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ENGINEERING MECHANICS

Fourth Fully Revised and Enlarged Edition

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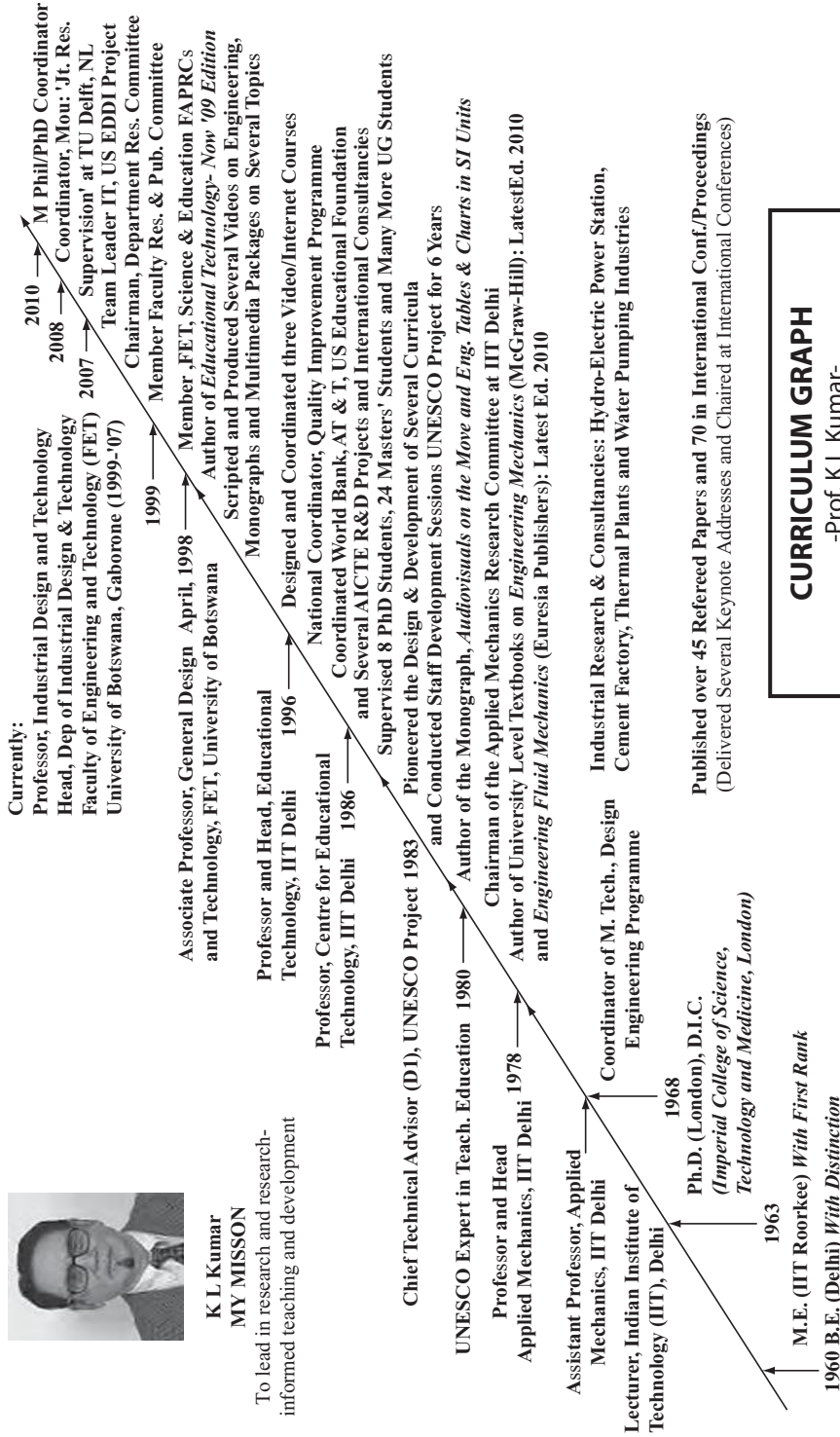


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K.L. Kumar
MY MISSION

To lead in research and research-informed teaching and development



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Fourth Fully Revised and Enlarged Edition

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Preface

This edition is a reincarnation of the popular textbook born in 1986! The book begins with glimpses of the fascinations in mechanics by way of some possible applications. It is new in all respects—computer-composed text and figures, all revised and improved additional solved examples and well-arranged experiments in order to make things clear to teachers and students alike. We would like to express our gratitude to all the teachers and students who provided useful feedback. This edition should make them glad they did!

I hope that students of Civil, Mechanical, Electrical, Electronics, Chemical, Aero, Agricultural and other engineering and design disciplines find this book a worthwhile possession in their initial course until design projects, and thereafter as reference in their working life.

Chapters on applications of Statics and Dynamics, e.g., S1, S2, ..., D1, D2, ... have been designed to 'stand alone' so that teachers and students may suit their curricula and gain greater flexibility in their sequence of teaching–learning. It is more comprehensive than the syllabus of any single university or institution, thereby enabling it to cater to the syllabi of different universities.

Let me also add that, in revising the text, I have retained all the positive aspects of the book, i.e., emphasis on concepts, abundance of examples, extent of coverage, review questions and problems, experimentation in mechanics, etc.

Here are some of the highlights of this edition:

- Glimpses of the Fascinations of Studying Mechanics at the beginning
- Terminal Objectives, i.e., Intended Student Abilities for all chapters
- Review of SI Units and Vector Operations for ease of use by the students
- Separate Chapters on Applications of Statics
- Completely revised figures and text for all chapters
- Separate Chapters on Applications of Dynamics
- More Solved Examples with alternative solutions and comments
- Experiments in mechanics in different chapters, as appropriate
- Concept-review questions and Multiple-choice questions for all chapters
- Hints for solutions to all Tutorial problems at the end of the book
- Your Feedback Form for the authors for subsequent editions
- Comprehensive Pedagogy including the following
 - 330 Figures
 - 15 Laboratory Experiments with relevant topics
 - 111 Concept Review Questions
 - 341 Tutorial Problems
 - 115 Multiple Choice Questions

To give you a brief overview of what you'll find if you mechanically flip through this book: is designed to refresh your memory with foundational concepts that you may have consigned to oblivion at vacation time. It introduces the subject formally and establishes Newton's second law as the 'law of motion'. The Review section should rake up your memory of SI units and vector operations. takes you on an exhaustive journey through forces and force systems—bringing out the concept of resultant forces and moments—while experiments with the conditions necessary for equilibrium. This chapter

lays the foundation of Statics and leads us to its applications covered in Section S on Simple Structures, Thin Beams and Systems with Friction including simple machines.

Discover why it's important to compute the centroid, centre of mass and centre of gravity of an object in which enables us to determine the central points for lengths, areas, volume, mass and weight! is where things begin to move and deals with their motion under the application of forces. In these chapters, among other things, enrich your knowledge of the kinematics and dynamics of a particle. Allow yourself the flexibility to analyze rigid bodies in s, and don't overlook 8 in a moment of inertia! Applications of dynamics are dealt with in section D; these are Impact of Two Bodies in Section D1 and Central Force Motion in Section D2.

Did you know that virtual work can be hard work too? is a little digression on Virtual Work, Potential Energy and Work-Energy principles. And finally, swing along to to create and investigate vibrations of some simple mechanical systems.

I invite you all to go through the book, use it and let me know if it helps you to learn the subject by way of building your *fundas*. I have added 'Hints to Tutorial Problems' for all problems at the end of different chapters and placed the same at the end. The idea is to give you some assistance in solving the problems as and when you need it! It would reduce the teacher's task a bit and it should promote self-learning by the students. That is really my intention in these days of self-reliance and fast pace of learning.

Acknowledgements are due to all those staff members, who have contributed by way of feedback and additional information; also to Tata McGraw Hill who have patiently worked to produce it elegantly.

Writing, revising and updating a good book is a massive job; I have taken Ms, a qualified scholar, as my coauthor and manager who has indeed done a great deal of work in bringing out this reincarnation. I trust that the readers will give her all the support she deserves to keep the book at the leading edge!

Enjoy learning the subject, dear students and teachers! If you like the book, tell everyone else. And if you don't, tell me!

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Publisher's Note

A number of experts took pains to provide valuable feedback about the book. Our heartfelt gratitude goes out to those whose names are given below.

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| | | |
|---|---|---|
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Terminal Objectives

(Intended Student Abilities on Learning the Subject)

The subject matter has been so developed and large number of illustrations, solved examples recapitulation, multiple-choice items and tutorial problems have been provided in order to enable the learner, i.e., the engineering student, to achieve a number of terminal objectives. Objectives of chapters on applications are clubbed with the main chapters. On successful completion of all the activities, planned and provided chapterwise, a learner will be able to:

1.0

- Explain the importance of Mechanics in the context of engineering;
- idealise a body as a particle, rigid body, continuum or a deformable substance;
- draw free body diagrams of the components of a given system and state the basic laws of mechanics;
- perform basic vector operations, e.g., addition, dot product, cross products, differentiation and integration, divergence and curl of vectors, and
- employ vector methods to describe the motion of a point.

2.0

- Identify the types of force fields and force systems, and
- determine the resultant of a concurrent force system, a parallel force system, a coplanar force system and a spatial force system.

3.0

- Derive the conditions of equilibrium of a statical system; i.e., a particle, a rigid body and a system of particles,
- solve simple equilibrium problems with plane and spatial force systems;
- identify simple just-rigid trusses,
- analyse the forces in the members of just-rigid trusses by the method of joints and the method of sections;
- define loading, shear force and bending moment at a cross section of a beam and relate them;
- draw the bending-moments diagram for a simply supported beam;
- state the laws of sliding friction;
- explain the mechanism of rolling; and
- solve simple mechanical problems with frictional forces.

4.0

- Explain the concepts of centroid, centre of mass and centre of gravity;
- determine the centroid of a given length, an area and a volume;
- locate the centre of mass and centre of gravity of a given body.

5.0

- Analyse the motion of a particle with reference to fixed rectangular coordinates, cylindrical polar coordinates or path coordinates;
- analyse the motion of a particle with reference to a moving frame of reference; and
- interpret the terms Coriolis acceleration, tangential acceleration and centripetal acceleration.

6.0

- Formulate the equation of motion for a particle or for the centre of mass of a system;
- solve the equation and describe the motion;
- derive the work-energy principle and hence the energy conservation principle in mechanics;
- derive the impulse-momentum principle and hence the momentum conservation principle in mechanics
- establish the moment of momentum equation and hence the conservation of moment of momentum principle; and
- apply different principles to study the motion of a particle or the motion of the centre of mass of a system.

7.0

- Classify the types of motions of a rigid body;
- locate the instantaneous centre of rotation of a body; and
- determine the relative velocity and acceleration for points on a rigid body.

8.0

- Determine the area moment of inertia of simple and composite sections;
- compute the mass moment of inertia for thin plates and solid bodies; and
- apply the parallel-axis theorem vis-a-vis area and mass moments of inertia.

9.0

- Analyse the motion of a rigid body in translation in fixed-axis rotation and in general plane motion vis-a-vis Newton's laws and the Euler's equation of motion;
- apply the work-energy principle to analyse the plane motion of a rigid body;
- describe the plane motion of a rigid body by employing the impulse-momentum principle;
- develop the equation of motion for a rigid body moving in space under the action of an external force system;
- describe the gyroscopic action for a spinning body undergoing precession;
- classify different types of impacts of two rigid bodies;
- define the coefficient of restitution;
- solve simple problems on direct and indirect central impact;
- analyse a problem with central force motion;

- describe the trajectory of a particle under a central force; and
- solve simple problems for synchronous satellites.

10.0

- Establish and apply the principle of virtual work to analyse the motion of simple mechanical systems, and
- develop and apply the principle of minimum potential energy and conditions of equilibrium for simple engineering problems.

11.0

- Identify the basic elements of a mechanical system and write their constitutive equations;
- solve simple problems on linear and angular free vibrations; and
- describe pendulum motions for simple, compound, torsional and conical pendulums.

S S S S S SS S

S S S S S S S S S

A video, 'MECHANICS—An Insight' developed by the author at IIT, Delhi goes a long way in clarifying some basic concepts in mechanics and it focuses on some common misconceptions in the subject. Copies of the video are available from the Educational Technology Facility, IIT, New Delhi 110 016, India. Website:

*<http://web.iitd.ac.in/eklavya/vediocourse/iitscourselist.pdf>
(Look up General Engineering and Sciences)*

Notation

Symbol

| | |
|-----------------|--|
| a | Acceleration |
| a | Constant; distance; semi-major axis of an ellipse |
| a_{BA} | Acceleration of B relative to A |
| a_{cor} | Coriolis acceleration |
| A | Area |
| B | Body force |
| b | Width; distance; semi-minor axis of an ellipse |
| c | Velocity of light |
| c | Constant |
| C | Couple |
| C | Centroid |
| C | Centre of mass |
| d | Distance |
| e | Coefficient of restitution; base of natural logarithms |
| e_n, e_t | Unit vectors along normal and tangent |
| e_r, e_θ | Unit vectors in radial and transverse directions |
| f | Frequency; scalar function |
| f, F | Force; friction force |
| g | Acceleration due to gravity |
| G | Centre of gravity; constant of gravitation |
| h | Height |
| h | Angular momentum |
| i, j | Unit vectors along coordinate axes |
| I_{xx} | Moment of inertia (about x axis) |
| I_{xy} | Moment of inertia (about x and y axes) |
| I | Instantaneous centre of rotation |
| k | Spring constant; Radius of gyration |
| l | Length |
| L | Length; span |
| m | Mass |
| m, M | Moment of a force |
| M_e | Mass of earth |
| M | Mass |
| N | Normal component of reaction |
| O, o | Origin of coordinates |
| p | Pressure |
| p | Momentum |
| P | Power; point |
| P | Position vector |
| r | Radius; distance |
| r | Reaction; distance |

xx

Notation

| | |
|-------------------|--|
| R | Radius of earth |
| s | Distance; length of arc |
| t | Time; thickness; tangential direction |
| t | Tension coefficient |
| T | Tension; kinetic energy |
| | Velocity |
| u | Speed along x -axis |
| | Velocity |
| v | Speed along y -axis |
| $v_{B/A}$ | Velocity of B relative to A |
| V | Volume; potential energy |
| w | Load per unit length |
| W, W | Weight; load |
| W | Work |
| x, y, z | Rectangular coordinates; distances |
| r, θ, z | Cylindrical coordinates |
| R, θ, ϕ | Spherical coordinates |
| θ, α | Angular acceleration |
| δ | Virtual displacement |
| δW | Virtual work |
| η | Efficiency |
| μ | Coefficient of friction; dynamic viscosity |
| ρ | Density; radius of curvature |
| τ | Period; periodic time; shear stress |
| ϕ | Angle of friction; angle |
| ω, ω | Angular velocity |
| ω | Circular frequency |
| Ω | Angular velocity of frame of reference |

Prefixes

| | |
|-----------------------|--|
| Δ, δ, d | Infinitesimal increment |
| $\partial/\partial x$ | Partial differential with respect to x |
| d/dx | Ordinary differentiation with respect to x |

Subscripts

| | |
|-------------------|--|
| x, y, z | for components along rectangular coordinates |
| r, θ, z | for components along cylindrical coordinates |
| R, θ, ϕ | for components along spherical coordinates |
| 1, 2, 3 | at, for or about |
| R | Reaction; e.g. R_1, R_2 |
| b | for bending |
| t | for torsion |
| a | for axial or normal |
| S | for shear |

| | | | | |
|---|---|---|---|---|
| B | s | s | | s |
| | s | s | s | s |
| | s | s | | |

The Fascinations of Studying Mechanics!

Prelude

Mechanics is perhaps the most fascinating core subject in engineering and technology. Once you know the things you can do by employing the basic principles of mechanics, you may want to learn more about it!!

Take, for example, a structure, say a space truss. You can work out the tensile and compressive forces in its members for any given loading conditions. You can even design one for a given situation. Analyse the forces in the members of cranes and lifting equipment and design those too!



A Timber Frame Truss



Cranes at Work

You can analyse beams and cantilevers; bridges and flyovers, at least by making some simplifying assumptions. You can work out the shearing forces and bending moments at different points, which are so crucial to their safe design.



A Beam Bridge over a Small River



Flyovers and Highways in Hawaii

Lifting machines! You can analyse the operation of all kinds of lifting machines, like screw jacks, winch pulley blocks, etc. and design one for your use. You can ensure that they have a high mechanical advantage and be self-locking.



An Automobile Lifted by a Jack



A Hoist Used in a Workshop

Just imagine the range of mechanisms employed in transmitting motion and force! You will be able to analyse their kinematics and dynamics. You may also experience the thrill of designing a new mechanism for a desired duty.

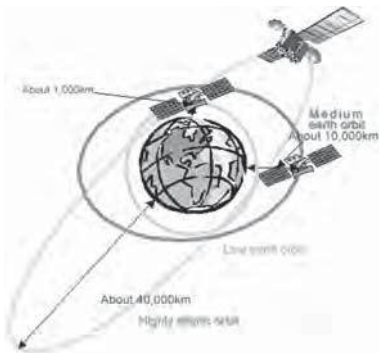


A Four-Bar Mechanism in Use

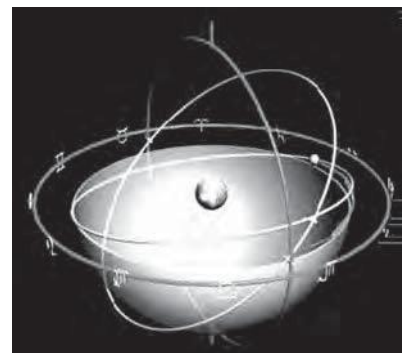


A Rotary Reversing Mechanism

You can analyse the motion of satellites and planetary orbits, which are special cases of central force motion. Use your knowledge as a launching pad into space mechanics by working out the escape velocity for a hyperbolic trajectory!



Low Earth Orbital Motions



Multiple Planetary Motions

Collisions! The origin of the earth is said to be from a Big Bang between two bodies. You can study the impact between two bodies and draw important conclusions to aid you in playing different games and outperforming other players!



Shooting a Ball in Billiards



Positioning the Bat for a Hit in Cricket

The human structural design is indeed the most elegant and perfect of all designs. So are human mechanisms, all the way from your shoulders and elbows down to your ankles and toes!

A sound conceptual understanding of these is essential to Biomechanics, in case you would like to design artificial limbs and support systems for the disabled. You can also improve the quality of life of people by designing ergonomic interfaces between humans and machines, e.g., a bio-signal supervision system providing medical care.

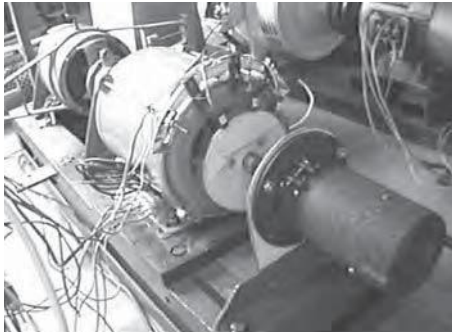


A High-Tech Artificial Leg



A Wheelchair in Use

Electrical, civil, chemical, aeronautical, agricultural and other students in engineering have no less reason to be fascinated. They're the ones who design machines and devices in their domain of work!



Electrical Machines at Work



Innovative Agricultural Equipment



A Super Fast Ferry in Action



A Concorde Being Positioned!

These are but a few of the things you can do during the course of learning mechanics. Just imagine what you would be able to do after completing the course! Maybe study a related course, like Fluid Mechanics or Solid Mechanics or Vibrations of Systems! Possibly design vibratory systems for health and relaxation!

Maybe even take up research and explore the frontiers of knowledge by taking up a course in Advanced Mechanics!



Vibrations Therapy



Vibrations Research Laboratory

Engineering Mechanics: What and Why?

1



1.1 ENGINEERING AND 'ENGINEERING SCIENCES'

Engineering is an activity concerned with the *creation of new systems* for the benefit of mankind. The process of creativity proceeds by way of research, design and development; new systems emerge from innovation and systems may be constituted by mechanical, electromechanical, hydraulic, thermal or other elements. Creation of new systems is thus basic to all engineering. The Living Webster Encyclopedic Dictionary aptly defines engineering as *the art of executing a partial application of scientific knowledge*.

It is important to understand the difference between engineering and science. Science is concerned with a systematic understanding and gathering of the facts, laws and principles governing natural phenomena. Engineering, on the other hand, is an art of utilisation of the established facts, laws and principles to create certain desired phenomena as shown in Fig. 1.1. The activities of science and engineering are thus mutually opposite. Both may proceed through similar ways and means of analysis and synthesis but are oppositely directed. The training of scientists and engineers should be correspondingly designed for their respective objectives.

The sets of core courses meant for engineering students are called *engineering sciences*. These are essentially basic sciences compartmentalised and labelled specially for engineering students with regard to their future responsibility. The existing laws and principles are conveyed

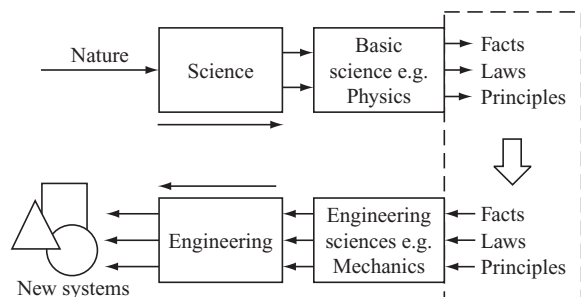


Fig.1.1 Role of engineering sciences

to the students by the engineering-science courses and emphasis is laid on their application to real-life problems. Some of the engineering-science courses being offered in India and abroad are: Mechanics, Manufacturing Processes, Energy Conversions, Transport Phenomena, Material Science and Design Engineering.

1.2 MECHANICS AND ITS RELEVANCE TO ENGINEERING

Mechanics is the physical science concerned with the statical and dynamical behaviour of material bodies in the presence of mechanical disturbances. Since such behaviour is of interest to mechanical, civil, electrical, chemical, aeronautical, textile, metallurgical and mining engineers, it is appropriate to conclude that the subject of mechanics lies at the core of all engineering analysis.

