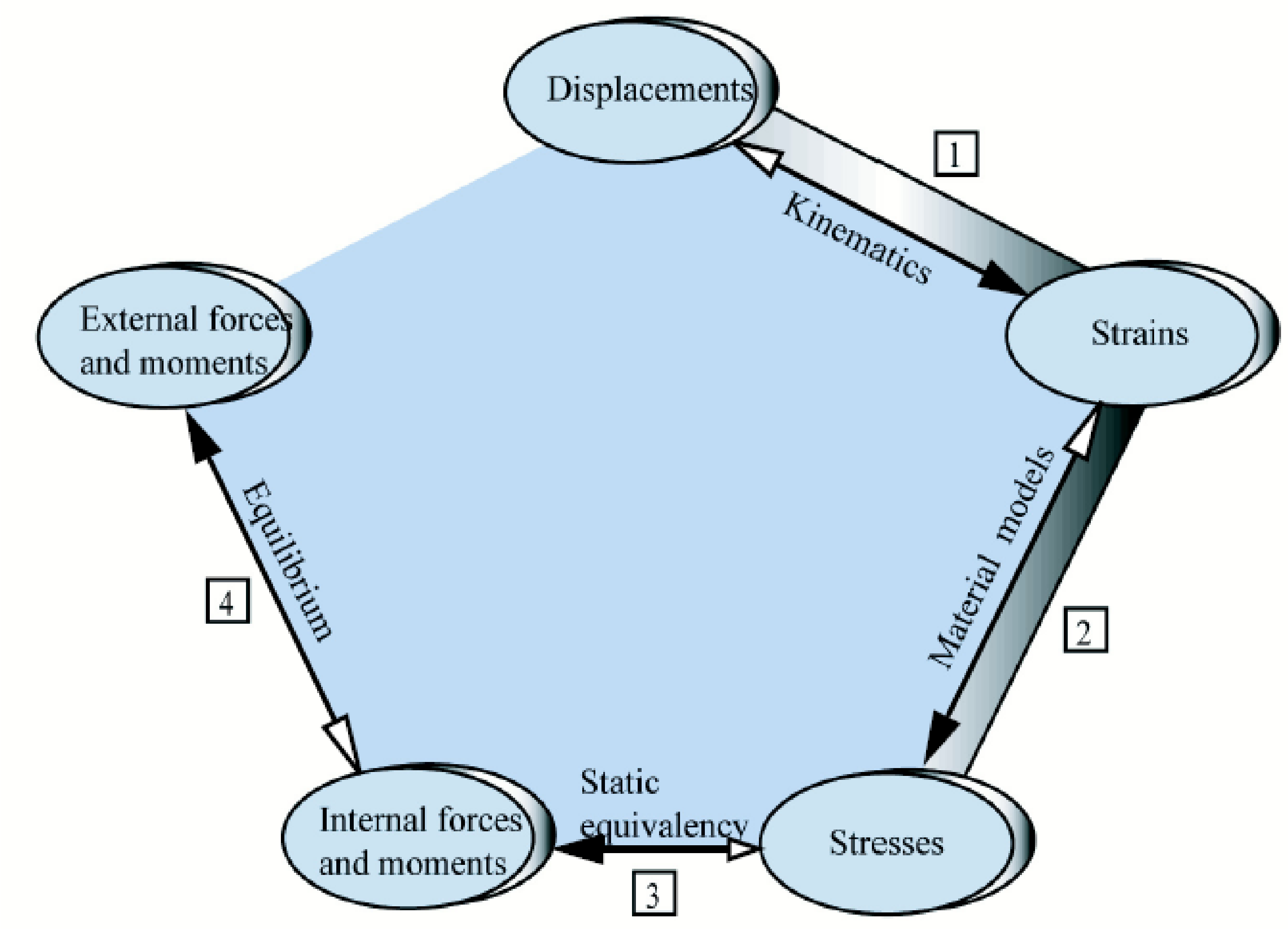
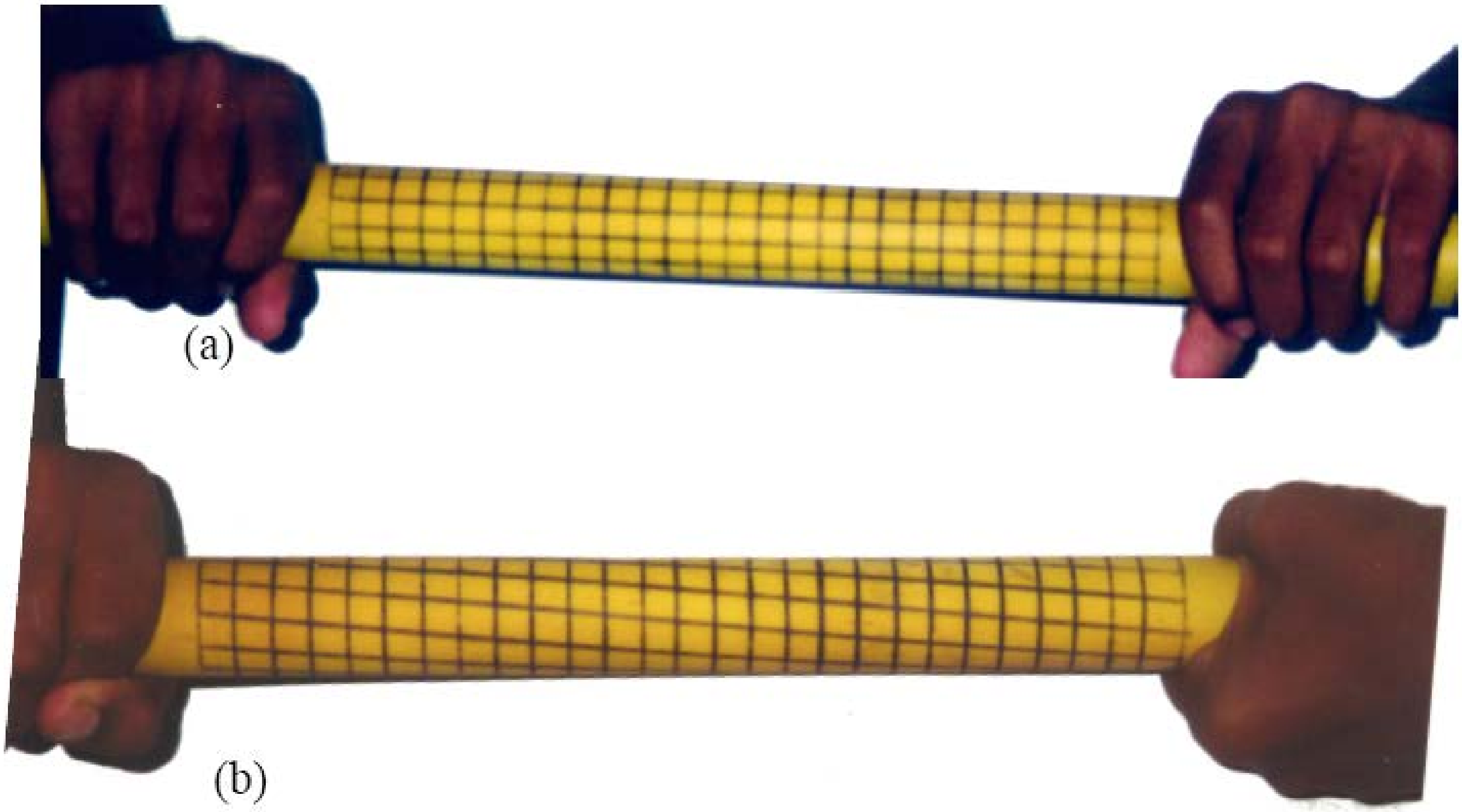
Torsion

The logic of the Mechanics of Materials



Torsion



Chapter 5 – Torsion

Figure: 05-01-COC

The torsional stress and angle of twist of this soil auger depend upon the output of the machine turning the bit as well as the resistance of the soil in contact with the shaft.

Torsional failures (ductile, buckling, buckling):

Bars subjected to Torsion

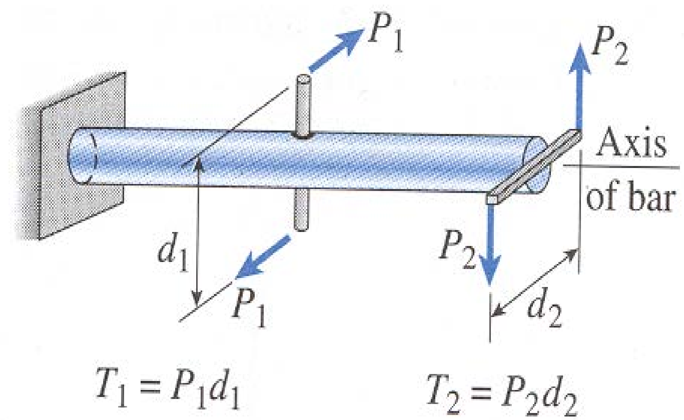
Let us now consider a straight bar supported at one end and acted upon by two pairs of equal and opposite forces.

$P_1 \quad P_2$

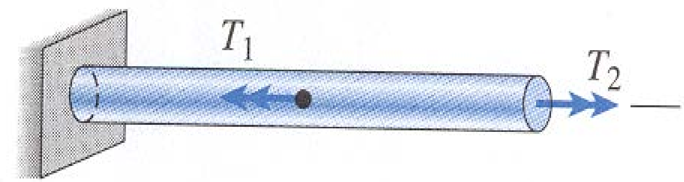
Then each pair of forces and form a couple that tend to twist the bar about its longitudinal axis, thus producing surface tractions and moments.

Then we can write the moments as

$$T_1 = P_1 d_1 \qquad\qquad T_2 = P_2 d_2$$



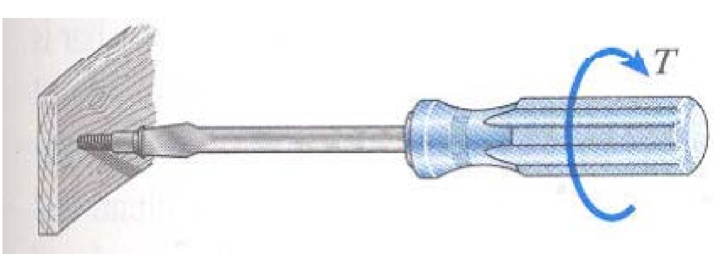
(a)

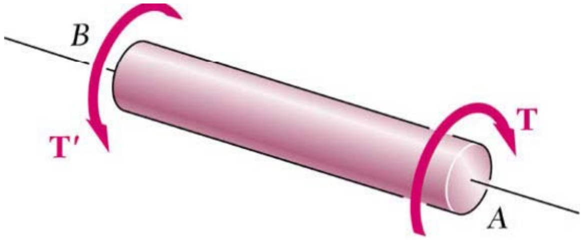


(b)

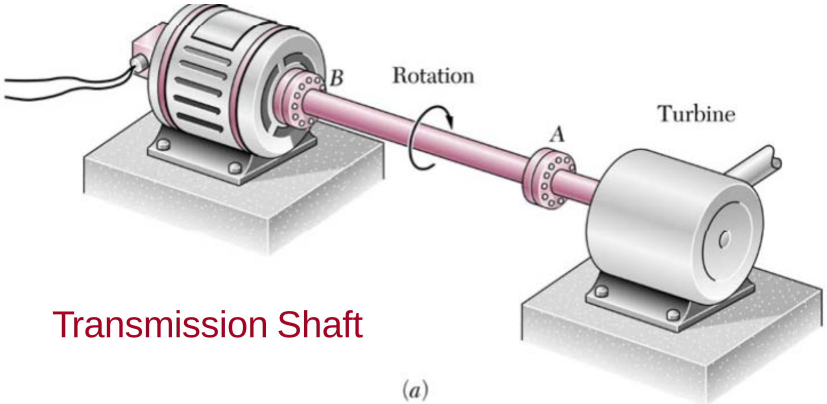
Torsion of Circular Shafts

- In this chapter, we will examine uniaxial bars subject to torque.



