Unit - II

Transportation Problems.

Mathematical Formulation of a Transportation Problem:

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.c
$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, 3 \ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, 3 \ldots n.$$

and $x_{ij} \geq 0$. for all i and j

Note 1: The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) (total demand)

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problem satisfying this condition are called balanced transportation problems.

Note 2: If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be unbalanced.

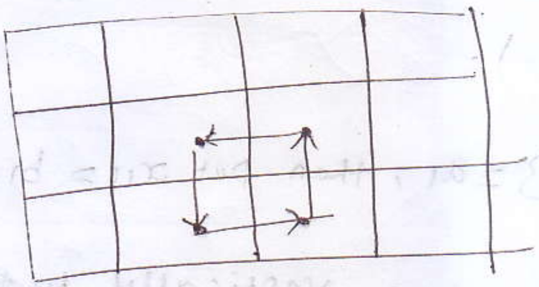
Note 3: For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Defn: A set of non-negative value x_{ij}, $i = 1, 2, \dots m$, $j = 1, 2, \dots n$. that satisfies the constraints is called a feasible solution to the transportation problem.

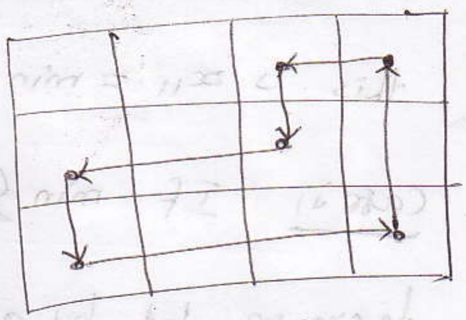
Defn: A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m+n-1$ non-negative allocations is called a basic feasible solution (BFS) to the transportation problem.

The allocation are said to be in independent positions. If it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to travel from any allocation back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example:

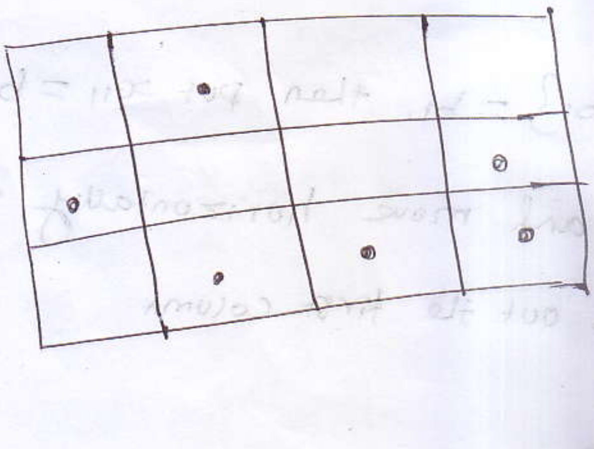
Non-independent positions



Non-independent positions



Independent positions



Methods for finding initial basic feasible solution:-

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced.

Method 1: North west corner Rule:-

Step 1. The first assignment is made in the cell occupying the upper left-hand corner of the transportation table. The maximum possible amount is allocated there. this is $x_{11} = \min(a_1, b_1)$.

Case i) If $\min\{a_1, b_1\} = a_1$, then put $x_{11} = b_1$, and decrease b_1 by a_1 and move vertically to the 2nd row (ie) to the cell $(2,1)$ cross out the first row.

Case ii) If $\min\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right to the cell $(1,2)$ cross out the first column

Case iii) If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell $(2,2)$ cross out the first row and the first column.

Step 2: Repeat the procedure until all the rim requirements are satisfied.

1) Determine basic feasible solution to the following transportation problem using North-west corner rule.