Engineering mechanics refers to a course in mechanics tailored exclusively for engineers. Essential features of such a course are:

1. The subject matter is not presented as rigorously as a course in analytical or axiomatic mechanics may demand.
2. On the other hand, the contents are not just a series of applications as implied by *Applied Mechanics* but a thorough grounding of the basic principles together with engineering applications.
3. The course is integrated to provide a sound foundation in engineering-science.
4. The course comprises the foundation for a number of courses that are to be built upon it. Some of them are shown in Fig. 1.2.

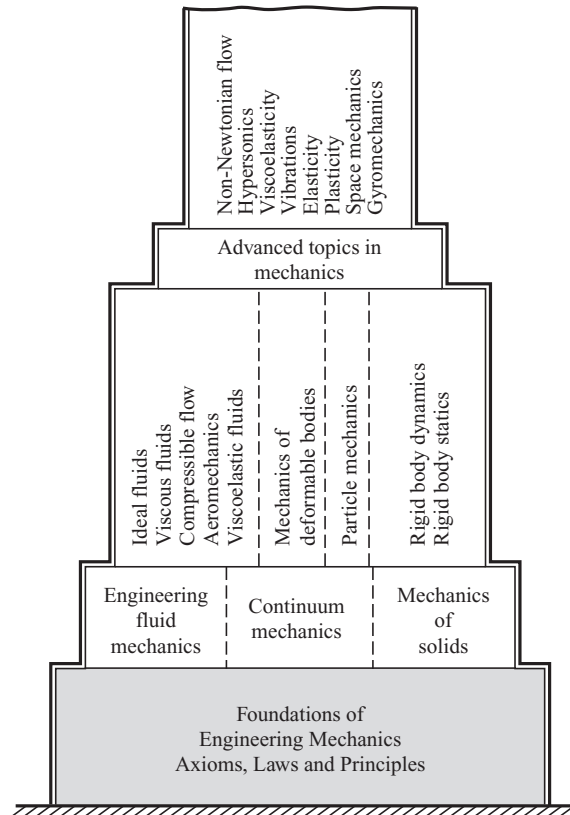


Fig. 1.2 Mechanics as the foundation of a number of courses

Table 1.1 *Pioneers of Mechanics*

| <i>Quantum or wave mechanics</i> | <i>Classical mechanics</i> | <i>Relativistic mechanics</i> |
|----------------------------------|---|--|
| Schrödinger (1887–1961) | ≡ Non-relativistic ≡ Newtonian | Lorentz (1853–1928) |
| Broglie (1892–1987) | Newton (1643–1727) Equivalent foundations by Lagrange (1736–1813) Hamilton (1805–1865) D’Alembert (1717–1783) Jacobi (1804–1851) Contributions mainly by Kepler (1571–1630) Archimedes (287–212 B.C.) Galileo (1564–1642) Euler (1707–1783) Coulomb (1736–1806) Coriolis (1792–1843) Foucault (1819–1868) Bernoulli (1700–1782) | Einstein (1879–1955) Bose (1894–1974) |

1.4 FOUNDATIONAL CONCEPTS

The axiomatic foundations of mechanics, have the following ingredients:



Some terms and concepts cannot be absolutely defined but are developed for axiomatic thinking and mutual understanding. Examples of such classical terms are: point, line and plane; space, time and matter; mass, force and energy.

Space refers to the unlimited general expanse of physical dimensions in which all material objects are located. Measurements and locations in space involve the concepts of *point*, *direction*, *length* and *displacement*. A *point*, for example, is just an exact indication of a location in space, requiring no space at all for itself. *Time* refers to the sequence of events. It is related to the concepts of *before*, *after* and *simultaneous* occurrence of two or more events. Measurement of time is made with the help of a clock. *Matter* refers to the substance of which physical objects are composed, the constituent substances are indeed the *atoms* and *molecules*. The quantity of matter associated with an object is measured as its *mass*. A physical object may consist of matter which is uniformly or non-uniformly distributed. Bodies with the same quantity of matter or the same mass can possess different shapes and sizes depending upon the distribution of matter in them.

Defined entities include momentum, moment of a force, impulse, work, equilibrium, rigid body, etc. Axioms are the relatively universal statements relating undefined concepts and defined concepts. Examples of axioms are Newton’s 2nd law, laws of friction and the law of gravitation.

Theorems and principles are derived from the axioms. Theorems and principles can be proved. Examples of principles are Work-Energy principle, Lami's theorem and Impulse-Momentum principle.

1.5 FRAMES OF REFERENCE

It is necessary to refer the motion of a body under study to some datum or reference space and clock. A reference frame, therefore, consists of a space and a clock to measure time. A reference frame should be such that the relative location of any two arbitrary points in it remains the same. It follows that the distance between any two points in the reference frame should remain invariant. A reference frame is called *fixed frame of reference* or *absolute frame* if each point on the frame is at 'absolute rest'. It is impossible to locate a fixed frame of reference in the universe. Rectangular Cartesian axes can be embedded in a frame of reference. The origin and orientation of axes can be according to convenience.

A reference frame is termed as *moving frame of reference* if each point on the frame is not at rest. A moving frame may be *inertial* or *non-inertial*. An inertial frame is one which moves at a constant velocity, i.e., the velocity of each point identified on the frame is the same and remains constant. Obviously, an inertial frame can move in a straight line at constant speed. In other words, an inertial frame can be defined as a frame which does not have any acceleration. An inertial frame is also known as *Galilean frame*.

The state of *rest* of a body refers to the absence of motion relative to some coordinate system. By *absolute rest* we mean a state of fixedness in space. Such a state could provide an absolute reference for the motion of other objects. However, it is doubtful if any such reference exists in the solar system or in the entire universe. It is, therefore, appropriate to speak of *relative rest* of a body with reference to a moving frame of reference. A reference frame fixed on the earth is both an approximation of the rest-frame and a convenient choice for all earth-bound objects for most engineering applications. A better choice from the point of view of physicists and mathematicians would be the centres of the earth, solar system, galaxy, and so on.

1.6 IDEALISATION OF BODIES

A *body* is a distinct mass, continuously distributed over a volume V enclosed in a surface S . An *element of a body* is referred to occupy a small volume ΔV and have a small mass Δm .

The words 'body' and 'system' are often used interchangeably. By general consensus, a body implies a single material configuration and a system refers to a combination of bodies. For example, a car is said to be a body if we were to consider the whole car as a single lump of mass but the car is referred to as a system of engine, chassis and wheels if we were to identify these items collectively.

It may be understood that the mathematical modelling of a system should be done for the specific purpose in view: different mathematical models of a system are made for different objectives of analysis. For example, an aeroplane may be regarded as (a) a concentrated mass with negligible dimensions for the purpose of tracing its trajectory when it is flying sky-high, (b) merely a wing with a large span for the analysis of its lifting characteristics, (c) a distributed mass system for the stability analysis under different flight conditions and (d) a deformable shell for the purpose of calculations of the strain when subjected to different pressures inside and outside the cabin.

Different idealisations of bodies bear standard nomenclature and have specific implications. These are:

P

s

s

R

Particle When the dimensions or size of a body are considered to be negligible and are irrelevant to the description of its motion, the body is modelled as a particle. A *particle* is a *point mass* or a *material point* in the abstract sense. A body is, therefore, represented as a particle if its dimensions are small compared to the coordinates describing its motion as shown in Fig. 1.4.

Examples A cricket ball as viewed by a spectator; a distant aeroplane tracked by a ground observer; a satellite orbiting the earth and seen by an observer on the earth; a planet as seen from another planet.

System of Particles When two or more bodies are represented by particles and are dealt with together, a system of particles is constituted. A system of particles is an idealisation of a collection of point masses. A body or a set of bodies is, therefore, represented as a system of particles if each part of the body or each body individually qualifies to be represented by a particle. A system of particles may comprise a rigid collection or a deformable collection in accordance with the criteria of rigid or deformable bodies which follow.

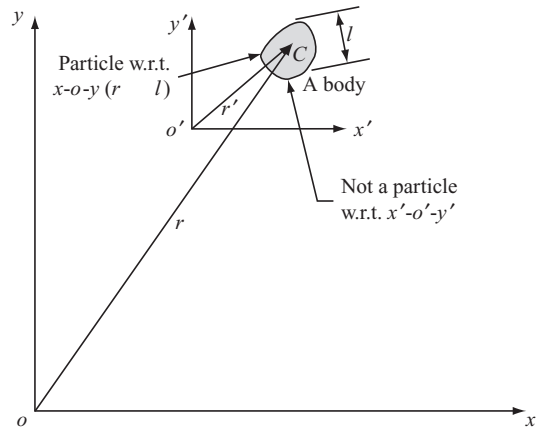


Fig. 1.4 Criteria of idealisation of bodies

Examples Billiard balls observed by a viewer in the gallery; sun-earth-moon system; electron-proton-neutron nature of atom.

Continuum When the microscopic nature of matter is disregarded and properties of the substance are defined assuming a continuous distribution of mass, the embodiment of matter so modelled is called a continuum. In a continuum, the gross effects of the actions of the molecules and atoms are conveyed by the concepts of density, pressure and temperature which simplify our study considerably.

The mass density ρ at a point P in a continuum is defined as the ratio of the mass element Δm to the volume ΔV enclosing the point, in the limit when ΔV tends to zero

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \quad (1.1a)$$

The expression of the mass of a body in terms of its density is, therefore,

$$m = \int \rho dV \quad (1.1b)$$

where ρ , the density of the continuum may be constant or may vary continuously with the space coordinates.

A body may, therefore, be represented as a continuum if the approximations about the continuity of mass and the continuous variation of the physical variables are acceptable in terms of analysis and results.

A continuum may be a rigid or a deformable medium in accordance with the definitions which follow.

Rigid Body When the dimensions, linear, volumetric or angular, of a body do not change during the course of events, the body is modelled as a rigid body. A rigid body, in other words, is the one in which the distance between any two arbitrary points is invariant. A body, therefore, qualifies to be represented as a rigid body if the deformation between its parts is negligible in the course of its analysis.

Examples An aeroplane observed in roll, pitch and yaw; a spinning top; a wheel of a cart; a body being pulled by applied forces.

Deformable Body When the dimensions, linear, volumetric or angular, of a body change during its analysis, the body is modelled as a deformable body. Deformation may be brought about in a variety of

ways; it may be temporary or permanent, instantaneous or continuous. A body is, therefore, represented as a deformable body if the relative deformation between its parts cannot be ignored in the course of its analysis.

Examples A bar deflecting under the application of a load; a shaft twisting under the application of a torque.

Fluid A substance which deforms continuously under the application of shear stresses, however small, is called a fluid. The process of continuous deformation is called a flow. A fluid must, therefore, flow when subjected to a shear stress. In the absence of shear stresses the fluids behave as static masses or as rigid bodies in motion.

Examples A liquid, e.g., water, oil or molten metal; a gas, e.g., air, oxygen or a vapour, blood, slurry, milk and beer.

Solid A substance characterised by some preferred configuration of its own, i.e., possessing a definite shape and a definite volume, is called a *solid*. Any change of shape or volume of a solid is accompanied by its tendency to regain its original configuration or stay in the new configuration with a change in some of its properties. Solids can be regarded as rigid bodies or as deformable bodies depending on their mathematical-modelling requirements.

Examples A straight metre scale for linear measurements is regarded as a rigid-solid body; a metre rod bent to draw a curve is regarded as a deformable solid.

1.7 EXTERNAL AND INTERNAL FORCES

A force is said to be ‘external’ if it is applied to a body from outside. An ‘internal’ force comes up from within the body in an attempt to resist the applied external force.

For example, if a bar is acted upon by axial forces P and Q , as shown in Fig. 1.5(a) trying to pull it apart; they are external forces. The bar resists the pull by generating internal forces p and q , as shown in Fig. 1.5(b).

Likewise, if a bar is pushed by two coaxial forces, the directions of the internal forces would be reversed.

External forces are those which act on a body or a system from outside. It is indeed the forces exerted on a body from outside that govern its state of motion.

If a component, say a gear, is to be considered for analysis, the forces exerted by the other gears and components on it are external forces.

It should be clear that a force is classified as internal or external depending upon the boundaries of the system. For example, the force between the earth and the moon is external if we were to consider the motion of the moon alone but the same force is internal if we were to consider the motion of the earth-moon system.

The concept of internal and external action is equally valid for moments also. Internal moments are those originating from inside a body or a system, whereas external moments are by virtue of sources outside the body or the system under consideration.

1.8 PRINCIPLE OF TRANSMISSIBILITY OF FORCE

The principle of transmissibility of force states that the condition of motion of a rigid body remains unchanged if a force P of a given magnitude, direction and sense acts anywhere along the same line

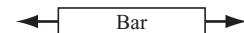


Fig. 1.5(a) External forces on a bar

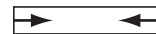


Fig. 1.5(b) Internal forces in the bar resisting the external forces

of action on the rigid body. For example a force acting at a_2 along the line of action a_1a_2 is equivalent to a force acting at a_1 along the same line of action a_1a_2 as shown in Fig. 1.6. Another example of a rigid body motion is provided if a mass m being lifted with an acceleration by means of a force applied differently at different places by along the same line of action passing through the centre of gravity C as shown in Fig. 1.7. A string with tension pulling it up or an upward force applied from below or a combination of the two such that

$$= = a_1 + a_2$$

result in the same motion of the body.

The principle of transmissibility applies only to a rigid body and is valid only from the point of view of the net external effect for the state of motion of the rigid body. It applies neither to the rigid body from the point of view of internal resistance or internal forces developed in a body nor to deformable bodies under any circumstances.

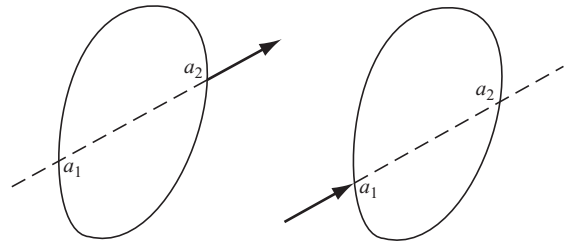


Fig. 1.6 Transmissibility of force

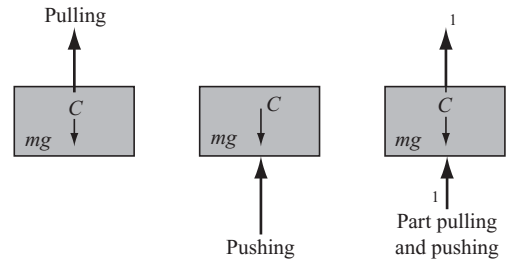


Fig. 1.7 Examples of transmissibility of force

1.9 CONCEPT OF FREE-BODY DIAGRAM

No system, natural or man-made, consists of a single body alone or is complete by itself. A single body or a part of the system can, however, be isolated from the rest by appropriately accounting for its effect. A free-body diagram (*fd*) consists of a diagrammatic representation of a single body or a subsystem of bodies isolated from its surroundings but shown under the action of forces and moments due to external actions.

Consider, for example, a book lying flat on a table Fig. 1.8 (a). the book exerts its weight on the table and the table exerts its own weight as well as transmits the weight of the book on the ground. A free-body diagram for the book alone would consist of its weight W acting through the centre of gravity and the reaction exerted on the book by the table top as shown in Fig. 1.8(b).

Let us consider one more case; a trailer of weight (and mass m) being pulled by a car of weight W (and mass M) by a tractive force produced by the engine on a level road as shown in Fig. 1.9(a).

Considering pure rolling of wheels, the free-body diagrams of the car and the trailer are shown in Fig. 1.9(b). Notice that the net forward force on the car is $(T - R)$ which must be accelerating the Car of mass M ; the net forward on the trailer is T , which must be accelerating the trailer of mass m with the same acceleration.

Drawing a free-body diagram (*fd*) of a body is an essential step, generally the first step, to analyse a given problem in mechanics. Whether the

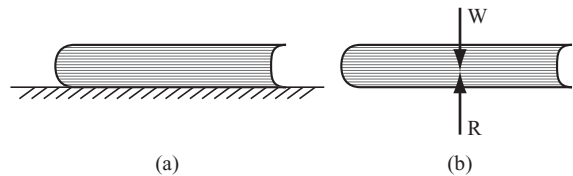


Fig. 1.8 (a) Book on a table top (b) Free-body diagram of the book

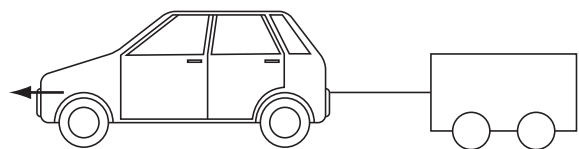


Fig. 1.9(a) A car pulling a trailer

body is at rest or in some kind of motion, an *fd* is equally important. In fact, it is the analysis following the *fd* which would decide the state of rest or motion of the body!

A free-body diagram may be drawn for any single member of a system, any subsystem of the system or the entire system irrespective of whether the system is in equilibrium: at rest, in uniform motion or in a dynamic state of motion.

The example of a book lying flat on a table is that of static equilibrium. In such cases the forces and moments acting on the body must be in conformity with the conditions of equilibrium which are dealt with in detail in Chapter 3.

Free-body diagram of a single member or a subsystem of a dynamic system, on the other hand, would reveal an unbalance of the forces and moments; the unbalanced resultants causing accelerations, linear or angular. Further discussion on the subject of free-body diagram for dynamic systems will be resumed in the chapters on dynamics and when the concept of *inertia forces* is introduced.

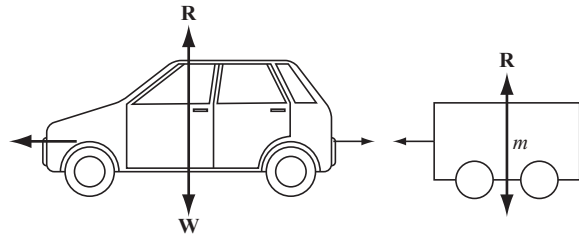


Fig. 1.9(b) Free-body diagram of the car and trailer

1.10 LAWS OF MOTION

The three laws of motion due to Newton, a literal translation from the original Latin ‘Principia Mathematica Philosophia Naturalis’ written in 1667 are collated:

Law 1 Every body perseveres in its state of rest, or of uniform motion in a right line unless it is compelled to change that state by forces impressed thereon.

Law 2 The alteration (acceleration) of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Law 3 To every action, there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

The laws due to Newton reproduced as above are indeed philosophical and useful but need some clarification:

1. The word ‘body’ is undefined. It either refers to a particle or to the centre of mass of a rigid body.
2. The term ‘motion in a right line’ appears in the first and second law but no attempt has been made to govern the rotational and general motion of the bodies of finite size.
3. Only the ‘forces’ have been considered; the action of a moment is not included.
4. The second law which relates acceleration to the forces impressed assumes the constancy of mass of the body.
5. If ‘force’ is recognised as a primitive concept, then the first law can be considered to be contained in the second law.
6. The action-reaction principle put forth by the third law can also be derived from the second law for rigid bodies.
7. The first and third laws are, therefore, not entirely independent of the second law; the message can be conveyed by the second law alone.

The second law applies to a body of mass m under the application of the motive force F . Mathematically, the acceleration

or

$$= k$$

where k is a constant of proportionality. This constant, determined experimentally, equals the mass of the system.

Hence,

$$= m = m \frac{d}{dt} \tag{1.2a}$$

It is obvious that the force and acceleration must be collinear, vector being m times vector . Further, the units of force are derived from the base units of mass and acceleration.

| | |
|--------------|---|
| Mass | kg |
| Acceleration | m/s^2 |
| Force | $kg \times m/s^2 = kg \ m/s^2 \equiv N$ or newton |

Quantitatively, a force of 1 N causes an acceleration of $1 \ m/s^2$ of a body of mass 1 kg.

The second law is not immediately applicable to the systems of variable mass. The law can, however, be reframed to cover the motion of constant-mass and variable-mass bodies by writing

$$= \frac{d}{dt}(m) \tag{1.2b}$$

The bracketed term (m) is the *momentum* p of the body of mass m moving at a velocity . The second law, in other words, states:

The rate of change of momentum of a body equals the force impressed upon it.

In view of the fact that the first and third laws are contained in this law, only this law will be retained and henceforth referred to as *Newton's law*.

In order to appreciate that the first and third laws of motion due to Newton are substantially contained in the second law, we proceed as follows:

From the second law,

$$= m = m \frac{d}{dt}$$

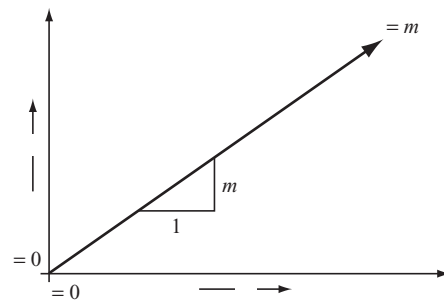
If $\mathbf{F} = 0, \mathbf{a} = 0 = \frac{d}{dt}$

whence = zero or constant.

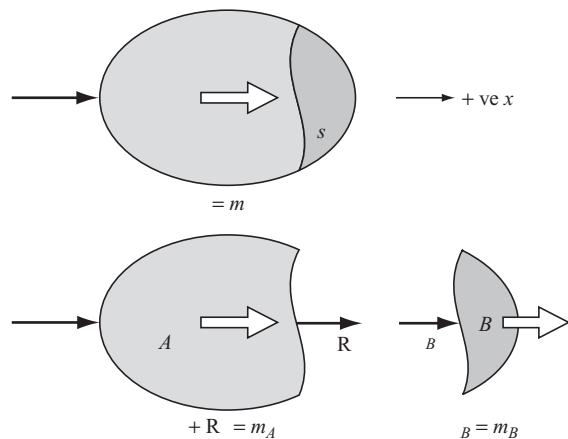
It follows that, in the absence of an external force, a body will continue to be in a state of rest or of uniform velocity. This is, in essence, the statement of the first law.