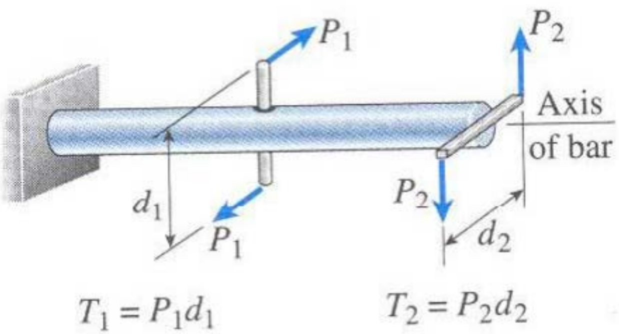


- Where does this occur?



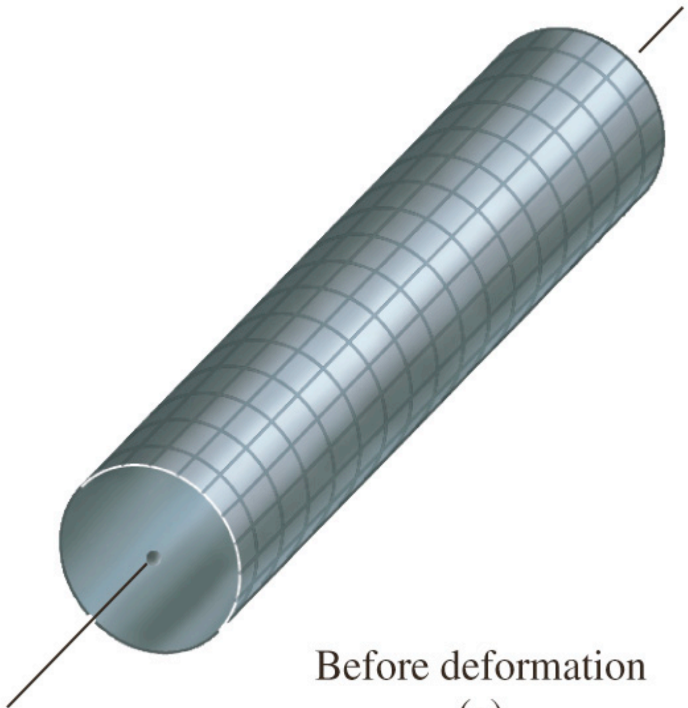
Transmission Shaft

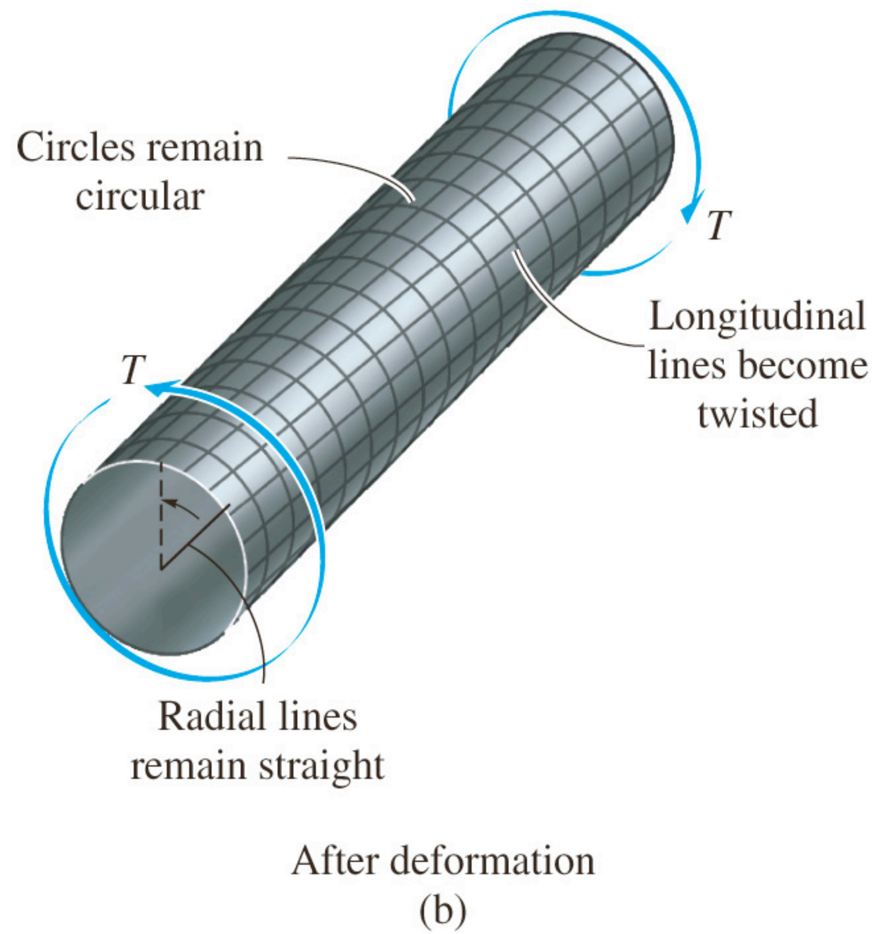
(a)



Force Couples

Torsion

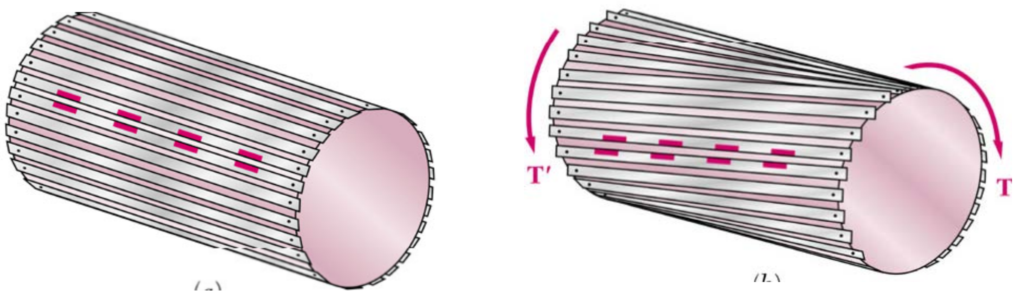




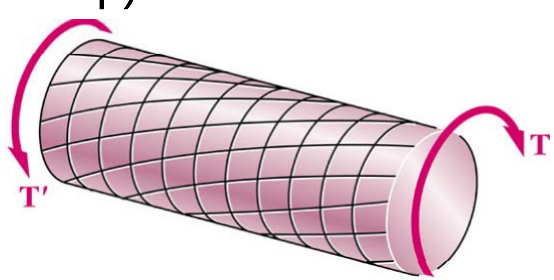
Recall: External loads (T) produce internal loads which produce deformation, strain and stress.

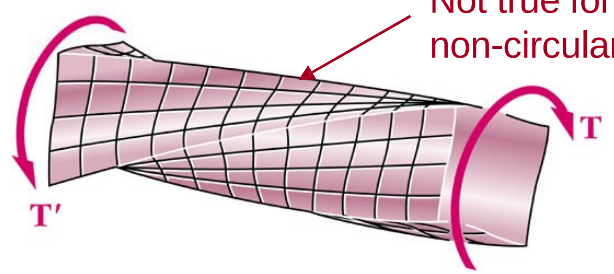
Torsion of Circular Shafts *cont'd*

- We assume
  - Bar is in pure torsion
  - Small rotations (the length and radius will not change)
- How does the bar deform?
  - Cross-section of the bar remains the same shape, bar is simply rotating.



  - Cross-section remains perpendicular to axis of cylinder (cylinder does not warp).



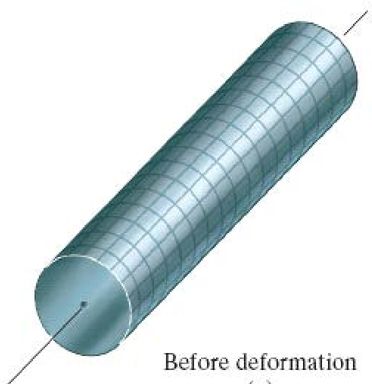


Not true for most non-circular bars

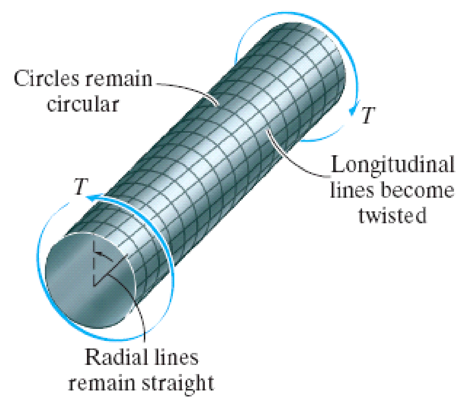
Torsional Deformation of a Circular Shaft

Torque is a moment that twists a member about its longitudinal axis.

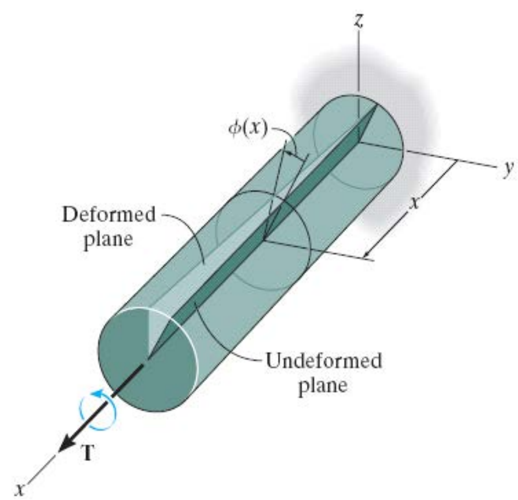
If the angle of rotation is *small*, the *length of the shaft* and *its radius* will *remain unchanged*.