Sink

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

Soln

Since supply = Demand = 21, the given problem is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

(3) 2	11	10	3	7	1 ~~4~~
1	4	7	2	1	8
3	9	4	8	12	9
~~3~~	3	4	5	6	

1) 11	10	3	7	~~1~~
4	7	2	1	8
9	4	8	12	9
~~3~~ 2	4	5	6	

2) 4	7	2	1	~~8~~ 6
9	4	8	12	9
~~2~~	4	5	6	

4) 7	2	1	~~8~~ 2
4	8	12	9
~~4~~	5	6	

2) 2	1	~~2~~
3) 8	6) 12	~~9~~
~~8~~ 3	6	

∴ The initial transportation cost =

Rs. $2\times3 + 11\times1 + 4\times2 + 7\times4 + 2\times2 + 8\times3$

$+ 12\times6$

$= 153/-$

Method 2: Least cost method (or) matrix minima method

Step 1 Identify the cell with smallest cost and allocate

$x_{ij} = \min\{a_i, b_j\}$

Case (i) : If $\min\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, go to step (2)

Case (ii) If $\min\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j go to step (2).

Case (iii) If $\min\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both go to step (2).

Step 2 Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

(1) Find the starting solution of the following transportation model.

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using least cost method.

Soln

Since $\Sigma a_i = \Sigma b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

1	2	6	7
(10) 0	4	2	~~12~~ 2
3	1	5	11
~~10~~	10	10	

2	6	7
4	2	2
(10) 1	5	~~11~~ 1
~~10~~	10	

(7) 6	7
(2) 2	2
(1) 5	1
10	

The starting solution is as shown in the following table.

1	2	(7) 6
(10) 0	4	(2) 2
3	(10) 1	(1) 5

$\therefore$ The initial Transportation cost

$= 6\times7 + 0\times10 + 2\times2 + 1\times10 + 5\times1$

$=$ Rs 61/-

Method 3: Vogel's approximation method (VAM)

Step 1 Find the difference between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step 2 Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3)

Step 3: Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

(1) Find the starting solution of the following transportation model.

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

using Vogel's approximation method.

Soln

Since $\Sigma a_i = \Sigma b_j = 30$, the given transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

1	2	6	7	(1)
0	4	(10) 2	~~12~~ 2	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	(3)		
		7		

1	2	7	(1)	
(2) 0	4	~~2~~	(4)	7
3	1	11	(2)	
~~10~~ 8	10			
(1)	(1)			

1	2	7	(1)
3	(10) 1	~~11~~ 1	7 (2)
8	~~10~~		
(2)	(1)		

(7) 1	7
(1) 3	1
8	

The starting solution is as shown in the following table.

$\boxed{7}$ 1	2	6
$\boxed{2}$ 0	4	$\boxed{10}$ 2
$\boxed{1}$ 3	$\boxed{10}$ 1	5

$\therefore$ The initial transportation cost =

Rs. $1\times7 + 0\times2 + 2\times10 + 3\times1 + 1\times10$

= Rs. 40/-

Unbalanced Transportation Problems

(1) Solve the transportation problem

Destination

Source	A	B	C	D	Supply
1	11	20	7	8	50
2	21	16	20	12	40
3	8	12	18	9	70
Demand	30	25	35	40	

Soln: Since the total supply ($\Sigma a_i = 160$) is greater than total demand ($\sum b_j = 130$), the given problem is an unbalanced transportation problem. To convert this into a balanced one. we introduce a dummy destination E with zero unit transportation costs and having demand equal to $160 - 130 = 30$ units.

∴ The given problem becomes

	A	B	C	D	E	
1	11	20	7	8	0	50
2	21	16	20	12	0	40
3	8	12	18	9	0	70
	30	25	35	40	30	

By using VAM, the initial solution is

11	20	(35) 7	(15) 8	0
21	16	20	(10) 12	(30) 0
(30) 8	(25) 12	18	(15) 9	0

∴ the initial transportation cost = RS: 1160/-

Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

Step 1: Find the initial basic feasible solution of the given problem by North corner rule (or) Least cost method (or) VAM.

Step 2: Check the number of occupied cells. If these are less than $m+n-1$, there exists degeneracy and we introduce a very small +ve assignment of ϵ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step 3: Find the set of values u_i, v_j ($i = 1, 2, \ldots, m$; $j = 1, 2, 3, \ldots, n$) from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i, j), by starting initially with $u_i = 0$ or $v_j = 0$. preferably for which the corresponding row or column has maximum number of individual allocations.

Step 4: Find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding cell (i, j).