This reduction may also be seen graphically by plotting vs as shown in Fig. 1.10. The resulting straight line with slope m passes through the origin where,

for = 0, = 0



(a) First Law from the Second Law



(b) Third Law from the Second Law

Fig. 1.10 Derivations from the Second Law

Again, consider a rigid body moving at an acceleration a under the application of a force F . By the second law,

$$F = ma \tag{i}$$

Imagine the body to be constituted of two sub-bodies A and B such that the surface of contact is S as shown in Fig.1.10. The sub-bodies have their masses m_A and m_B for A and B respectively such that

$$m_A + m_B = m \tag{ii}$$

Since the whole body was moving at an acceleration a , every part of the body must have the same acceleration a . The total applied force F is, however, distributed over the parts of the body to bring about this state. In particular, let the force acting to accelerate the part B be F_B such that, by the second law,

$$F_B = m_B a \tag{iii}$$

In other words, the action of the body A on B is the force F_B . Let the reaction of B on A be R . Consequently, the net external force acting on A is given by

$$F + R = m_A a \tag{iv}$$

Adding (iii) and (iv),

$$F_B + F + R = m_B a + m_A a = (m_A + m_B) a$$

Employing (ii),

$$F_B + F + R = m a$$

and comparing with (i),

$$F_B + R = 0$$

it follows that

$$F_B + R = 0$$

whence

$$R = -F_B$$

and

$$R = -F_B \tag{v}$$

Relation (v) proves that the reaction force R by the body B is equal in magnitude and direction to the action F_B exerted on it but is opposite in sense. In other words, to an action F_B , there is an equal and opposite reaction R . This is, in essence, the third law of motion due to Newton.

While claiming to prove that the second law contains the first and third laws one must not underestimate the conditions of validity:

1. The first law would have served to define the terms ‘force’, ‘frame of reference’ and ‘state of rest’. These terms need to be defined axiomatically if the first law is regarded derivable from the second law.
2. The third law can be derived from the second law under the condition of transmissibility of force which is only valid for rigid bodies.
3. The third law as stated by Newton does not restrict the action and reaction principle to forces only. Since the first two laws relate to forces and their actions only and the concept of moment was not introduced by Newton, the third law as stated by Newton refers to the action and reaction of forces alone.
4. The action and reaction principle, in general, is valid for moments also. The concept of reaction may, therefore, be introduced axiomatically referring both to forces and moments as actions.

Having recognised that Newton’s laws are, by themselves, inadequate to govern the general motion of finite-size bodies under the action of forces and moments and also that only the restated second law is carried over, it is but natural to decide and state a *complete set of laws* for the general motion of observable bodies. The complete set of laws should include the laws governing the behaviour of mass, momentum and energy. The basic assumptions in classical mechanics are that the mass must be conserved and the energy must be conserved separately. Rates of changes of linear momentum and angular momentum must be governed by the laws of motion.

Table 1.2 Laws of Mass, Momentum and Energy

| Entity | Law | Statement | Mathematical formulation |
|------------------|-------------------------------|--|---|
| Mass | Law of conservation of mass | Mass can neither be created nor destroyed by any physical or chemical means. | $\frac{d}{dt}(m) = 0$ |
| Linear momentum | Newton's law | The rate of change of momentum of a body equals the force impressed upon it. | $\mathbf{F} = \frac{d}{dt}(m \mathbf{v}) = \frac{d}{dt}(m \mathbf{v}) = \dot{\mathbf{p}}$ |
| Angular momentum | Euler's law | The rate of change of angular momentum of a body about an origin O equals the moment impressed upon it about the origin. | $= \frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) = \frac{d}{dt} \mathbf{M} = \dot{\mathbf{M}}$ |
| Energy | Law of conservation of energy | The rate of change of internal energy and kinetic energy of any mechanical system equals the sum of the rates of work done by the external forces and the energy flux across the boundary as well as the energy developed within the system. | |

In quite the same way as Newton's law governs the motion of a particle (or of the centre of mass of a body), Euler's law governs the motion of a rigid body. Euler's law states that

$$\mathbf{M} = \frac{d}{dt}(\mathbf{L}) = \dot{\mathbf{L}}$$

where \mathbf{L} is the angular momentum of the body about a point and \mathbf{M} is the moment of the external forces acting on the body about that point.

It is interesting to note the similarity of the Newton's and Euler's laws:

Newton's Law $\mathbf{F} = \frac{d}{dt}(\mathbf{p}) = \dot{\mathbf{p}}$ (1.3)

Euler's Law $\mathbf{M} = \frac{d}{dt}(\mathbf{L}) = \dot{\mathbf{L}}$ (1.4)

The role of force in the rate of change of linear momentum is similar to the role of moment in the rate of change of angular momentum. In fact, both the laws relate to the rate of change of momentum; Newton's for the linear momentum and Euler's for the angular momentum. The force and the moment refer to the external action; the force for translational motion and the moment for rotational motion or tendencies thereof. A general statement to include both the laws may be made thus:

The rate of change of momentum of a body is proportional to the external action impressed upon it.

It should be clear that the word 'action' implies 'force' or 'moment' and the corresponding 'momentum' is linear or angular.

The law of conservation of energy may at first appear redundant to the problems in mechanics. This is not true because the law explains a number of dissipative phenomena on the one hand and degenerates to a simple form on the other hand for reversible phenomena. A pictorial representation of the laws of mechanics is given in Fig. 1.11.

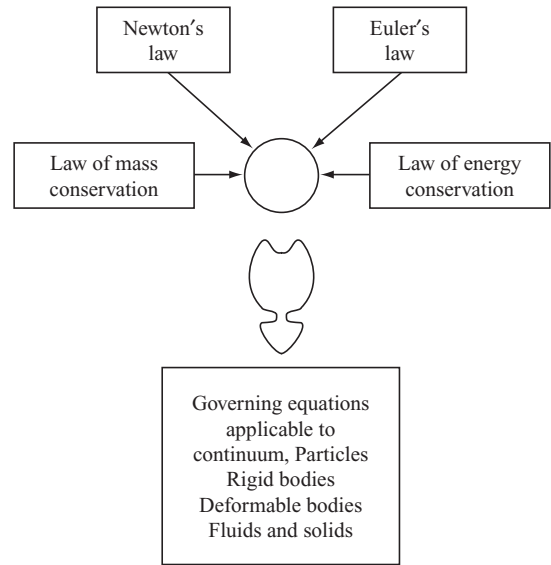


Fig. 1.11 Law of mechanics

1.11 LAW OF GRAVITATION—WEIGHT OF A BODY

In addition to the fundamental laws of motion, there are some more laws concerned with the origin and nature of forces. The law of gravitation due to Newton is perhaps the closest to the foundational laws and is discussed below.

Any two particles will be attracted towards each other along a line connecting their centres with a mutual force whose magnitude is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The law of gravitation requires that the force of attraction between two particles of masses m_1 and m_2 separated by a distance r as shown in Fig. 1.12 is given by

$$F = G \frac{m_1 m_2}{r^2} \quad (1.5)$$

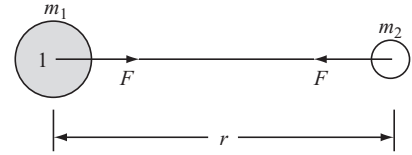


Fig. 1.12 Concept of gravitation

where G is the universal constant of gravitation; its value being $6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ or $\text{m}^2/\text{kg s}^2$. Quantitatively, an attractive force of $6.67 \times 10^{-11} \text{ N}$ is exerted by a body of mass 1 kg on another body of mass 1 kg at 1 m distance from it. Obviously, the attractive force of reaction by the other body on it must also be equal to the same value.

The law of gravitation helps in defining the *weight* of a body. The weight of a body is the force exerted on it by the planet. For an earth-bound object of mass m , the weight is approximately given by

$$W = G \frac{M_e m}{r^2} \quad (1.6)$$

where M_e is the mass of the earth = $5.9761 \times 10^{24} \text{ kg}$ and r is the radial distance between the centres of the earth and the object.

It is customary to write

$$W = mg \quad (1.7)$$

where

$$g = \frac{GM_e}{R_e^2} = 9.806 \text{ 65 m/s}^2 \quad (1.8)$$

and R_e = mean radius of the earth $\approx 6371 \text{ km}$.

Since g is a constant for a planet and, when multiplied by the mass of a body, it provides the force on the body, it is termed as acceleration due to gravity. It is indeed the acceleration acquired by a body falling freely, i.e., without resistance, in the gravitational field of the planet.

Example 1.1

The mass of a body is 10 kg.

- Determine its weight, assuming $g = 9.81 \text{ m/s}^2$.
- Estimate its weight by employing the Newton's law of gravitation, considering the following values
 G , gravitational constant = $6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$
 M_e , mass of the earth = $5.98 \times 10^{24} \text{ kg}$
 r , the radius of the earth = 6370 km

Solution

- The weight of a body W is related to its mass m by the equation, $W = mg$
Hence, $W = 10 \times 9.81 = 98.1 \text{ N}$.

(b) By Newton's law of gravitation, the force between the body and the earth is also the weight,

$$\begin{aligned} W = F &= G \frac{mM_e}{r^2} \\ &= 6.67 \times 10^{-11} \times \frac{10 \times 5.98 \times 10^{24}}{(6370 \times 1000)^2} \\ &= 98.0 \text{ N} \end{aligned}$$

The two results are the same, within permissible error of calculations!

Example 1.2

A body weighs 200 N on the earth's surface.

(a) What would it weigh on the moon's surface?

Assume that the mass of the moon is 0.0123 times the mass of the earth and its radius is 0.273 times that of the earth.

(b) Estimate the value of g at the moon.

Solution

(a) On the surface of the earth,

$$W = mg; m = 200/9.81 = 20.39 \text{ kg}$$

From the Newton's law of gravitation,

$$F = G \frac{m_1 m_2}{r^2}$$

We notice that the force exerted on a body is directly proportional to the mass of the 'planet' m_2 and inversely proportional to the square of the radius of the planet.

Hence, weight of the body on the moon is given by

$$= \frac{0.0123}{(0.273)^2} \times W = 0.169 W = 0.169 \times 200 = 33.37 \text{ N}$$

(b) Once we know the weight and mass of a body, the value of g is estimated from

$$W = mg$$

Hence, $g = 33.37/20.39 = 1.64 \text{ m/s}^2$ at the moon.

Concept Review Questions

Comment on the scope of classical mechanics vis-a-vis other formulations in mechanics.

Comment on the need to idealise a body as a particle, a rigid body, a deformable body or a continuum.

What is the advantage of drawing a free-body diagram? Is it possible to draw a free-body diagram

of a body or a system undergoing acceleration? Give examples.

State Newton's second law of motion and show that the first and third laws are contained in it.

What is meant by the state of equilibrium of a body? State the dynamical conditions of equilibrium and comment whether the conditions are both necessary and sufficient or not.

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

In all engineering problems a frame of reference at rest with respect to the earth is taken as an inertial frame. The assumption is valid because

- (a) the centrifugal force on the earth and the force of attraction between the earth and the sun balance each other
- (b) the acceleration and angular velocity of the earth is so small that the error caused is negligible
- (c) the error due to the acceleration of the earth is taken care of by the experimental calculation of the value of g
- (d) the earth does not have any acceleration

Zero work done by a system of forces acting on a body implies that

- (a) the resultant of the system of forces is zero
- (b) the cross product of the resultant of the system of forces and the vector in the direction of motion of the body is zero
- (c) the body does not have any motion
- (d) the motion of the body is in a direction perpendicular to the direction of the simplest resultant of the system of forces

An inertial frame of reference is one which necessarily has

- (a) fixed directions of its coordinate axes but the origin can move with constant speed
- (b) fixed directions of its axes but the origin can move with constant velocity
- (c) a fixed origin but directions can change with time
- (d) fixed origin and fixed directions of its axes

A free-body diagram of a body shows a body

- (a) isolated from all external effects
- (b) isolated from its surroundings and all external forces acting on it
- (c) isolated from its surroundings and all external actions acting on it
- (d) separately from its surroundings and all external and internal forces acting on it

The free-body diagram of a satellite rotating about the earth will show the satellite isolated from its surroundings and

- (a) no force acting on it
- (b) its velocity shown on it
- (c) the force of gravity and centrifugal force acting on it
- (d) only the force of gravity acting on it

An implication of Newton's law is that

- (a) the total momentum (linear + angular) of the body is conserved
- (b) the linear and angular momentum of the body are conserved separately
- (c) only the linear momentum of the body is conserved
- (d) a rigid body will tend to rotate if a force is applied at a point other than the centre of mass of the body

An implication of Euler's law is that

- (a) a rotating wheel will not change the orientation of its axis of rotation unless acted upon by an external torque
- (b) a rotating body will not change its angular velocity unless a couple is applied to it
- (c) a stationary body cannot be made to rotate by the application of a single force only
- (d) the total momentum of a body is conserved

8 The momentum of a particle

- (a) does not depend on the frame of reference at all
- (b) does not depend on the frame of reference so long as it is an inertial frame of reference
- (c) is zero if no external force is acting on it
- (d) is conserved under all circumstances

If a body moving in a horizontal line with a certain velocity starts ejecting mass downwards at a constant rate, the horizontal velocity of the body will

- (a) remain unchanged
- (b) start decreasing at a constant rate
- (c) start increasing at a constant rate
- (d) start increasing at an increasing rate

The force of gravitation between two bodies will be inversely proportional to the square of the distance between their centre of masses if the bodies

- (a) are of constant densities
- (b) are symmetrical about their centres of mass
- (c) are of any arbitrary shape
- (d) are of same shape, size and orientation

A man falling down from a height h starts rotating mid-way of his fall. The vertical velocity with which he will touch the ground will be

- (a) $\sqrt{2gh}$
- (b) less than $\sqrt{2gh}$
- (c) more than $\sqrt{2gh}$
- (d) less or greater but never equal to $\sqrt{2gh}$

Answers to Multiple-Choice Questions

- | | | | | | |
|-------|-------|-------|--------|--------|-------|
| 1 (b) | 2 (e) | 3 (b) | 4 (c) | 5 (c) | 6 (d) |
| 7 (b) | 8 (a) | 9 (a) | 10 (b) | 11 (b) | |

Review Section

In this section, an attempt has been made to review the SI units and vector operations which are required throughout the study of mechanics:

-
- **R1 REVIEW OF SI UNITS**
 - **R2 REVIEW OF VECTORS**
-

Review of SI Units

| | | |
|---|----|--------|
| | | s |
| | m | length |
| | kg | mass |
| s | s | time |

R1

A review of the Système International d'Unités, abbreviated as SI Units with special reference to mechanics is presented as follows:

R1.1 SI UNITS AND BASE UNITS

| <i>Name</i> | <i>Symbol</i> | <i>Measure</i> | <i>Definition/Justification</i> |
|-------------|---------------|--------------------|---|
| s | m | length | 0.1 millionth of the distance from the earth's equator to the North Pole measured through Paris |
| | kg | mass | The mass of one litre of water |
| | s | time | 1/(24 × 60 × 60) of the day |
| | A | electric current | |
| | K | temperature | One hundredth of the difference between the boiling and freezing points of water |
| | mol | quantity of matter | One gram per atomic mass unit |
| | cd | luminous intensity | Average candle intensity |

| Base units | Unit symbol | Dimensions | Physical quantity |
|----------------------------|-------------|------------|---------------------|
| kilogram | kg | M | mass |
| metre | m | L | length |
| second | s | T | time |
| kelvin | K | t | temperature |
| ampere | A | I | current |
| candela | cd | | luminous intensity |
| mole | mol | | amount of substance |
| <i>Supplementary Units</i> | | | |
| radian | rad | — | plane angle |
| steradian | sr | — | solid angle |

R1.2 SI DERIVED UNITS WITH NEW NAMES

| Derived unit | Unit symbol | Physical quantity |
|--------------|--|--------------------|
| newton | $N = \text{kg m/s}^2$ | force |
| joule | $J = \text{Nm} = \text{kg m}^2/\text{s}^2$ | energy, work, heat |
| watt | $W = J/s = N \text{ m/s} = \text{kg m}^2/\text{s}^3$ | power |
| pascal | $\text{Pa} = \text{N/m}^2 = \text{kg/ms}^2$ | pressure, stress |
| hertz | $\text{Hz} = \text{s}^{-1}$ | frequency |

Notes:

- Note that kilogram is written as kg and not as kgm, kgf, etc. Similarly second as s, not sec or sec., etc. No full stops, plurals, dots or dashes should be used. For example, torque is in Nm, not N.m, N-m, Nms, etc.
- The unit of force is newton with symbol N and there is no such thing as kilogram force in SI units; just N and its multiple and submultiples. The unit of energy in any form is joule, $J = \text{Nm}$
No horsepower or metric horsepower; just watt, $W = J/s = N \text{ m/s}$ and its multiples and submultiples.
- Always leave a space between the number and the unit symbol, e.g., 23.2 cm and 2500 N.
- For numbers less than unity, zero must be put on the left of the decimal and for larger numbers exceeding five figures, one space after every three digits counting from the right end must be left blank without any commas, e.g. 0.23 cm and 15 232 756 are the correct ways of writing these numbers.
- It is permissible and perhaps advisable that one space be left between any two unit symbols and no space be left after a multiple or submultiple symbol, e.g., $\text{kg m}^2/\text{s}$, kJ/kg K.

R1.3 UNITS OF SOME COMMON PHYSICAL QUANTITIES

| Physical quantity | Unit | Unit symbol |
|----------------------|----------------------------|------------------|
| Acceleration | metre/second ² | m/s^2 |
| Angular acceleration | radian/second ² | rad/s^2 |

(Continued)

| <i>Physical uantity</i> | <i>Unit</i> | <i>Unit symbol</i> |
|-------------------------|---|--|
| Angular displacement | radian | rad |
| Angular momentum | kilogram metre ² /second | kg m ² /s |
| Angular velocity | radian/second | rad/s |
| Area | Square metre | m ² |
| Couple, moment | newton metre | N m |
| Density | kilogram/metre ³ | kg/m ³ |
| Discharge | metre ³ /second | m ³ /s |
| Displacement | metre | m |
| Energy | joule or newton metre | J or N m |
| Force | newton or kilogram metre/second ² | N or kg m/s ² |
| Frequency | hertz or per second | Hz or /s |
| Length | metre | m |
| Mass | kilogram | kg |
| Moduli of elasticity | pascal or newton/metre ² | Pa or N/m ² |
| Moment | newton metre | N m |
| Momentum | kilogram metre/second or newton second | kg m/s or Ns |
| Moment of inertia | kilogram metre ² | kg m ² |
| Plane angle | radian | rad |
| Power | watt or newton metre/second | W or N m/s |
| Pressure, Stress | pascal or newton/metre ² | Pa or N/m ² |
| Specific energy | joule/kilogram | J/kg |
| Specific volume | kilogram/metre ³ | Kg/m ³ |
| Speed | metre/second | m/s |
| Time | second | s |
| Torque | newton metre | N m |
| Velocity | metre/second | m/s |
| Velocity potential | metre ² /second | m ² /s |
| Viscosity (dynamic) | newton second/metre ² or pascal second or kilogram/metre second | N s/m ² or Pa s kg/m s |
| Volume | metre ³ | m ³ |
| Weight | newton or kilogram metre/second ² | N or kg m/s ² |
| Work | joule or newton metre | J or N m |

R1.4 MULTIPLES AND SUBMULTIPLES

| | | | | | |
|-------|---|------------------|-------|---|-------------------|
| tera | T | 10 ¹² | milli | M | 10 ⁻³ |
| giga | G | 10 ⁹ | micro | μ | 10 ⁻⁶ |
| mega | M | 10 ⁶ | nano | n | 10 ⁻⁹ |
| kilo | k | 10 ³ | pico | p | 10 ⁻¹² |
| deci | d | 10 ⁻¹ | femto | f | 10 ⁻¹⁵ |
| centi | c | 10 ⁻² | atto | a | 10 ⁻¹⁸ |

R1.5 SOME CONVERSION FACTORS

| To convert the following | Into | Multiply by | Conversely multiply by |
|--------------------------|--------------------|-------------------------|------------------------|
| inch | cm | 2.5400 | 0.3937 |
| foot | m | 0.3048 | 3.2808 |
| mile | km | 1.6093 | 0.6214 |
| gallon | m ³ | 4.546×10^{-3} | 220 |
| pint | m ³ | 0.5683×10^{-3} | 1.76×10^3 |
| gallon (US) | m ³ | 3.785×10^{-3} | 264.2 |
| degree | rad | 0.017 45 | 57.2957 |
| pound (lb) | kg | 0.4536 | 2.2046 |
| ton | kg | 1016.0 | 9.842×10^{-4} |
| tonne | kg | 1000.0 | 10^{-3} |
| knot | m/s | 0.5144 | 1.943 |
| r.p.m. | rad/s | 0.1047 | 9.550 |
| pound/foot ³ | kg/m ³ | 16.02 | 0.0624 |
| cusec | m ³ /s | 0.0283 | 35.31 |
| g.p.m. | m ³ /s | 0.0758×10^{-9} | 13.20×10^9 |
| lb _f | N | 4.448 | 0.2248 |
| kg _f | N | 9.807 | 0.1019 |
| ton _f | kN | 9.964 | 0.1003 |
| kgf/cm ² | kPa | 98.07 | 0.0102 |
| p.s.i. | kPa | 6.895 | 1.1450 |
| inch (water gauge) | kPa | 0.2491 | 4.015 |
| inch (Mercury) | kPa | 3.386 | 0.2953 |
| torr | kPa | 1.333 | 0.7502 |
| foot pound | J | 1.356 | 0.7375 |
| h.p. | kW | 0.7457 | 1.341 |
| poise | N s/m ² | 0.1 | 10 |
| stoke | m ² /s | 10^{-4} | 10^4 |

R1.6 VALUES OF SOME USEFUL CONSTANTS

| Constant quantity | Symbol | Value | SI units |
|-------------------------------------|-----------------------|--------------------------|-----------------------------------|
| Speed of light in vacuum | <i>c</i> | $2.997\,925 \times 10^8$ | m/s |
| Planck's constant | <i>h</i> | 6.6253×10^{-34} | J s |
| Gravitational constant | <i>G</i> | 6.670×10^{-11} | N m ² /kg ² |
| Universal gas constant | <i>R</i> ₀ | 8314.4 | J/K mol |
| Zero Celsius (centigrade) | 0°C | 273.15 | K |
| Triple point of water | tr | 273.16 | K |
| Characteristic gas constant for air | <i>R</i> _a | 287 | J/kg K |
| Mean molecular weight of air | <i>M</i> _a | 28.966 | |
| Mean ICAO air density | <i>ρ</i> _a | 1.225 | kg/m ³ |
| Mean ICAO air viscosity | <i>μ</i> _a | 18×10^{-6} | Ns/m ² |

(Continued)

| Constant quantity | Symbol | Value | SI units |
|---|----------|------------------------|--------------------------------|
| Mean density of dry air (S.T.P.) | ρ | 1.293 | kg/m ³ |
| Standard atmosphere (pressure) | atm | 101.325 | kN/m ² |
| Standard atmosphere (temperature) | T_s | 288.15 | K |
| Lapse rate for standard atmosphere | L | 6.5 | K/km |
| Mass of atmosphere | | 5.27×10^{18} | kg |
| Voltage gradient, fine weather; average | | 100 | V/m |
| Solar constant for earth | S_c | 1400 | J/m ² s |
| Sonic speed in air at STP | a | 340.3 | m/s |
| Gravitational parameter | GM | 3.986×10^{14} | m ³ /s ² |
| Mass of the earth | M | 5.976×10^{24} | kg |
| Standard gravitational acceleration | g_c | 9.806 65 | m/s ² |
| Mean radius of the earth | r_e | 6371 | km |
| Mean density of the earth | ρ_e | 5515 | kg/m ³ |
| Escape velocity at the surface | V_e | 11.2 | km/s |
| Rotational velocity at the equator | | 465 | m/s |
| Mean velocity in orbit | | 29.78 | km/s |
| Approximate age of the earth | | 4.5×10^9 | years |
| Area of land surface | | 148.9×10^{12} | m ² |
| Area of water surface | | 362.2×10^{12} | m ² |
| Height of Mount Everest | | 8847.7 | m |
| Depth of Marianas Trench | | 11.033 | km |

Acceleration $g = 9.80616 - 0.025928 \cos 2\lambda + 0.000069 \cos^2 \lambda - 0.000003h \text{ m/s}^2$ at a place with latitude λ and at height h metres above the sea level for the earth.