Before deformation
(a)



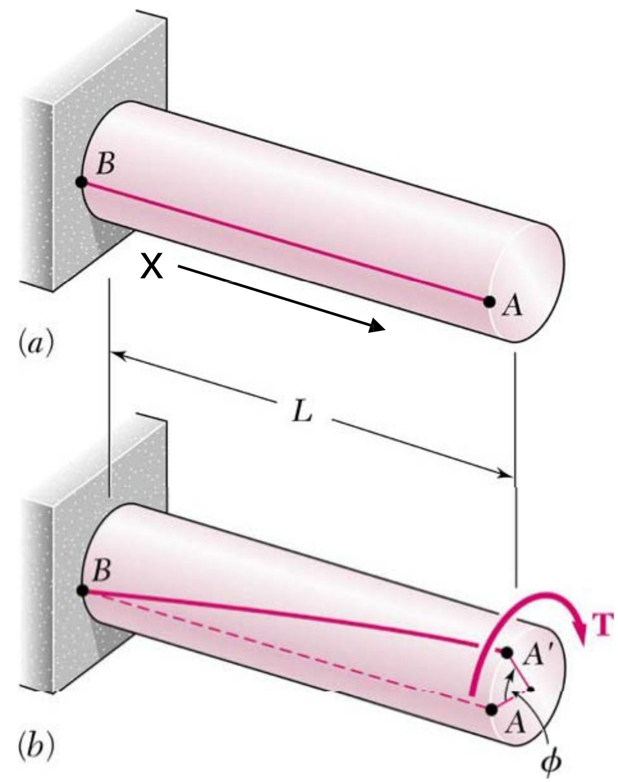
After deformation
(b)



The angle of twist $\phi(x)$ increases as x increases.

Angle of Twist

- Deformation of a circular shaft subjected to pure torsion
  - Fix left end of shaft
  - A moves to A'
  - ϕ = angle of twist (in radians)



- What are the boundary conditions on ϕ?
  - $\phi(x) = 0$ at $x = 0$
  - $\phi(x) = \phi$ at $x = L$

- For pure torsion, ϕ is linear.

$$\phi(x) = \frac{\phi x}{L}$$

Shearing Strain

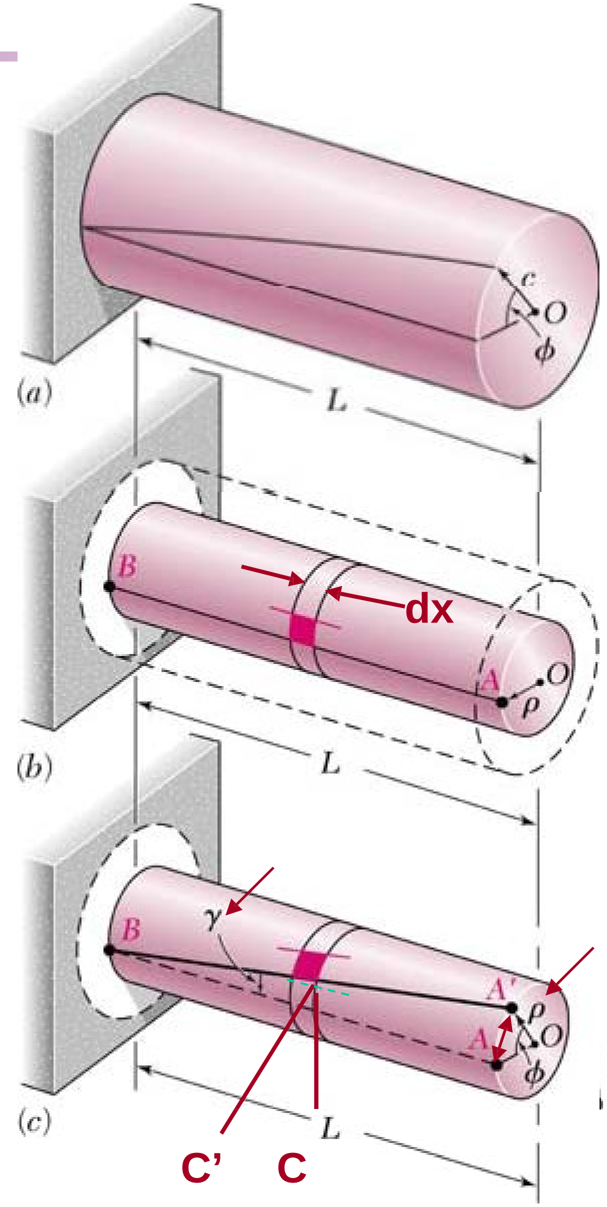
- Calculate the surface shear strain in the cylinder.
- Consider an element of length dx.
- Recall we assume small ϕ & small γ.

$$\gamma = \frac{\overline{C'C}}{dx} \qquad \overline{C'C} = \rho d\phi \qquad \gamma = \rho \frac{d\phi}{dx}$$

$$\frac{d\phi}{dx} = \textit{rate of change of angle of twist along the bar}$$

- This equation applies to any function ϕ (x).
- For pure torsion ϕ (x) = ϕ x / L, so

$$\boxed{\gamma = \frac{\rho\phi}{L}}$$



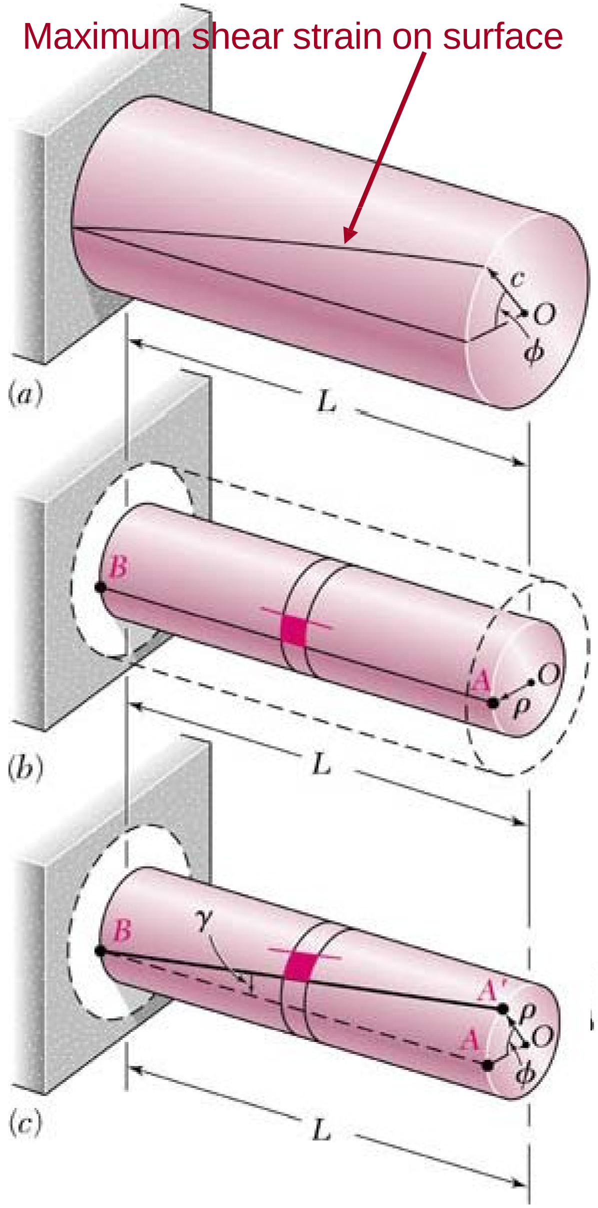
Shearing Strain *cont'd*

- The maximum shear strain on the surface of the cylinder occurs when ρ=c.

$$\gamma_{max} = \frac{c\phi}{L}$$

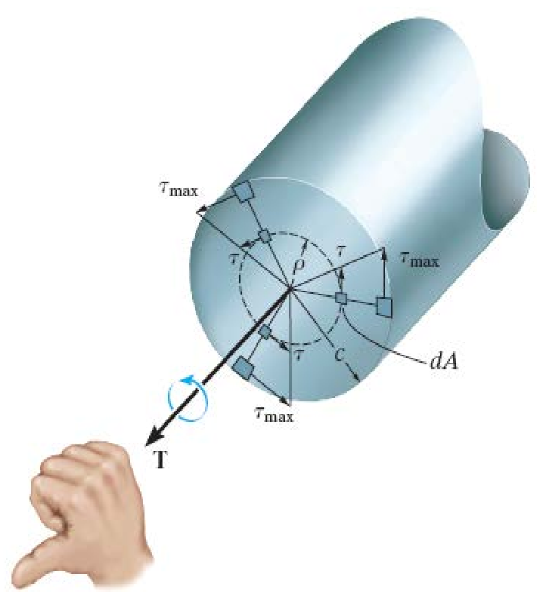
- We can express the shearing strain at any distance from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{max}$$



The Torsion Formula

When material is linear-elastic, Hooke's law applies. A ***linear variation in shear strain*** leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section.



Shear stress varies linearly along each radial line of the cross section.

$$\tau_{max} = \frac{Tc}{J} \quad \text{or} \quad \tau = \frac{Tp}{J}$$

τ_{max} = maximum shear stress in the shaft

τ = shear stress

T = resultant internal torque

J = polar moment of inertia of cross-sectional area

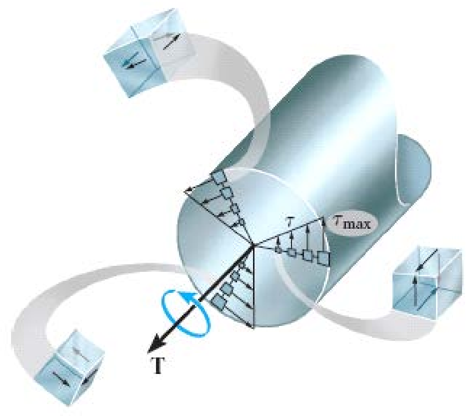
c = outer radius of the shaft

p = intermediate distance

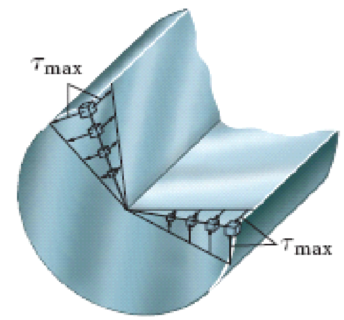
The Torsion Formula

If the shaft has a solid **circular** cross section,

$$J = \frac{\pi}{2} c^4$$



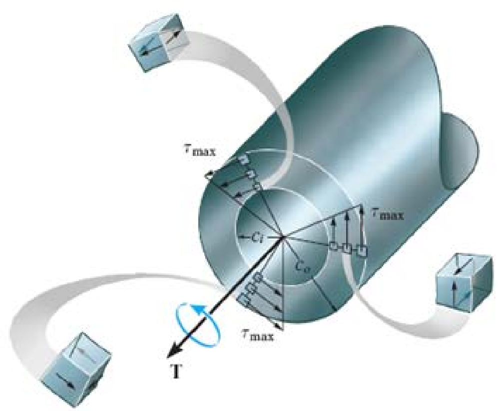
(a)



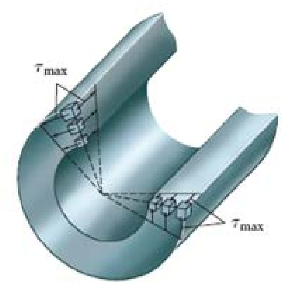
Shear stress varies linearly along each radial line of the cross section.
(b)