Step 5: Find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (d_{ij} = upper left − upper right) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

Step 6 Examine the cell evaluations d_{ij} for all unoccupied cells (i,j) and conclude that

i) If all $d_{ij} \geq 0$, then the solution under the test is optimal and unique.

ii) If all $d_{ij} > 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternative optimal solution exists.

iii) If atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step 7:

Form a new BFS by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied ~~cell~~ cell empty. For that draw a closed path consisting of horizontal and vertical lines begining and ending at the cell for which d_{ij} is most negative and having its other corners at some allocated cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and substract this minimum allocation from the allocation to the cells with $-\theta$.

Step 8 Repeat step (2) to (6) to test the optimality of this new basic feasible solution.

Step 9 continue the above produre till an optimum solution is attained.

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1. Obtain an optimum basic feasible solution to the following transportation problem:

To

From				Available
	7	3	2	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	

Soln since $\Sigma a_i = \Sigma b_j = 10$, the given transportation problem is balanced.

By VAM, the initial solution is as shown in the following table.

7	3	(2) 2	(1)	(5)	—
2	(1) 1	(2) 3	(1)	(1)	(1)
(4) 3	4	(1) 6	(1)	(3)	(3)
(1)	(2)	(1)			
(1)	—	(1)			
(1)	—	(3)			

$\therefore$ The initial transportation cost

$= Rs.\ 2\times2+1\times1+3\times2+3\times4+6\times1$

$= Rs.\ 29/-$

For optimality :-

Step 1 Since the number of non-negative independent allocations is $m+n-1$, (ie) $3+3-1$

ie, $6-1=5$

we apply MODI method.

Step 2 To find u_i, v_j value for occupied cells

*	*	2	$v_1 = 2$
*	1	3	$v_2 = 3$
3	*	6	$v_3 = 6$

$u_1 = -3 \quad u_2 = -2 \quad u_3 = 0$

Step 3: To find $u_i + v_j$ value for non-occupied cells:

-1	0	*	$v_1 = 2$
0	*	*	$v_2 = 3$
*	4	*	$v_3 = 6$

$u_1 = -3 \quad u_2 = -2 \quad u_3 = 0$

Step 4 To find $d_{ij} = c_{ij} - (u_i + v_j)$

8	3	*
2	*	*
*	0	*

Since all $d_{ij} > 0$, $d_{32} = 0$, the current soln. is optimal and there exists an alternative optimal solution. ∴ The optimum transportation cost = Rs: 29/-

2, Find the optimal transportation cost of the following matrix using Least cost method for finding the critical solution

Market

		A	B	C	D	E	Available
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
Demand		40	50	70	90	90	

Soln

Since $\Sigma a_i = \Sigma b_j = 340$, the given transportation problem is balanced

4	(50) 1	(50) 2	6	9
(10) 6	4	(20) 3	5	(90) 7
(30) 5	2	6	(90) 4	8

∴ The initial transportation cost

= Rs. $1\times 50 + 2\times 50 + 6\times 10 + 3\times 20 + 7\times 90 + 5\times 30 + 4\times 90$

= Rs. 1410/-

For optimality

Step 1:- Since the number of non-negative independent allocation is $(m+n-1)$

$5+3-1$

$= 8-1 = 7$

we apply MODI method.

Step 2 To find u_i, v_j for occupied cells

*	1	2	*	*	$u_1 = -1$
6	*	3	*	7	$u_2 = 0$
5	*	*	4	*	$u_3 = -1$

$v_1 = 6$ $v_2 = 2$ $v_3 = 3$ $v_4 = 5$ $v_5 = 7$

Step 3:- To find $u_i + v_j$ for non-occupied cells

5	*	*	4	6	$u_1 = -1$
*	2	*	5	*	$u_2 = 0$
*	1	2	*	6	$u_3 = -1$

$v_1 = 6$ $v_2 = 2$ $v_3 = 3$ $v_4 = 5$ $v_5 = 7$

Step 4:- $d_{ij} = c_{ij} - (u_i + v_j)$

$(+\theta)$ -1	*	$(-\theta)$ *	2	3
$(-\theta)$ *	2	$(+\theta)$ *	0	*
*	1	4	*	2

From the two cells (1,3) (2,1) having $-\theta$, we find that the minimum of the allocations 50, 10, is 10. Add this 10 to the cells with $+\theta$ and Subtract this 10 to the cells with $-\theta$.