International Civil Aviation Organisation.

R1.7 PROPERTIES OF WATER, MERCURY AND AIR

| Fluid properties | Water | Mercury | Air |
|--------------------------------------|--------------------|-----------------------|---------------------|
| Density kg/m ³ (at 20 °C) | 1000 | 13546 | 1.20 |
| Viscosity, N s/m ² | 1×10^{-3} | 1.55×10^{-3} | 18×10^{-6} |
| Surface tension, N/m | 0.073 | 0.472 | — |
| Melting point, K | 273 | 234 | — |
| Boiling point, K | 373 | 630 | 83 |
| Sonic speed m/s (at 1 bar) | 1410 | 1370 | 340 |

Test Your SI Power

Write down the seven base units and the two supplementary units in SI. Show, by way of expressing the following physical quantities in terms of these units, that this is a complete set of the base and supplementary units:

- (a) force
- (b) energy
- (c) pressure
- (d) charge
- (e) illuminance

(s kg, m, s, K, A, cd, mol; rad, sr: kg m/s², kg m²/s², kg/ms², As, cd sr/m²)

Recognise the following units and express them in equivalent symbolic forms: newton, joule, watt, pascal, poise, tesla.

(s N ≡ kg m/s², J ≡ N m ≡ kg m²/s²
 W ≡ J/s ≡ N m/s ≡ kg m²/s³, Pa ≡ N/m² ≡ kg/m s²
 P ≡ g/Ms ≡ 0.1 kg/m s ≡ 0.1 N s/m² ≡ 0.1 Pa s
 T ≡ Wb/m² ≡ V s/m² ≡ Ws A m²)

Pick up the correct SI abbreviations:

N m/S, N-m/s, N/m².s, Ns/m².s, N s/m², Pa-s, WS/A², N.m/S, kg_f/m², kg_fm, kg/s².m, kg m/s², Cd sr., N m/kg_f, N m/N, C, K.
 (s N s/m², kg m/s², N m/N)

Convert the following quantities into coherent SI units:

1 kg_f/cm², 20 kN/cm², 2 grams/cm³, 5 foot pounds, 20 Chu, 10 metric horse power, 20 knots, 3000 r.p.m., 2 quintals, 0.05 cumecs, 2 centipoise, 10 centistokes, 20 C 1 kcal/kg C, 10 lumens/foot², 2 light years.
 (s 98.07 × 10³ N/m²; 200 × 10⁶ N/m², 2000 kg/m³, 6.78 J, 38 × 10³ j.

7.355 × 10³ W, 10.288 m/s, 314.1 rad/s, 200 kg, 0.05 m³/s, 0.002 N s/m², 10 × 10⁻⁶ m²/s, 293.15 K, 4.187 × 10³ J/kg K, 107.64 lx, 18.921 × 10¹⁵ m)

State the value of the universal gas constant in SI units and hence express the characteristic gas constants for the following gases:

| Gas | Molecular Weight |
|--------------------|------------------|
| (a) air | 28.966 |
| (b) carbon dioxide | 44.01 |
| (c) oxygen | 32.00 |
| (d) hydrogen | 2.016 |

(s 8314.4 J/k mol; 287, 188.9, 259.8, 4124.2 J/kg K)

Write down the accurate mean value of *g*, the acceleration due to gravity on the earth. What is the approximation usually made by engineers? State the circumstances under which a quantity should be multiplied by *g* in SI units?

(s 9.806 65 m/s², 9.81 m/s²; only to calculate the weight of a given mass, e.g., 1 kg mass weighs 1 × 9.81 kg m/s² or 9.81 N)



Review of Vectors

R2

R2.1 NUMBERS, SCALARS AND VECTORS

The magnitude of a physical variable in terms of a pre-determined unit of measurement is expressed in *numbers*.

The quantities which are specified completely by the magnitude and units are called *scalars* or *scalar quantities*.

Vector quantities are those which are specified completely by the *magnitude with units, direction and sense*. Vectors must, in addition, obey the laws of vector operations and, in particular, the parallelogram law of addition.

Examples

Numbers 1, 2, 3.14159, 9.80665, 10, 20

Scalars 2 kg mass, 3.14159 m length, 1 s time, 10 m/s speed

Vectors 20 N force vertically downwards, 2 m/s velocity along the forward tangent to the path,
9.80665 m/s² acceleration directed towards the centre of the earth,
10 N m torque about the positive z-axis.

A vector is represented by a bold-faced letter such as \mathbf{A} and \mathbf{B} in print and by overbars in handwriting such as \overline{A} and \overline{B} .

Geometrically, a vector is represented by a bold line segment with an arrow at one end such that the length of the line represents the magnitude A with units, the orientation of the line shows the direction of the vector and the arrow mark specifies the sense of the vector, i.e., to or from a point.

A vector \mathbf{A} is geometrically represented as in Fig. R2.1(a). In a right-handed system or dextral system of coordinates, a vector represented by an arrowed-line segment in a certain direction

may also imply its rotational character governed by the right-handed screw-rule. For example, a vector representing angular velocity, angular acceleration or moment would imply the sense by the right-handed screw rule as shown in Fig. R2.1(b).

Vectors are categorised as sliding, free, or bound as follows:

A *sliding vector* or *transmissible vector* may be applied anywhere along its line of action; the line segment can be taken anywhere on the line of action so that the magnitude, direction and sense as well as the line of action remain the same. A force acting on a rigid body and producing acceleration is a transmissible vector. The principle of transmissibility of force is taken up further in Art. 2.6. A *free vector* may be moved anywhere in space provided its magnitude, direction and sense remain the same. A *bound vector* must be specified with a point of application; a bound vector has the magnitude, direction and sense as well as the point of application specified. A representation of these concepts is made in Figs. R2.1(c) and (d).

A vector is said to be a *unit vector* if its magnitude equals unity. A unit vector may, therefore, be chosen in any direction and with any sense. In particular, the unit vector along a vector or in the direction of the vector must be

$$= \frac{\text{vector}}{A} = \frac{1}{A} \quad (\text{R2.1})$$

which is in the same direction as but with a magnitude of unity as shown in Fig. R2.1(e).

The unit vectors along the coordinate axes are given a special status.

In the rectangular coordinates,

- \hat{i} \equiv unit vector along the x -axis
- \hat{j} \equiv unit vector along the y -axis
- \hat{k} \equiv unit vector along the z -axis

In the cylindrical coordinates,

- \hat{r} \equiv unit vector radially outwards in the x - y plane
- $\hat{\theta}$ \equiv unit vector in circumferential direction in the x - y plane.
- \hat{z} \equiv unit vector along the z -axis

In the spherical coordinates,

- \hat{R} \equiv unit vector radially outwards in space
- $\hat{\theta}$ \equiv unit vector in circumferential direction referred to the z -axis
- $\hat{\phi}$ \equiv unit vector in circumferential direction in the x - y plane.

The unit vectors in different coordinate systems are illustrated in Fig. R2.2(a), (b) and (c).

A *null* or *zero vector* is defined as a vector whose magnitude is zero. The role of a zero vector in vector operations is equivalent to the role of zero value in scalar operations. Interestingly, a zero vector may be thought of as parallel to any direction for convenience since a zero vector must be parallel to all directions simultaneously.

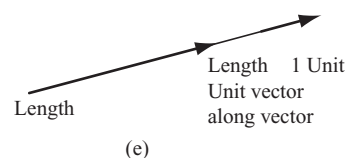
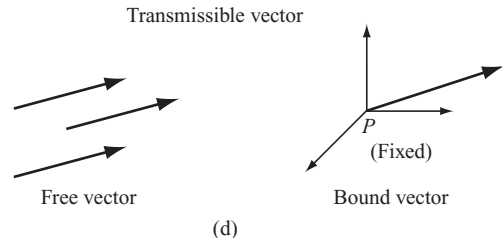
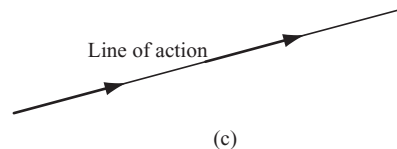
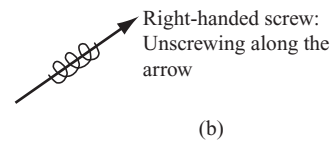
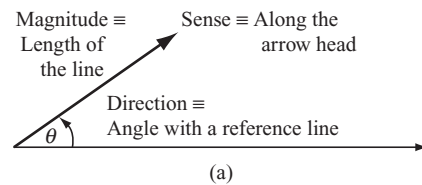


Fig. R2.1 Vector representation

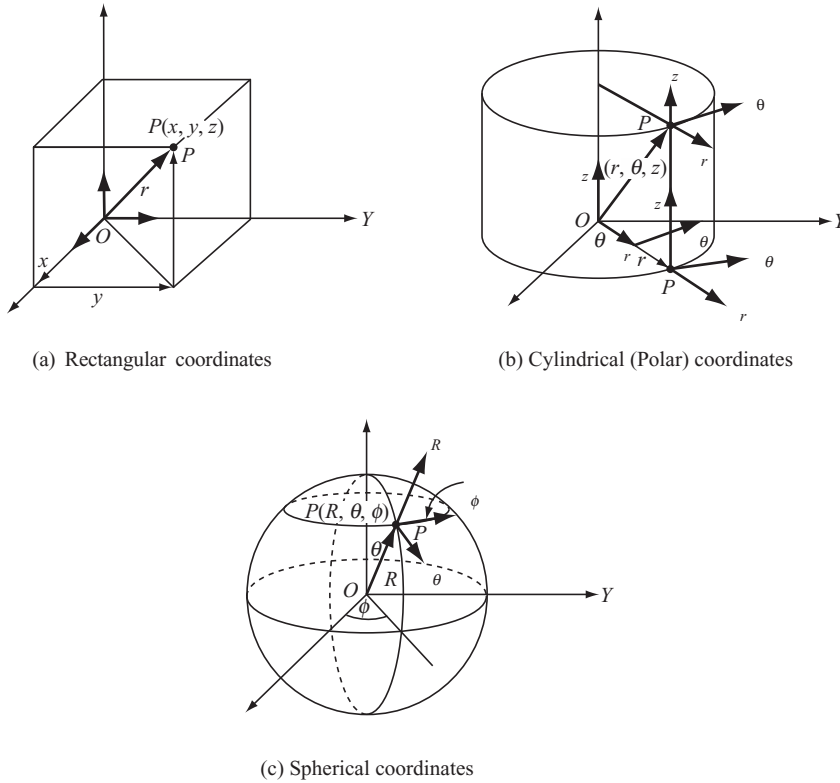


Fig. R2.2 Unit vectors in different coordinate systems

Two vectors are said to be *e ual vectors* if their magnitudes, directions and sense are the same. Two vectors are said to be *e uivalent* or *e uipollent vectors* if, in a certain sense, they produce the same effect. It may be mentioned that the equality of vectors does not necessarily mean their equivalence of effect. A vector is said to be *negative* of another vector, if they have the same magnitude and direction but are opposite in sense.

It is necessary to understand the concept of the angle between two vectors. The included angle is defined as the angle, restricted to the interval $0 \leq \theta \leq \pi$, formed *between two vectors when both are taken from or towards a common initial point*. In other words, it is the smaller of the two angles formed between the directions of the vectors when their arrows point out or towards a common point. Clearly, the concept of the angle between two vectors is restricted to a pair of coplanar vectors. Vectors may be mutually orthogonal or parallel if the angle between them is $\pi/2$ or 0 respectively. Two vectors are said to be skew if a common plane cannot be passed through them.

The correct angle between two vectors and B has been shown as in different situations in Fig. R2.3. In particular, B_3 is perpendicular to , B_5 is parallel to and B_6 is antiparallel or parallel and opposed to .

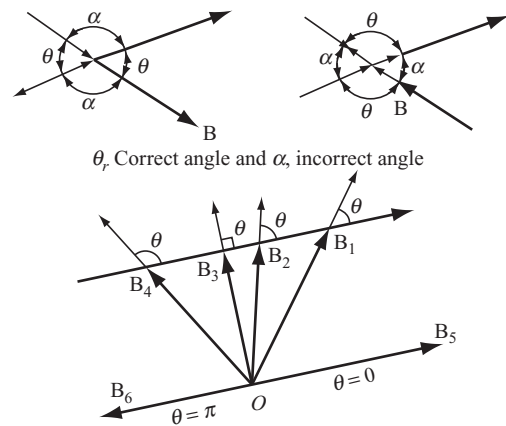


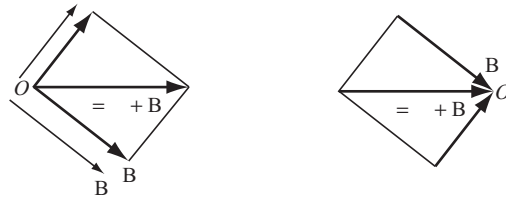
Fig. R2.3 Angle between two vectors

R2.2 ADDITION OF VECTORS

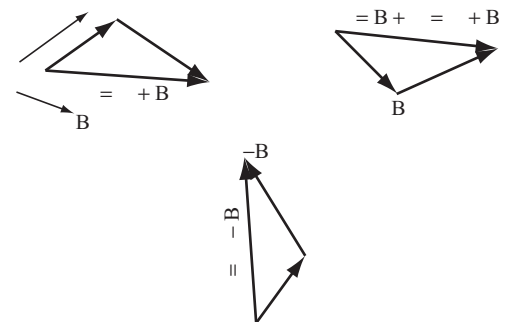
The most fundamental law of vector algebra is the parallelogram law of vector addition; so much so that the quantities possessing direction, magnitude and sense may be denied the vectorial status, if they do not obey the parallelogram law. Conformity with the law may as well be incorporated in the definition of the vector quantities.

The parallelogram law of vector addition states that if two vectors comprise the adjacent sides of a parallelogram, pointing towards or away from the point of intersection, then the diagonal of the parallelogram passing through the same point and with the same sense represents the sum of the two vectors. The addition of vectors \mathbf{A} and \mathbf{B} requires that

1. \mathbf{A} and \mathbf{B} be placed together to point towards or away from a point O
2. A parallelogram be made with \mathbf{A} and \mathbf{B} as adjacent sides
3. The diagonal of the parallelogram passing through O with the arrow pointing towards or away from O as the case may be, represents the sum of two vectors \mathbf{A} and \mathbf{B} as shown in Fig. R2.4(a).



(a) Parallelogram Law of Vector Addition



(b) Triangle Law of Vector Addition

A corollary of the parallelogram law is the law of triangle of vectors illustrated in Fig. R2.4(b). The additive vectors \mathbf{A} and \mathbf{B} are placed one after the other in the same sense to constitute two sides of a triangle, the third side of which, drawn from the initial point of \mathbf{A} to the final point of \mathbf{B} , represents the sum of the vectors \mathbf{A} and \mathbf{B} . It can be seen that \mathbf{A} added to \mathbf{B} or \mathbf{B} added to \mathbf{A} results in the same vector, i.e.,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

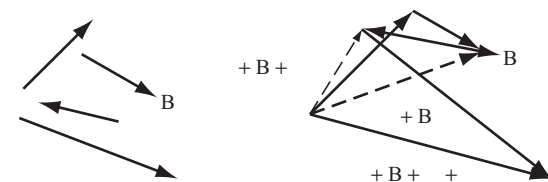
The difference of two vectors can be obtained by adding the additive vector to the negative of the subtractive vector, i.e.,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

If it is desired to add more than two vectors, then the parallelogram law can be used to continue adding two at a time or the triangle law can be extended to comprise the polygon law as demonstrated in Fig. R2.4(c).

The triangle resulting from the triangle law of vectors addition may be solved by invoking the sine law and cosine law from trigonometry:

$$\text{sine law : } \frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

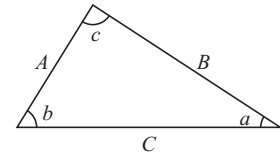


(c) Polygon Law of Vector Addition

Fig.R2.4 Addition of vectors

$$\text{cosine law : } C = \sqrt{A^2 + B^2 - 2AB \cos c}$$

The addition of a vector to itself results in a vector twice its magnitude but the same in direction and sense. In general, a vector is n times another vector if its magnitude is n times that of the other and the direction and sense are the same. The laws of vector addition and rules of multiplication by scalars are given below.



Sine and cosine laws

$$\begin{aligned} A + B &= B + A && \text{(Commutative law of addition)} \\ A + (B + C) &= (A + B) + C && \text{(Associative law of addition)} \end{aligned} \tag{R2.2}$$

$$\begin{aligned} n(A + B) &= nA + nB && \text{(Distributive)} \\ (n + m)A &= nA + mA && \text{(Distributive)} \\ m(nA) &= (mn)A && \text{(Associative)} \\ n(mA) &= (nm)A && \text{(Commutative)} \end{aligned} \tag{R2.3}$$

where n and m are scalars.

R2.3 RESOLUTION OF VECTORS

The resolution of a vector into its constituent vectors is defined as the reverse action of addition of the component vectors to result in the given vector. Thus if $A + B = C$, i.e., if A and B can add to give C , then C can be resolved to give A and B , i.e., $C = A + B$.

In general, a vector can be resolved into an infinite pair of constituent vectors but the resolution of a vector in any two stipulated directions coplanar with the given vectors is unique. Just as a number of vectors can be added to comprise a resultant, a given vector can be resolved into a number of constituent vectors.

In particular, it is important to understand the resolution of a vector into three mutually orthogonal component vectors. A vector A is resolved into three components corresponding to its projections along the three orthogonal coordinate axes, i.e., A_x , A_y and A_z along the x , y and z axes respectively as shown in Fig. R2.5. It can be seen that A_x and A_y add up to constitute OQ which when added to A_z results in vector OP . Conversely, the vector OP is considered resolved into A_z and OQ and OQ further resolved into A_x and A_y giving rise to A_x , A_y and A_z as the three orthogonal components.

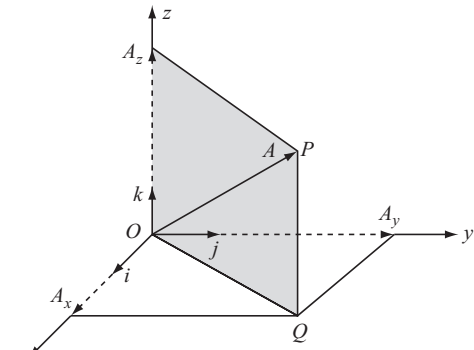


Fig. R2.5 Components of a vector

In terms of the unit vectors i , j and k in the Cartesian system of coordinates,

$$\begin{aligned} A_x &= A \cdot i \\ A_y &= A \cdot j \\ A_z &= A \cdot k \end{aligned}$$

We may, therefore, state that the scalar components of a vector A are A_x , A_y and A_z along the x , y and z directions respectively. The scalar components are generally referred to as the components of the vector. It also follows by the Pythagoras theorem that

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (R2.4)$$

It may be noted here that the process of arriving at the components of a vector by our analysis has been long and requires the use of the parallelogram law of vectors. In fact, it was stated that the parallelogram law is fundamental to the existence of the vectors and that it could be included in the definition of a vector quantity. This is indeed the modern approach where the parallelogram law is not talked of and the vector components are defined straightaway. By definition, then, a vector is a quantity possessing n components, i.e.,

$$= (r_1, r_2, \dots, r_i, \dots, r_n) \quad (R2.5)$$

such that the components commute, associate, etc., according to a set of rules. The components can then be specialized for the orthogonal systems as arrived at here.

A vector can be expressed in terms of its scalar components as

$$= A_x + A_y + A_z$$

Similarly

$$B = B_x + B_y + B_z$$

$$= C_x + C_y + C_z$$

and

$$= D_x + D_y + D_z$$

(i) If

$$= + B$$

then

$$= C_x + C_y + C_z$$

$$= (A_x + B_x) + (A_y + B_y) + (A_z + B_z)$$

because the scalar components can be added numerically.

Hence,

$$C_x = A_x + B_x$$

$$C_y = A_y + B_y$$

$$C_z = A_z + B_z$$

(ii) If

$$= B$$

then

$$= D_x + D_y + D_z$$

$$= (A_x - B_x) + (A_y - B_y) + (A_z - B_z)$$

and

$$D_x = A_x - B_x$$

$$D_y = A_y - B_y$$

$$D_z = A_z - B_z$$

In general, if $R = + B - \dots$

$$R_x + R_y + R_z = (A_x + B_x - \dots) + (A_y + B_y - \dots) + (A_z + B_z - \dots) \quad (R2.6)$$

Example R2.1

A vector of magnitude 10 units is directed 30 degrees north of east. Represent it graphically and analytically and determine its components due east and north.