If a shaft has a **tubular** cross section,

$$J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right)$$



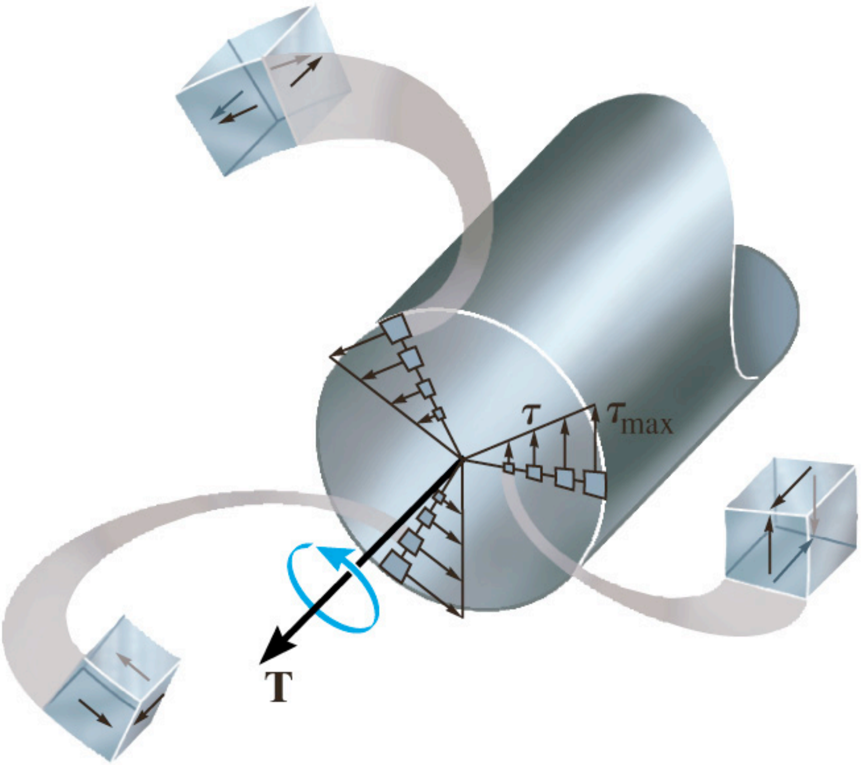
(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

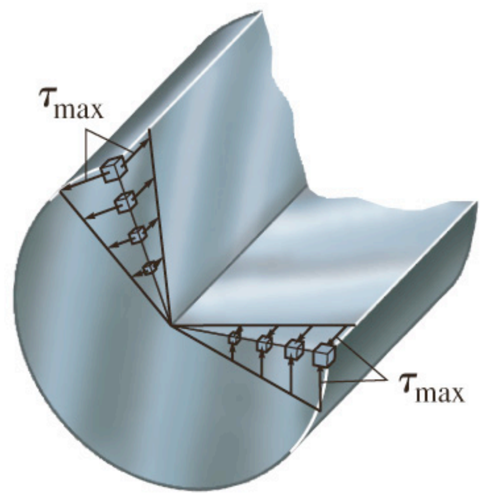
Stress Profiles:

Shear stress profile – YOU MUST UNDERSTAND THIS!!!!



(a)

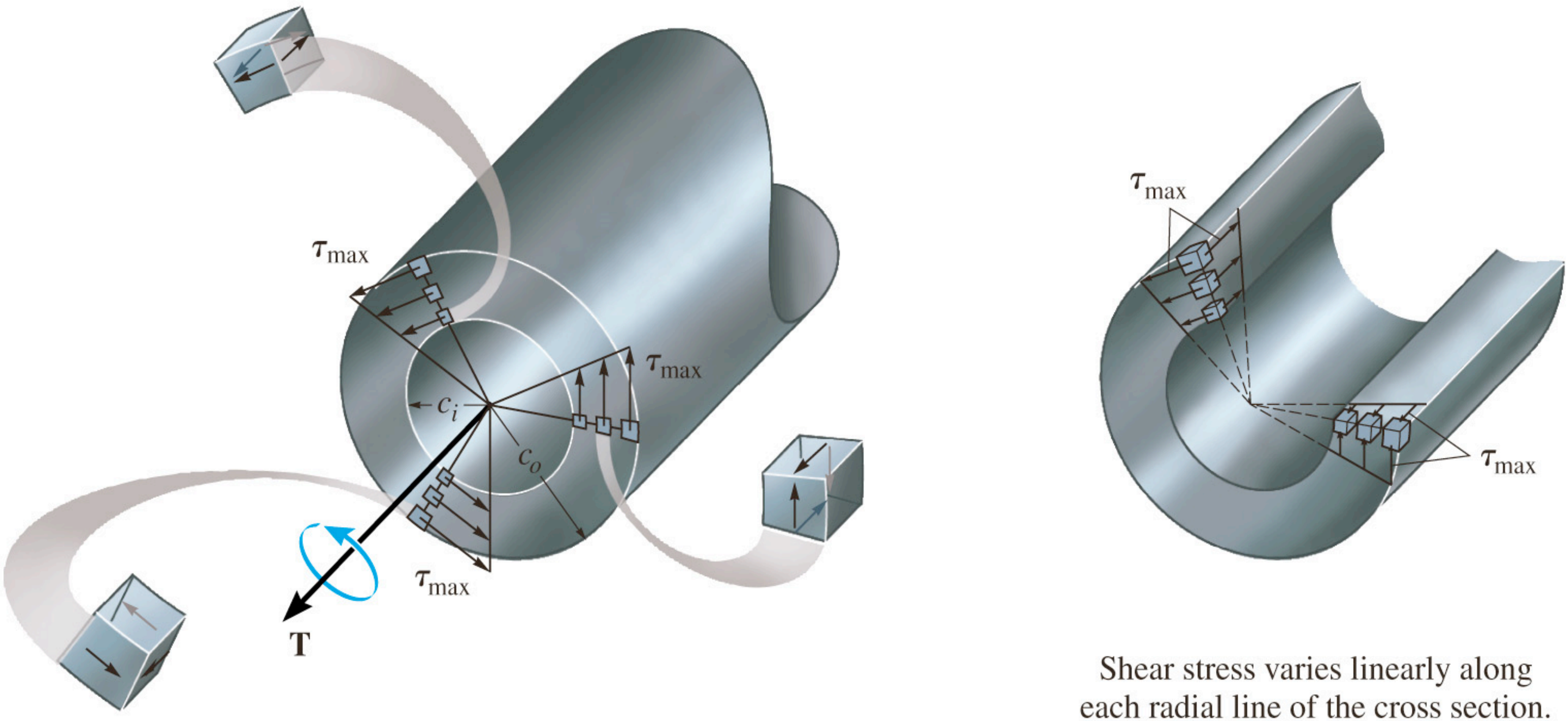
Where is shear stress max? zero? How does it vary along the length and circumference?



Shear stress varies linearly along each radial line of the cross section.

(b)

Stress Profiles:



Shear stress varies linearly along each radial line of the cross section.

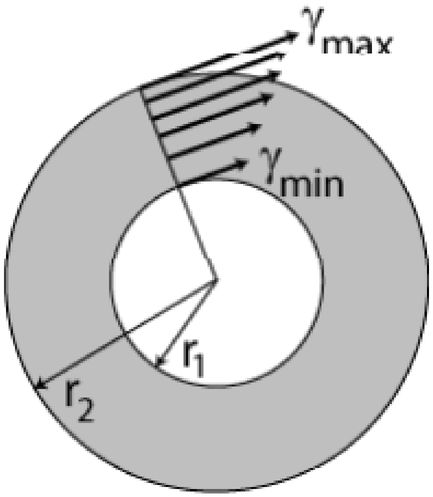
(b)

Shearing Strain *cont'd*

- We can also apply the equation for maximum surface shear strain to a hollow circular tube.

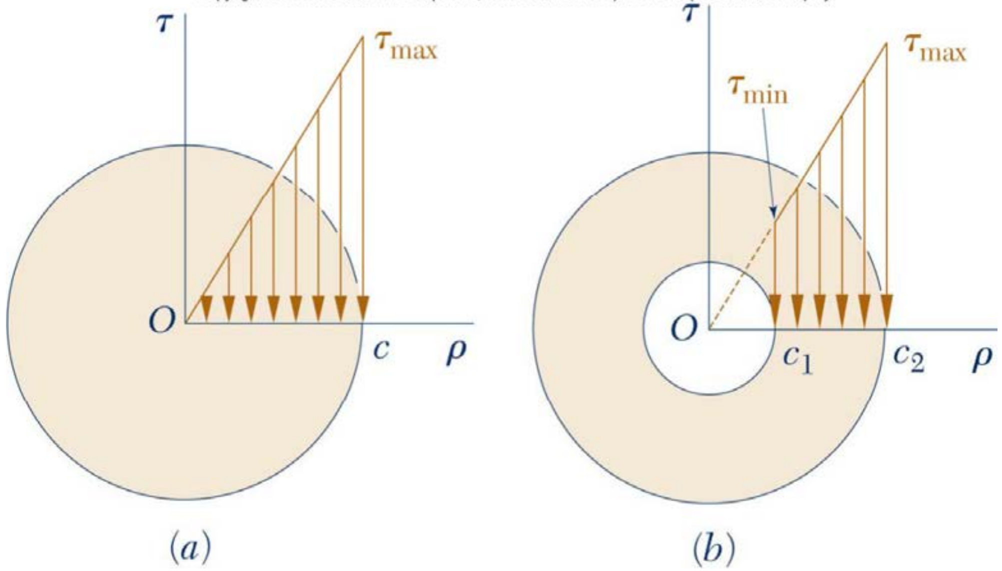
$$\gamma_{\min} = \frac{c_1 \phi}{L}$$

$$\gamma_{\max} = \frac{c_2 \phi}{L}$$



- This applies for all types of materials: elastic, linear, non-linear, plastic, etc.

Elastic Shearing Stress



- Calculate shear stress in a bar made of linearly elastic material.
- Recall Hooke's Law for shearing stress: $\tau = G\gamma$

$$\tau_{max} = G\gamma_{max} = \frac{Gc\phi}{L} \Rightarrow \boxed{\tau = \frac{\rho}{c}\tau_{max}}$$

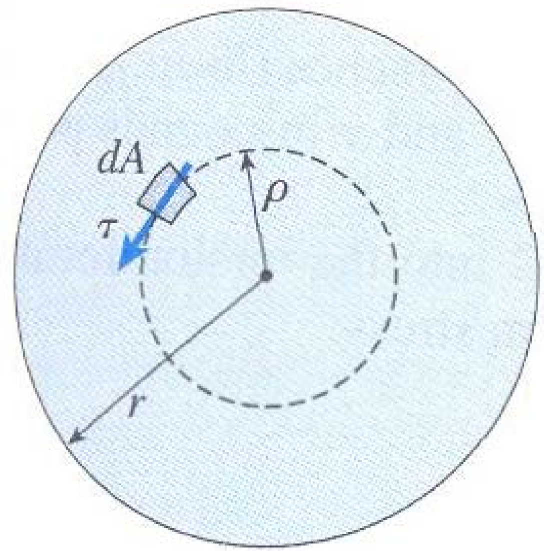
Torque

- We still need to relate τ to the applied torque T, which is generally the known, applied load.
- First, find the resultant moment acting on a cross-section and set this equal to T.

$$\tau = \frac{\rho}{c}\tau_{\max}$$

$$dM = \tau\rho dA = \frac{\rho^2}{c}\tau_{\max} dA$$

$$T = \int_A \frac{\rho^2}{c}\tau_{\max} dA = \frac{\tau_{\max}}{c}\int_A \rho^2 dA$$