(10) 4	(50) 1	(40) 2	6	9
6	4	(30) 3	5	(90) 7
(30) 5	2	6	(90) 4	8

Step 1

$m+n-1$

$5+3-1$

$= 8-1 = 7$. So, we apply MODI method

Step 2 To find u_i, v_j for occupied cells

4	1	2	*	*	$u_1 = 0$
*	*	3	*	7	$u_2 = 1$
5	*	*	4	*	$u_3 = 1$

$v_1 = 4$ $v_2 = 1$ $v_3 = 2$ $v_4 = 3$ $v_5 = 6$

Step 3:- To find $u_i + v_j$ for non-occupied cells

*	*	*	3	6	0
5	2	*	4	*	1
*	2	3	*	7	1
4	1	2	3	6	

Step 4:- To find $d_{ij} = c_{ij} - (u_i + v_j)$

*	*	*	3	3
1	2	*	1	*
*	0	3	*	1

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ Optimal solution $= 4\times10 + 1\times50 + 2\times40 + 3\times30$

$+ 7\times50 + 5\times30 + 4\times90$

$=$ Rs $1400/-$

Degeneracy in Transportation Problems:-

In a transportation problem, whenever the number of non-negative independent allocations is less than m+n-1, the transportation problem is said to be a degenerate one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero) to one or more empty cells of the transportation table so that the total number of occupied cells becomes (m+n-1) at independent positions. We denote this small amount by ϵ. (epsilon)

(1) Solve the following transportation problem to minimize the total cost of transportation.

Destination

Origin	1	2	3	4	Supply
1	14	56	48	27	70
2	82	35	21	81	47
3	99	31	71	63	93
Demand	70	35	45	60	

Soln Since $\Sigma a_i = \Sigma b_j = 210$, the given transportation problem is balanced.

By using VAM, the initial solution is

(70) 14	56	48	27
82	35	(45) 21	(2) 81
99	(35) 31	71	(58) 63

Since the number of non-negative allocation is 5, which is lessthan $(m+n-1) = (3+4-1) = 6$ this basic feasible solution is degenerate.

we allocate a very small quantity ϵ.

(70) 14	56	48	(ϵ) 27
82	35	(45) 21	(2) 81
99	(35) 31	71	(58) 63

Step 1

$m+n-1 = 6$, we apply MODI method

Step 2 To find u_i, v_j for occupied cells

14	*	*	27	27
*	*	21	81	81
*	31	*	63	63
-13	-32	-60	0	

Step 3 To find $u_i + v_j$ for non-occupied cells

*	-5	-33	*	27
68	49	*	*	81
50	*	3	*	63
-13	-32	-60	0	

Step 4 $d_{ij} = c_{ij} - (u_i + v_j)$

*	61	81	*
14	-14	*	*
49	*	68	*

Since $d_{22} = -14 < 0$, the

solution under the test is not optimal.

*	61	81	*
14	+θ −14	*	* −θ
49	−θ *	68	* +θ

From the two cells (2,4) & (3,2) having −θ, we find that the minimum of the allocations 2, 35, is 2.

Add this 2 to the cells with +θ

Subtract this 2 to the cells with −θ

(70) 14	56	48	(ε) 27
82	(2) 35	(45) 21	81
99	(33) 31	71	(60) 63

we see that the above table satisfies the rim conditions with (m+n−1) non-negative allocations at independent positions. we apply MODI method for optimality.

Step 2

To find u_i, v_j for occupied cells

14	*	*	27	-40
*	35	21	*	0
*	31	*	63	-4
54	35	21	67	

Step 3: To find $u_i + v_j$ for non-occupied cells.

*	-5	-19	*	-40
54	*	*	67	0
50	*	17	*	-4
54	35	21	67	

Step 4: To find $d_{ij} = c_{ij} - (u_i + v_j)$

*	61	67	*
28	*	*	14
49	*	54	*

Since all $d_{ij} > 0$, the solution under is optimal.

