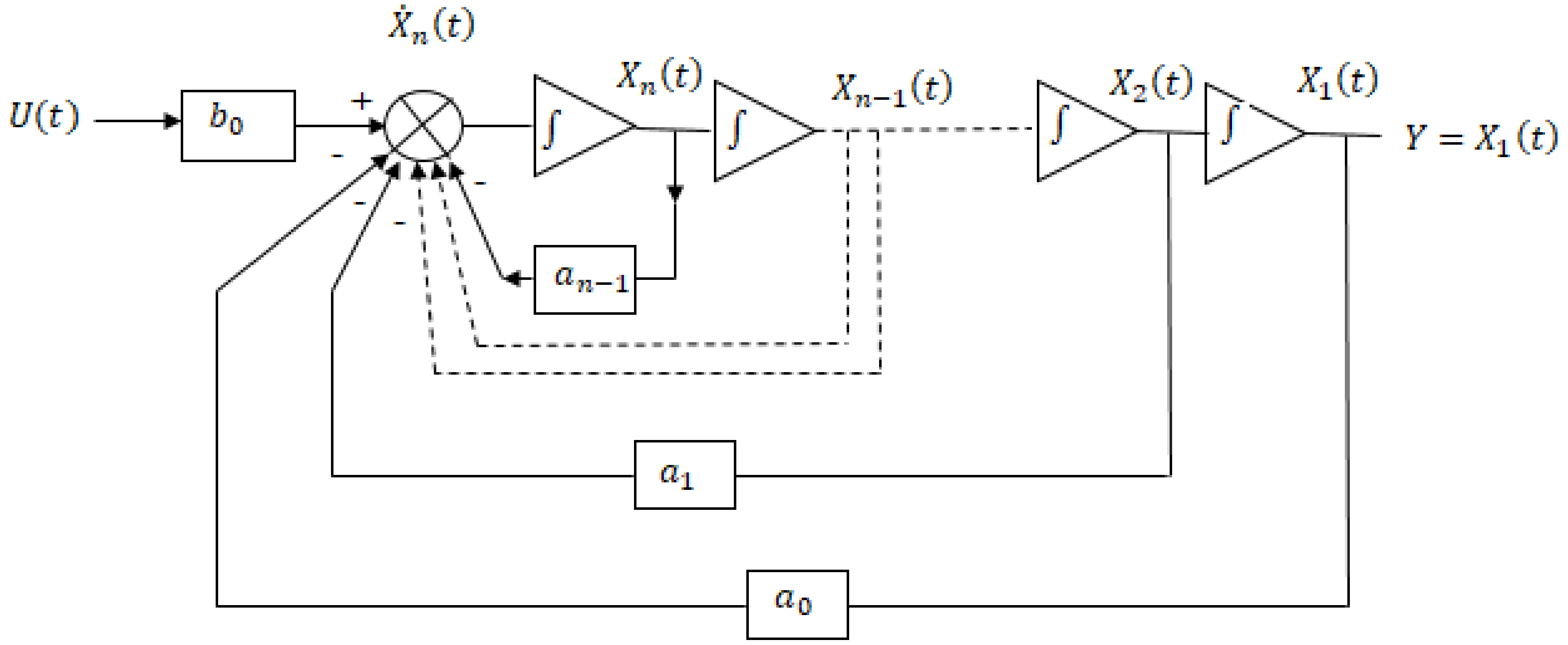
- The output equation is $Y(t) = X_1(t)$

- Therefore

$$Y(t) = \begin{bmatrix} 1 & 0 & \ldots. & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}$$

i.e. $Y(t) = CX(t)$ where $\mathrm{D} = 0$

- The model in the Bush form is shown in the state diagram as in below figure.
- The output of each integrator is a state variable



State diagram for phase variable form

- Observe that the transfer function of the blocks in the various feedback paths are the coefficients existing in the original differential equation.

Q3. Construct the state model using phase variables if the system is described by differential equation.

$$\frac{d^3Y(t)}{dt^3} + 4\frac{d^2Y(t)}{dt^2} + 7\frac{dY(t)}{dt} + 2Y(t) = 5U(t)$$

Draw the state diagram.

- **Solution:** Choose the output Y(t) as the state variable $X_1(t)$ and successive derivatives of it gives us the remaining state variables.
- As the order of equation is 3, only 3 state variables are allowed.

$$X_1(t) = Y(t)$$

$$X_2(t) = \dot{X_1}(t) = \dot{Y}(t) = \frac{dy(t)}{dt}$$

$$X_3(t) = \dot{X}_2(t) = \ddot{Y}(t) = \frac{d^2y(t)}{dt^2}$$

Thus $\dot{X}_1(t) = X_2(t)$ ......(1)

$\dot{X}_2(t) = X_3(t)$ ......(2)

To obtain the $\dot{X}_3(t)$, substitute the state variables obtained in the differential equation.

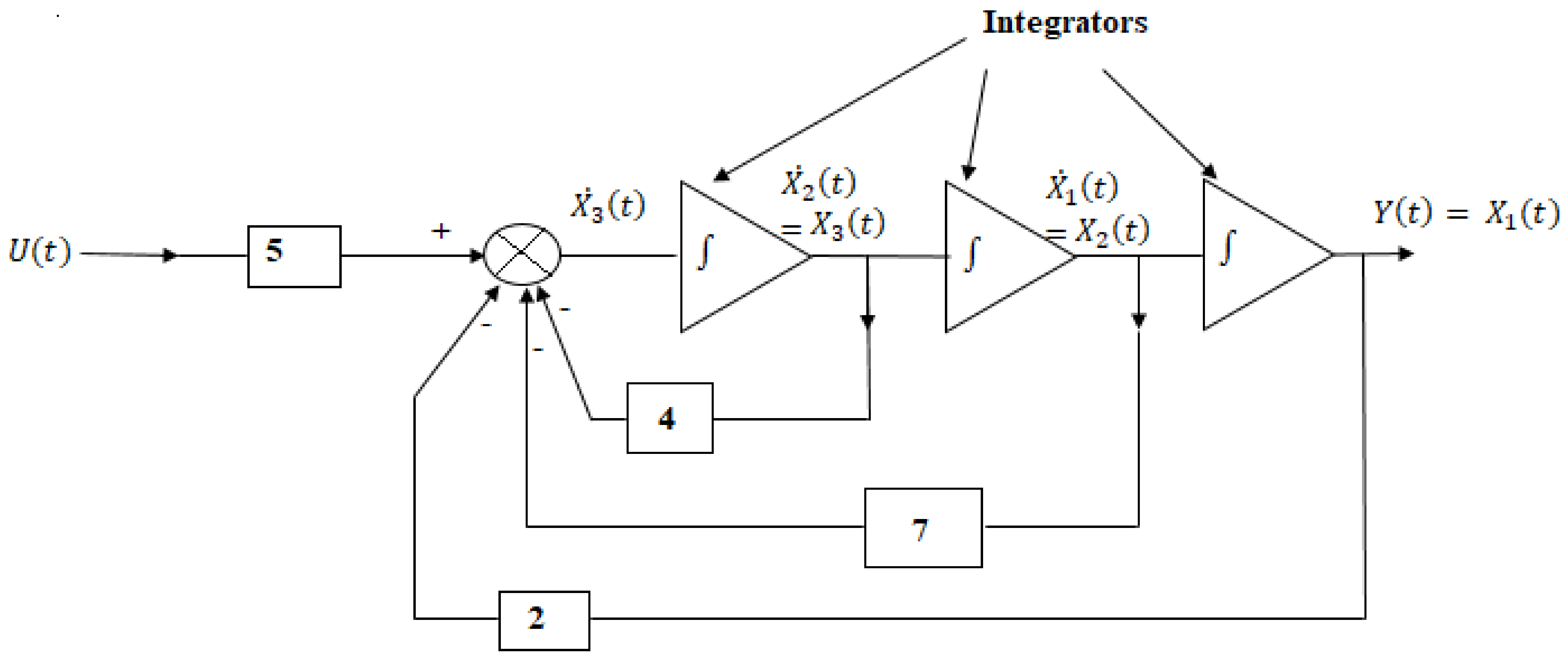
$$\frac{d^3Y(t)}{dt^3} = \dddot{Y}(t) = \frac{d}{dt}\left[\ddot{Y}(t)\right] = \frac{dX_3}{dt} = \dot{X}_3(t)$$

$$\therefore \quad \dot{X}_3(t) + 4X_3(t) + 7X_2(t) + 2X_1(t) = 5U(t)$$

$$\dot{x}_3(t) = -2X_1(t) - 7X_2(t) - 4X_3(t) + 5U(t) \qquad \dots (3)$$

- The equations (1),(2) and (3) gives us required state equation.
- $\dot{X} = A\,X(t) + B\,U(t)$
- Where
- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \quad and \qquad B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$
- The output is $Y(t) = X_1(t)$
- Therefore $\quad Y(t) = C\,X(t) + D\,U(t)$
- Where $\qquad C = [1 \quad 0 \quad 0], \quad D = 0$
- This is the required state model using phase variables.

The state diagram:



State model from transfer function:

- Consider a system characterized by the differential equation containing the derivatives of the input variable U(t) as,

- $\boldsymbol{Y^n + a_{n-1}Y^{n-1} + \cdots + a_1\,\dot{Y} + a_0\,Y(t) = b_0\,U + b_1\,\dot{U} + \cdots + b_{m-1}\,U^{m-1} + b_m\,U^m \qquad \ldots.(1)}$

- In such cases, it is advantageous to the transfer function, assuming zero initial conditions.
- Taking Laplace transform of both sides of equation (1) and neglecting initial conditions we get,

- $\boldsymbol{Y(s)\left[s^n + a_{n-1}\,s^{n-1} + \cdots + a_1\,s + a_0\right] = \left[\,b_0 + sb_1 + \cdots. + b_{m-1}s^{m-1} + b_m s^m\right]U(s)}$

- Therefore $\frac{Y(s)}{U(s)} = T(s) = \frac{b_0 + sb_1 + \cdots + b_{m-1}s^{m-1} + b_m s^m}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n}$ ....(2)
- Practically in most of the control systems m < n, but for general case, let us assume m = n.
- Therefore $T(s) = \frac{b_0 + sb_1 + \cdots + b_{m-1}s^{m-1} + b_m s^m}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n}$
- From such transfer function state model is obtained and then zero initial conditions can be replaced by the practical initial conditions to get required result.

There are *two methods obtaining state model from the transfer function*

1. Using signal flow graph approach

2.Using direct decomposition of transfer function.

1) Using signal flow graph approach

- The mason`s gain formula for signal flow graph states that,

$$T(s) = \frac{\sum T_K \Delta_K}{\Delta}$$

Where T_K = Gain of K^{th} forward path

Δ = system determinant.

$= 1 - \{\sum all\ loop\ gains\} +$

$\{\sum Gain * Gain\ product\ of\ all\ combinations\ a\ two\ non-touching\ loops\} - \cdots$

Δ_K = value of eliminating those loop gains and products which are touching to K^{th} forward paths.

- According to this formula construct the signal flow graph, state model can be obtained.
- While obtaining signal flow graph, try to get the gains of branches as 1/s representing the integrators.

$$T(s)=\frac{b_0+sb_1\ +\ s^2b_2+s^3b_3}{a_0+sa_1\ +\ s^2a_2+s^3}=\frac{Y(s)}{U(s)}$$

- Divide both numerator and denominator by highest power of s.