Torque *cont'd*

- Continuing from previous slide:

$$T = \frac{\tau_{max}}{c}\int_A \rho^2 dA = \frac{\tau_{max}}{c} J \Rightarrow \boxed{\tau_{max} = \frac{Tc}{J}}, \quad \boxed{\tau = \frac{T\rho}{J}}$$

  – Where J is the polar moment of inertia of the cross section of the bar

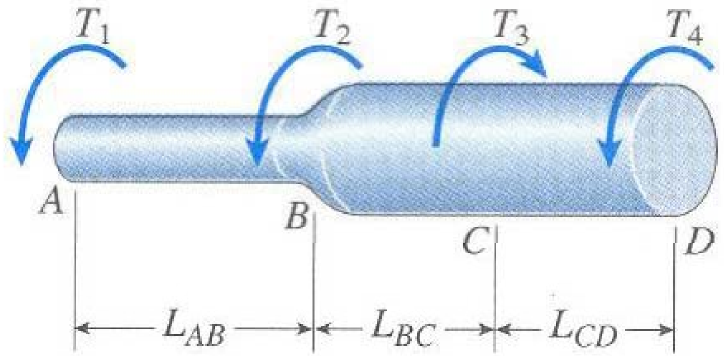
- Plug this into the equation for τ_{max}:

$$\tau_{max} = \frac{Gc\phi}{L} \rightarrow \frac{Gc\phi}{L} = \frac{Tc}{J} \Rightarrow \boxed{\phi = \frac{TL}{GJ}}$$

Torque *cont'd*

- For a non-uniform bar

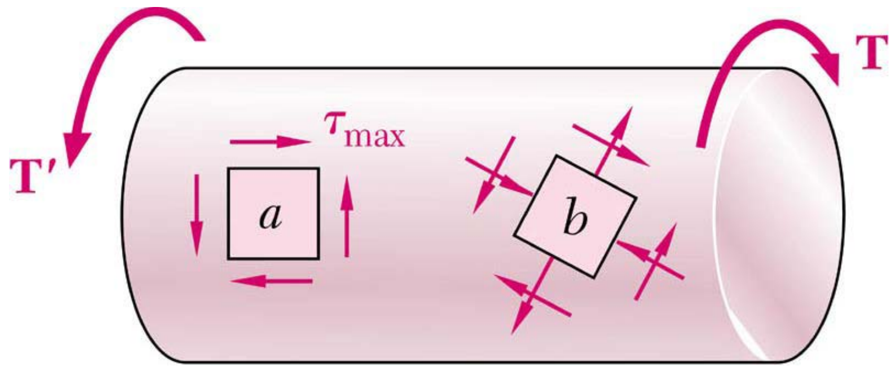
$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$



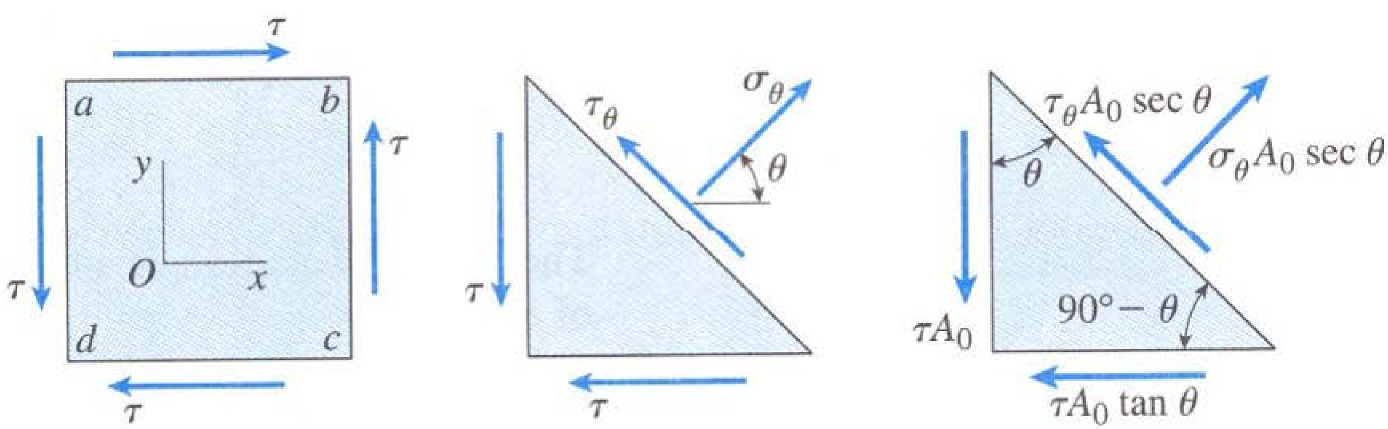
- For a continuously varying bar

$$\phi = \int_0^L \frac{T(x)}{GJ(x)} dx$$

Inclined Plane



- Cut a rectangular element along the plane at an angle θ.



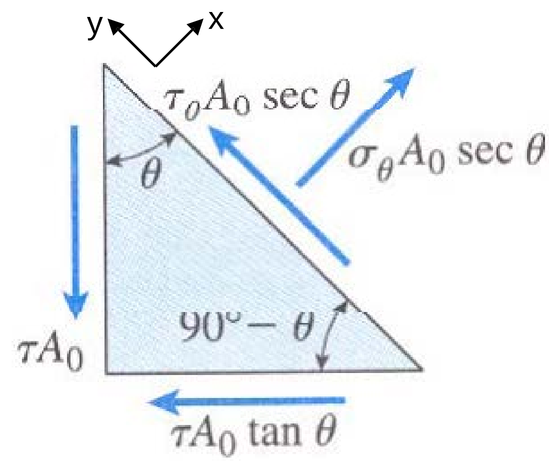
Inclined Plane *cont'd*

- Sum forces in x-direction.

$$\sigma_\theta A_0 \sec\theta - \tau A_0 \sin\theta - \tau A_0 \tan\theta \cos\theta = 0$$

$$\sigma_\theta = \tau \sin\theta \cos\theta + \tau \sin\theta \cos\theta$$

$$\boxed{\sigma_\theta = 2\tau \sin\theta \cos\theta = \tau \sin 2\theta}$$



- Sum forces in y-direction.

$$\tau_\theta A_0 \sec\theta - \tau A_0 \cos\theta + \tau A_0 \tan\theta \sin\theta = 0$$

$$\tau_\theta = \tau \cos^2\theta - \tau \sin^2\theta$$

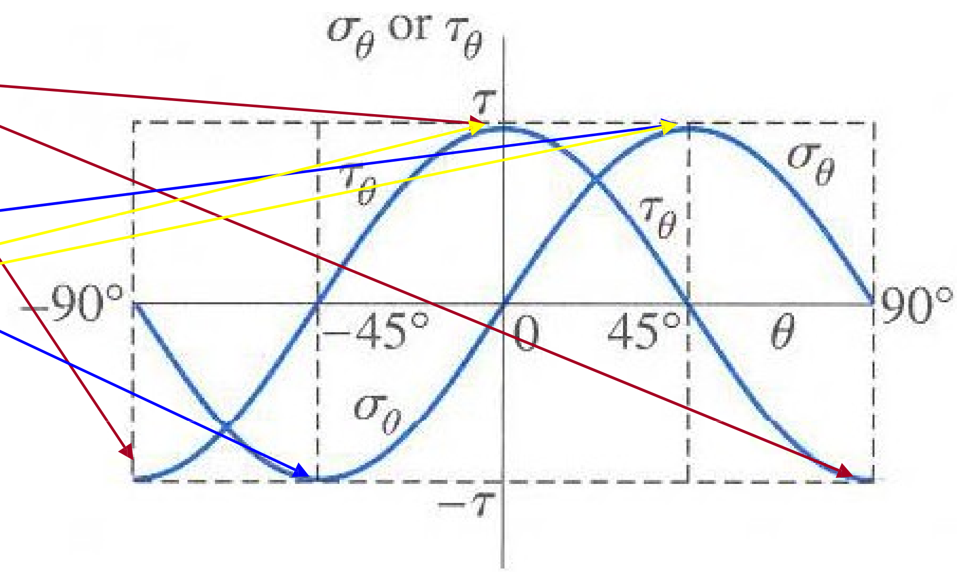
$$\boxed{\tau_\theta = \tau \cos 2\theta}$$

Inclined Plane *cont'd*

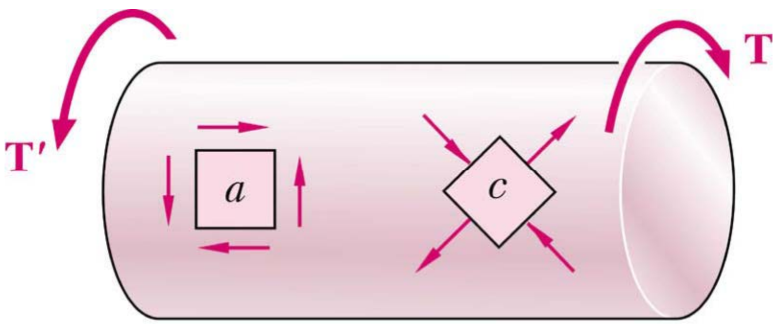
τ_{max} occurs at $\theta = 0^o, \pm 90^o$

- σ_{max} occurs at $\theta = \pm 45^o$

$\tau_{max} = \sigma_{max}$



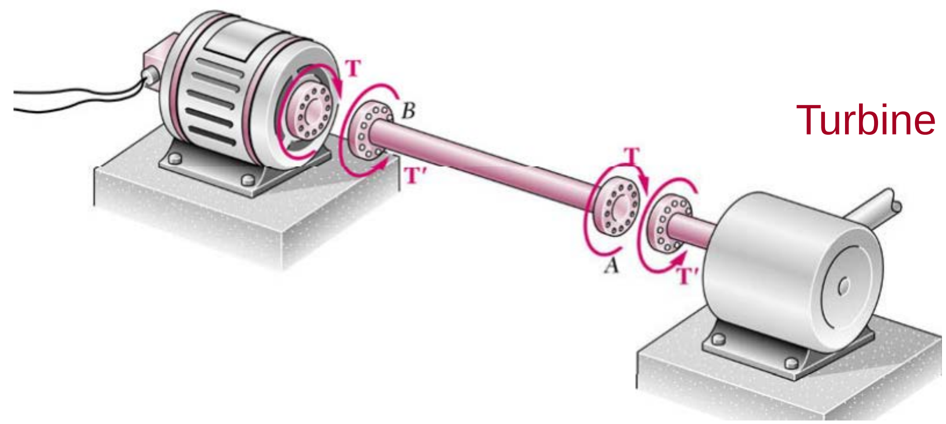
- When σ_θ is max, $\tau_\theta = 0$, and when τ_θ is max, $\sigma_\theta = 0$.



$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm\frac{Tc}{J}$$

Transmission Shafts

- In a transmission, a circular shaft transmits mechanical power from one device to another.