Optimum Transportation cost = 6798/- as $\epsilon \to 0$.

Assignment problems

The assignment problem is a particular case of the Transportation problem in which the objective is to assign a number of tasks to an equal number of facilities at a minimum cost (or maximum profit)

Suppose that we have 'n' jobs to be performed on 'm' machines and our objective is to assign the jobs to the machines at the minimum cost under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of m×n matrix (c_{ij}) called a cost matrix where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

Jobs

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & \cdots & n \\
\hline
1 & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\
2 & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\
3 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
m & c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mn} \\
\hline
\end{array}$$

Machines

Mathematical formulation of an assignment problem.

Consider an assignment problem of assigning n jobs to n machines. Let c_{ij} be the unit cost be the unit cost of assigning j^{th} machine to the i^{th} job and

$$\text{Let } x_{ij} = \begin{cases} 1, & \text{If } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine.} \end{cases}$$

The assignment model is the given by the following LPP.

$$\min z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.c

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots n$$

and $x_{ij} = 0$ (or) 1.

Hungarian Assignment algorithm:-

Step 1 If the number of rows are not equal to the number of columns and vice versa, then a dummy row or dummy column must be added with zero cost elements.

Step2 Find the smallest cost in each row of the cost matrix. subtract this smallest cost element from each element in that row. Therefore, there will be at least one zero in each row of this new matrix which is called the first reduced cost matrix.

Step3 In the reduced cost matrix, find the smallest element in each column. Substract the smallest cost element from each element in that column. As a result, there would be at least one zero in each row and column of the second reduced cost matrix.

Step4 Determine an optimum assignment as follows:

i) Examine the rows successively until a row with exactly one zeros is found. Box around the zero element as an assigned cells and cross out all other zero in its column. Proceed in this manner until all the rows have been examined. If there are more than one zero in any row, then do not consider that row and pass onto the next row.

ii) Repeat the procedure for the columns of the reduced cost matrix. If there is no single zero in any row or column of the reduced matrix. then arbitrarily choose a row or column having the minimum number of zeros. Arbitrarily, choose zero in the row or column and cross the remaining zeros in that row or column.

Repeat steps i) and ii) until all zeros are either assigned or crossed out.

Step 5 An optimal assignment is found, if the number of assigned cells equals the number of rows. If a zero cell is arbitrarily chosen, there may be an alternate optimum. If no optimum solution is found, then go to next step.

Step 6: Draw the minimum number of horizontal and/or vertical lines through all the zeros as follows.

i) mark (✓) to those rows where no assignment has been made.

ii) mark (✓) to those columns which have zeros in the marked rows.

iii) mark (✓) rows which have assignments in marked columns.

iv) The process may be repeated until no more rows or colums can be checked.

v) Draw straight lines through all unmarked rows and marked columns

Step 7 If the minimum number of lines passing through all the zeros is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crossed in step 3. Otherwise go to next step.

Step 8: Revise the cost matrix as follows.

i) Find the elements that are covered by a line. Choose the smallest of these elements and subtract this element from all the uncrossed elements and add the same at the point of intersection of the two lines.

ii) Other elements crossed by the lines remain unchanged.

Step 9 Go to step 4 and repeat the procedure till an optimum solution is attained.

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① The assignment cost of assigning any one operator to any one machine is given in the following table.

Operators

$$\begin{array}{c c} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{bmatrix} \end{array}$$

Machine

Find the optimal assignment.

Step1 Row wise solved

$$\begin{bmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$$

Step2 column wise solved

$$\begin{bmatrix} 5 & 0 & 7 & 10 \\ 0 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{bmatrix}$$

Step3 Make assignment.

$$\begin{bmatrix} 5 & \boxed{0} & 7 & 10 \\ \cancel{0} & 6 & 14 & \boxed{0} \\ 8 & 5 & \boxed{0} & \cancel{0} \\ \boxed{0} & 6 & 3 & 2 \end{bmatrix}$$

∴ The current assignment is optimal

∴ The optimum schedule is

A — II , B — IV , C → III , D → I

and the optimum assignment cost is

Rs[5+3+3+5]. = Rs 16 /-

2. A department has five employees with five jobs to be performed. The time (in hours) each man will take to perform each job is given in the cost matrix.