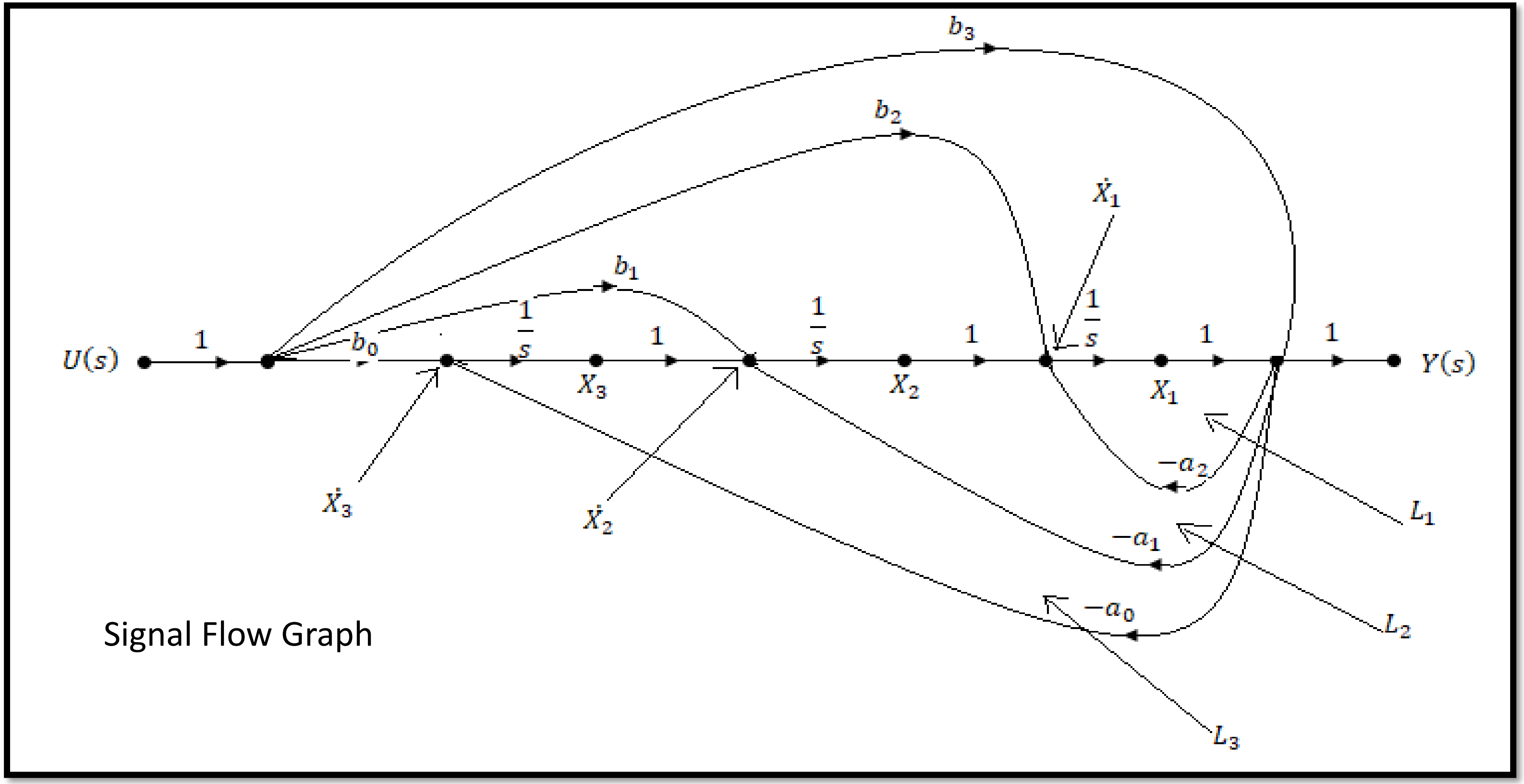
$$T(s)=\frac{\frac{b_0}{s^3}+\frac{b_1}{s^2}+\frac{b_2}{s}+b_3}{1+\frac{a_2}{s}+\frac{a_1}{s^2}+\frac{a_0}{s^3}}=\frac{b_3+\frac{b_2}{s}+\frac{b_1}{s^2}+\frac{b_0}{s^3}}{1-[-\frac{a_2}{s}-\frac{a_1}{s^2}-\frac{a_0}{s^3}]}$$

$$=\frac{T_1\triangle_1+T_2\triangle_1+T_3\triangle_3+T_4\triangle_4}{1-[\sum All\ loop\ gains]+[\sum Gain\ products\ of\ 2\ non-touching\ loops]\ldots\ldots}$$

- Assuming that there are no combination of 2 and more non touching loops.

- Therefore Loop gains
- $L_1 = -\frac{a_2}{s}, \quad L_2 = -\frac{a_1}{s^2}, \quad L_3 = -\frac{a_0}{s^3}$
- And let $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1,$
- i. e. all loops are touching to all forward paths .

- Hence forward path gains are,
- $T_1 = b_3, \quad T_2 = \frac{b_2}{s}, \quad T_3 = \frac{b_1}{s^2}, \quad T_4 = \frac{b_0}{s^3}$
- Involving various branches having gains 1/s, the signal flow graph can be obtained as,



Signal Flow Graph

- Each branch with gain 1/s represents an integrator. Output of each integrator is a state variable.
- According to signal flow graph value of the variable at the node is an algebraic sum of all the signals entering at the node.

Outgoing branches does not affect the value of variable. Hence from the signal flow graph,

- $\dot{X}_1 = \ b_2U + \ X_2 - \ a_2Y$
- $\dot{X}_2 = \ b_1U + \ X_3 - \ a_1Y$
- $\dot{X}_3 = \ b_0U \ - \ a_0Y$
- $Y \ = \ b_3U \ - \ X_1$

and Substituting Y in all equations,

- $\dot{X}_1 = -\,a_1\,X_1 + X_2 + [b_2 - \ a_2b_3]U(t) \qquad \ldots.(1a)$
- $\dot{X}_2 = -\,a_1\,X_1 + X_3 + [b_1 - \ a_1b_3]U(t) \qquad \ldots.(1b)$
- $\dot{X}_1 = -\,a_0\,X_1 + [b_0 - \ a_0b_3]U(t) \qquad \ldots.(1c)$

- These equations give the required state model. Therefore

- $\dot{X} = A\,X(t) + B\,U(t)$

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \quad and \qquad B = \begin{bmatrix} b_2 - a_2 b_3 \\ b_1 - a_1 b_3 \\ b_0 - a_0 b_3 \end{bmatrix}$$

- $Y(t) = C\,X(t) + D\,U(t)$

$$C = [1 \quad 0 \quad 0], \qquad D = b_3$$

- Now as Y(0), $\dot{Y}(0)\ddot{Y}(0)$,U(0), $\dot{U}(0)\ and\ \ddot{U}(0)$ are known initial conditions, using the derived state model, the initial conditions $X_1(0), X_2(0)$ and $X_3(0)$ can be obtained.

Q4. A feedback system is characterized by the closed loop transfer function,

$$T(s) = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$$

draw a suitable signal flow graph and obtain the state model.

Solution: Divide numerator and denominator by s^3

$$T(s) = \frac{\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3}}{1 + \frac{2}{s} + \frac{3}{s^2} + \frac{1}{s^3}} = \frac{\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3}}{1 - \left[-\frac{2}{s} - \frac{3}{s^2} - \frac{1}{s^3}\right]}$$

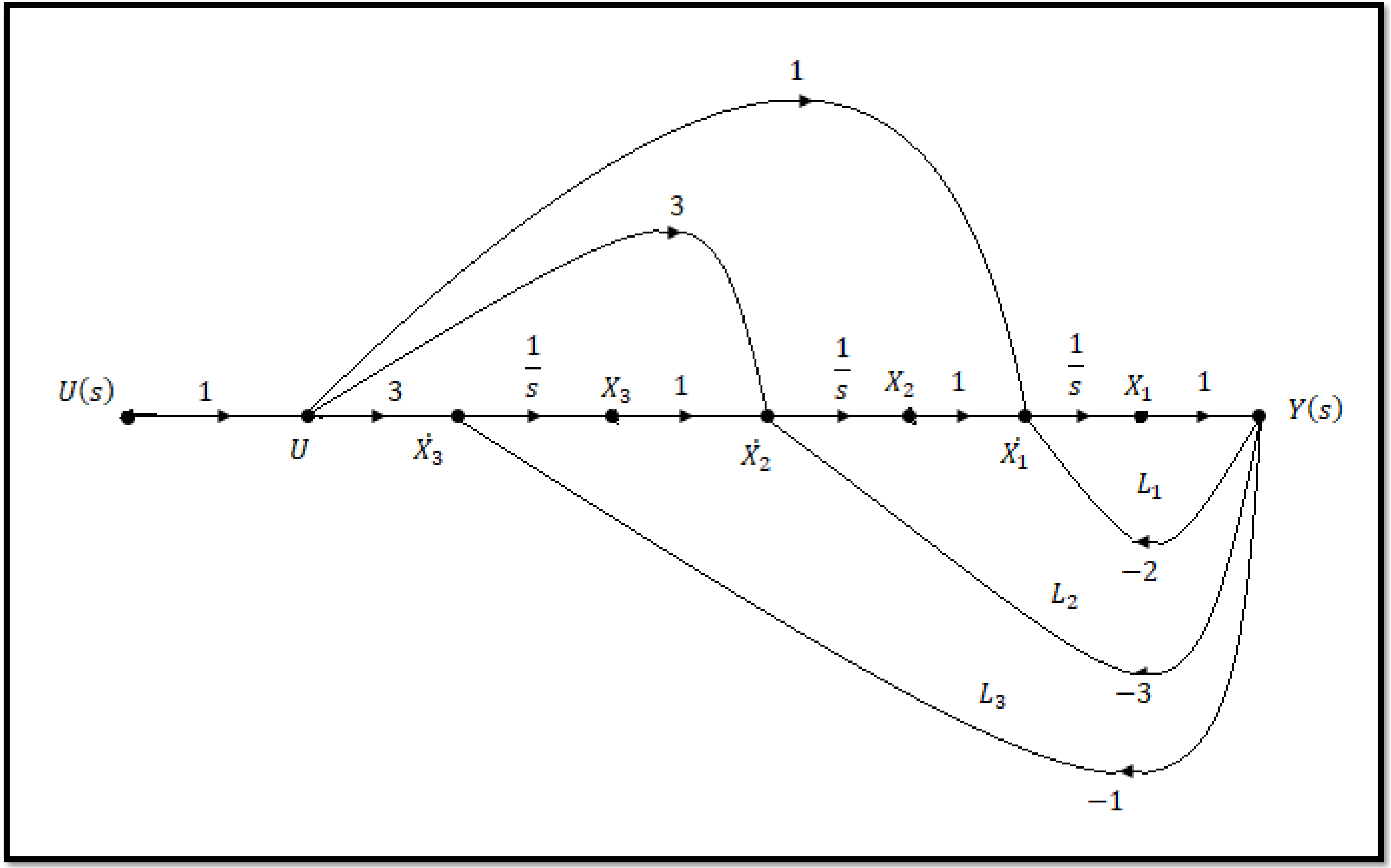
- Therefore

- $L_1 = -\frac{2}{s}, \quad L_2 = -\frac{3}{s^2}, \quad L_3 = -\frac{1}{s^3}$
- $T_1 = \frac{1}{s}, \quad T_2 = \frac{3}{s^2}, \quad T_3 = \frac{3}{s^3}$

- And $\Delta_1 = \Delta_2 = \Delta_3 = 1$, with no combinations of non-touching loops.