  - ω = angular speed of rotation of the shaft
  - The shaft applies a torque T to another device
  - To satisfy equilibrium the other device applies torque T to the shaft.
  - The power transmitted by the shaft is

$$P = T\omega$$

Transmission Shafts *cont'd*

- Units for P=Tω
  - ω = rad/s
  - T = N·m (SI)
  - T = ft·lb (English)
  - P = Watts (1 W = 1 N·m/s) (SI)
  - P = ft·lb/s (1 horsepower = hp = 550 ft·lb/s) (English)

- We can also express power in terms of frequency.

$$\omega = 2\pi f \qquad f = Hz = s^{-1}$$

$$\boxed{P = 2\pi f T}$$

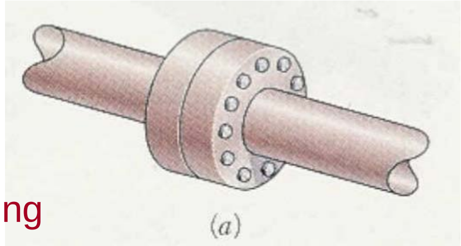
Stress Concentrations in Circular Shafts

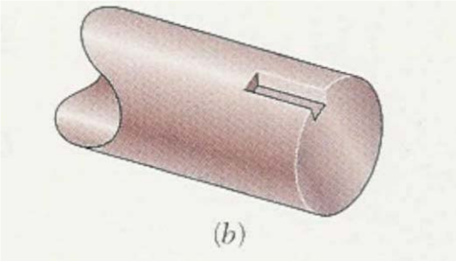
- Up to now, we assumed that transmission shafts are loaded at the ends through solidly attached, rigid end plates.

- In practice, torques are applied through flange couplings and fitted keyways, which produce high stress concentrations.

Fitted keyway

Flange coupling





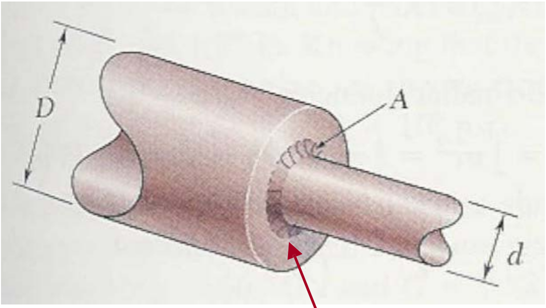
- One way to reduce stress concentrations is through the use of a fillet.

Stress Concentrations in Circular Shafts *cont'd*

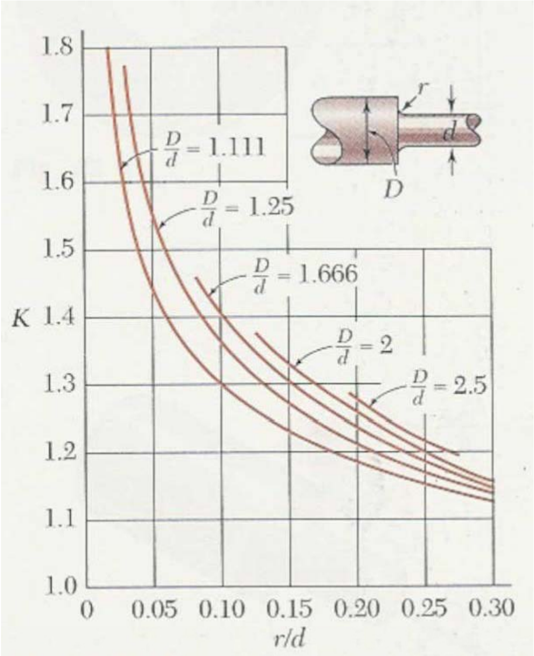
- Maximum shear stress at the fillet

$$\tau_{\max} = K \frac{Tc}{J}$$

  - Tc/J is calculated for the smaller-diameter shaft
  - K = stress concentration factor



Fillet



Στρέψη – περιορισμοί για άξονες κυκλικής διατομής

- Το μήκος του μέλους που υπόκειται σε στρέψη είναι σημαντικά μεγαλύτερο από τη μεγαλύτερη διάσταση της διατομής.
- Η περιοχή που εξετάζεται δεν περιέχει συγκεντρώσεις τάσεων.
- Η μεταβολή της εξωτερικής στρέψης ή η μεταβολή των εμβαδών των διατομών είναι βαθμιαία, με εξαίρεση τις περιοχές συγκέντρωσης τάσεων.
- ΟΙ εξωτερικές στρεπτικές ροπές δεν εξαρτώνται από το χρόνο, το πρόβλημα είναι στατικό.
- Η διατομή είναι κυκλική, επιτρέποντας τη χρήση της συμμετρίας ως προς τον άξονα για τη μείωση της παραμόρφωσης.

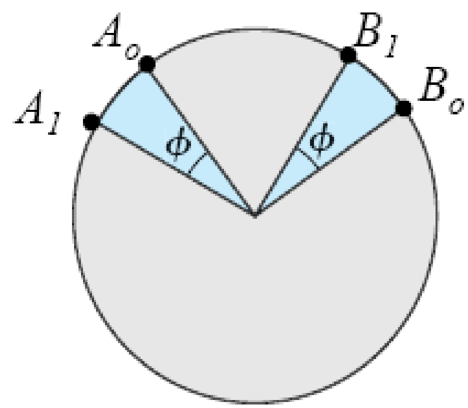
Στρέψη - Παραδοχές

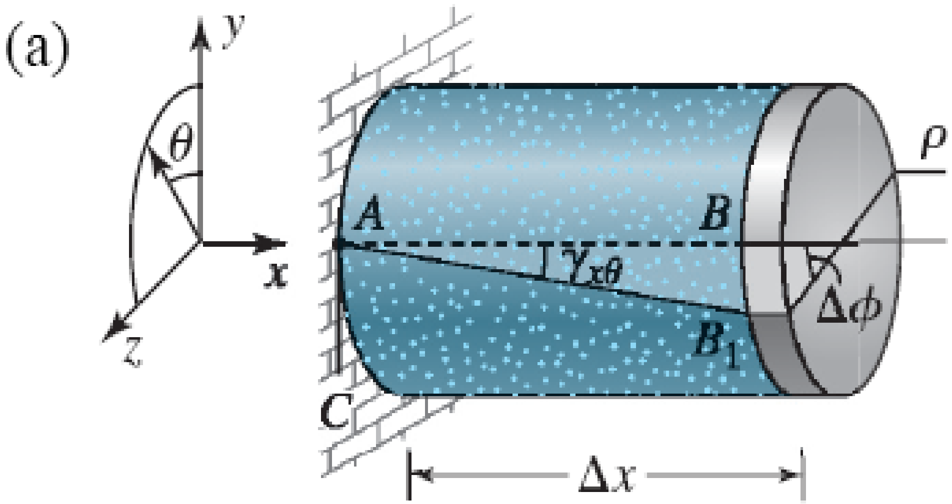
1. Τα επίπεδα τμήματα που είναι κάθετα στον άξονα περιστροφής παραμένουν επίπεδα κατά τη διάρκεια της παραμόρφωσης.
2. Σε μια διατομή, όλες οι ακτινικές γραμμές περιστρέφονται κατά ίσες γωνίες κατά τη διάρκεια της παραμόρφωσης.
3. Οι ακτινικές γραμμές παραμένουν ευθείες κατά τη διάρκεια της παραμόρφωσης.
4. Οι παραμορφώσεις είναι μικρές.
5. Το υλικό είναι γραμμικά ελαστικό.
6. Το υλικό είναι ισότροπο.
7. Το υλικό είναι ομογενές κατά μήκος της περιοχής της διατομής.

$$\phi = \phi(x)$$

A_o, B_o —Initial position

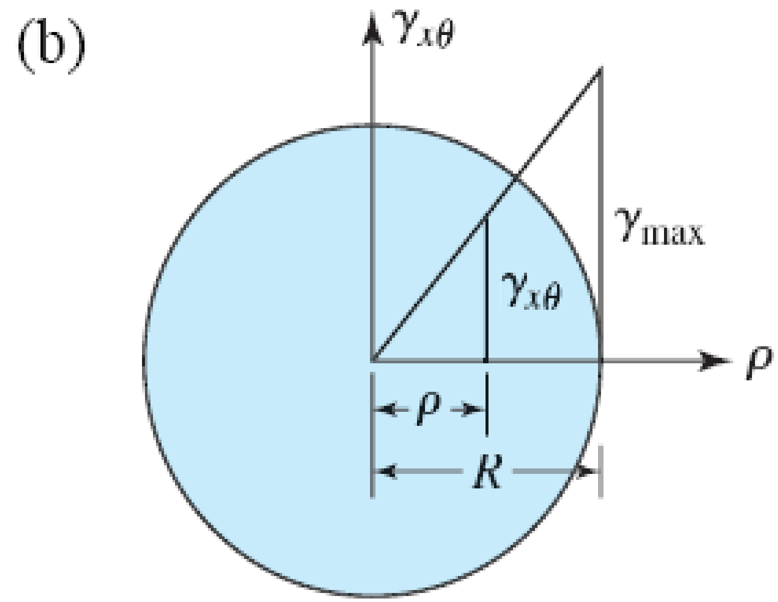
A_1, B_1 —Deformed position





$$\tan\gamma_{x\theta} \approx \gamma_{x\theta} = \lim_{AB\to 0}\left(\frac{BB_1}{AB}\right) = \lim_{\Delta x\to 0}\left(\frac{\rho\Delta\phi}{\Delta x}\right) \quad \text{or}$$

$$\gamma_{x\theta} = \rho\frac{d\phi}{dx}$$



$$\gamma_{x\theta} = \frac{\gamma_{max}\rho}{R}$$

$$\tau = G\gamma, \qquad \tau_{x\theta} = G\rho\frac{d\phi}{dx}$$

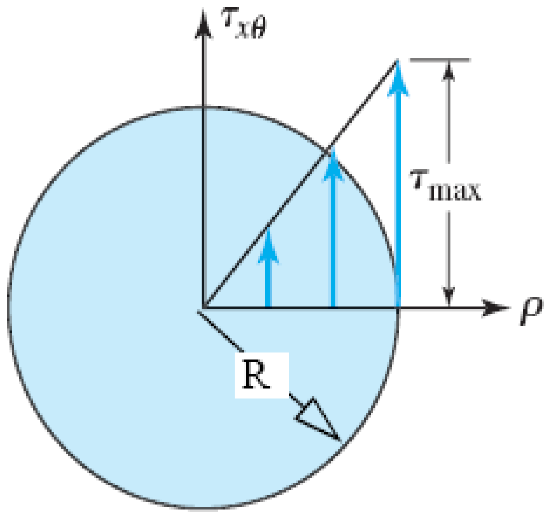
$$\boldsymbol{T} = \int_A G\rho^2 \frac{d\phi}{dx}\, dA = \frac{d\phi}{dx}\int_A G\rho^2\, dA$$

$$\boldsymbol{T} = G\frac{d\phi}{dx}\int_A \rho^2\, dA = GJ\frac{d\phi}{dx}$$

$$J = \int_A \rho^2\, dA = \frac{\pi}{2}R^4 = \frac{\pi}{32}D^4$$

$$\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ}$$

$$\tau_{x\theta} = \frac{\boldsymbol{T}\rho}{J}$$



$$\phi_2 - \phi_1 = \int_{\phi_1}^{\phi_2} d\phi = \int_{x_1}^{x_2} \frac{T}{GJ}\, dx$$