Solution Vector shown in Fig. Ex. R2.1 with its length OP of 10 units to a chosen scale, direction 30 with the east, the x -axis and arrow to indicate its sense O to P , is the required vector. Its components due

east and north, found by measuring its projections on the x and y axes respectively to the same scale, are 8.66 units and 5 units as shown. Alternatively, its component due east is

$$OP \cos 30 = A \cos 30 = 10 \times \cos 30 = 8.66 \text{ units}$$

and its component due north is

$$OP \sin 30 = A \sin 30 = 10 \times \sin 30 = 5.00 \text{ units}$$

The vector may therefore be expressed analytically as

$$= 8.66 \mathbf{i} + 5 \mathbf{j}$$

where \mathbf{i} and \mathbf{j} are the unit vectors along x and y axes respectively. The analytical representation implies that the magnitude of the vector is

$$A = \sqrt{8.66^2 + 5^2} = 10 \text{ units}$$

and that the vector makes an angle θ with the x -axis.

$$\begin{aligned} \theta &= \sin^{-1} \left(\frac{5.00}{10} \right) = \cos^{-1} \left(\frac{8.66}{10} \right) = \tan^{-1} \left(\frac{5}{8.66} \right) \\ &= 30^\circ = \pi/6 \text{ rad} \end{aligned}$$

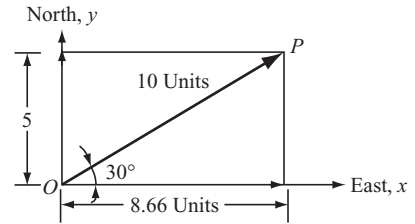


Fig. Ex. R2.1 (Solution)

Example R2.2

A vector of magnitude 100 units makes an angle of 30° with the z -axis and its projection on the x - y plane makes an angle of 45° with the x -axis. Determine (a) the components of the vector and (b) the angles of the vector with the axes.

Solution The vector represented in Fig. Ex. R2.2 has the components represented by OZ along the z -axis, OY along the y -axis and OQ along the x -axis.

The projection of OP on the x - y plane is OQ which is composed of OZ and OY component.

$$OZ = A \cos 30 = 100 \times 0.86606 = 86.6 \text{ units}$$

$$OQ = A \sin 30 = 100 \times 0.500 = 50.0 \text{ units}$$

whence, by further resolution,

$$OY = OQ \cos 45 = 50 \times 0.707 = 35.35 \text{ units}$$

$$OX = OQ \sin 45 = 50 \times 0.707 = 35.35 \text{ units}$$

The components of the vector along the x , y and z axes are 35.35, 35.35 and 86.6 units respectively.

$$= 35.35 \mathbf{i} + 35.35 \mathbf{j} + 86.6 \mathbf{k}$$

It may be checked that

$$\sqrt{(35.35)^2 + (35.35)^2 + (86.6)^2} = 100 \text{ units as expected}$$

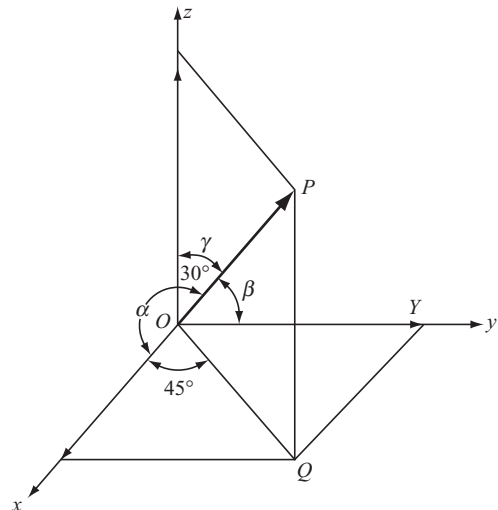


Fig. Ex. R2.2 (Solution)

The direction cosines are determined as follows:

$$l = \cos \alpha = \frac{O}{OP} = \frac{35.35}{100} = 0.3535$$

$$m = \cos \beta = \frac{OY}{OP} = \frac{35.35}{100} = 0.3535$$

$$n = \cos \gamma = \frac{O}{OP} = \frac{86.6}{100} = 0.866$$

It follows that the angles in the respective axes are

$$\alpha = 69.3, \quad \beta = 69.3, \quad \gamma = 30$$

It may be checked that

$$\sqrt{0.3535^2 + 0.3535^2 + 0.866^2} = 1$$

in accordance with the relationship

$$l^2 + m^2 + n^2 = 1$$

Example R2.3

The coordinates of the initial and terminal points of a vector are (3, 1, -2) and (4, -7, 10) respectively. Determine the components of the vector and its angles with the axes. Specify the vector.

Solution The components of the vector are

$$\begin{aligned} 4 - 3 &= 1 && \text{along the } x\text{-axis} \\ -7 - 1 &= -8 && \text{along the } y\text{-axis} \\ 10 - (-2) &= 12 && \text{along the } z\text{-axis} \end{aligned}$$

The magnitude of the vector is, therefore,

$$A = \sqrt{1^2 + (-8)^2 + 12^2} = 14.46$$

and its direction cosines are

$$l = \cos \alpha = \frac{1}{14.46} = 0.069$$

$$m = \cos \beta = \frac{-8}{14.46} = -0.553$$

$$n = \cos \gamma = \frac{12}{14.46} = 0.830$$

whence, $\alpha = 86.04$, $\beta = 123.57$, $\gamma = 33.9$

The vector is specified as

$$= 1 \quad -8 \quad +12$$

or, alternatively stated as a vector of magnitude 14.46 units making angles of 86.04°, -56.43° and 33.9° with the x , y and z axes respectively with its sense from the initial to the terminal point.

Example R2.4

Two vectors A and B are added and subtracted to comprise vectors C and D . Determine these vectors and the unit vector along them.

$$A = 2\mathbf{i} + 3\mathbf{j}$$

$$B = 3\mathbf{i} - 2\mathbf{j}$$

Evaluate also the magnitude of vector $E = 2C + 0.75D$.

Solution

For A , $A_x = 2$ $A_y = 3$
 For B , $B_x = 3$ $B_y = -2$
 For C , $C_x = A_x + B_x = 5$ $C_y = A_y + B_y = 1$
 For D , $D_x = A_x - B_x = -1$ $D_y = A_y - B_y = 5$

Hence, $C = 5\mathbf{i} + 1\mathbf{j}$
 $D = -1\mathbf{i} + 5\mathbf{j}$

The magnitude of C is given by

$$C = \sqrt{5^2 + 1^2} = 5.1 \text{ units}$$

Hence the unit vector along C must be given by

$$\frac{C}{C} = \frac{5}{5.1}\mathbf{i} + \frac{1}{5.1}\mathbf{j} = 0.981\mathbf{i} + 0.196\mathbf{j}$$

Clearly, the magnitude of the unit vector can be checked to be

$$\sqrt{0.981^2 + 0.196^2} = 1, \text{ as expected}$$

Similarly, the unit vector along D is given by

$$\frac{D}{D} = \frac{-1}{\sqrt{1^2 + 5^2}}\mathbf{i} + \frac{5}{\sqrt{1^2 + 5^2}}\mathbf{j} = -0.196\mathbf{i} + 0.981\mathbf{j}$$

which is also unity in magnitude.

Graphically, C and B are added to yield E by the parallelogram method of addition. B is subtracted from C if $-B$ is added to C to comprise D by the same procedure.

$$E = 2C + 0.75D$$

$$= 2(5\mathbf{i} + 1\mathbf{j}) + 0.75(-1\mathbf{i} + 5\mathbf{j})$$

$$= 9.25\mathbf{i} + 5.75\mathbf{j}$$

$$\text{Magnitude } E = \sqrt{9.25^2 + 5.75^2} = 10.89 \text{ units}$$

The unit vector along E is given by

$$\frac{9.25}{10.89}\mathbf{i} + \frac{5.75}{10.89}\mathbf{j} = 0.849\mathbf{i} + 0.528\mathbf{j}$$

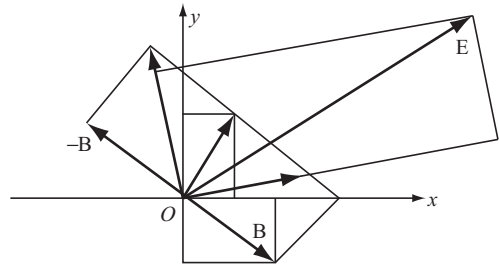


Fig. Ex. R2.4 (Solution)

Example R2.5

Find the resultant of the four given vectors:

$$\begin{aligned} &= 3 \quad + 2 \\ \mathbf{B} &= 2 \quad + 3 \\ &= 3 \quad 4 \\ &= 5 \quad 2 \end{aligned}$$

Solution Analytically, the components of the resultant vector R are given by

$$\begin{aligned} R_x &= A_x + B_x + C_x + D_x \\ &= 3 + 2 + 3 - 5 = 3 \\ R_y &= A_y + B_y + C_y + D_y \\ &= 2 + 3 - 4 - 2 = -1 \\ \mathbf{R} &= 3 \end{aligned}$$

Hence,

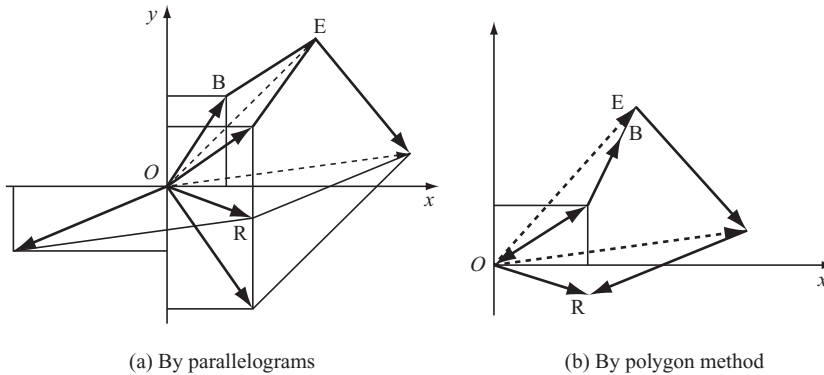


Fig. Ex. R2.5 (Solution)

Graphically, \mathbf{A} and \mathbf{B} are added to comprise \mathbf{E} which when added to \mathbf{C} provides \mathbf{D} and that added to \mathbf{D} results in the final vector \mathbf{R} as shown in Fig. Ex. R2.5(a). Alternatively, the polygon method of vector addition as shown in Fig. Ex. R2.5(b) requires vector \mathbf{B} to be placed at the tip of vector \mathbf{A} , \mathbf{C} at the tip of \mathbf{B} and \mathbf{D} at the tip of \mathbf{C} . The closing line of the polygon directed from the starting point O is the resultant \mathbf{R} .

Example R2.6

Three vectors \mathbf{A} , \mathbf{B} and \mathbf{C} are given as

$$\begin{aligned} \mathbf{A} &= 2 \quad + 3 \quad 4 \\ \mathbf{B} &= 3 \quad 4 \quad + 5 \\ \mathbf{C} &= 2 \quad 3 \end{aligned}$$

Determine (a) the resultant vector \mathbf{R} , (b) the vector \mathbf{E} to make the sum of \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{E} zero and (c) the vector $\mathbf{D} = (\mathbf{A} + \mathbf{B} - 2\mathbf{C})$.

Solution

(a) Writing \mathbf{A} , \mathbf{B} and \mathbf{C} components columnwise,

$$\begin{array}{r} \mathbf{A} = 2 \quad + 3 \quad 4 \\ \mathbf{B} = 3 \quad 4 \quad + 5 \\ \mathbf{C} = 2 \quad + 0 \quad 3 \\ \hline \mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} = 7 \quad 1 \quad 2 \end{array}$$

- (b) Since $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{R}$
 E must be equal to $-\mathbf{R}$ so that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{E} = \mathbf{0}$$

Hence, $\mathbf{E} = -\mathbf{R} = 7\mathbf{i} + 2\mathbf{j}$

- (c) Writing \mathbf{A} , \mathbf{B} and $-\mathbf{2C}$ columnwise,

$$\begin{array}{r} \mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + 3\mathbf{j} + 4\mathbf{k} \\ \mathbf{B} = \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix} + 5\mathbf{k} \\ -2\mathbf{C} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} \\ \hline \mathbf{A} + \mathbf{B} - 2\mathbf{C} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} \end{array}$$

R2.4 CONCEPT OF PRODUCTS

Numbers, scalars and vectors can be multiplied to yield meaningful results:

- (a) *Product of a scalar by a number* means the magnitude multiplication of the scalar.

Example 10 m length multiplied by 4.2 means 42 m length

(b) *Product of a scalar by a scalar* means the multiplication of the magnitudes as well as that of the units to result in the units of the resulting physical quantity.

Example 10 m/s speed for 5 s time results in a distance of 50 m.

(c) *Product of a vector by a number* means that the magnitude of the vector is multiplied by the number, the direction and sense of the vector remaining the same.

Example 5 times the acceleration of 9.81 m/s^2 means an acceleration of 49.05 m/s^2 in the same direction and sense.

(d) *Product of a vector by a scalar* implies the multiplication of their magnitudes as well as that of the units resulting in a new vector quantity with its magnitude as the product of their magnitudes and the direction and sense the same as before.

Example A velocity of 10 m/s for 5 s time produces a displacement of 50 m directed forward along the velocity vector.

(e) *Product of a vector by a vector* should also provide meaningful results. There are two types of vector products of interest to us, i.e., *scalar or dot product* and *vector or cross product*.

(e1) *Scalar or dot product* of two coplanar vectors \mathbf{A} and \mathbf{B} denoted by $\mathbf{A} \cdot \mathbf{B}$ and read as \mathbf{A} dot \mathbf{B} implies a scalar quantity equal to (a) the magnitude of \mathbf{A} times the magnitude of the projection of \mathbf{B} on \mathbf{A} or (b) the magnitude of \mathbf{B} times the magnitude of the projection of \mathbf{A} on \mathbf{B} or (c) the product of the magnitudes of \mathbf{A} and \mathbf{B} and the cosine of the smaller angle between them.

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \tag{R2.7}$$

as shown in Fig. R2.6(a).

The dot product of two skew vectors is not defined. The dot product of two collinear or parallel vectors must result in a scalar quantity equal to the multiplication of the magnitudes of the two vectors since $\cos \theta = 1$ for $\theta = 0$.

$$\mathbf{A} \cdot n\mathbf{A} = nA^2$$

The dot product of two equal vectors results into the square of its magnitudes

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{R2.8}$$

The dot product of two orthogonal vectors must be zero since $\cos 90^\circ = 0$.

In particular, the dot products of the unit vectors are:

$$\begin{aligned} \hat{i} \cdot \hat{i} &= 1 \times 1 \cos 0 = 1 \\ \hat{j} \cdot \hat{j} &= 1 \times 1 \cos 0 = 1 \\ \hat{k} \cdot \hat{k} &= 1 \times 1 \cos 0 = 1 \\ \hat{i} \cdot \hat{j} &= \hat{j} \cdot \hat{i} = 1 \times 1 \cos 90 = 0 \\ \hat{i} \cdot \hat{k} &= \hat{k} \cdot \hat{i} = 1 \times 1 \cos 90 = 0 \\ \hat{j} \cdot \hat{k} &= \hat{k} \cdot \hat{j} = 1 \times 1 \cos 90 = 0 \end{aligned} \quad (R2.9)$$

In terms of the rectangular components,

$$\begin{aligned} \mathbf{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \\ \mathbf{B} &= B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \\ \mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y + A_z B_z = \mathbf{A} \cdot \mathbf{B} \end{aligned} \quad (R2.10)$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and

$$\begin{aligned} \hat{i} \cdot \hat{j} &= \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = \dots = 0 \\ A^2 &= A_x^2 + A_y^2 + A_z^2 \end{aligned}$$

The scalar or dot product is both commutative and distributive

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (n\mathbf{B}) &= n \mathbf{A} \cdot \mathbf{B} \end{aligned} \quad (R2.11)$$

If $\mathbf{A} \cdot \mathbf{B} = 0$, then either $\mathbf{A} = 0$ or $\mathbf{B} = 0$ or both are zero (trivial case), or \mathbf{A} and \mathbf{B} are mutually perpendicular vectors. The angle between two vectors \mathbf{A} and \mathbf{B} is given by

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

The magnitude of a vector \mathbf{A} is given by

$$A = \sqrt{A^2} = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

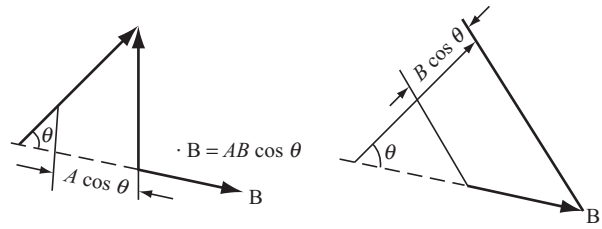
Example If a force $F = 10$ N acts upon a particle causing a displacement $S = 3$ m at an angle of 60° to the direction of the force, the work done on the particle equals

$$10 \times 3 \times \cos 60 = 15 \text{ N m}$$

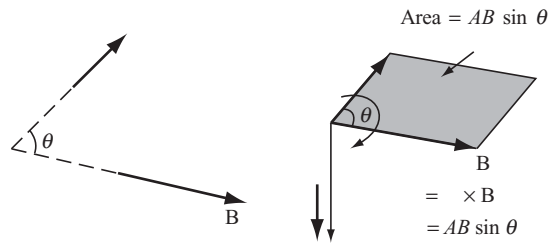
(e2) *Vector or cross product* of two coplanar vectors \mathbf{A} and \mathbf{B} is a vector \mathbf{C} denoted by $\mathbf{A} \times \mathbf{B}$ and read as cross \mathbf{B} such that (a) its magnitude equals the product of the magnitudes of the vectors \mathbf{A} and \mathbf{B} and the sine of the smaller angle between them, (b) its direction is perpendicular to the plane of \mathbf{A} and \mathbf{B} , and (c) its sense is given by the right-handed screw rule.

$$|\mathbf{C}| = \mathbf{A} \times \mathbf{B} = AB \sin \theta \quad (R2.12)$$

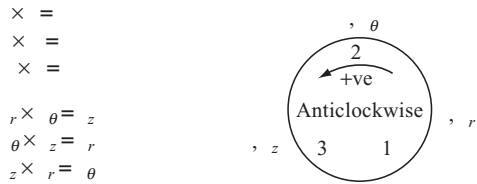
Geometrically, the magnitude of the cross product vector \mathbf{C} equals the area of the parallelogram bounded by the vectors \mathbf{A} and \mathbf{B} as the adjacent sides as shown in Fig. R2.6(b). The direction of vector \mathbf{C} is perpendicular to this area with an arrowhead decided by the right-handed screw rule.



(a) Scalar or Dot Product



(b) Vector or Cross Product



Aid to Memory: Cross Product

Fig. R2.6 Products of vectors

The cross product of two skew vectors is not definable.

The cross product of two collinear or parallel vectors must vanish since $\sin 0 = 0$.

The cross product of two orthogonal vectors must be a vector directed along the third orthogonal axis given by the cross-product rule.

In particular, the cross products of the unit vectors are:

$$\begin{array}{l} \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \end{array} \quad (\text{R2.13})$$

In terms of the rectangular components,

$$\begin{aligned} \mathbf{A} &= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \\ \mathbf{B} &= B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \\ \mathbf{A} \times \mathbf{B} &= (A_y B_z - A_z B_y) \mathbf{i} + (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ \mathbf{A} \times \mathbf{A} &= \mathbf{0} \end{aligned} \quad (\text{R2.14})$$

The vector or cross product is *not commutative* but obeys the distributive law:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &\neq \mathbf{B} \times \mathbf{A} \\ \mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \\ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ n(\mathbf{A} \times \mathbf{B}) &= n \mathbf{A} \times \mathbf{B} = \mathbf{A} \times n\mathbf{B} \end{aligned} \quad (\text{R2.15})$$

If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, then either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or both are zero (trivial case), or \mathbf{A} is parallel to \mathbf{B} , or \mathbf{A} and \mathbf{B} are collinear.

The angle between two vectors \mathbf{A} and \mathbf{B} is given by

$$\sin \theta = \frac{|\mathbf{A} \times \mathbf{B}|}{AB} \quad (\text{R2.16})$$

(f) *Triple products of vectors* can also be meaningfully defined as follows:

(f1) *Scalar triple product* of three vectors \mathbf{A} , \mathbf{B} and \mathbf{C} is defined as

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad (\text{R2.17})$$

where

$$\begin{aligned} \mathbf{A} &= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \\ \mathbf{B} &= B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \\ \mathbf{C} &= C_x \mathbf{i} + C_y \mathbf{j} + C_z \mathbf{k} \end{aligned}$$

The scalar triple product results in a scalar quantity represented by the volume of the parallelepiped having \mathbf{A} , \mathbf{B} and \mathbf{C} as the adjacent edges.

This can be shown with reference to Fig. R2.7

$$|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = BC \sin \theta$$

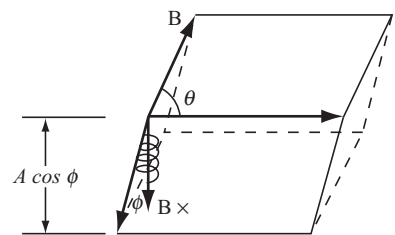


Fig. R2.7 Representation of $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = BC \sin \theta A \cos \phi$$

= Area of the parallelogram contained by \mathbf{B} and \mathbf{C} multiplied by the perpendicular distance between the face $OB\mathbf{C}$ and the one parallel to it

= Volume of the parallelepiped contained by the three vectors \mathbf{A} , \mathbf{B} and \mathbf{C}

Volume is a scalar quantity. One can arrive at the volume of a parallelepiped by multiplying the area of its faces by the perpendicular distance between that face and the one parallel to it. Hence the order of vectors in a scalar triple product is immaterial. Hence

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad (\text{R2.18})$$

If the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is zero, then either \mathbf{A} or \mathbf{B} or \mathbf{C} is zero singly or in combination, or they are coplanar vectors or two or all are collinear vectors.