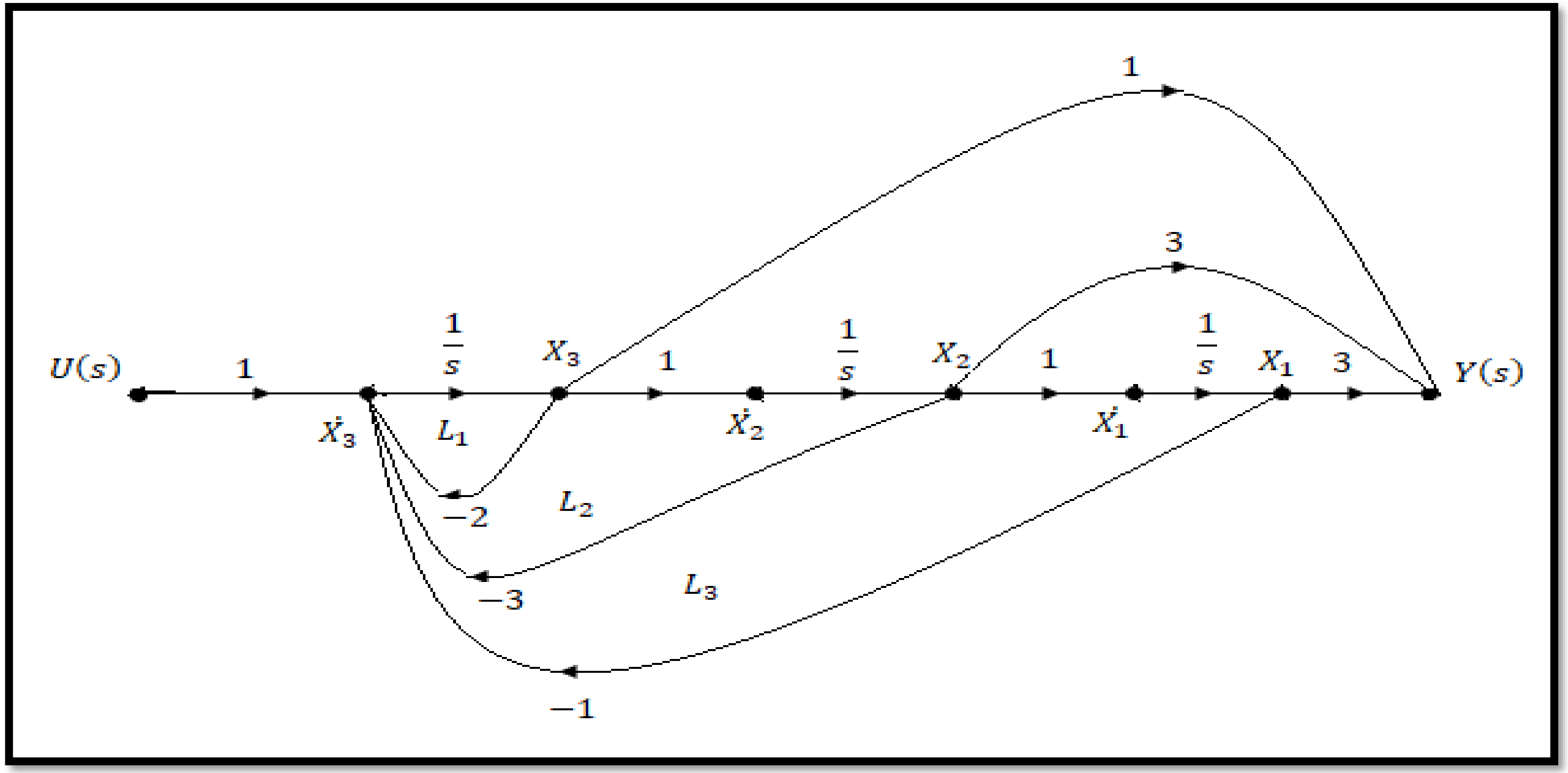
- There are many signal flow graphs which can be obtained to satisfy the transfer function.

- Method 1: the signal flow graph is as shown in figure below



- From signal flow graph,
- $Y = X_1$
- $\dot{X}_1 = X_2 + U - 2Y = -2X_1 + X_2 + U$
- $\dot{X}_2 = X_3 + 3U - 3Y = -3X_1 + X_3 + 3U$
- $\dot{X}_3 = 3U - Y = -X_1 + U$
- Hence the state model is
- $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} U(t)$
- $\dot{X} = A\,X(t) + B\,U(t)$

- And output $Y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$
- i.e. $Y(t) = X(t) \quad with\, D = 0$
- **Method 2**: when m ≠ n and m < n then signal flow graph can be constructed so as to obtain matrix A in phase variable form. This is shown in the figure.

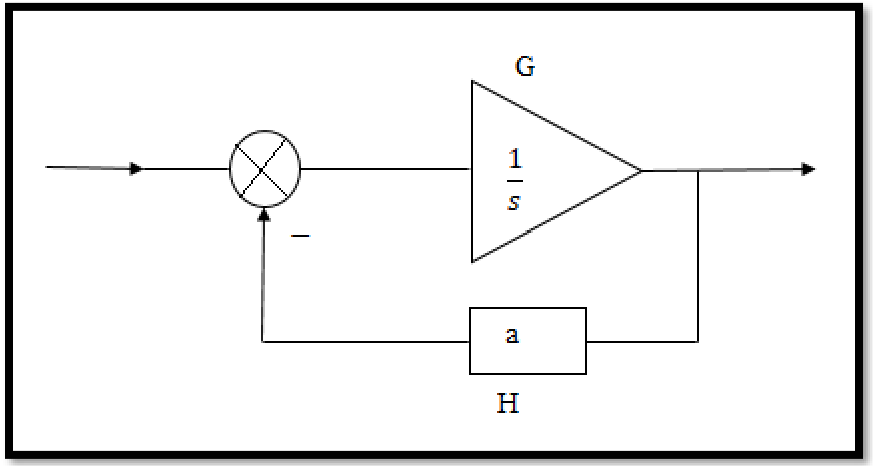


- From the signal flow graph,
- $\dot{X}_1 = X_2$
- $\dot{X}_2 = X_3$
- $\dot{X}_3 = -X_1 - 3X_2 - 2X_3 + U$
- $Y = 3X_1 + 3X_2 + X_3$
- Hence the state model has,
- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$
- $D = 0$
- The matrix A is called phase variable form.

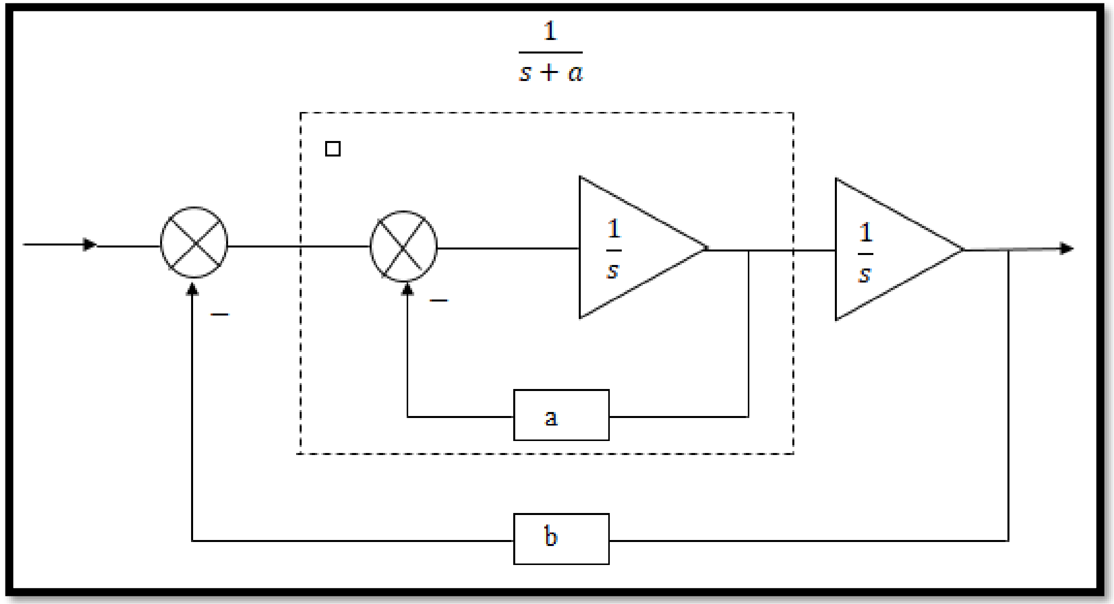
2) Using direct decomposition of transfer function

- This is also called direct programming. In this method, **denominator of transfer function is rearranged** in a specific form.
- To understand the rearrangement, consider an element with transfer function $\frac{1}{s+a}$. From block diagram algebra, the transfer function of minor feedback loop is $\frac{G}{1+GH}$ for negative feedback.
- Let
- $\frac{1}{s+a} = \frac{\frac{1}{s}}{1+\frac{a}{s}} = \frac{G}{1+GH}$
- Where $G = \frac{1}{s} = integrator \ \ and \ \ H = a$

- The feed back is negative and the transfer function can be simulated as shown in figure, with a minor feedback loop.



- Now such loop is added in the forward path of another such loop then we get the block diagram as shown in the below figure.



- The transfer function now becomes

$$= \frac{\frac{1}{s(s+a)}}{1+\frac{b}{s(s+a)}} = \frac{1}{s(s+a)+b} = \frac{1}{s\,X+\,b}$$

Where $X = (s + a)$

- If now the entire block is shown above figure is added in the forward path of another minor loop with an integrator and feedback gain `c', we get the transfer function as,

$$= \frac{1}{sY+c} \quad \text{where } Y = sX + b$$

- Thus the denominator of transfer function becomes

$$s(sX + b) = [s(s(s + a) + b)] = s^3 + as^2 + \text{bs}$$

- Thus denominator of any order can be directly programmed.

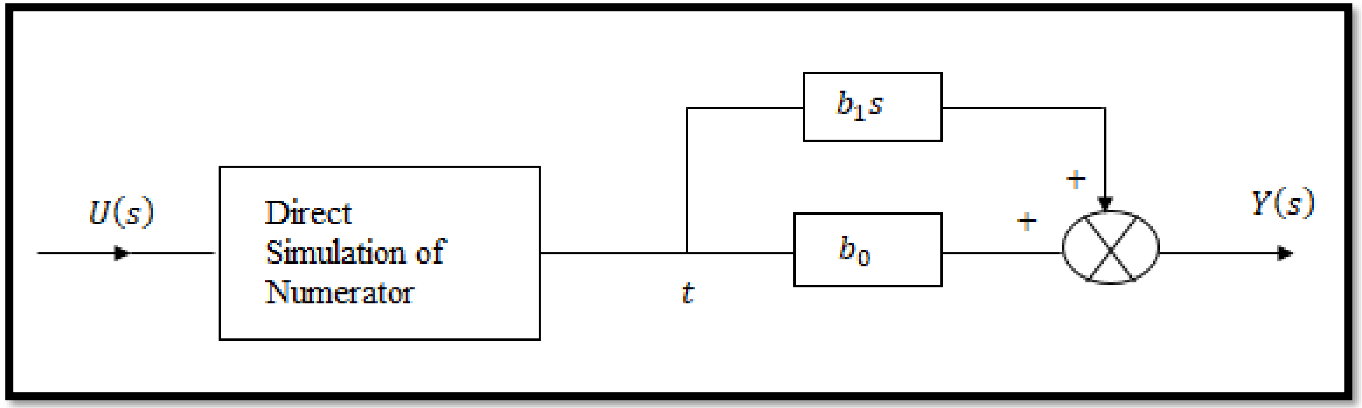
$$s^2 + as + b \implies \{ s (s + a) + b\}$$

$$s^3 + as^2 + bs + c \implies \{[(s + a)s + b]s + c\}$$

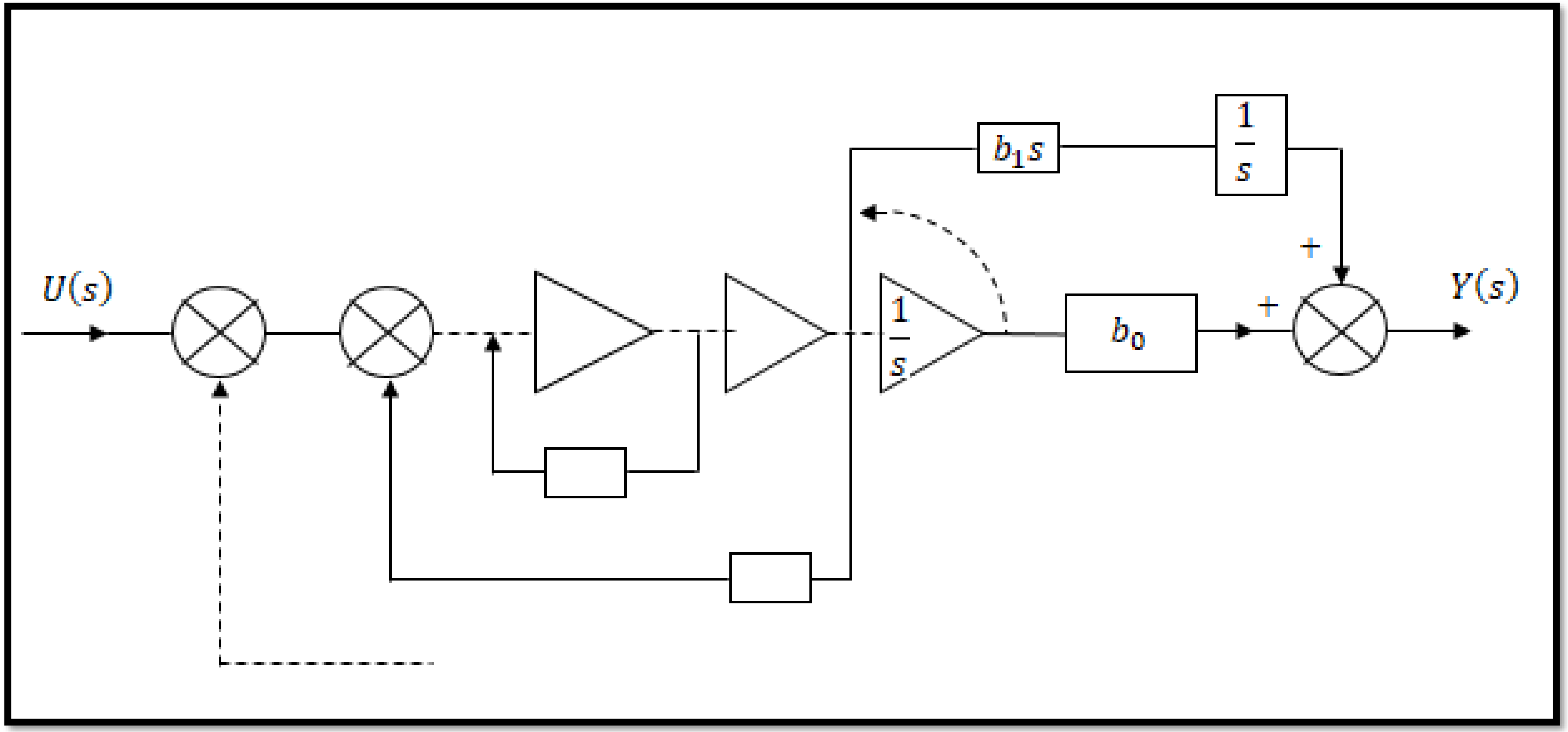
$$s^4 + as^3 + bs^2 + cs + d \implies \{([(s + a)s + b]s + c)s + d\}$$