Employees

$$\begin{array}{cc} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \text{Jobs} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 10 & 5 & 13 & 15 & 16 \\ 3 & 9 & 18 & 13 & 6 \\ 10 & 7 & 2 & 2 & 2 \\ 7 & 11 & 9 & 7 & 12 \\ 7 & 9 & 10 & 4 & 12 \end{bmatrix} \end{array}$$

How should the jobs be allocated, one per employee, so as to minimise the total man-hours?

Soln Row wise reduced

$$\begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 10 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 5 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix}$$

Column

Column wise Reduced & Make assignment

$$\begin{bmatrix} 5 & \boxed{0} & 8 & 10 & 11 \\ \boxed{0} & 6 & 15 & 10 & 3 \\ 8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & 4 & 2 & \cancel{0} & 5 \\ 7 & 5 & 6 & \boxed{0} & 8 \end{bmatrix}$$

$$\begin{bmatrix} 7 & \boxed{0} & 8 & 12 & 11 \\ \boxed{0} & 4 & 13 & 10 & 1 \\ 10 & 5 & \cancel{0} & 2 & \boxed{0} \\ \cancel{0} & 2 & \boxed{0} & \cancel{0} & 3 \\ 3 & 3 & 4 & \boxed{0} & 6 \end{bmatrix}$$

The minimum total time for this assignment schedule is $5+3+2+9+4 = 23$ hours.

Variations of the Assignment Problem:-

1. Non - Square matrix (unbalanced assignment problem)

2. Maximisation Problem

3. multiple optimal solutions.

4. Restrictions on assignments (or) impossible assignment

Unbalanced Problem

The Hungarian method of assignment requires that the number of columns and rows in the assignment matrix must be equal. when the given cost matrix is not a square matrix, then the problem is called an unbalanced problem. In this case dummy row(s) or column(s) with zeros cost is added to make it a square matrix.

(1) Assign four trucks 1, 2, 3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 4 & 7 & 3 & 7 \\ 8 & 2 & 5 & 5 \\ 4 & 9 & 6 & 9 \\ 7 & 5 & 4 & 8 \\ 6 & 3 & 5 & 4 \\ 6 & 8 & 7 & 3 \end{bmatrix} \end{array}$$

Soln

Since the number of rows is more than the number of columns, the given assignment problem is unbalanced. To make it balanced, let us introduce two dummy columns with zero costs. We get

$$\begin{bmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{bmatrix}$$

Step 1 Row wise reduced

$$\begin{bmatrix} 0 & 5 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 7 & 3 & 6 & 0 & 0 \\ 3 & 3 & 1 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 0 & 0 \end{bmatrix}$$

Step 2 Column wise reduced :- & Step 3 Make assignment

$$\begin{bmatrix} \cancel{0} & 5 & \boxed{0} & 4 & \cancel{0} & \cancel{0} \\ 4 & \boxed{0} & 2 & 2 & \cancel{0} & \cancel{0} \\ \boxed{0} & 7 & 3 & 6 & \cancel{0} & \cancel{0} \\ 3 & 3 & 1 & 5 & \boxed{0} & \cancel{0} \\ 2 & 1 & 2 & 1 & \cancel{0} & \boxed{0} \\ 2 & 6 & 4 & \boxed{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

∴ The optimum assignment schedule is given by

$A \to 3,\ B \to 2,\ C \to 1,\ D \to 5,\ E - 6,\ F \to 4$

and the optimum (minimum) distance.

$= (3+2+4+0+0+3)$

units of distance $= 12$ / units of distance.

Maximisation problem:

There may be problems of maximising the profit, revenue, and so on. Such problems may be solved by converting the given maximisation problem into a minimisation problem before the Hungarian method is applied. The transformation may be done in the following two ways.

i) by substracting all the elements from the highest element of the matrix.

ii) by multiplying the matrix elements by -1.