(f2) Vector triple product of three vectors \mathbf{A} , \mathbf{B} and \mathbf{C} is defined as

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} \end{aligned} \quad (\text{R2.19})$$

and

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

If the vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 0$ then either \mathbf{A} or \mathbf{B} or \mathbf{C} is zero singly or in combination, or is in the plane containing \mathbf{B} and \mathbf{C} .

Example R2.7 Determine the components of the 500 N force shown along the aa and bb axes.

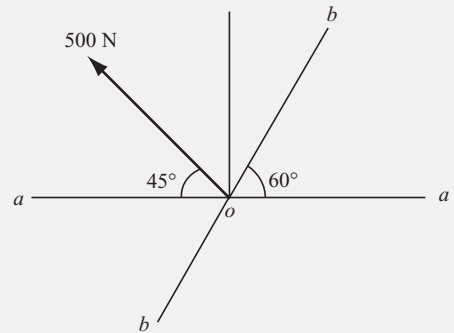


Fig. Ex. R2.7

Solution Let the x axis be along oa and the y axis perpendicular to it.

Unit vector along oa is \mathbf{i} and the unit vector along ob is $\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j} = 0.5 \mathbf{i} + 0.866 \mathbf{j}$.

The 500 N force is expressed as

$$(-500 \cos 45^\circ \mathbf{i} + 500 \sin 45^\circ \mathbf{j}) \text{ N}$$

or $(-353.5 \mathbf{i} + 353.5 \mathbf{j}) \text{ N}$

The component of the force along oa is

$$(-353.5 \mathbf{i} + 353.5 \mathbf{j}) \cdot \mathbf{i} = -353.5 \text{ N}$$

and the component along ob is

$$(-353.5 \mathbf{i} + 353.5 \mathbf{j}) \cdot (0.5 \mathbf{i} + 0.866 \mathbf{j}) = -176.8 + 306.2 = 129.4 \text{ N}$$

One can as well determine the desired components geometrically by estimating the projections of the 500 N force along oa and ob respectively. The projection along oa is $-500 \cos 45^\circ$, i.e., -353.5 N and the projection along ob is $500 \cos (90^\circ - 60^\circ + 45^\circ)$, i.e., $500 \cos 75^\circ$ or 129.4 N .

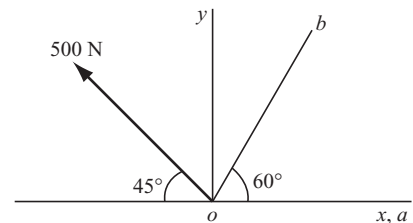


Fig. Ex. R2.7 (Solution)

Example R2.8

Determine the component of the vector $(3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$ along the vector $(4\mathbf{i} - 3\mathbf{j})$.

Solution The dot product of a vector with another results in the product of the projection of one vector along the other and the magnitude of the other vector.

$$\mathbf{a} \cdot \mathbf{B} = A \cos \theta \quad B = AB \cos \theta$$

where θ is the angle between \mathbf{a} and \mathbf{B} .

If \mathbf{e} is unit vector along \mathbf{B} then

$$\mathbf{a} \cdot \mathbf{e} = A \cos \theta$$

the component of \mathbf{a} along \mathbf{B} .

The unit vector along $(4\mathbf{i} - 3\mathbf{j})$ is

$$\mathbf{e} = \frac{(4\mathbf{i} - 3\mathbf{j})}{\sqrt{4^2 + 3^2}} = 0.8\mathbf{i} - 0.6\mathbf{j}$$

The desired component is

$$(3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \cdot (0.8\mathbf{i} - 0.6\mathbf{j}) = 2.4 - 1.2 = 1.2 \text{ units}$$

The component vector is, however, given by

$$1.2(0.8\mathbf{i} - 0.6\mathbf{j}) = 0.96\mathbf{i} - 0.72\mathbf{j}$$

Example R2.9

Two vectors \mathbf{A} and \mathbf{B} are given as

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{B} = 3\mathbf{i} - \mathbf{j}$$

Determine (a) the dot product $\mathbf{A} \cdot \mathbf{B}$, (b) the cross product $\mathbf{A} \times \mathbf{B}$ and (c) the angle between \mathbf{A} and \mathbf{B} .

Solution

(a) The dot product is given by

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y = 2 \times 3 + 3 \times (-1) = 3 \text{ units}$$

(b) The cross product is expressed as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= 2 \times (-1) - 3 \times 3 = -11\mathbf{k}$$

This is represented in Fig. Ex. R2.9.

(c) From the definition of the dot product,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

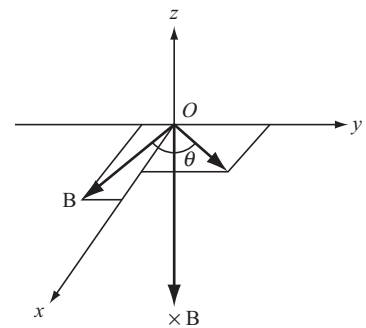


Fig. Ex. R2.9 (Solution)

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{3}{\sqrt{2^2 + 3^2} \times \sqrt{3^2 + 1^2}} = \frac{3}{3.61 \times 3.16} = 0.264$$

and $\theta = 74.7$

Alternatively, from the definition of cross product,

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta$$

$$\sin \theta = \frac{|\mathbf{A} \times \mathbf{B}|}{AB} = \frac{11}{3.61 \times 3.16} = 0.964$$

and $\theta = 74.7$

Example R2.10

Two vectors \mathbf{A} and \mathbf{B} are given:

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 1\mathbf{k}$$

$$\mathbf{B} = 3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

Determine (a) their dot product, (b) their cross product and the unit vector along it, and (c) the included angle between vector \mathbf{A} and the vector resulting from the cross product.

Solution

(a) $\mathbf{A} \cdot \mathbf{B} = 2 \times 3 + 3 \times (-3) + 1 \times 4 = 1$

(b) $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{vmatrix}$

$$= 3 \times 4 - (-3) \times 1 \mathbf{i} - 2 \times 4 - 3 \times 1 \mathbf{j} + 2 \times (-3) - 3 \times 3 \mathbf{k}$$

$$= 15 \mathbf{i} - 5 \mathbf{j} - 15 \mathbf{k}$$

The magnitude of this product is

$$\sqrt{15^2 + 5^2 + 15^2} = \sqrt{475} = 21.8$$

The unit vector along it is, therefore, given by

$$\frac{15}{21.8} \mathbf{i} - \frac{5}{21.8} \mathbf{j} - \frac{15}{21.8} \mathbf{k} = 0.688 \mathbf{i} - 0.229 \mathbf{j} - 0.688 \mathbf{k}$$

(c) The included angle between the vector resulting from the cross product and either of the constituent vectors must be $90^\circ = \pi/2$. Examining the same for the present case,

$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 15 & -5 & -15 \end{vmatrix}$$

$$= (-45 + 5) - (-30 - 15) + (-10 - 45)$$

$$= -40 + 45 - 55$$

The magnitude of which is

$$\sqrt{40^2 + 45^2 + 55^2} = 81.55$$

and

$$\sin \theta = \frac{81.55}{3.74 \times 21.8} = 1$$

whence θ , the angle between \mathbf{A} and $\mathbf{A} \times \mathbf{B}$, is 90° .

R2.5 DERIVATIVES AND INTEGRALS OF VECTORS

(a) Derivatives of Vectors

The derivative of the vector \mathbf{A} with respect to a scalar, say time t , is expressed as

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{A}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} \quad (\text{R2.20})$$

where the vector \mathbf{A} is a function of the scalar t and $\Delta \mathbf{A}$ refers to the changes in \mathbf{A} during intervals of scalar Δt as shown in Fig. R2.8a.

If $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$

then
$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt} \mathbf{i} + \frac{dA_y}{dt} \mathbf{j} + \frac{dA_z}{dt} \mathbf{k} \quad (\text{R2.21})$$

The derivative of a given vector with respect to a scalar results in a vector representing the rate of change of the given vector with respect to the scalar.

An extension of the usual rules of differential calculus leads to the following rules:

$$\begin{aligned} \frac{d}{dt}(n \mathbf{A}) &\equiv n \frac{d\mathbf{A}}{dt} \\ \frac{d}{dt}(\mathbf{A} + \mathbf{B}) &\equiv \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \\ \frac{d}{dt}(\phi \mathbf{A}) &\equiv \phi \frac{d\mathbf{A}}{dt} + \frac{d\phi}{dt} \mathbf{A} \\ \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &\equiv \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} \\ \frac{d}{dt}(\mathbf{A} \times \mathbf{B}) &\equiv \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \end{aligned} \quad (\text{R2.22})$$

where n is a real number and ϕ is a scalar function of t .

It may be noted that the derivative of a constant vector \mathbf{A} with respect to any scalar must be zero, i.e.,

$$\frac{d\mathbf{A}}{dt} = 0$$

(b) Derivatives of Unit Vectors

Derivatives of the unit vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} along the fixed space coordinates x , y and z respectively must be zero with respect to any scalar.

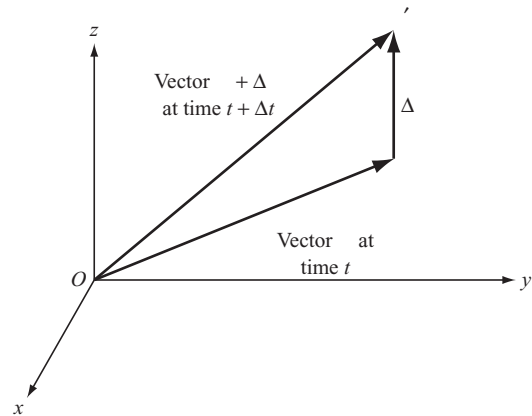


Fig. R2.8(a) Change in vector \mathbf{A}

$$\frac{d}{dt} = 0 = \frac{d}{dt} = \frac{d}{dt} \dots$$

Let us consider the derivatives of the unit vectors in radial and tangential direction, i.e., of e_r and e_θ as shown in Fig. R2.8(b)

$$e_r = \cos \theta \, e_x + \sin \theta \, e_y$$

$$\frac{d e_r}{dt} = (-\sin \theta \, e_x + \cos \theta \, e_y) \frac{d\theta}{dt}$$

Since
and

$$e_\theta = -\sin \theta \, e_x + \cos \theta \, e_y$$

$$\omega = \frac{d\theta}{dt}$$

$$\frac{d e_r}{dt} = \omega e_\theta$$

Similarly,

$$\frac{d e_\theta}{dt} = (-\cos \theta \, e_x - \sin \theta \, e_y) \frac{d\theta}{dt}$$

$$= -\omega e_r$$

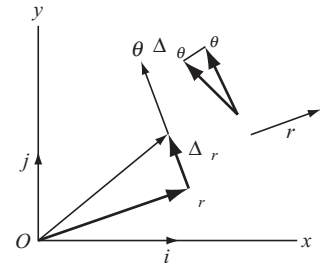


Fig. R2.8(b) Change in unit vectors

(c) Integrals of Vectors

If a vector A is a function of a scalar variable, say time t ,

$$A(t) = A_x \, e_x + A_y \, e_y + A_z \, e_z$$

then the integral of A over the range t_1 to t_2 is

$$\int_{t_1}^{t_2} A(t) dt = \int_{t_1}^{t_2} (A_x \, e_x + A_y \, e_y + A_z \, e_z) dt$$

$$= \int_{t_1}^{t_2} A_x \, dt \, e_x + \int_{t_1}^{t_2} A_y \, dt \, e_y + \int_{t_1}^{t_2} A_z \, dt \, e_z \quad (R2.23)$$

Example R2.11

Assuming $r = (r_0 \sin \omega t) \, e_x + (r_0 \cos \omega t) \, e_y$ evaluate (a) r , the magnitude of the vector at any time t , (b) $\frac{d}{dt}$, (c) $\frac{d^2}{dt^2}$ and (d) the integral of r from $t = 0$ to $t = \pi/\omega$.

Solution

(a) $r = \sqrt{(r_0 \sin \omega t)^2 + (r_0 \cos \omega t)^2} = r_0$

(b) $\frac{d}{dt} = (r_0 \omega \cos \omega t) \, e_x - (r_0 \omega \sin \omega t) \, e_y$

the magnitude of which is given by

$$\sqrt{(r_0 \omega \cos \omega t)^2 + (r_0 \omega \sin \omega t)^2} = \omega r_0$$

$$(c) \frac{d^2}{dt^2} = -(r_0 \omega^2 \sin \omega t) - (r_0 \omega^2 \cos \omega t) = -\omega^2$$

the magnitude of which is given by ω^2

$$\begin{aligned} (d) \int_0^{\pi/\omega} dt &= \int_0^{\pi/\omega} r_0 \sin \omega t \, dt + \int_0^{\pi/\omega} r_0 \cos \omega t \, dt \\ &= -\left. \frac{r_0 \cos \omega t}{\omega} \right|_0^{\pi/\omega} + \left. \frac{r_0 \sin \omega t}{\omega} \right|_0^{\pi/\omega} \\ &= -\frac{r_0}{\omega} (-1 - 1) + \frac{r_0}{\omega} (0 - 0) \\ &= \frac{2r_0}{\omega} \end{aligned}$$

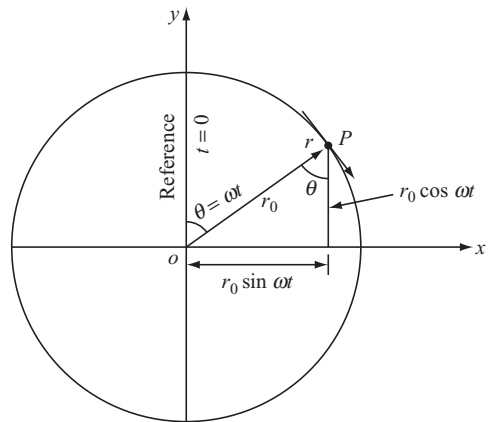


Fig. Ex. R2.11 (Solution)

Example R2.12

If $\mathbf{A} = 2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k}$ and $\mathbf{B} = \sin t \mathbf{i} + \cos t \mathbf{j}$ evaluate (a) $\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$

- (b) $\frac{d}{dt}(\mathbf{A} \times \mathbf{B})$ (c) $\frac{d}{dt}(\mathbf{A} \times \mathbf{B})$ (d) $\frac{d}{dt}(\mathbf{A} \times \mathbf{B})$

Solution

$$\begin{aligned} (a) \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} \\ &= (2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k}) \cdot (\cos t \mathbf{i} - \sin t \mathbf{j}) + (4t \mathbf{i} + \mathbf{j} - 3t^2 \mathbf{k}) \cdot (\sin t \mathbf{i} + \cos t \mathbf{j}) \\ &= 3t \sin t + 2t^2 \cos t + \cos t \end{aligned}$$

$$\begin{aligned} (b) \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} = 2 \cdot \frac{d}{dt} = 2(2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k}) \cdot (4t \mathbf{i} + \mathbf{j} - 3t^2 \mathbf{k}) \\ &= 16t^3 + 2t + 6t^5 \end{aligned}$$

$$\begin{aligned} (c) \frac{d}{dt}(\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \\ &= (2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k}) \times (\cos t \mathbf{i} - \sin t \mathbf{j}) + (4t \mathbf{i} + \mathbf{j} - 3t^2 \mathbf{k}) \times (\sin t \mathbf{i} + \cos t \mathbf{j}) \\ &= -t^3 \sin t \mathbf{i} - t^3 \cos t \mathbf{j} - (2t^2 \sin t + t \cos t) \mathbf{k} + 3t^2 \cos t \mathbf{i} - 3t^2 \sin t \mathbf{j} + (4t \cos t - \sin t) \mathbf{k} \\ &= (-t^3 \sin t + 3t^2 \cos t) \mathbf{i} - (t^3 \cos t + 3t^2 \sin t) \mathbf{j} + (3t \cos t - (1 + 2t^2) \sin t) \mathbf{k} \end{aligned}$$

$$(d) \frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d}{dt}(0) = 0$$

Alternatively, the bracketed vector operations can be performed first and differentiation done later. For example, in part (b),

$$\begin{aligned} \frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \frac{d}{dt}[(2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k}) \cdot (2t^2 \mathbf{i} + t \mathbf{j} - t^3 \mathbf{k})] = \frac{d}{dt}(4t^4 + t^2 + t^6) \\ &= 16t^3 + 2t + 6t^5 \end{aligned}$$

which is the same as obtained earlier.

R2.6 GRADIENT, DIVERGENCE AND CURL

If a vector \mathbf{A} and a scalar ϕ are functions of the space coordinates x, y and z then a *vector differential operator* ∇ called 'del',

$$\nabla \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad (\text{R2.24})$$

can be operated to define the following:

(a) Gradient of a Scalar ϕ

$$\begin{aligned} \text{grad } \phi &= \nabla \phi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \phi \\ &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \end{aligned} \quad (\text{R2.25})$$

Physically, the gradient of a scalar is a vector in the direction of the steepest variation of ϕ with respect to the space coordinates. The component of $\text{grad } \phi$ in any direction is the rate of change of ϕ in that direction.

(b) Divergence of a Vector \mathbf{A}

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (\text{R2.26})$$

The divergence of a vector refers to the net efflux of the vector at a point in space.

(c) Curl of a Vector \mathbf{A}

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad (\text{R2.27})$$

R2.7 SOME VECTOR OPERATIONS

Some useful results in vector algebra and calculus are summarized in Table R2.1.

R2.8 POSITION VECTOR, DISPLACEMENT, VELOCITY AND ACCELERATION

The position of a moving point A in space is described by a *position vector*, a vector directed from an origin to the point. Thus \mathbf{r} is the position vector of A in Fig. R2.8(a) at an instant. At a later instant after an interval of time Δt , the position vector will be \mathbf{r}' corresponding to the new position of the moving point. The change in the position from A to A' is called the displacement vector or just the displacement.

$$\text{Displacement} = \mathbf{r}' - \mathbf{r} = \Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k} \quad (\text{R2.28})$$

Since this displacement is brought about in a time interval Δt , the *velocity* of the point is given by

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Table R2.1 Vector operations in different coordinate systems

| | Cartesian | Cylindrical | Spherical |
|---|--|--|---|
| Vector | $A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$ | $A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_z \mathbf{e}_z$ | $A_R \mathbf{e}_R + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi$ |
| $ A $ | $(A_x^2 + A_y^2 + A_z^2)^{1/2}$ | $(A_r^2 + A_\theta^2 + A_z^2)^{1/2}$ | $(A_R^2 + A_\theta^2 + A_\phi^2)^{1/2}$ |
| $\mathbf{A} \cdot \mathbf{B} = AB \cos \alpha$ | $A_x B_x + A_y B_y + A_z B_z$ | $A_r B_r + A_\theta B_\theta + A_z B_z$ | $A_R B_R + A_\theta B_\theta + A_\phi B_\phi$ |
| $\mathbf{A} \times \mathbf{B} = AB \sin \alpha$ | $\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ | $\begin{vmatrix} r & \theta & z \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$ | $\begin{vmatrix} R & \theta & \phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$ |
| $(\mathbf{A} \cdot \mathbf{B}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A})$ | $\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$ | $\begin{vmatrix} A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \\ C_r & C_\theta & C_z \end{vmatrix}$ | $\begin{vmatrix} A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \\ C_R & C_\theta & C_\phi \end{vmatrix}$ |
| Operator $\nabla \equiv$ | $\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$ | $\frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z$ | $\frac{\partial}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} \mathbf{e}_\phi$ |
| $\nabla p =$ | $\frac{\partial p}{\partial x} \mathbf{i} + \frac{\partial p}{\partial y} \mathbf{j} + \frac{\partial p}{\partial z} \mathbf{k}$ | $\frac{\partial p}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \mathbf{e}_\theta + \frac{\partial p}{\partial z} \mathbf{e}_z$ | $\frac{\partial p}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial p}{\partial \theta} \mathbf{e}_\theta + \frac{1}{R \sin \theta} \frac{\partial p}{\partial \phi} \mathbf{e}_\phi$ |
| $\text{div } \mathbf{A} = \nabla \cdot \mathbf{A}$ | $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ | $\frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$ | $\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$ |
| $\mathbf{A} \cdot \nabla \times \mathbf{B}$ | $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$ | $\begin{vmatrix} r & \theta & z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & r A_\theta & A_z \end{vmatrix}$ | $\begin{vmatrix} R & \theta & R \sin \theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R \sin \theta & A_\phi \end{vmatrix}$ |

$$= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = \dot{x} + \dot{y} + \dot{z} \tag{R2.29}$$

Similarly, the rate of change of velocity is the *acceleration* given by

$$\begin{aligned} \mathbf{a} &= \frac{d}{dt} \left(\frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) \\ &= \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} + \frac{d^2z}{dt^2} \mathbf{k} \\ &= \ddot{x} \mathbf{i} + \ddot{y} \mathbf{j} + \ddot{z} \mathbf{k} \end{aligned} \tag{R2.30}$$

It is thus noted that (with respect to time):

The derivative of the position vector is the velocity and the derivative of the velocity is acceleration.

Alternatively (with respect to time),

The integral of the velocity is the position vector and the integral of the acceleration is the velocity

$$\begin{aligned} \int_{t_1}^{t_2} \mathbf{v}(t) dt &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= \text{change in position} = \text{displacement} \end{aligned} \tag{R2.31}$$

$$\begin{aligned} \int_{t_1}^{t_2} \mathbf{a}(t) dt &= \mathbf{v}_2 - \mathbf{v}_1 \\ &= \text{change in velocity in the interval} \end{aligned} \tag{R2.32}$$

Example R2.13 Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$ where C is a constant. Hence find the unit vector normal to the surface $3xz^2 - 3xy - 4x = 7$ at the point $(1, 2, -2)$ and an equation to the tangent plane at this point.

Solution Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the surface $\phi(x, y, z) = C$. Vector $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ must, therefore, be along a tangent to the surface at that point.