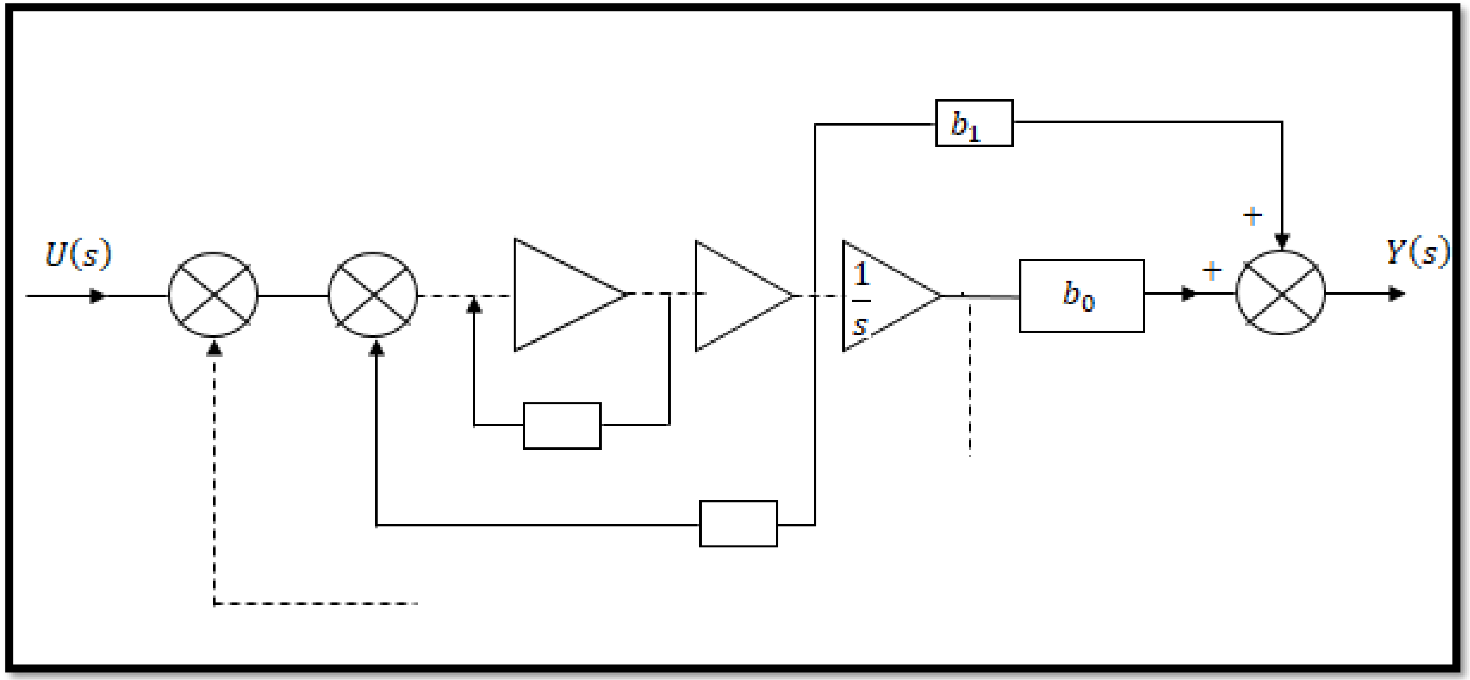
And so on.

- Now if numerator is $b_1 s + b_0$ and denominator. Simulation is obtained directly then, the block diagram is as shown in figure.

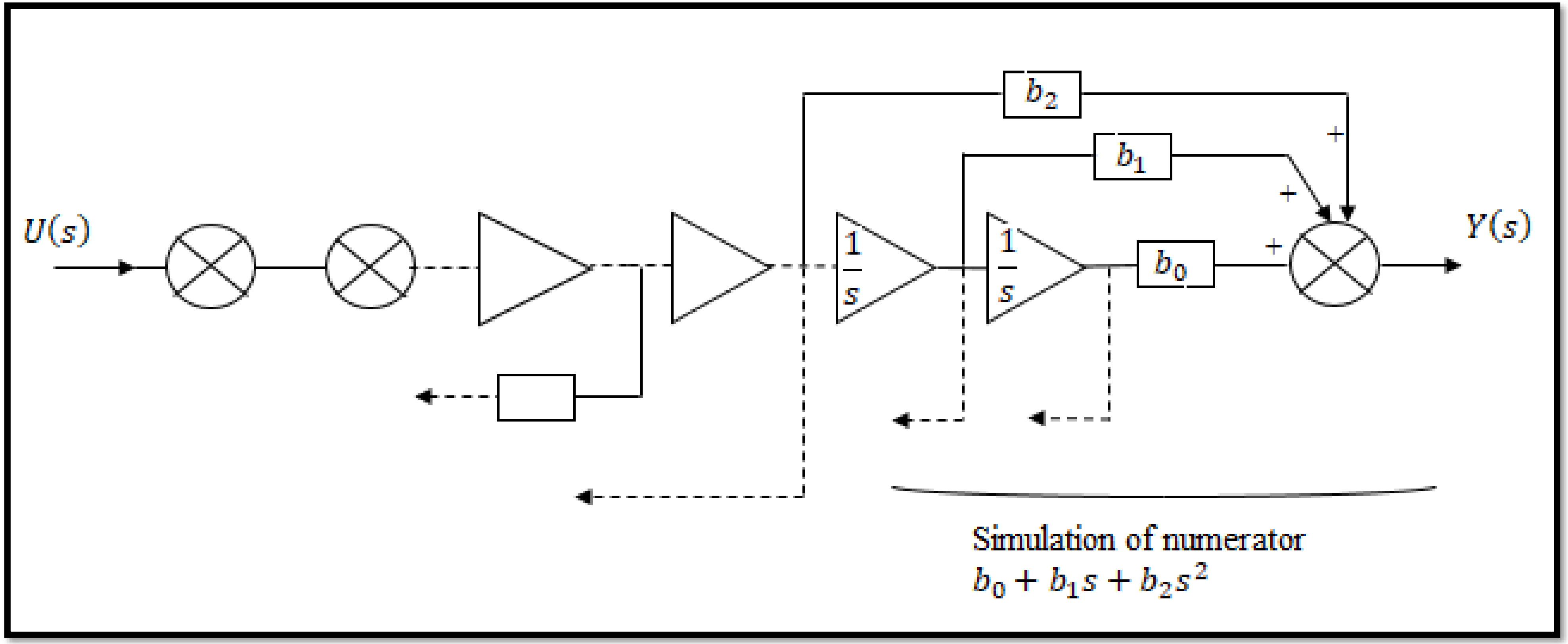


- But s = $\frac{d}{dt}$, which is differentiator and is not used to obtain state model.
- In such a case, take off point `t’ is shifted before the last integrator block.





- According to block diagram reduction rule, while shifting take off point before the block, the take off signal must be multiplied by transfer function of block before which it is to be shifted.
- Thus we get block of b_1 with take off from input of last integrator.
- Similarly if there is term $b_2 s^2$ in the numerator then shift take off point before one more integrator as shown in below figure.



- Thus of any order of numerator, complete simulation of the transfer function can be achieved.
- Then assigning output of each integrator as the state variable, state model in the phase variable form can be obtained.

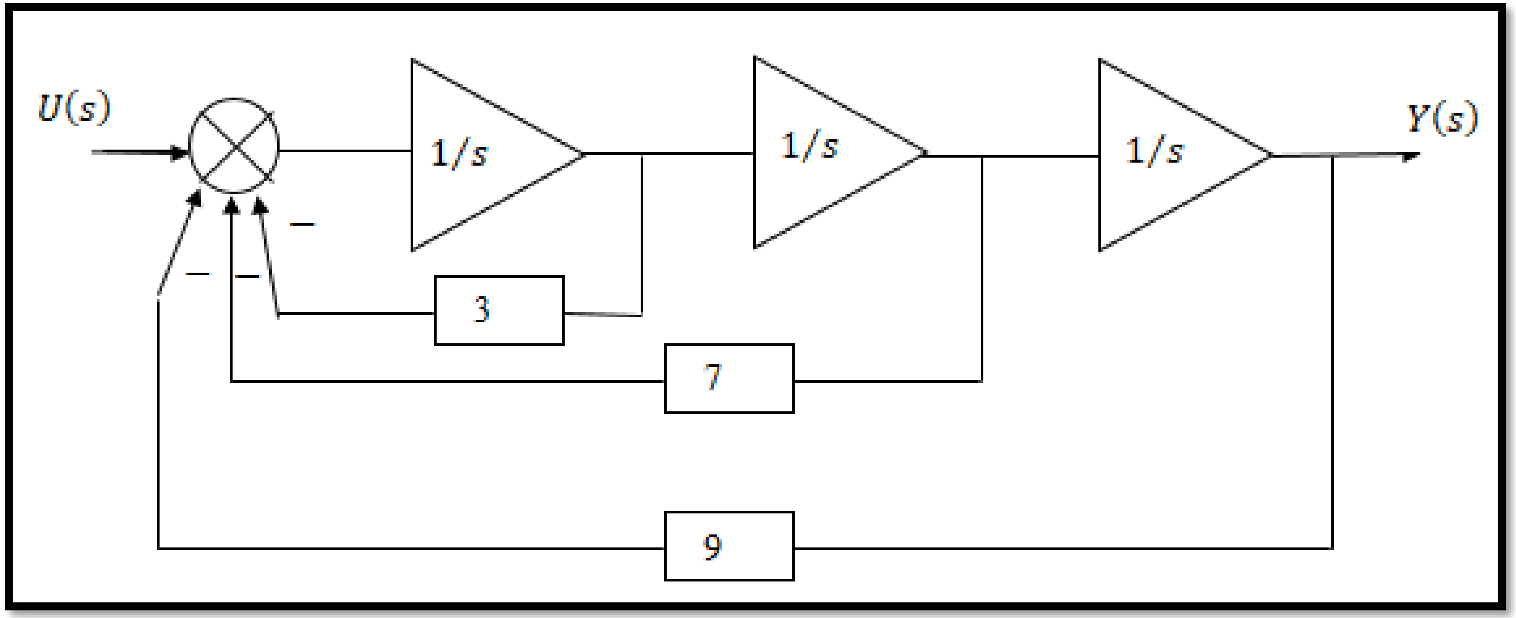
Q5. Obtain the state model by direct decomposition method of a system whose transfer function is