ΠΑΡΑΔΟΧΕΣ

- Το υλικό είναι ομογενές μεταξύ x_1 και x_2 (G=σταθερό)
- Ο άξονας δεν λαμβάνει κωνοειδές σχήμα (J =σταθερή)
- Η εξωτερική (και επομένως και η εσωτερική) στρέψη δεν μεταβάλλεται με το x μεταξύ x_1 και x_2 (T = σταθερό)

$$\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}$$

NON-UNIFORM TORSION

- Uniform/Pure torsion – torsion of prismatic bar subjected to torques acting only at the ends
- Non-uniform torsion– the bar need not be prismatic and the applied torque may act anywhere along the axis of bar
- Non-uniform torsion can be analysed by
  - Applying formula of pure torsion to finite segments of the bar then adding the results
  - Applying formula to differential elements of the bar and then integrating

NON-UNIFORM TORSION

- CASE 1: Bar consisting of prismatic segments with constant torque throughout each segment

$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_p)_i}$$

- CASE 2: Bar with continuously varying cross sections and constant torque

$$\phi = \int_0^L d\phi = \int_0^L \frac{T dx}{G I_p(x)}$$

NON-UNIFORM TORSION

- CASE 3: Bar with continuously varying cross sections and continuously varying torque

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)dx}{GI_p(x)}$$

NON-UNIFORM TORSION

- Limitations
  - Analyses described valid for bar made of linearly elastic materials
  - Circular cross sections (Solid /hollow)
- Stresses determined from the torsion formula valid in region of the bar away from stress concentrations (diameter changes abruptly/concentrated torque applied
  - For the case above, Angle of twist still valid
  - Changes in diameter is are small and gradually (angle of taper max 10o)

SOLID NON-CIRCULAR SHAFTS

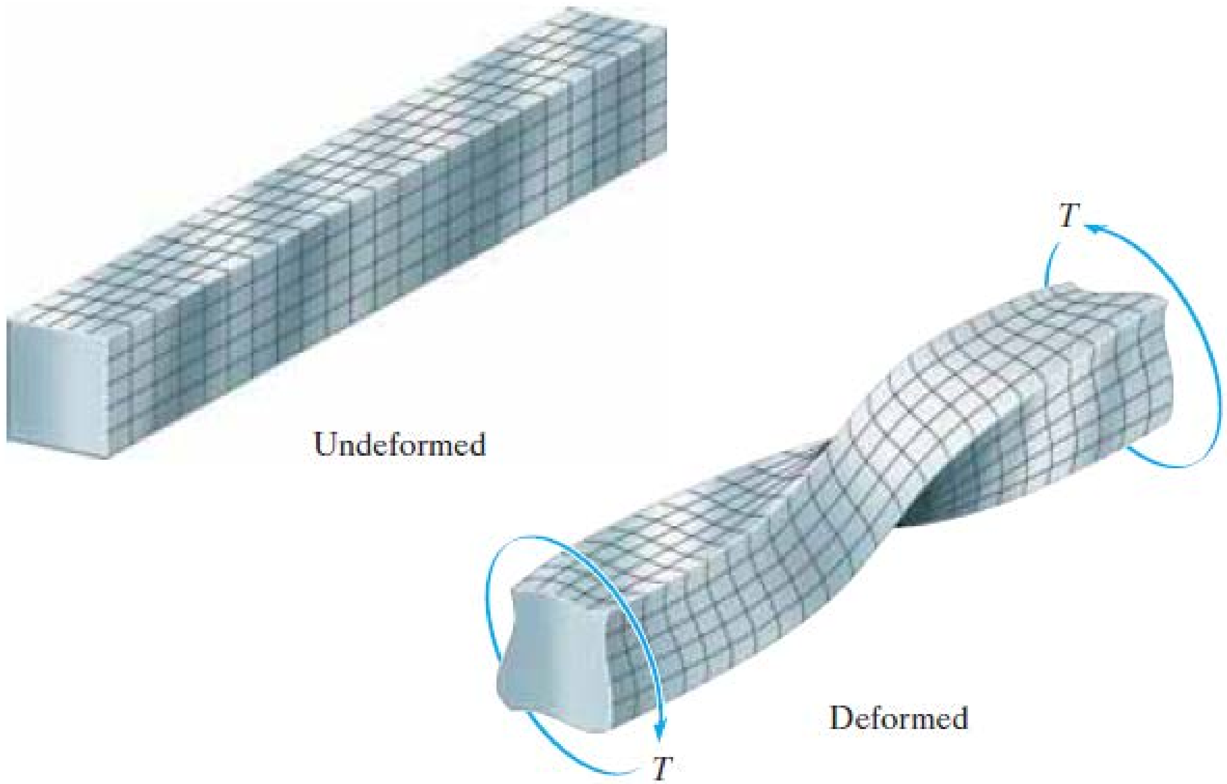
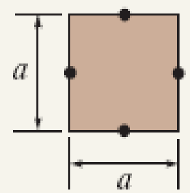
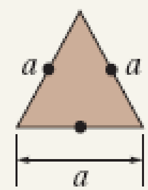
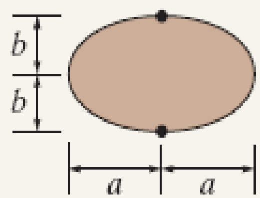


Fig. 5–25

Solid Noncircular Shafts

The maximum shear stress and the angle of twist for solid noncircular shafts are tabulated as below:

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,T}{a^4G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

Comparison of Axial and Torsion formulae.

AE =Axial rigidity

Axial Stiffness

$$k_A = \frac{AE}{L}$$

Axial Flexibilty: $f_A = k_A^{-1}$

Axial displacement

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Axial stress

$$\sigma = \frac{P}{A}$$

GJ =Torsional rigidity

Torsional Stiffness

$$k_T = \frac{GJ}{L}$$

Torsional Flexibilty: $f_T = k_T^{-1}$

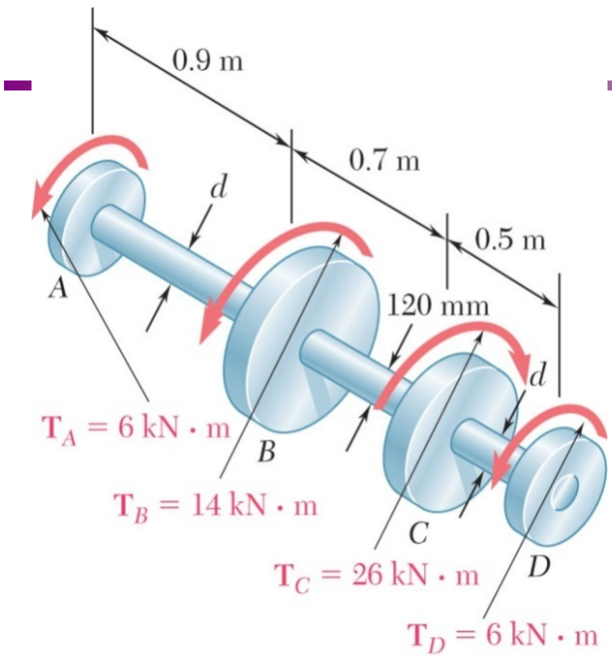
Torsional displacement

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Torsional stress

$$\tau = \frac{T\rho}{J}$$

Sample Problem 1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid and of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION:

Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.

Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.

Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

Sample Problem 1

SOLUTION:

Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.

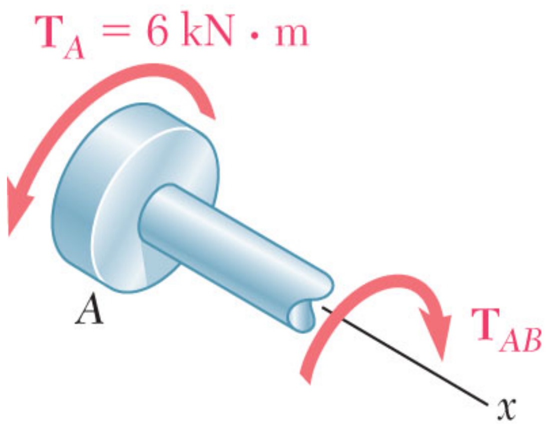


Fig. 1 Free-body diagram for section between A and B.

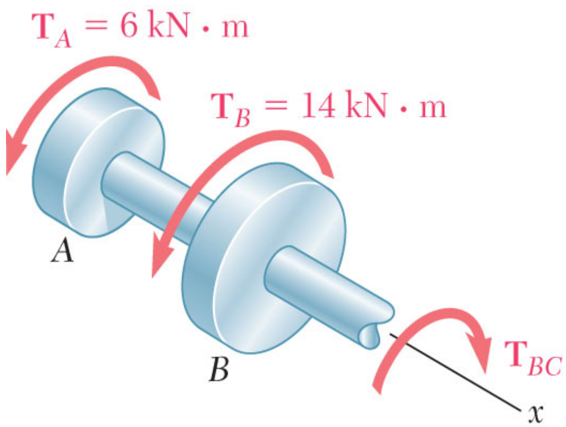


Fig. 2 Free-body diagram for section between B and C.

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$

$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$

$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

Sample Problem 1

Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.

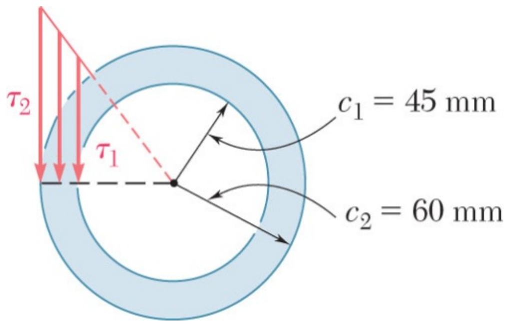


Fig. 3 Shearing stress distribution on cross section.

$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\text{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\text{kN}\cdot\text{m})(0.060\,\text{m})}{13.92\times10^{-6}\,\text{m}^4}$$

$$= 86.2\,\text{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\text{MPa}} = \frac{45\,\text{mm}}{60\,\text{mm}}$$

$$\tau_{min} = 64.7\,\text{MPa}$$

$$\tau_{max} = 86.2\,\text{MPa}$$

$$\tau_{min} = 64.7\,\text{MPa}$$

Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

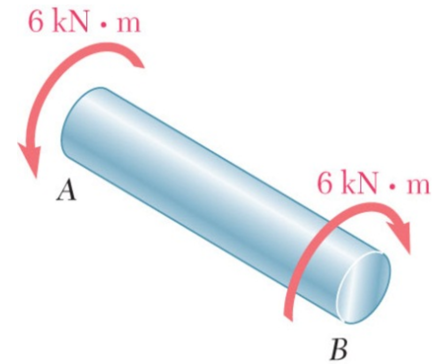


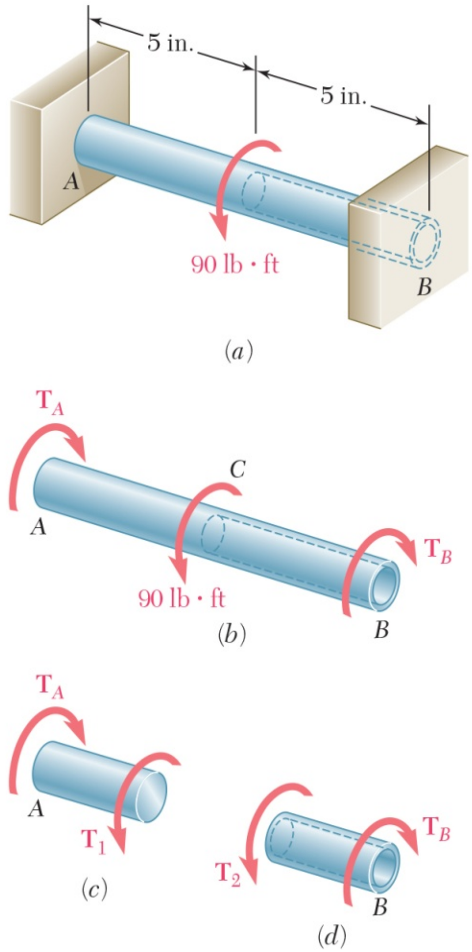
Fig. 4 Free-body diagram of shaft portion AB.