① A company has a team of four salesman and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit ~~the~~ per day in rupees for each salesman in each salesman in each district is as below.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & 16 & 10 & 14 & 11 \\ B & 14 & 11 & 18 & 15 \\ C & 15 & 15 & 13 & 12 \\ D & 13 & 12 & 14 & 15 \end{array}$$

Find the assignment of salesman to various districts which yield maximum profit.

Step 1 Convert the maximisation problem into a minimisation problem by subtracting all the elements from the highest element 16.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix} \end{array}$$

Step 2
Apply Hungarian algorithm

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \boxed{0} & 6 & 2 & 5 \\ 1 & 4 & \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & 2 & 3 \\ 2 & 3 & 1 & \boxed{0} \end{bmatrix} \end{array}$$

The schedule is $A \to 1$, $B \to 3$, $C \to 2$, $D \to 4$

Profit per day $= \text{Rs}\,[16 + 15 + 15 + 15]$

$= \text{Rs}\ 61.$

3. Multiple optimal solutions:-

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off certain number of zeros. Such situation leads to multiple solutions with the same optimal value of objective function. In such cases the most suitable solution may be considered by the decision - maker.

① Solve the minimal assignment problem whose cost matrix is given below.

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 8 \\ 3 & 5 & 8 & 4 \end{bmatrix} \end{array}$$

Step 1 Row wise reduced and Step 2 Column wise reduced

$$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

Since single zeros do not exist either in the columns or in the row, we get the following alternative

$$\begin{bmatrix} \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} \\ \boxed{0} & 1 & 3 & \cancel{0} \end{bmatrix} \quad \begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & 2 \\ \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} \\ \cancel{0} & 1 & 3 & \boxed{0} \end{bmatrix}$$

$$\begin{bmatrix} \cancel{0} & \cancel{0} & \boxed{0} & 2 \\ \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} \\ \boxed{0} & 1 & 3 & \cancel{0} \end{bmatrix}$$

The possible optimal Solutions with each of costs Rs 20 are.

I → 2, II → 3, III → 4, IV → 1

I → 1, II → 2, III → 3, IV → 4

I → 3, II → 2, III → 1, IV → 4

I → 3, II → 2, III → 4, IV → 1

I → 2, II → 3, III → 1, IV → 4

4 Restrictions on Assignment (or) impossible assignment

Cells in which assignment are not allowed are assigned a very heavy cost (written as M or ∞) Such cells are prohibited to enter into the final solution.

(1) Four new machines M1, M2, M3, M4 are to be installed in a machine shop. There are five vacant places A, B, C, D, E that are available. Because of limited space, machine M2 cannot be placed at C and M1 cannot be placed at A. The cost Matrix is shown below.

$$\begin{array}{c} \\ M1 \\ M2 \\ M3 \\ M4 \end{array}
\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \left[\begin{array}{ccccc} 4 & 6 & 10 & 5 & 4 \\ 7 & 4 & - & 5 & 4 \\ - & 6 & 9 & 6 & 2 \\ 9 & 3 & 7 & 2 & 3 \end{array}\right] \end{array}$$

Find the optimal assignment schedule.

→ The given assignment problem is unbalanced. So, balance it by adding a dummy row with costs 0 as shown below.

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & E \\
\hline
M1 & 4 & 6 & 10 & 5 & 4 \\
M2 & 7 & 4 & M & 5 & 4 \\
M3 & M & 6 & 9 & 6 & 2 \\
M4 & 9 & 3 & 7 & 2 & 3 \\
M5 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Applying Hungarian method,

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & E \\
\hline
M1 & \boxed{0} & 2 & 6 & 1 & 2 \\
M2 & 3 & \boxed{0} & M & 1 & \cancel{0} \\
M3 & M & 4 & 7 & 4 & \boxed{0} \\
M4 & 7 & 1 & 5 & \boxed{0} & 1 \\
M5 & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0}
\end{array}
$$

The optimal assignment is :

$M1 \rightarrow A$, $M2 \rightarrow B$, $M3 \rightarrow E$, $M4 \rightarrow D$

$M5 \rightarrow C$ (C will remain vacant)

Total assignment cost $= Rs\ (4+4+2+2)$

$= Rs.\ 12$