$$\frac{Y(S)}{U(s)} = \frac{5s^2 + 6s + 8}{s^3 + 3s^2 + 7s + 9}$$

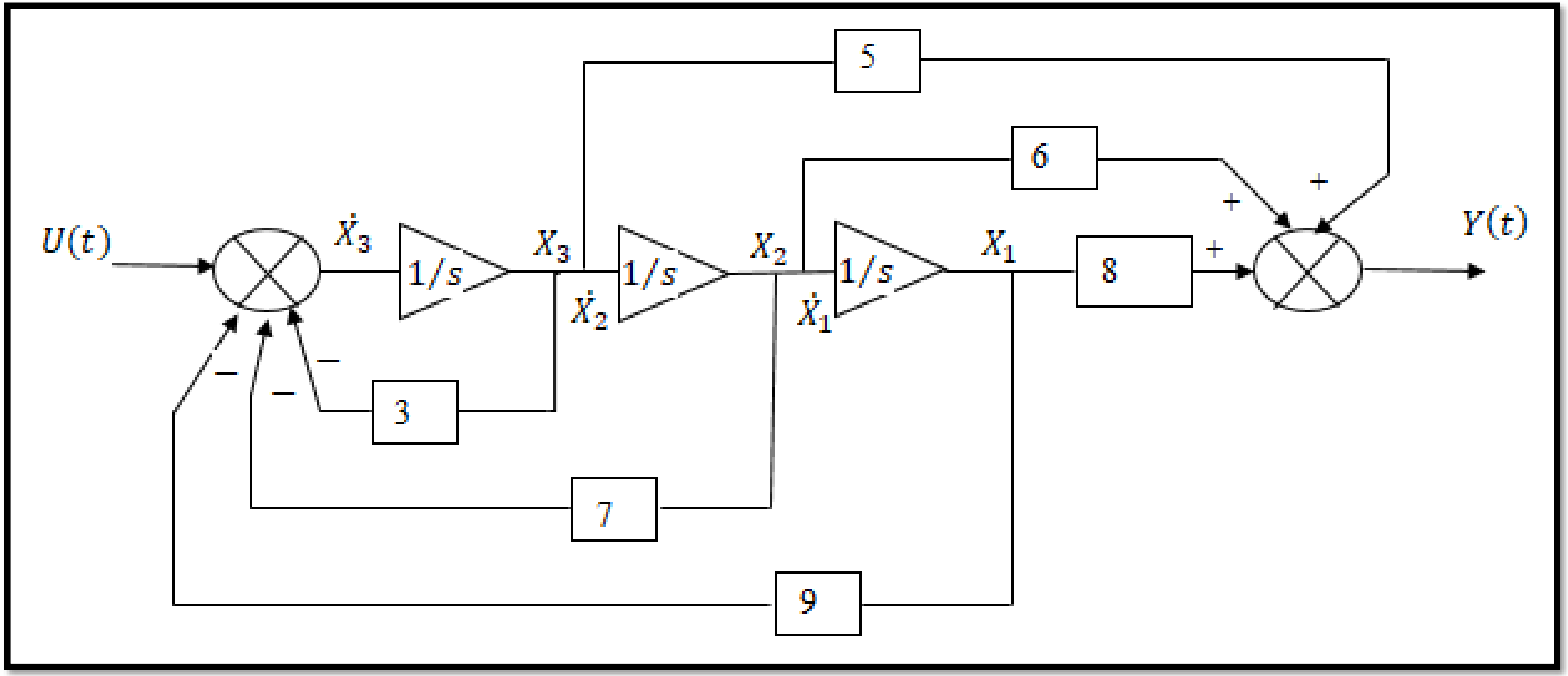
Solution: Decomposer denominator as below,

$$s^3 + 3s^2 + 7s + 9 = \{([s + 3]s + 7)s + 9\}$$

Its simulation starts from (s+3) in denominator.



- To simulate numerator, shift take-off point once for 6s and shift twice for $5s^2$.
- Therefore complete state diagram can be obtained as follows.



- As output of each integrator as the state variable. $\dot{X}_1 \doteq X_2$ $\dot{X}_2 \doteq X_3$

- $\dot{X}_3 = U(t) - 9X_1(t) - 7X_2(t) - 3X_3(t)$
- $while\ output,$
- $Y(t) = 8X_1(t) + 6X_2(t) + 5X_3(t)$

$\therefore \quad State\ model\ is,$

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$
$$Y(t) = C\,X(t) + D\,U(t)$$

- $where$
- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -7 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [8 \quad 6 \quad 5], \quad D = [0]$
- The matrix A is obtained is in Bush form or Phase variable form

Advantages:

The various advantages of phase variable i.e. direct programming method are,

1. Easy to implement.
2. The phase variable need not be physical variables hence mathematically powerful to obtain state model.
3. It is easy to establish the link between the transfer function design and domain design using phase variables.
4. In many simple cases, just by inspection, the matrices A,B,C and D can be obtained.

Limitations:

1. The phase variables are not the physical variables hence they loose the practical significance. They have mathematical importance.
2. The phase variables are mathematical variables hence not available for the measurement point of view.
3. Also these variables are not available from control point of view.
4. The phase variables are the output and its derivatives, if derivatives of input are absent. But it is difficult to obtain second and higher derivatives of output.
5. The phase variable form, though special, does not offer any advantage from the mathematical analysis point of view.

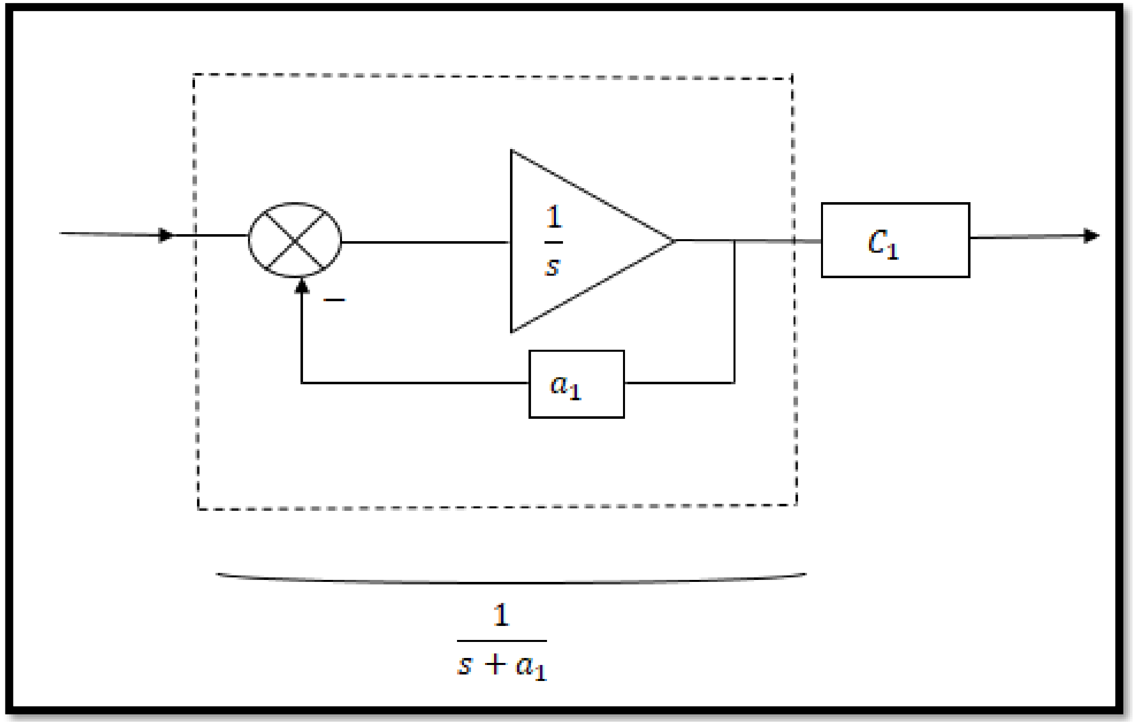
Due to all these disadvantages, canonical variables are very popularly used to obtain the state model.

State space representation using canonical variables

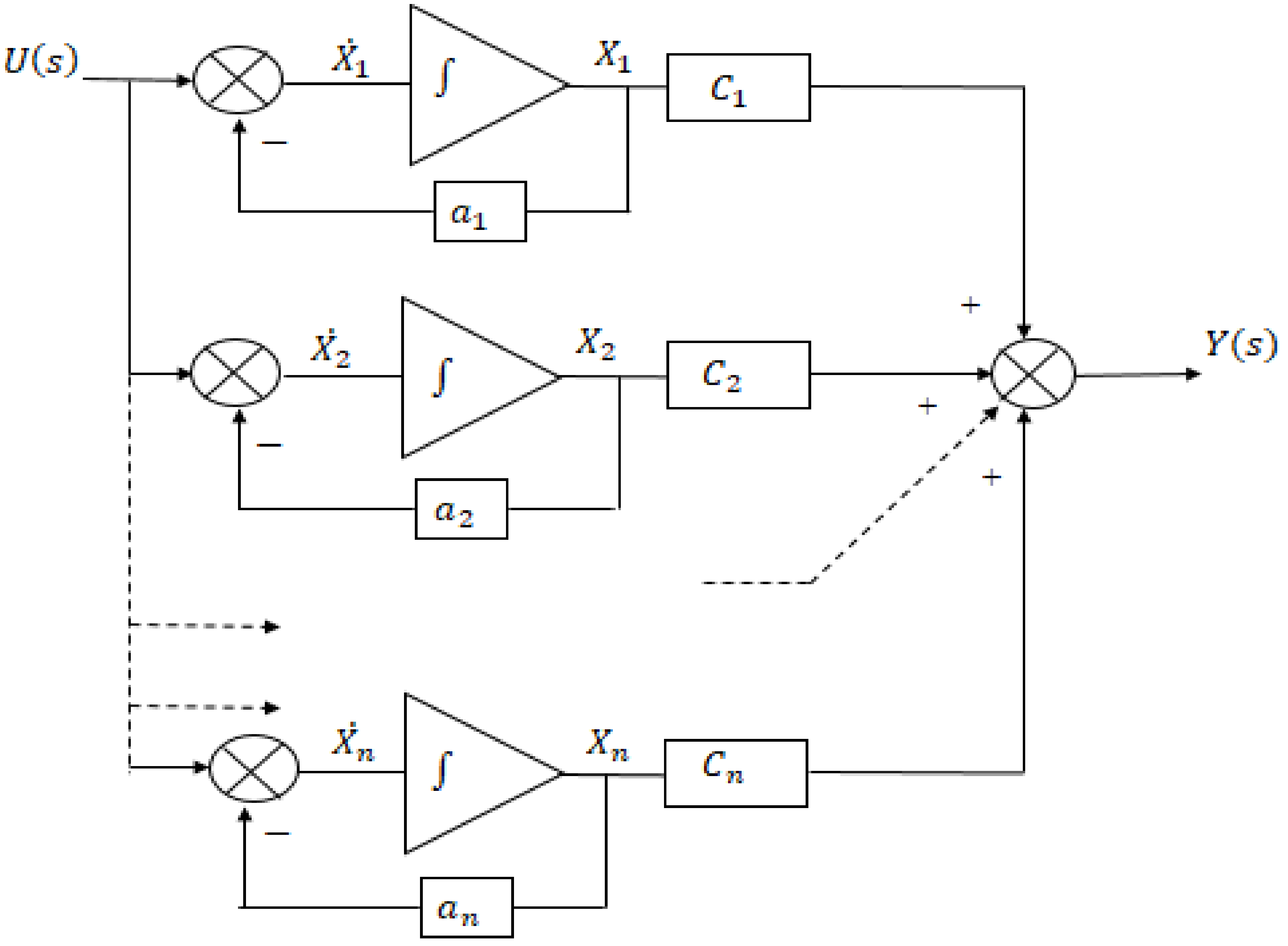
- This method of obtaining the state model using the canonical variables is also called parallel programming method and matrix A obtained using this method is said to have canonical form, normal form or Foster`s form.
- The matrix A in such a case is a diagonal matrix and plays an important role in the state space analysis.
- This method is basically based on partial fraction expansion of the given transfer function T(s).
- Consider the transfer function T(s) as,
- $T(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_m}{(s+a_1)(s+a_2)\ldots\ldots(s+a_n)}$

Case 1: if the degree `m' is less than `n' (m<n)

- Then T(s) can be expressed using partial fraction expansion as,
- $T(s) = \frac{C_1}{s+a_1} + \frac{C_2}{s+a_2} + \cdots . + \frac{C_n}{s+a_n} = \sum_{i=1}^{n} \frac{C_i}{s+a_i}$
- Now each group $\frac{c1}{s+a1}$ can be simulated using the minor loop in state diagram as shown in the figure.