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\text{kN}\cdot\text{m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times10^{-3}\,\text{m}$$

$$d = 2c = 77.8\,\text{mm}$$

Statically Indeterminate Shafts



(a) Shaft with central applied torque and fixed ends. (b) free-body diagram of shaft AB. (c) Free-body diagrams for solid and hollow segments.

Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.

From a free-body analysis of the shaft,

$$T_A + T_B = 90\,\text{lb} \cdot \text{ft}$$

which is not sufficient to find the end torques. The problem is *statically indeterminate*.

Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90\,\text{lb} \cdot \text{ft}$$

Example 2

The tapered shaft is made of a material having a shear modulus *G*. Determine the angle of twist of its end *B* when subjected to the torque.

Solution:

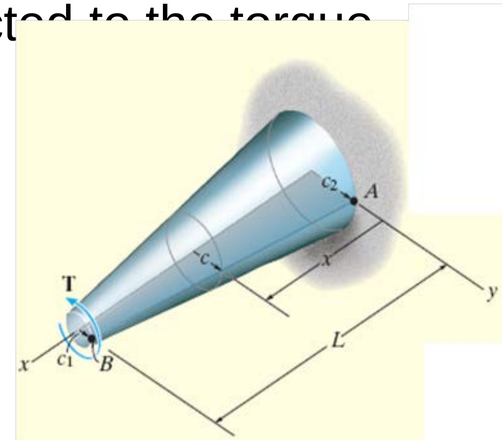
From free body diagram, the internal torque is *T*.

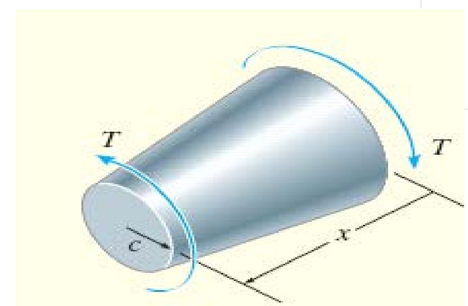
$$\frac{c_2 - c_1}{L} = \frac{c_2 - c}{x} \Rightarrow c = c_2 - x\left(\frac{c_2 - c_1}{L}\right)$$

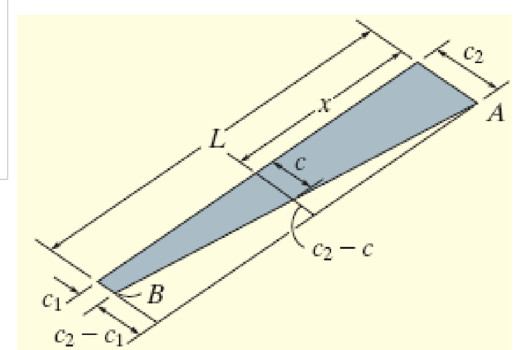
Thus, at *x*, $J(x) = \dfrac{\pi}{2}\left[c_2 - x\left(\dfrac{c_2 - c_1}{L}\right)\right]^4$

For angle of twist,

$$\phi = \frac{2T}{\pi G}\int_0^L \frac{dx}{\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right)\right]^4} = \frac{2TL}{3\pi G}\left(\frac{c_2^2 + c_1 c_2 + c_1^2}{c_1^3 c_2^3}\right) \quad \textbf{(Ans)}$$







STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS

Procedure for analysis:

use both equilibrium and compatibility equations

Equilibrium

Draw a free-body diagram of the shaft in order to identify all the torques that act on it. Then write the equations of moment equilibrium about the axis of the shaft.

Compatibility

To write the compatibility equation, investigate the way the shaft will twist when subjected to the external loads, and give consideration as to how the supports constrain the shaft when it is twisted.

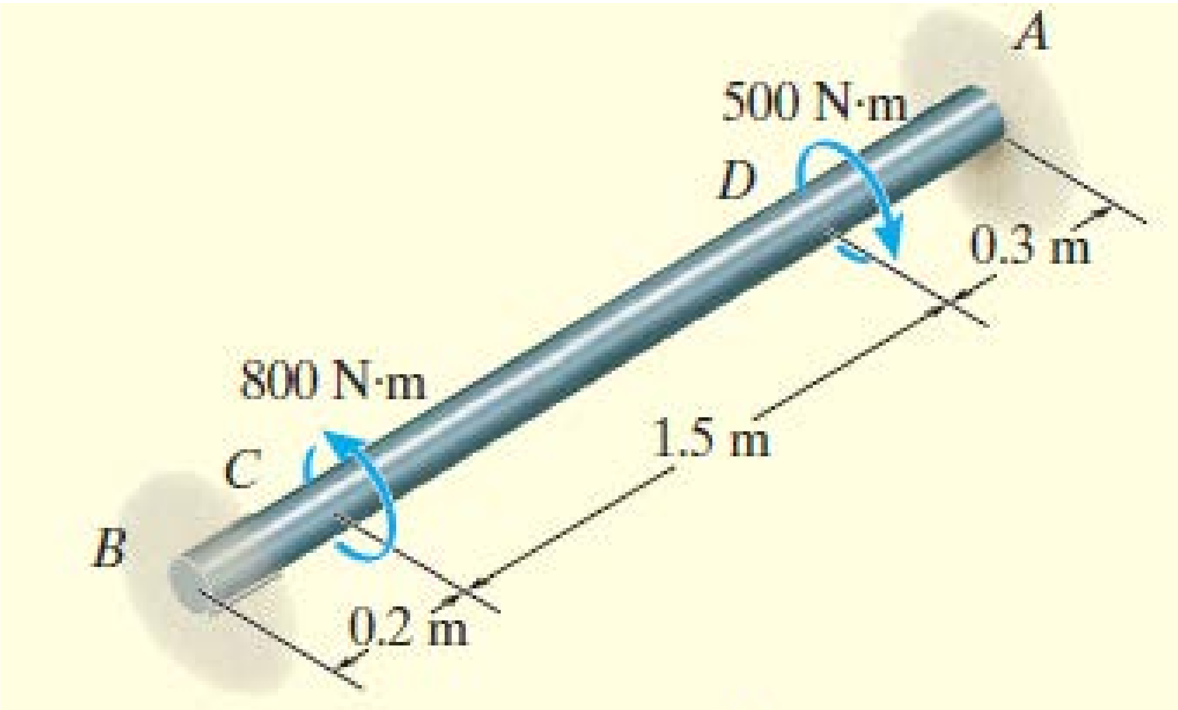
STATICALLY INDETERMINATE TORQUE-LOADED MEMBERS (cont)

Express the compatibility condition in terms of the rotational displacements caused by the reactive torques, and then use a torque-displacement relation, such as $\Phi = TL/JG$, to relate the unknown torques to the unknown displacements.

Solve the equilibrium and compatibility equations for the unknown reactive torques. If any of the magnitudes have a negative numerical value, it indicates that this torque acts in the opposite sense of direction to that indicated on the free-body diagram.

EXAMPLE 3

The solid steel shaft shown in Fig. has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports *A* and *B*.



(a)

EXAMPLE 3 (cont)

Solution

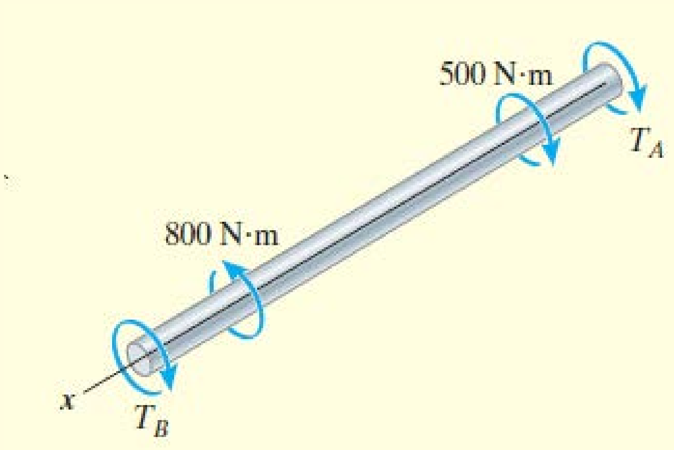
It is seen that the problem is statically indeterminate since there is only *one* available equation of equilibrium and there are 2 unknowns

$$\sum M_x = 0$$

$$-T_b + 800 - 500 - T_A = 0 \qquad (1)$$

Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero.

$$\phi_{A/B} = 0$$



EXAMPLE 3 (cont)

Solution

Using the sign convention established,

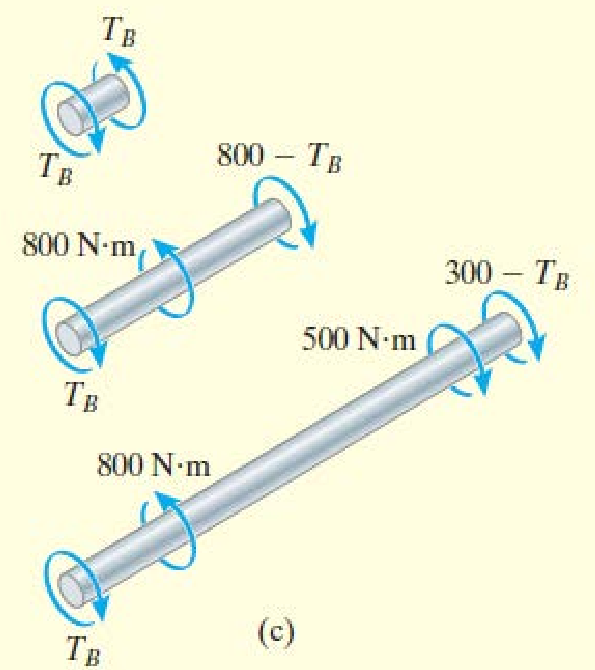
$$\frac{-T_B(0.2)}{JG} + \frac{(800-T_B)(1.5)}{JG} + \frac{(300-T_B)(0.3)}{JG} = 0$$

$$T_B = 645 \text{ N} \cdot \text{m} \quad (\text{Ans})$$

Using Eq. 1,

$$T_A = -345 \text{ N} \cdot \text{m}$$

The negative sign indicates that acts in the opposite direction of that shown in Fig. (b)



(c)