By differential calculus,

$$\partial\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

or

$$\left(\frac{\partial\phi}{\partial x} \mathbf{i} + \frac{\partial\phi}{\partial y} \mathbf{j} + \frac{\partial\phi}{\partial z} \mathbf{k} \right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = 0$$

or

$$\nabla\phi \cdot d\mathbf{r} = 0$$

which implies that $\nabla\phi$ must be perpendicular to $d\mathbf{r}$. Since $d\mathbf{r}$ is tangential to the surface, vector $\nabla\phi$ must be normal to it.

Another interesting point worth noting is that

since

$$\begin{aligned} d\phi &= \nabla\phi \cdot d\mathbf{r} \\ \frac{d\phi}{ds} &= \nabla\phi \cdot \frac{d\mathbf{r}}{ds} \end{aligned}$$

where s is an arbitrary space direction. For $d\phi/ds$ to be maximum, $\nabla\phi$ and d/ds must be collinear and d/ds being unity, $d\phi/ds = \nabla\phi$. In other words, the greatest rate of change of ϕ , i.e., the maximum directional derivative takes place in the direction of, and has the magnitude of the vector $\nabla\phi$.

For the given surface,

$$3xz^2 - 3xy - 4x = 7 = \phi(x, y, z)$$

$$\nabla\phi = \nabla(3xz^2 - 3xy - 4x)$$

$$= (3z^2 - 3y - 4) \mathbf{i} - 3x \mathbf{j} + 6xz \mathbf{k}$$

At $(1, 2, -2), \nabla\phi = 2 \mathbf{i} - 3 \mathbf{j} - 12 \mathbf{k}$, $|\nabla\phi| = \sqrt{2^2 + (-3)^2 + (-12)^2} = 12.53$

Since $\nabla\phi$ is normal to the surface, the unit vector normal to the surface is given by

$$\frac{2}{12.53} \mathbf{i} - \frac{3}{12.53} \mathbf{j} - \frac{12}{12.53} \mathbf{k} = 0.16 \mathbf{i} - 0.24 \mathbf{j} - 0.96 \mathbf{k}$$

If d is tangential to the surface

or $\nabla\phi \cdot d = 0$

$$(2 \mathbf{i} - 3 \mathbf{j} - 12 \mathbf{k}) \times (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) - (2 \mathbf{i} - 3 \mathbf{j} - 12 \mathbf{k}) = 0$$

which provides $2(x - 1) - 3(y - 2) - 12(z + 2) = 0$

as the equation of the tangent plane at the point $(1, 2, -2)$.

Example R2.14

The motion of a point is expressed as $x = 2t^3, y = t^2 + 4t, z = 3t - 5$ in terms of the time parameter t . At time $t = 2$, determine (a) the velocity and acceleration, (b) the components of the velocity and acceleration in the direction of $(4 \mathbf{i} + 3 \mathbf{j})$, and (c) the unit tangent at the point. Also, determine the displacement of the point from $t = 0$ to $t = 2$.

Solution

(a) The position vector \mathbf{r} is expressed as

$$\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

$$= 2t^3 \mathbf{i} + (t^2 + 4t) \mathbf{j} + (3t - 5) \mathbf{k}$$

$$\text{Velocity } \mathbf{V} = \frac{d\mathbf{r}}{dt} = 6t^2 \mathbf{i} + (2t + 4) \mathbf{j} + 3 \mathbf{k}$$

$$= 24 \mathbf{i} + 8 \mathbf{j} + 3 \mathbf{k} \quad \text{at } t = 2.$$

$$\text{Acceleration } \mathbf{a} = \frac{d\mathbf{V}}{dt} = 12t \mathbf{i} + 2 \mathbf{j}$$

$$= 24 \mathbf{i} + 2 \mathbf{j} \quad \text{at } t = 2$$

(b) The unit vector along $(4 \mathbf{i} + 3 \mathbf{j})$ would be

$$\frac{4}{\sqrt{4^2 + 3^2}} \mathbf{i} + \frac{3}{\sqrt{4^2 + 3^2}} \mathbf{j} = 0.8 \mathbf{i} + 0.6 \mathbf{j}$$

The velocity component at $t = 2$ along this direction is

$$(24 \mathbf{i} + 8 \mathbf{j} + 3 \mathbf{k}) \times (0.8 \mathbf{i} + 0.6 \mathbf{j}) = 19.2 \mathbf{i} + 4.8 \mathbf{j} = 24 \text{ units.}$$

The acceleration component at $t = 2$ along this direction is

$$(24 + 2) \times (0.8 + 0.6) = 19.2 + 1.2 = 20.4 \text{ units.}$$

(c) The velocity vector must be tangential to the curve at the point and time considered.

Since
$$V = \sqrt{24^2 + 8^2 + 3^2} = \sqrt{649} = 25.48$$

The unit tangent at that instant must be

$$\frac{24}{25.48} + \frac{8}{25.48} + \frac{3}{25.48} = 0.94 + 0.31 + 0.12$$

(d) Since

$$\frac{d}{dt} = 6t^2 + (2t + 4) + 3$$

$$d = (6t^2 + (2t + 4) + 3) dt$$

Displacement

$$\begin{aligned} (x_2 - x_1) &= \int_{t_1}^{t_2} d \\ &= \int_0^2 (6t^2 + (2t + 4) + 3) dt \\ &= 2t^3 + (t^2 + 4t) + 3t \Big|_0^2 \\ &= 16 + 12 + 6 \end{aligned}$$

The displacement may alternatively be determined by finding the initial and final positions:

At the initial position, $t = 0$,

$$x_1 = 2t^3 = 0; y_1 = t^2 + 4t = 0; z_1 = 3t - 5 = -5$$

At the final position, $t = 2$,

$$x_2 = 2t^3 = 16; y_2 = t^2 + 4t = 12; z_2 = 3t - 5 = 1$$

The displacement is given by

$$\begin{aligned} x_2 - x_1 &= (16 - 0) + (12 - 0) + (1 - (-5)) \\ &= 16 + 12 + 6 \end{aligned}$$

Example R2.15

Determine and sketch the curve traced by a point such that $x = 200t$ and $y = 1000 - 4t^2$.

Also, comment on the salient features of the motion if the distances are in metres and time is in seconds.

Solution Eliminating t from the parametric equations,

$$\begin{aligned} t &= \frac{x}{200}, \quad y = 1000 - 4\left(\frac{x}{200}\right)^2 \\ y &= 1000 - 0.0001x^2 \end{aligned}$$

This is the equation of the curve to be traced.

| | | | | | |
|-------------|---|------|------|------|------|
| x (metre) | 0 | 1000 | 2000 | 2236 | 3162 |
| y (metre) | 0 | 900 | 600 | 500 | 0 |

The curve is sketched in Fig. Ex. R2.15

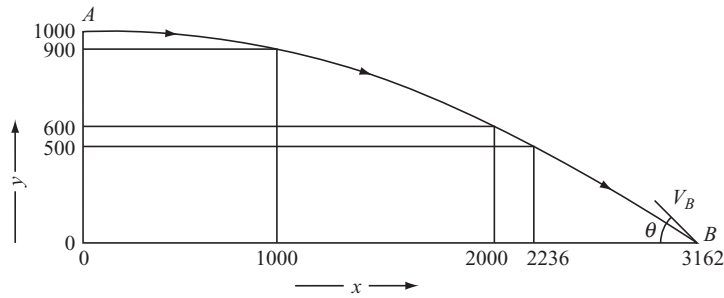


Fig. Ex R2.15 (Solution)

At $t = 0$, $x = 0$ and $y = 1000$ m correspond to point A where the observation starts. In general, the velocity components are

$$V_x = \frac{dx}{dt} = 200 \text{ m/s}$$

$$V_y = \frac{dy}{dt} = -8t \text{ m/s}$$

At $t = 0$, $V_x = 200$ m/s and $V_y = 0$ which shows that the initial velocity at A is wholly horizontal and equals 200 m/s, whereas with the passage of time, the horizontal component stays at 200 m/s and the vertical component increases downwards linearly with time. The acceleration is

$$a_y = \frac{dv_y}{dt} = -8 \text{ m/s}^2$$

which is approximately 80% of the gravitational acceleration due to the earth. The case in hand closely resembles the motion of a bomb released from the low-altitude bomber aircraft flying at 200 m/s parallel to the ground. The bomb, when dropped, travels horizontally and, with the passage of time, acquires a vertical velocity by virtue of the acceleration due to gravity and is resisted by the aerodynamic drag.

It reaches the base at B where $y = 0$ and $x = 3162$ m. The time taken to reach the base is $t = 3162/200 = 15.81$ seconds. The velocity components at the base B are

$$V_x = 200 \text{ m/s}, V_y = -8 \times 15.81 = -126.48 \text{ m/s}$$

or

$$\mathbf{V}_B = 200 \text{ -- } 126.48 \text{ m/s}$$

where

$$V_B = \sqrt{200^2 + 126.48^2} = 236.64 \text{ m/s}$$

and

$$\theta = \tan^{-1} \left(\frac{126.48}{200} \right) = \tan^{-1} 0.6324 = 32.32^\circ$$

R2.9 SOME OTHER VECTOR QUANTITIES

In addition to certain vector quantities already referred to, a number of other important vector quantities are listed in this section. No attempt is made here to define the quantities but merely to recognise them as vector quantities. It may also be noted that only the *position vector* and the *force* be conferred the vector status axiomatically; rest of the vector quantities can be proved to be vectors by virtue of their definitions.

Force Force is an action exerted on a body which changes or tends to change the state of rest or of rectilinear motion of the body.

Moment Moment refers to the turning effect of a force about a point O

$$= \times$$

The magnitude of the moment is given by

$$M = rF \sin \theta = F \cdot p \tag{R2.33}$$

where is the position vector of any point on the line of action of the force and p is the perpendicular distance from the point to the line of action of the vector . The moment is directed perpendicular to the plane containing and as shown in Fig. R2.9.

The principle of moments due to Varignon called Varignon theorem, states that *The moment of a force equals the sum of the moments due to its components.*

Consider a force and moment of the force about an origin O . By definition,

$$= \times$$

Let be resolved into components $i_1, i_2, \dots, i_i, \dots, i_n$ such that

$$= i_1 + i_2 + \dots + i_i + \dots + i_n$$

$$= \sum_{i=1}^n i_i$$

and the position vector refers to each of these forces as well.

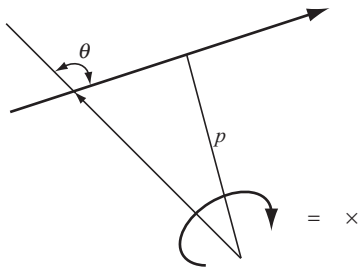


Fig. R2.9 Force and moment

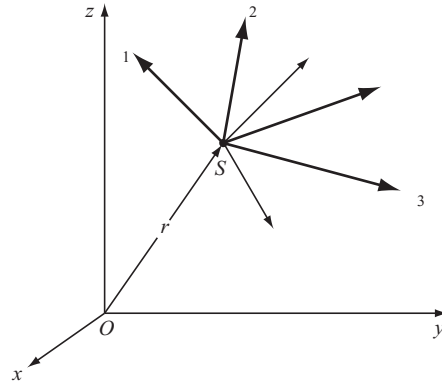


Fig. R2.10 Concurrent forces

Then,

$$= \times$$

$$= \times (i_1 + i_2 + i_3 + \dots + i_i + \dots + i_n)$$

$$= i_1 + i_2 + \dots + i_i + \dots + i_n$$

$$= \sum_{i=1}^n i_i \times i_i = \sum_{i=1}^n i_i$$

Therefore,

It will be noticed that this theorem has no outstanding concept to offer. The theorem was proposed by the French mathematician long before the introduction of vector algebra.

Linear Momentum Linear momentum of a body of mass m moving at a velocity at a certain instant is defined as the product of the mass and velocity

$$\mathbf{L} = m \mathbf{v} \times \mathbf{r} \tag{R2.34}$$

Angular Momentum Angular momentum or moment of momentum is expressed as the cross product of the position vector and linear momentum or the moment of the linear momentum $m \mathbf{v}$ about a point O as shown in Fig. R2.11.

$$\mathbf{L} = \mathbf{r} \times m \mathbf{v} \tag{R2.35}$$

The angular momentum vector is directed perpendicular to the plane of \mathbf{r} and \mathbf{v} , i.e., perpendicular to the instantaneous plane of motion.

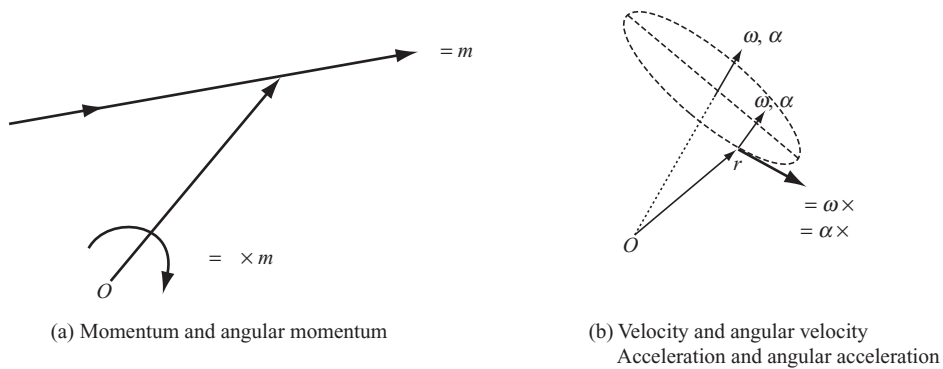


Fig. R2.11 Some other vector quantities

Infinitesimal Rotation Infinitesimal rotation $d\theta$, defined as a infinitesimally small amount of the angle of rotation, is a vector quantity. It is important to note here that finite rotation ‘ θ ’ is not a vector quantity. The reason is that the finite rotations θ_1 , and θ_2 , although possessing magnitudes, directions and sense, do not obey the commutative law of addition, i.e.,

$$\theta_1 + \theta_2 \neq \theta_2 + \theta_1 \tag{R2.36}$$

This has been demonstrated in Fig. R2.12(a) where (a) foot rule is shown subjected to rotations about the x and y axes respectively. An initial rotation $\theta_1 = \pi/2$ about the x -axis and a subsequent rotation $\theta_2 = \pi/2$ about the y -axis bring the sheet into positions (Aa) and (Aab). On the other hand, an initial rotation $\theta_2 = \pi/2$ about the y -axis and a subsequent rotation $\theta_1 = \pi/2$ about the x -axis result in positions (Ab) and (Aba) as shown. Obviously, the final positions (Aab) and (Aba) are not the same showing that the commutative law for finite rotations fails; admission disqualifying them from being put under the category of vector quantities.

The case of addition of infinitesimal rotations $d\theta_1$ and $d\theta_2$ is also shown in Fig. R2.12(b). The position of the foot rule after an initial rotation $d\theta_1$ is shown as (Ba) whereas that after an initial rotation $d\theta_2$ is shown as (Bb). The final orientation of the foot rule after both $d\theta_1$ and $d\theta_2$ have been imparted, in either

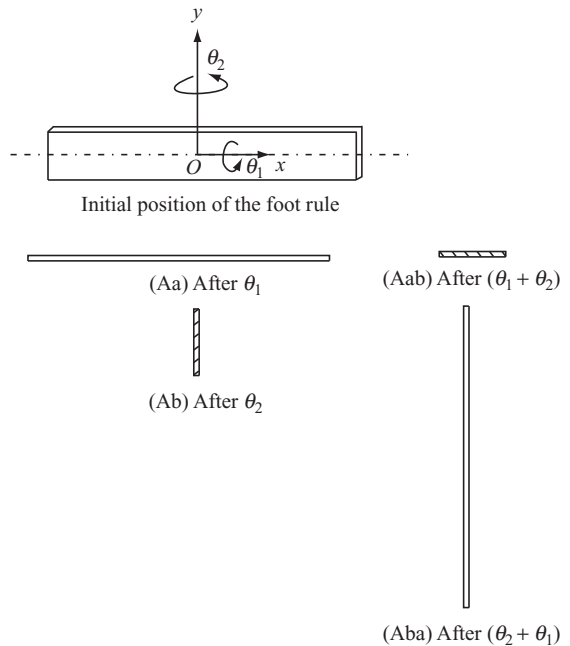


Fig. R2.12(a) Summation of finite rotations θ_1 and θ_2

order, is approximately the same in the limiting case shown in the same figure. Hence,

$$\Delta\theta_1 + \Delta\theta_2 \approx \Delta\theta_2 + \Delta\theta_1$$

and in the limit, $d\theta_1 + d\theta_2 = d\theta_2 + d\theta_1$ (R2.37)

Angular Velocity Angular velocity ω is defined as the rate of change of angular displacement or rotation of a body about an axis

$$\omega = \frac{d\theta}{dt} \quad (R2.38)$$

It follows from the vector nature of infinitesimal rotation $d\theta$ and the scalar nature of time interval dt that the angular velocity must be a vector quantity.

If a point is located with a position vector \mathbf{r} with respect to an origin O on the axis of rotation, the linear velocity \mathbf{v} of the point is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad (R2.39)$$

as shown in Fig. R2.11(b).

Angular Acceleration Angular acceleration α is defined as the rate of change of angular velocity, i.e.,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \quad (R2.40)$$

The linear acceleration \mathbf{a} of a point with a positive vector \mathbf{r} with respect to an origin O on the axis of rotation is given by

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} \quad (R2.41)$$

as also shown in Fig. R2.11(b).

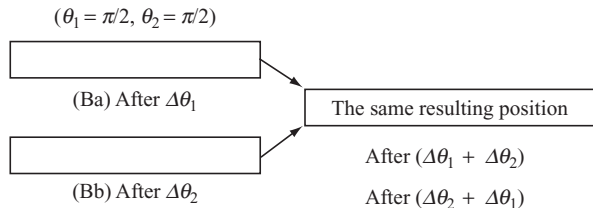


Fig. R2.12(b) Summation of infinitesimal rotations $\Delta\theta_1$ and $\Delta\theta_2$

Example R2.16

A force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j}$ passes through a point $(0, 2)$ with respect to an origin O . Determine the moment of the force about the origin and establish its uniqueness with respect to arbitrary position vectors.

Solution The fact that the force passes through a point $(0, 2)$ suggests that the vector $\mathbf{r} = 2\mathbf{j}$ is a position vector.

Hence
$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = 2\mathbf{j} \times (3\mathbf{i} + 2\mathbf{j}) = -6\mathbf{k}$$

Alternative position vectors, e.g., $(1.5\mathbf{i} + 3\mathbf{j})$ and $(-3\mathbf{i})$ would result in the same answer

$$(1.5\mathbf{i} + 3\mathbf{j}) \times (3\mathbf{i} + 2\mathbf{j}) = 3\mathbf{k} - 9\mathbf{k} = -6\mathbf{k}$$

$$(-3\mathbf{i}) \times (3\mathbf{i} + 2\mathbf{j}) = -6\mathbf{k}$$

If, instead, the perpendicular distance p from O to the line of action of \mathbf{F} is known,

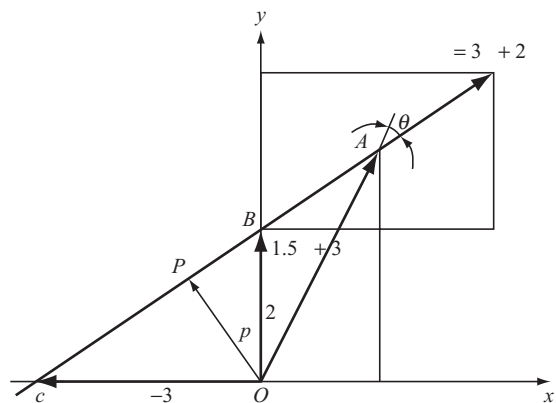


Fig. Ex. R2.16 (Solution)

$$= \cdot p \text{ (clockwise)}$$

$$= \sqrt{3^2 + 2^2} \times 1.66(-) = -6$$

In fact,

$$= \times = rF \sin \theta$$

$$= F \cdot r \sin \theta$$

$$= F p$$

which means that the cross product takes care of the included angle between the constituent vectors and allows no discrepancy in the result for different choices of the position vector.

Example R2.17

A vertical pole is guyed by three cables PA , PB and PC tied at a common point P 10 m above the ground. The base points of the cables are:

$$A(-4, -3, 0), B(5, 1, -1) \text{ and } C(-1, 5, 0)$$

If the tensile forces in the cables are adjusted to be 15, 18 and 20 kN, find the resultant force on the pole at P .

Solution Since P is 10 m above the ground at O , the forces in the cables must be directed along PA , PB and PC such that

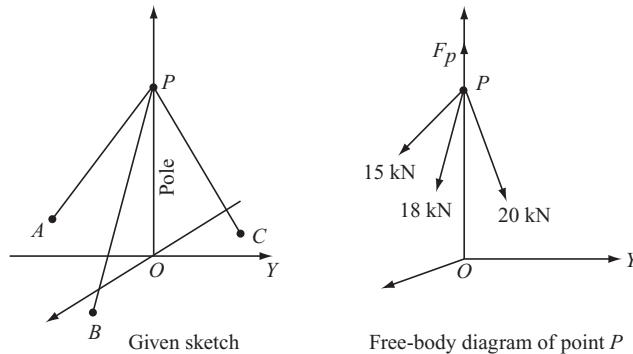


Fig. Ex. R2.17 (Solution)

$$P = -4 \quad -3 \quad -10 \qquad PA = \sqrt{4^2 + 3^2 + 10^2} = 11.18$$

$$PB = 5 \quad +1 \quad -11 \qquad PB = \sqrt{5^2 + 1^2 + 11^2} = 12.12$$

$$PC = -1 \quad +5 \quad -10 \qquad PC = \sqrt{1^2 + 5^2 + 10^2} = 11.22$$

The unit vectors along these directions are

$$e_1 = -\frac{4}{11.18} \quad -\frac{3}{11.18} \quad -\frac{10}{11.18} = -0.358 \quad -0.268 \quad -0.894$$

$$e_2 = -\frac{5}{12.12} + \frac{1}{12.12} - \frac{11}{12.12} = 0.412 + 0.082 - 0.907$$

$$e_3 = \frac{-1}{11.22} + \frac{5}{11.22} - \frac{10}{11.22} = -0.089 + 0.445 - 0.891$$

The forces in the cables are, therefore, given by

$$15 \quad F_1 = -5.37 \quad -4.02 \quad -13.41$$

$$18 \quad F_2 = 7.42 \quad +1.48 \quad -16.33$$

$$20 \quad F_3 = -1.78 \quad +8.90 \quad -17.82$$

Resultant force at $P = 0.27 \quad +6.36 \quad -47.56$

This is 48 kN in magnitude and acts predominantly downwards to hold the pole in position.