- The outputs of all such groups are to be added to obtain the resultant output.
- To add the outputs, all the groups must be connected in parallel with each other.
- The input U(s) to all of them is same. Hence the method is called parallel programming.
- The overall state diagram is shown in the figure.



Foster's form simulation

- The assign output of each integrator as a state variable and write the state equation as,

$$\dot{X}_1 = -a_1 X_1 + U$$

$$\dot{X}_2 = -a_2 X_2 + U$$

$$\vdots$$

$$\dot{X}_n = -a_n X_n + U$$

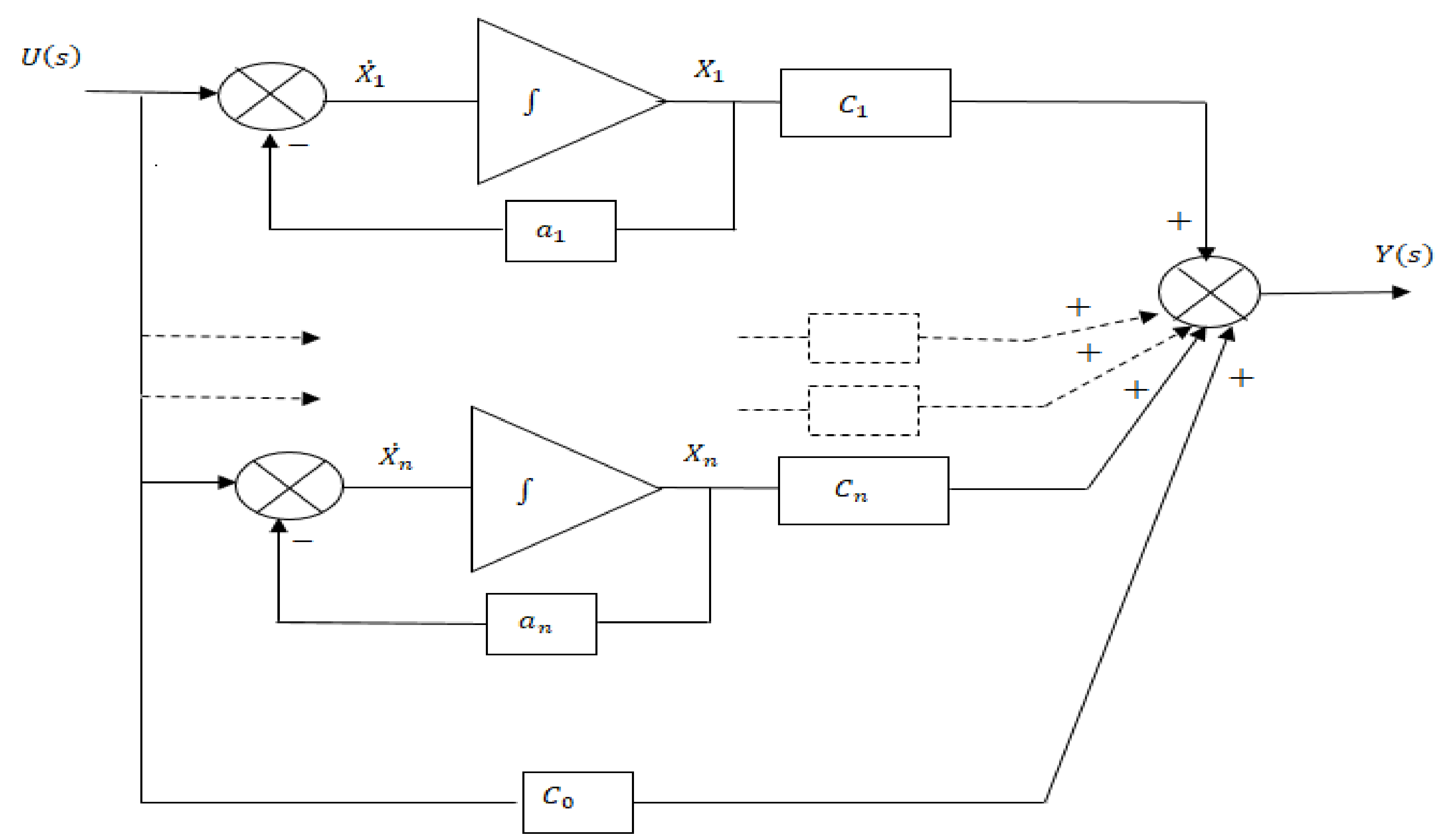
- When $Y = C_1 X_1 + C_2 X_2 + \ldots + C_n X_n$
- Therefore $\dot{X} = AX + BU \quad and \quad Y = CX + DU$

- Where
- $A = \begin{bmatrix} -a_1 & 0 & 0 & \dots. & 0 \\ 0 & -a_2 & 0 & \dots. & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots. & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
- $C = [C_1 \quad C_2 \quad \dots. \quad C_n] \quad and \quad D = 0$

Case 2: if the degree m = n

- i.e. numerator and denominator have same degree then first divide the numerator by denominator and then obtain partial fractions of remaining factor.
- Therefore $T(s) = \frac{N(s)}{D(s)} = C_0 + \sum_{i=1}^{n} \frac{C_i}{s+a_i}$

- Where c_0= Constant obtain by dividing N(s) by D(s).
- In such a case, the state diagram for partial fraction, terms remains same as before and in addition to all inputs, c_0 U(t) gets added to obtain the resultant output as shown in figure.



State diagram for T(s) with m=n

- Thus the state model consists of the matrices as,
- $A = \begin{bmatrix} -a_1 & 0 & 0 & \dots. & 0 \\ 0 & -a_2 & 0 & \dots. & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots. & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
- $C = [C_1 \quad C_2 \quad \dots. \quad C_n] \quad and \quad D = C_0$

- **Thus for m = n, direct transmission matrix D exists in the state model.**

When the denominator D(s) of the transfer function T(s) has non-repeated roots then the matrix A obtained by parallel programming has the following features,

1. Matrix A is diagonal i.e. in canonical form or normal form.
2. The diagonal consists of the elements which are the gains of all the feedback paths associated with the integrators.
3. The diagonal elements are the poles of the transfer function T(s).

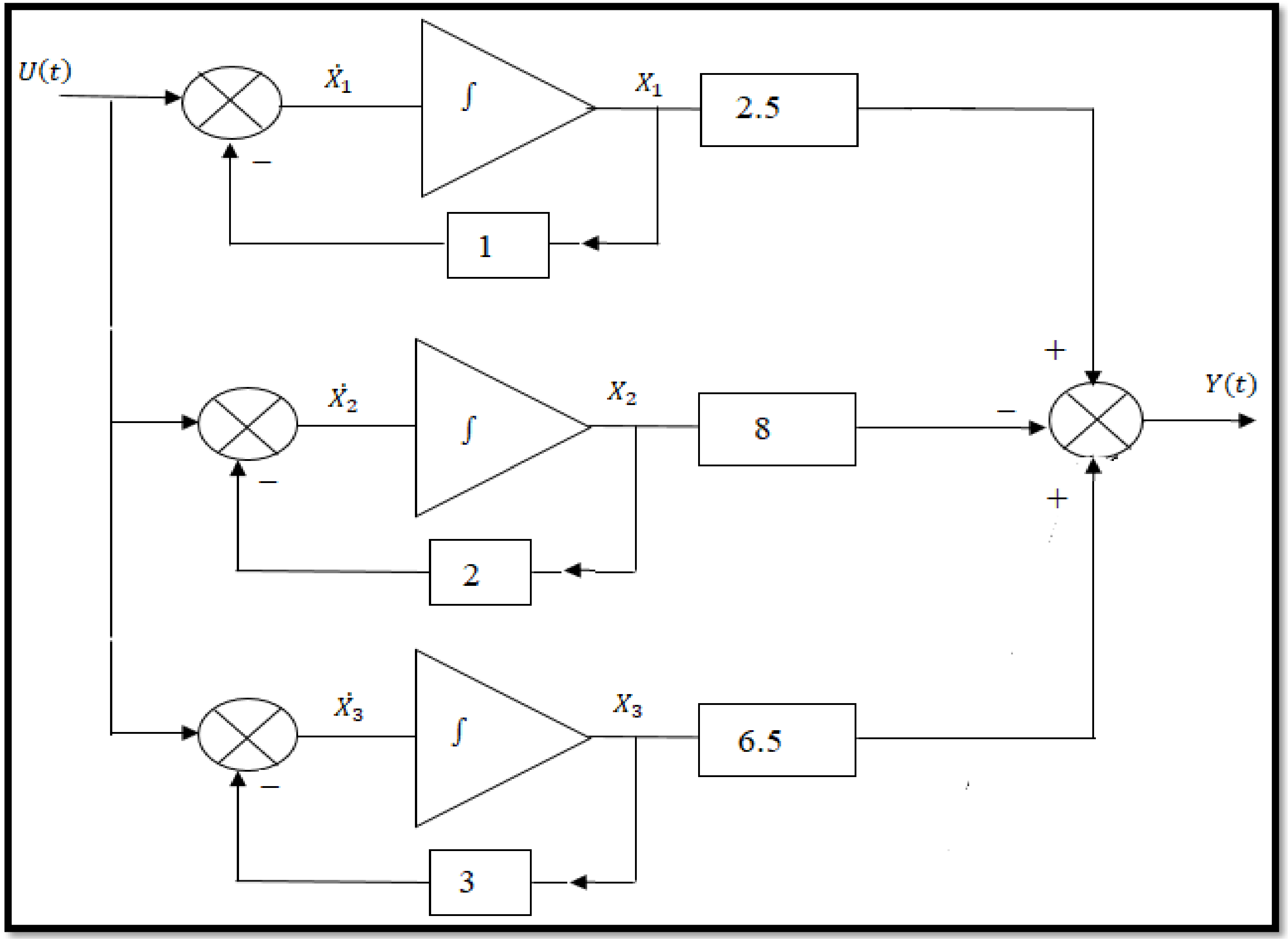
Q6. Obtain the state model by foster`s form of a system whose T.F is

$$\frac{s^2 + 4}{(s+1)(s+2)(s+3)}$$

- **Solution:**
- Find out partial fraction expansion of it,

$$\frac{s^2 + 4}{(s+1)(s+2)(s+3)} = \frac{2.5}{s+1} - \frac{8}{s+2} + \frac{6.5}{s+3}$$

- Therefore total state diagram as shown in below figure,



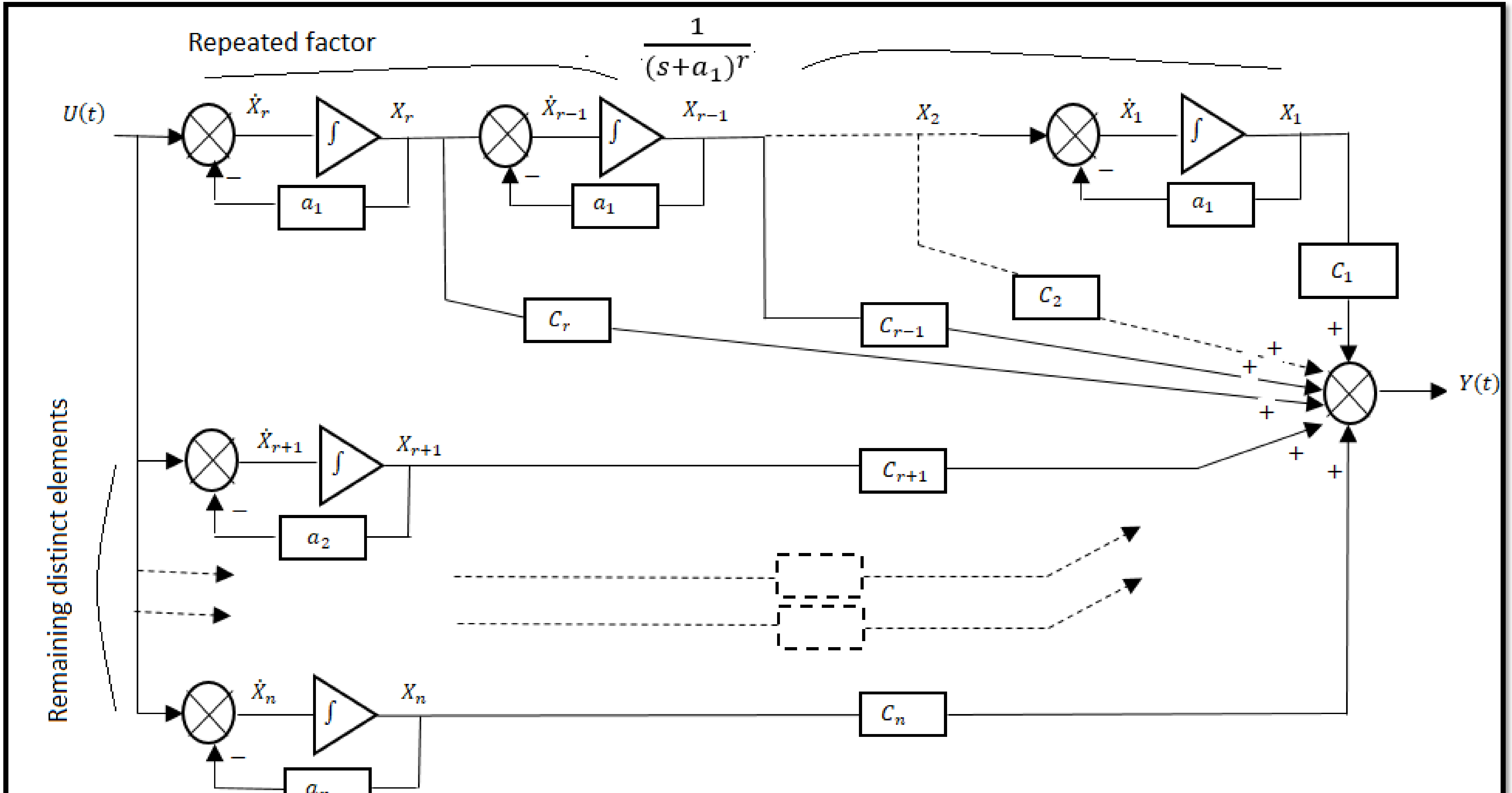
- $\dot{X}_1 = U(t) - X_1(t),$
- $\dot{X}_2 = U(t) - 2X_2(t),$
- $\dot{X}_3 = U(t) - 3X_3(t)$
- $Y(t) = 2.5X_1(t) - 8X_2(t) + 6.5X_3(t)$

- Therefore, the state model
- $\dot{X} = AX + BU$
- $Y = CX + DU$
- $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- $C = \begin{bmatrix} 2.5 & -8 & 6.5 \end{bmatrix} \qquad D = 0$

Jordan`s Canonical Form

- Let T(s) has pole at $s = -a_1$ which is repeated for r times as,
- $T(s) = \frac{N(s)}{(s+a_1)^r(s+a_2)\ldots(s+a_n)}$
- The method of obtaining partial fraction for such a case is,
- $T(s) = \frac{C_1}{(s+a_1)^r} + \frac{C_2}{(s+a_1)^{r-1}} + \cdots . + \frac{C_r}{(s+a_1)} + \frac{C_{r+1}}{(s+a_2)} + \cdots . + \frac{C_n}{(s+a_n)} \quad \ldots . m < n$
- If the degree of N(s) and D(s) is same i.e. m = n we get additional constant c_0 as,
- $T(s) = C_0 + \frac{C_1}{(s+a_1)^r} + \cdots . + \frac{C_r}{(s+a_1)} + \frac{C_{r+1}}{(s+a_2)} + \cdots . + \frac{C_n}{(s+a_n)} \quad \ldots . m = n$

- This can be mathematically expressed as,
- $T(s) = \sum_{i=1}^{r} \frac{C_i}{(s+a_1)^{r-i+1}} + \sum_{i=r+1}^{n} \frac{C_i}{(s+a_i)} \quad \ldots \; m < n$
- And $\quad T(s) = C_0 + \sum_{i=1}^{r} \frac{C_i}{(s+a_1)^{r-i+1}} + \sum_{i=r+1}^{n} \frac{C_i}{(s+a_i)} \quad \ldots \; m = n$
- Key point: Note that in partial fraction expression, a separate coefficient is assumed for each power of repeated factor.
- In simulating such an equation by parallel programming, $\frac{1}{(s+a_1)^r}$ is simulated by connecting $\frac{1}{(s+a_1)}$ groups, r times in series first.
- While all other distinct factors are simulated by parallel programming as before. The components of each power of $\frac{1}{(s+a_1)}$ to be added to get output is to be taken from output of each integrator which are connected in series. This is shown in the Fig.



State diagram for Jordon's form for m < n

- For series integrator, the state equations are,

$$\dot{X}_1 = -a_1 X_1 + X_2$$
$$\dot{X}_2 = -a_1 X_2 + X_3$$
$$\vdots$$
$$\dot{X}_{r-1} = -a_1 X_{r-1} + X_r$$
$$\dot{X}_r = -a_1 X_r + U(t)$$

- While for parallel integrators, the state equations are,

$$\dot{X}_{r+1} = -a_2 X_{r+1} + U(t)$$
$$\vdots$$
$$\dot{X}_n = -a_n X_n + U(t)$$

$$Y(t) = C_1 X_1 + C_2 X_2 + \cdots + C_r X_r + +C_{r+1} X_{r+1} + + \cdots + C_n X_n$$

- Key Point: Y(t) has additional c_0 U(t) term if m = n.
- Hence the state model has matrices in the form,

$-a_1$ for r times

Element
1 for r-1 times

A =
Jordon
block

$$
A = \left[\begin{array}{cccccc|cccc}
-a_1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\
0 & -a_1 & 1 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 \\
0 & 0 & -a_1 & \cdots & 1 & \vdots & 0 & \cdots\cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -a_1 & 1 & 0 & \cdots\cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -a_1 & 0 & \cdots\cdots & 0 & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & a_n
\end{array}\right]
$$

only diagonal
elements

$$B = \left[\begin{array}{c}0\\0\\\vdots\\0\\1\\1\\1\\1\end{array}\right]\begin{array}{l}\left.\vphantom{\begin{array}{c}0\\0\\\vdots\\0\end{array}}\right\}\text{r-1 times}\\\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\end{array} \qquad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_r & C_{r+1} & \cdots & C_n \end{bmatrix}$$

- The matrix D is zero if m < n is and is \`c_0' if m = n.
- The matrix A in such a case has Jordan block for repeated factor and many times A is denoted as J.
- Therefore $\dot{X}(t) = JX(t) + BU(t)$ ....for repeated roots.
- Note that matrix B has \` r-1 ' zeros and all other elements as unity.
- The matrix C has all the partial fraction coefficients $c_1 c_2 \ldots.. c_n$.

Advantages of canonical variables:

- The matrix A is diagonal.
- The diagonal element is very important in the mathematical analysis.
- Due to diagonal feature, the decoupling between the state variable is possible. This means all the n differential equations are independent of each other.

Disadvantages of canonical variables:

- These are not the physical variables hence practically difficult to measure and control.
- Hence such variables are not practically advantageous, though mathematically are very important.

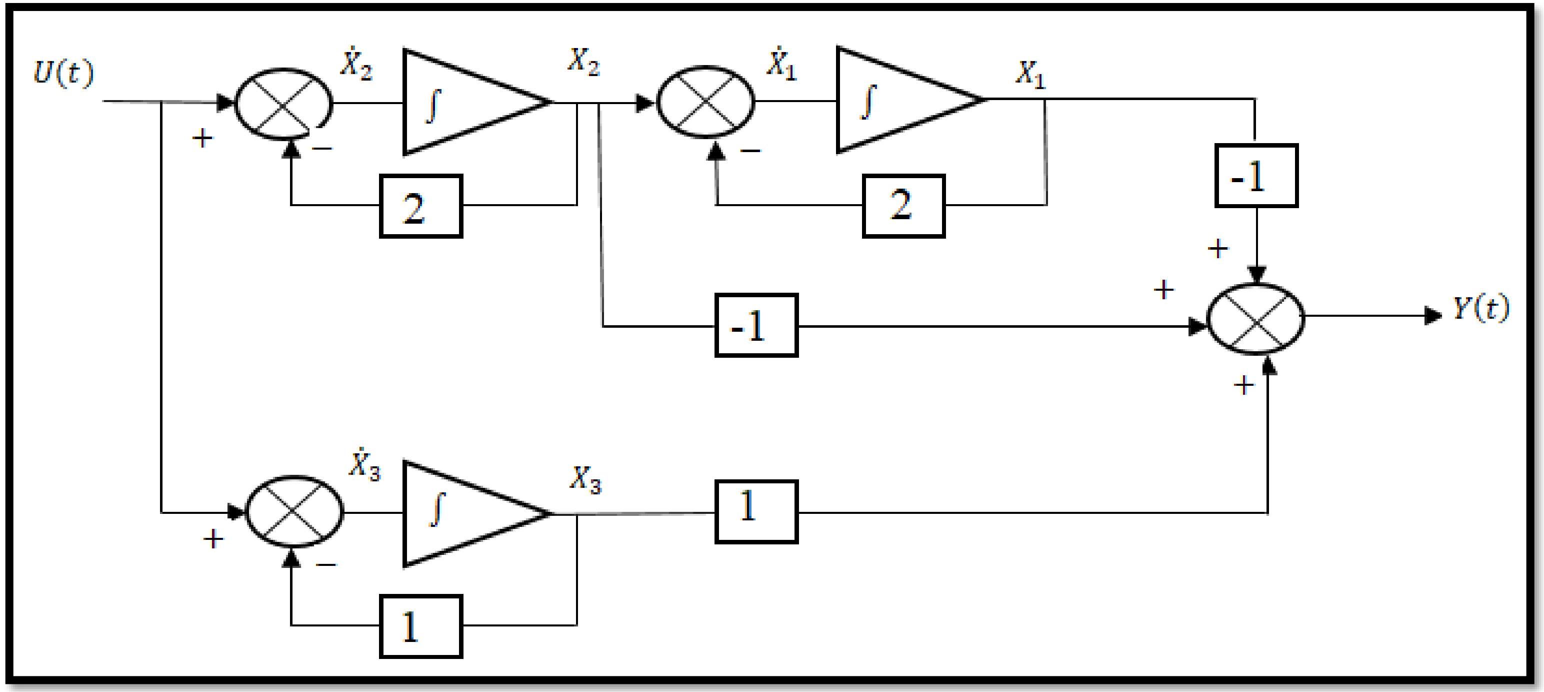
- Q7. Obtain the state model in Jordon`s form of a system using a system whose T.F of

$$\frac{1}{(s+2)^2(s+1)}$$

- Solution: Finding partial fraction expansion,
- $T(s) = \frac{A}{(s+2)^2} + \frac{B}{(s+2)} + \frac{C}{(s+1)}$
- Take LCM on right hand side and equate numerator with numerator of T(s),
- $\therefore \quad A(s+1) + B(s+2)(s+1) + C(s+2)^2 = 1$

- Equate coefficients of all powers of s on both sides,
- $\therefore \quad B + C = 0, \; \ldots .from\ power\ of\ s^2$
- $A + 3B + 4C = 0, \; \ldots .from\ power\ of\ s$
- $A + 2B + 4C = 0, \; \ldots .from\ constant\ term$
- Solving we get, $A = -1, B = -1, C = 1$
- $T(s) = \frac{-1}{(s+2)^2} - \frac{1}{(s+2)} + \frac{1}{(s+1)}$

- Simulate the first term by series integrators while other non repeated terms by parallel integrator.



- Total simulation is,
- $\dot{X}_1 = -2X_1 + X_2 \qquad \dot{X}_2 = U(t) - 2X_2 \qquad \dot{X}_3 = U(t) - X_3(t)$
- $Y(t) = -X_1(t) - X_2(t) + X_3(t)$

- $\therefore State\ model\ is, \dot{X} = AX + BU \quad and \quad Y = CX + DU$
- $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- $C = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}, \quad D = 0$
- The matrix A consists of Jordan block.
- **Important note:**
- If some of the poles of T(s) are complex in nature, then a mixed approach can be used.
- The quadratic or higher order polynomial having complex roots can be simulated by direct decomposition, while real distinct roots can be simulated by the parallel programming using canonical variable, the example below explains this procedure.

- Q8. Combination of direct decomposition and Foster`s form. Obtain the state model for the given T.F

$$T(s) = \frac{2}{(s+3)(s^2+2s+2)}$$

- Solution: Obtain partial fractions as,

$$T(s) = \frac{2}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2}$$

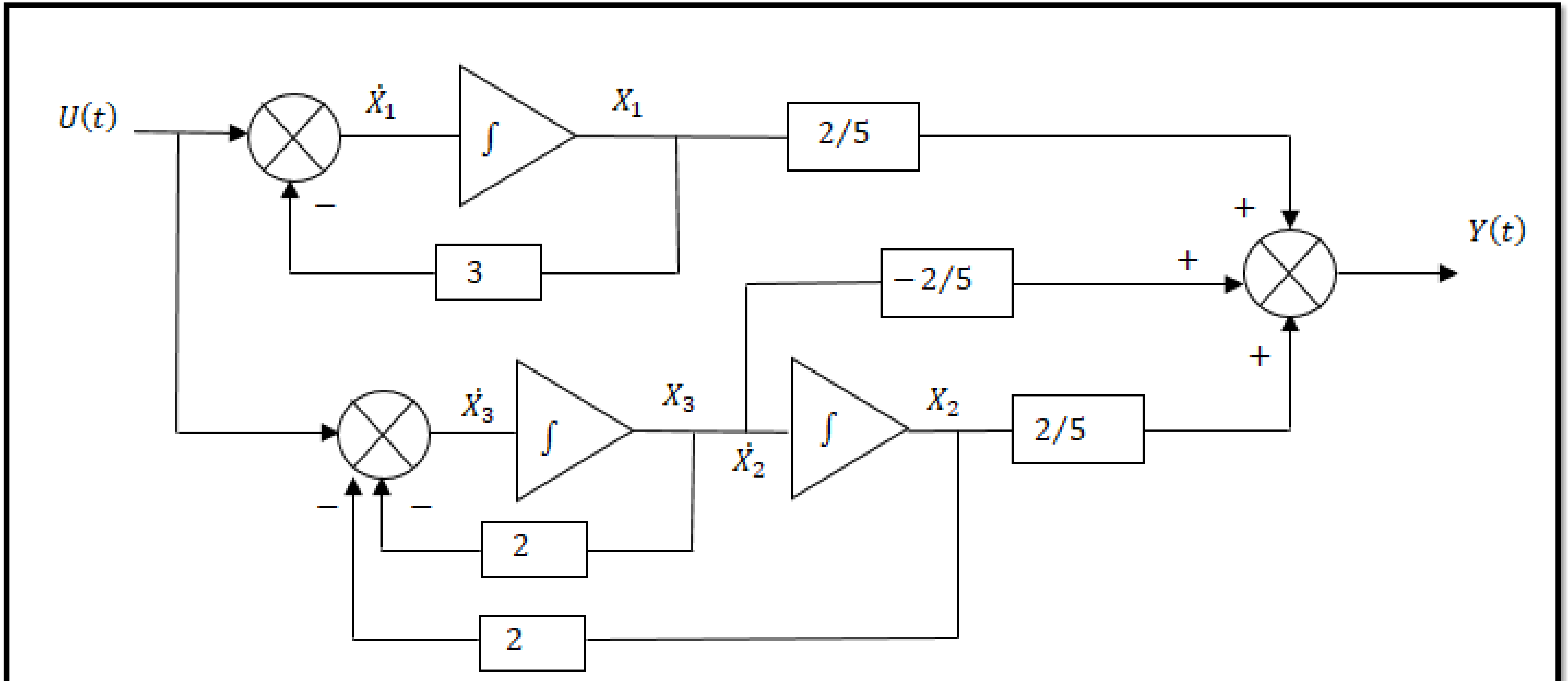
- Find LCM on right side and equate numerators of both sides,
- $\therefore \quad A(s^2+2s+2) + (Bs+C)(s+3) = 2$
- $\therefore \quad As^2 + 2As + 2A + Bs^2 + Cs + 3Bs + 3C = 2$

- Equating coefficients of all powers of s,
- $A + B = 0 \qquad 2A + C + 3B = 0 \qquad 2A + 3C = 2$
- $2A + C - 3A = 0 \qquad \therefore \quad 2A + 3A = 2$
- $C - A = 0$
- $C = A$
- Solving $\quad A = \frac{2}{5}, \quad B = -\frac{2}{5}, \quad C = \frac{2}{5}$

- $T(s) = \frac{\frac{2}{5}}{(s+3)} + \frac{-\frac{2}{5}s+\frac{2}{s}}{s^2+2s+2} = \frac{\frac{2}{5}}{s+3} + \frac{-\frac{2}{5}s+\frac{2}{5}}{\{(s+2)s+2\}}$

- The quadratic $s^2 + 2s + 2$ having complex roots is decomposed directly.

- ∴ Complete state diagram is as shown in the Fig.



Output of each integrator is state variable.

- $\dot{X}_1 = U(t) - 3X_1$
- $\dot{X}_2 = X_3$
- $\dot{X}_3 = U(t) - \dot{X}_1 = U(t) - 2X_2 - 2X_3$
- And $Y(t) = \frac{2}{5}X_1(t) + \frac{2}{5}X_2(t) - \frac{2}{5}X_3(t)$

- $\therefore \quad State\ model\ is, \ \ \dot{X} = AX + BU \quad and\ Y = CX$
- Where $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- $C = \begin{bmatrix} \frac{2}{5} & -\frac{2}{5} & \frac{2}{5} \end{bmatrix}, \quad D = 0$
- Note that in such a case matrix A does not have any specific form.

Sate model by cascade Programming

- This is also called pole-zero form or Gullemin’s form. This can be effectively used when both numerator and denominator can be factorised.
- In such form, group a pole and zero together and arrange given transfer function as the product of all such groups.
- Then simulate each group separately and connect all such simulations in cascade to get complete simulations.
- Then assigning output of each integrator as a state variable obtain a state model in standard form.
- Simulation of group: consider $\frac{s+a}{s+b}$

- First simulate denominator $\frac{1}{s+b}$ as in the figure(a) and then as discussed earlier simulate the numerator (s+a) as shown in figure(b).

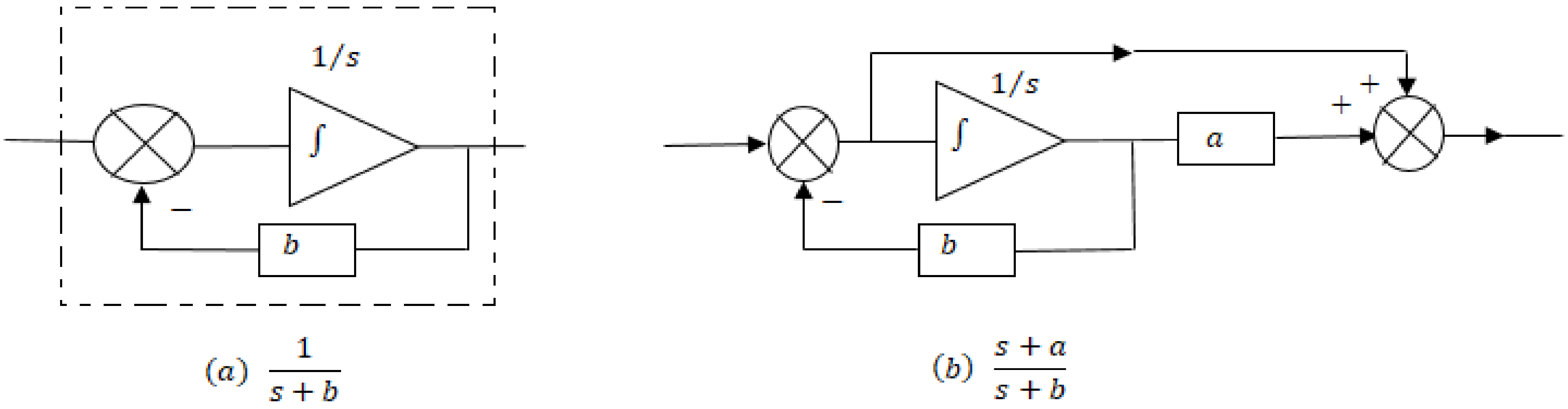


Fig. Simulation of a group of pole-zero

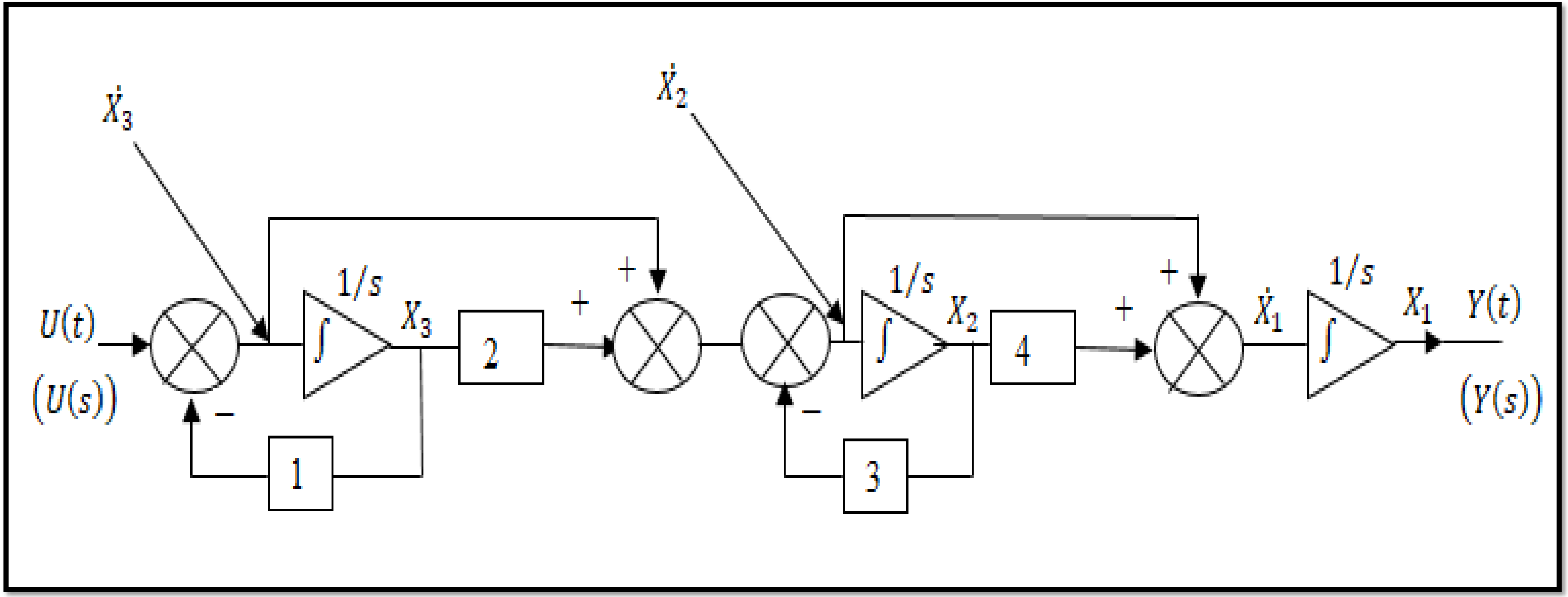
Q9. Obtain the state model of a system by cascade programming whose transfer function is $T(s) = \frac{Y(s)}{U(s)} = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$

- Solution: Arrange the given T.F. as below

$$\frac{Y(s)}{U(s)} = \frac{(s+2)}{(s+1)} \times \frac{(s+4)}{(s+3)} \times \frac{1}{s}$$

$$= \underset{\text{Group } 1}{\downarrow} \quad \underset{\text{Group } 2}{\downarrow} \quad \underset{\text{Group } 3}{\downarrow}$$

- Now simulate each group as discussed and connect all of them in series to obtain T(s).
- The complete simulation is shown in the below Fig.



- Now $\dot{X}_1 = X_2 + 4X_2\,,\quad \dot{X}_2 = -3X_2 + 2X_3 + \dot{X}_3$
- And $\dot{X}_3 = U(t) - X_3$

- Substituting $\dot{X}_3$ into $\dot{X}_2$ equation,
- $\dot{X}_2 = -3X_2 + 2X_3 + U(t) - X_3 = U(t) - 3X_2 + X_3$

- Substituting $\dot{X}_2$ into $\dot{X}_1$ equation,
- $\dot{X}_1 = U(t) - 3X_2 + X_3 + 4X_2 = U(t) + X_2 + X_3$

- Model becomes
- $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U(t)$
- And $\qquad Y(t) = X_1(t)$

- $i.e.$ $$Y(t) = [1 \quad 0 \quad 0]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

- So $$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $C = [1 \quad 0 \quad 0] \quad D = 0$