Example R2.18

A force of 1000 N in a particular direction must be applied to tow a boat. For some reason, it is not possible to apply the force in that direction but two forces can be applied to 30° and 45° on either side of it in the same plane containing the given force. Determine the magnitudes of the forces required along these directions.

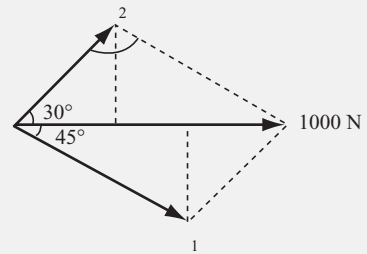


Fig. Ex. R2.18

Solution This is an example on the resolution of a force into two components at desired inclinations to it.

$$1000 = F_1 + F_2$$

Sum of the components of F_1 and F_2 along x should add to 1000 N, whereas their components perpendicular to it must cancel.

$$F_1 \cos 30^\circ + F_2 \cos 45^\circ = 1000$$

$$F_1 \sin 30^\circ = F_2 \sin 45^\circ$$

From these two equations,

$$F_1 = 732.1 \text{ N} \quad \text{and} \quad F_2 = 517.6 \text{ N}$$

Alternatively, the components can be determined geometrically by realising that 1000 N force must form the diagonal of the parallelogram with adjacent sides F_1 and F_2 . Completion of the parallelogram and measurement of F_1 and F_2 on the same scale to which the 1000 N force is drawn yields F_1 and F_2 .

Alternatively, vectorially,

$$1000 = F_1 e_1 + F_2 e_2$$

where e_1 and e_2 are the unit vectors along the respective forces as shown in Fig. Ex. R2.18(b).

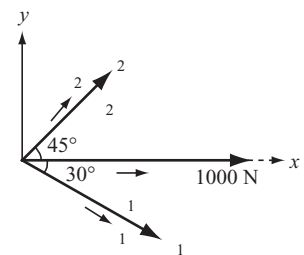


Fig. Ex. R2.18(a) (Solution)

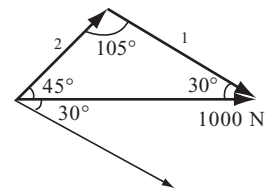


Fig. Ex. R2.18(b) (Solution)

Choosing x -axis along the 1000 N force, and y -axis perpendicular to it as shown,

$$\begin{aligned} &= \\ F_1 &= \cos 30^\circ - \sin 30^\circ \\ F_2 &= \cos 45^\circ + \sin 45^\circ \end{aligned}$$

Hence,

$$\begin{aligned} 1000 &= F_1 (\cos 30^\circ - \sin 30^\circ) + F_2 (\cos 45^\circ + \sin 45^\circ) \\ &= (F_1 \cos 30^\circ + F_2 \cos 45^\circ) + (F_2 \sin 45^\circ - F_1 \sin 30^\circ) \end{aligned}$$

and the two component equations are

$$\begin{aligned} 1000 &= F_1 \cos 30^\circ + F_2 \cos 45^\circ \\ 0 &= F_2 \sin 45^\circ - F_1 \sin 30^\circ \end{aligned}$$

which are indeed the two equations set up in the first method and give

$$F_1 = 732.1 \text{ N and } F_2 = 517.6 \text{ N}$$

It may be commented that if the x -axis was chosen in some other direction, say along F_1 then the equations will be different. For x -axis along F_1 the equations are

$$\begin{aligned} F_1 + F_2 \cos 75^\circ &= 1000 \cos 30^\circ \\ F_2 \sin 75^\circ &= 1000 \sin 30^\circ \end{aligned}$$

which again provide the same answer.

Alternatively, the components can be determined by observing that the components F_1 and F_2 add to result in 1000 N by way of a vector triangle as shown in Fig. Ex. R2.18(c). Applying the sine rule,

$$\frac{F_1}{\sin 45^\circ} = \frac{F_2}{\sin 30^\circ} = \frac{1000}{\sin 105^\circ}$$

whence,

$$F_1 = \frac{1000 \sin 45^\circ}{\sin 75^\circ} = 732.1 \text{ N}$$

$$F_2 = \frac{1000 \sin 30^\circ}{\sin 75^\circ} = 517.6 \text{ N}$$

Another point can be discussed with reference to this example. If the two components desired were not constrained to lie in the plane containing the given force as stated in the question, they will still have to lie in the same plane for equivalence. Hence, the phrase ‘in the same plane containing the given force’ is indeed superfluous. If, instead, the given force was to be resolved into three or more components in the same plane, then an infinite number of combinations of the magnitudes would be possible even if their directions were specified. On the other hand, if the given force was to be resolved into three space components at given inclinations, it would be possible to determine their magnitudes uniquely. Again, an infinite combination of the magnitudes would be possible for four or more number of space components even in the specified directions.

Example R2.19 A running person has, at an instant, his right foot oriented as shown. The muscular force he exerts is $m = 200$ N directed at 60° with the ground. Calculate the moment of the muscular force about the point of contact P with the ground.

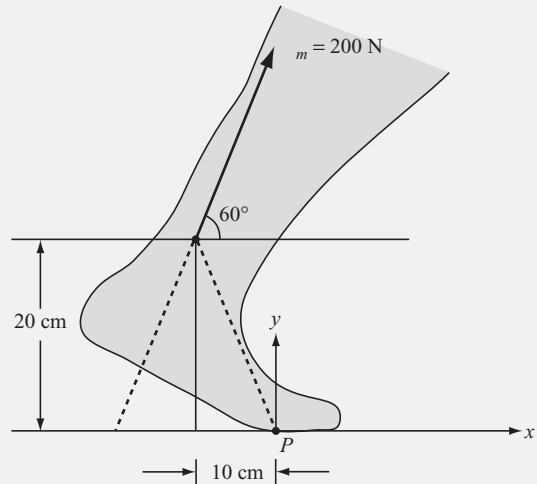


Fig. Ex R2.19

Solution Considering P as the origin, the position vector of the force is

$$\begin{aligned} \mathbf{r}_P &= -10\mathbf{i} + 20\mathbf{j} \\ \mathbf{m} &= 200(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j}) = 100\mathbf{i} + 173.2\mathbf{j} \text{ N} \end{aligned}$$

The moment of the force about P is, therefore, given by

$$\begin{aligned} M_P &= \mathbf{r}_P \times \mathbf{m} = (-10\mathbf{i} + 20\mathbf{j}) \times (100\mathbf{i} + 173.2\mathbf{j}) \\ &= -1732\mathbf{k} - 2000\mathbf{k} = -3732\mathbf{k} \end{aligned}$$

The negative sign shows that the moment is clockwise, i.e., in the direction of running.

Concept Review Questions

Match the following terms with the statements.

| <i>Terms</i> | <i>Statements</i> |
|-------------------------|---|
| (a) Sliding vector | (a) Adding a reversed vector to a vector |
| (b) Bound vector | (b) Division of vector quantities |
| (c) Undefined | (c) A vector divided by its magnitude |
| (d) Zero vector | (d) The transmissibility principle of vectors |
| (e) Unit vector | (e) Equivalence of effect |
| (f) Equipollent vectors | (f) Application at a unique point |

If a force $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ acts on a body with a position vector $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ in the xyz plane, which of the following expressions would result in the moment by the force about the origin.

- (i) $\mathbf{F} \times \mathbf{r}$
- (ii) $\mathbf{r} \cdot \mathbf{F}$
- (iii) $\mathbf{r} \times \mathbf{F}$
- (iv) $\mathbf{r} \cdot \mathbf{r}$

Discuss why

- (a) the parallelogram law must be obeyed by the vector quantities
- (b) the commutation rule fails for the vector products
- (c) the scalar triple product results in the volume of a parallelepiped
- (d) the position vector and the force must be vector quantities.

Show geometrically or otherwise that

- (a) $\mathbf{A} + \mathbf{B} + n\mathbf{C} = n\mathbf{A} + \mathbf{B} + \mathbf{C}$
- (b) $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
- (c) $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$

A vector may vary in magnitude and in direction with the passage of time. Illustrate the concepts of

- (a) derivative of a vector with respect to time
 - (b) integral of a vector with respect to time
- Give one example for each of the applications of differentiation and the integration operations.

Define the vector operator del , ∇ , and illustrate the physical significance of

- (a) the gradient of a scalar ϕ
- (b) the divergence of a vector
- (c) the curl of a vector \mathbf{B} .

Prove that

- (a) $d\phi \equiv \nabla\phi \cdot d$
- (b) $\nabla \times \nabla\phi \equiv 0$
- (c) $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$

where ϕ is a scalar, \mathbf{A} is a vector and d is a differential displacement vector.

Tutorial Problems

- R A vector quantity of 100 units acts along a line OP , terminating at P . If the coordinates of O and P are $(3, -1, 2)$ and $(10, 5, 8)$ respectively specify the vector quantity in terms of the unit vectors.
(s 63.6 + 54.5 + 54.5)
- R An object must be lifted from the ground point $P(0, 0)$ to a point $(0, 10)$ vertically above P . It is feasible to lift the object directed from $(0, 0)$ to $(3, 4)$ as far as necessary and then transport it horizontally to the destination. How long is the feasible path?
(s 20 units)
- R A point is located as $(-5, 2, 14)$ with respect to an origin $(0, 0, 0)$. Specify its position vector:
(a) in terms of the rectangular components,
(b) in terms of its direction cosines and
(c) in terms of its unit vector.
(s (a) $= -5\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$ (b) $= 15l\mathbf{i} + 15m\mathbf{j} + 15n\mathbf{k}$; $l = -0.33, m = 0.13, n = 0.93$ (c) $= 15r$; $r = -0.33\mathbf{i} + 0.13\mathbf{j} + 0.93\mathbf{k}$)
- R For a triangle with sides of lengths a, b and c and the angles facing these sides A, B and C respectively, prove that
(a) $c^2 = a^2 + b^2 - 2ab \cos C$ (cosine law)
(b) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ (sine law)
- R An object is projected at a velocity of 70 m/s perpendicular to the plane containing vectors $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Express the velocity in terms of the unit vectors.
(s 30 - 20 + 60 m/s)
- R Two vectors \mathbf{A} and \mathbf{B} have an included angle of 60° . If $\mathbf{A} = 4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$ and \mathbf{B} is recorded as $3\mathbf{i} + 3\mathbf{j} + (\)\mathbf{k}$ where the coefficient of \mathbf{k} is missing, calculate this coefficient.
(s -1.7)
- R If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{C} = 2\mathbf{i}$, evaluate the following:
(a) $\mathbf{A} \cdot \mathbf{B}$ (b) $\mathbf{A} \times \mathbf{B}$
(c) $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$ (d) $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B})$
(e) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ (f) $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} - \mathbf{B}) \times \mathbf{C}$
(s (a) -8 (b) 10 + 3 + 11
(c) -7 (d) -20 - 6 - 22
(e) 20 (f) -6 - 2)
- R 8 Show that
(a) $(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k})$, $(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})$ and $(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k})$ comprise a set of orthogonal unit vectors.
(b) $(2\mathbf{i} - \mathbf{j} + \mathbf{k})$, $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ are noncoplanar vectors.
- R If $\mathbf{A} = 3t\mathbf{i} + 2t^2\mathbf{j} + 4t^{-1}\mathbf{k}$, and $\mathbf{B} = 2t^2\mathbf{i} + 3t\mathbf{j} + t^4\mathbf{k}$, compute

(a) $\frac{d}{dt}$ (b) $\frac{d}{dt}(\cdot B)$

(c) $\int_{T_1}^{T_2} A(t)dt$ and (d) $\frac{d}{dt}(\times B)$

(s (a) $3 + 4t - 4t^{-2}$ (b) $48t^2$

(c) $3/2(T_2^2 - T_1^2) + 2/3(T_2^3 - T_1^3)$
 $+ 4 \log(T_2/T_1)$

(d) $12t^5 + (8 - 15t^4) + (18t - 16t^3)$)

R The motion of a point is described by the position vector

$$= (2 + 5t^2) + (4 - 3t^3)$$

the distance being in metres for the time lapsed in seconds. Compute the velocity and acceleration at the instant $t = 2$ seconds.

$$(\text{ s } = (20 \quad -36) \text{ m/s}$$

$$= (10 \quad -36) \text{ m/s}^2$$

R A vertical pole is guyed by three cables PA , PB and PC tied at a common point P at 8 m above the ground. The base points of the cables are: $A(4, 0, 0)$, $B(-1, 4, 0)$ and $C(-2, -3, 0)$ m. If the tension in PA is 20 kN, calculate the tensions to be provided in PB and PC so that the resultant force exerted on the pole is vertical. Find the force exerted on the pole.

$$(\text{ s } 22 \text{ kN and } 28.6 \text{ kN, } 63 \text{ kN}$$

R A force 300 N in magnitude acts through a point $P(1, 6, -5)$ directed towards another point $Q(0, 4, -3)$. Calculate the moment of the force about a point $O(1, 0, -1)$, if the distances are in metres.

$$(\text{ s } (400 \quad +400 \quad +600) \text{ N m}$$

R If r is the position vector of a point, show that

(a) $\text{div } r = 3$

(b) $\text{curl } r = 0$

(c) $\text{div}(r^n) = (n+3)r^{n-1}$

R Interpret the significance of the following relations:

(a) $r \cdot dr = 0$

(b) $r \times dr = 0$

(c) $\nabla \times r = 0$

where r is a vector and d its differential change.

R In a fluid flow the velocity of a particle is given by

$$= x^2 y + xy^2 \text{ m/s}$$

where the distances are measured in metres. Referred to the origin $(0, 0, 0)$ compute the cross product $r \times v$ for a fluid particle located at $(2, 3, 4)$.

$$(\text{ s } 42 \quad +4 \quad -24)$$

R In a magnetic field the velocity of an electron is given by

$$= 100 + 25 \text{ m/s}$$

and the magnetic flux density by

$$B = 0.01 - 0.001 \text{ Wb/m}^2$$

Compute the cross product $v \times B$ for the electron.

$$(\text{ s } -0.35 \text{ Wb/s m}$$

R The Indian satellite Aryabhata is imparted steadiness in its orbit by spinning it about the spin-axis as shown in Fig. Prob. R2.17. If the spin is maintained at 50 revolutions per minute, calculate the linear velocities of the solar cells

$C_1(0.3 \text{ m}, 0.0 \text{ m}, 0.5 \text{ m})$ and $C_2(0.6 \text{ m}, 0.2 \text{ m}, 0.5 \text{ m})$

with respect to the spin-axis.

$$(\text{ s } -2.618 \text{ m/s, } 1.047 - 2.618 \text{ m/s}$$

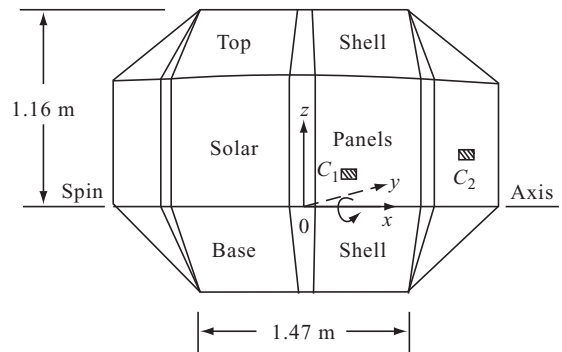


Fig. Prob. R2.17

R 8 A force of 200 N must be replaced by two forces inclined at right angles to it on one

side and at 45° on the other. Show that the magnitude of these forces should be 200 N and 282.6 N respectively.

Also show that, if the components are equally inclined to the given force, the magnitude of each component increases as the inclination with the given force increases until the inclination approaches $\pi/2$.

- R The vertical mast of a flag is positioned by three ropes tied to a common point on the mast with their other ends fixed on the ground. If the angles between the ropes and the mast are 30° , 25° and 30° respectively and they are spaced equally apart in the plan

view, determine the magnitudes of the forces in the ropes if the net downward force at the common point is 1000 N.

(Answers: 422, 357, 357 N)

- R Determine the magnitude of each of the three forces F_1 , F_2 and F_3 which when put together at a point result in a single force 100 N in a given direction ($\hat{i} = 0.6\hat{i} + 0.8\hat{j}$). The unit vectors in the respective directions are as follows:

$$\hat{u}_1 = 0.5\hat{i} + 0.5\hat{j} + 0.707\hat{k}$$

$$\hat{u}_2 = 0.707\hat{i} + 0.707\hat{j}$$

$$\hat{u}_3 = 0.8\hat{i} - 0.6\hat{j}$$

(Answers: -363, 363 and 169 N)

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

- Two vectors are equal if
- their magnitudes are equal
 - their magnitudes and directions are the same
 - they are equal in magnitude and are collinear
 - their magnitudes, direction and the sense are the same and they may lie anywhere in space

- The magnitude of a vector quantity is
- the dot product of the vector with itself
 - the cross product of the vector with unit vector along itself
 - the dot product of the vector with unit vector along itself
 - the cross product of the vector with itself

- Orthogonality of two vectors demands that
- their dot product equals unity
 - the magnitude of the dot and cross products are equal
 - their cross product vanishes
 - their cross product equals unity

- If the dot product of two vectors is zero, then
- either of the vectors or both must be zero
 - the vectors must be perpendicular to each other
 - either (a) or (b) is satisfied
 - the vectors must be concurrent

- If the cross product of two vectors is zero, then
- either of the vectors or both must be zero
 - the vectors must be parallel to each other
 - the vectors must be perpendicular to each other
 - the vectors must be collinear

The derivative of a vector with respect to a scalar must be

- in the direction of the given vector
- perpendicular to the given vector
- zero if the vector has a constant magnitude
- zero if the vector is a constant vector

The integral of a vector with respect to a scalar

- results in a vector in the direction of the given vector

- is called the line integral
- must be a definite integral
- may have a direction other than that of the given vector

- 8 The gradient of a scalar function
- must be a scalar quantity
 - must be a vector in the direction of the steepest variation of the scalar
 - must be a vector with its magnitude equal to the scalar
 - is undefinable

The divergence of a vector

- implies the net efflux of the vector at a point in space
- must be a vector quantity
- is less than the curl of the vector
- implies the continuity of the vector space

The curl of a vector

- may or may not be a vector
- refers to the rotationality of the vector field
- refers to the efflux of the vector
- vanishes if the vector has a constant magnitude

The linear momentum of a particle

- must be directed along the velocity of the particle
- is the dot product of the mass with its velocity
- is the cross product of the mass with its velocity
- is a scalar quantity

The angular momentum of a particle is the

- linear momentum per unit angle
- product of the mass with its angular velocity
- moment of the product of the mass and the angular velocity about an origin
- cross product of the position vector and the linear momentum

The simplest resultant of a plane force system is always

- a single force
- a wrench
- a single moment
- a single force or a single moment

A force acts in the plane of a paper from top to bottom. An anticlockwise moment is applied on it. The line of action of the force will shift parallel to itself to get the simplest resultant

- above the plane of the paper
- under the plane of the paper
- to the right of the given line of action of force in the same plane
- to the left of the given line of action of force in the same plane

A plane system of forces has a single force resultant if the sum of the moments

- about the origin is zero
- about any point on the plane is zero
- about any point on a particular line in that plane is zero
- about any point on or outside the plane is zero.

For a plane system of forces to have the simplest resultant as a single moment

- the forces must be parallel
- the force system must constitute moments and/or couples only
- the force system cannot have the forces as concurrent
- the force system cannot have an odd number of forces

Answers to Multiple-Choice Questions

- | | | | | | | | |
|-------|--------|--------|--------|--------|-------|--------|--------|
| 1 (d) | 2 (c) | 3 (e) | 4 (c) | 5 (b) | 6 (d) | 7 (d) | 8 (b) |
| 9 (a) | 10 (b) | 11 (a) | 12 (d) | 13 (d) | 14(c) | 15 (c) | 16 (b) |

Forces and Force Systems

Resultant, Equivalence and Origin of Forces



2

2.1 FORCE AND MOMENT CONCEPTS

It has been stated in Chapter 1 that an ‘action’ which changes or tends to change the state of rest or of uniform motion of a body must be a force or a moment. A force when exerted on the centre of mass of a body causes or tends to change the state of rest or of uniform rectilinear motion of the body. Action of a moment causes or tends to cause a rotational motion of a body. The moment of a force about a point has been defined as the turning effect of the force about that point

$$M = r \times F \quad (2.1)$$

$= r F \sin \theta$ or $F d$, perpendicular to the plane containing r and F

where F is the force, r , the position vector of any point on the line of action of the force with the origin at the point about which the moment is desired, d the perpendicular distance from the point to the line of action of the force and θ the angle between r and F as shown in Fig. 2.1.

Force and moment concepts are most important in the study of Newtonian mechanics. As a matter of fact, the laws of Newton and Euler lay the foundation of force-based mechanics. An alternative formulation called energy-based mechanics is based on energy concepts where force and moment do not play the primary role. Since we are concerned with the Newtonian formulation of mechanics, we shall have to allot special status to force and moment concepts.

A body may be subjected to a single force or a moment or a system of forces under the action of

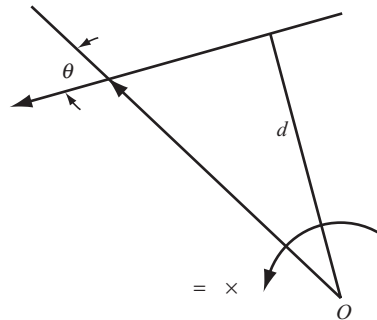


Fig. 2.1 Concept of moment due to a force

which the body may stay at rest, be in uniform motion or in general motion. It is important, therefore, to recognise the total effect of a system of forces acting on a body. This is represented by the concepts of resultant and equivalent systems which are discussed at length for different force systems acting on a particle or rigid body.

Forces may originate from a variety of circumstances. It is neither possible nor desirable to cover all the possible modes of origin and the nature of forces but it should be helpful to introduce the concepts of gravitational, electromagnetic or hydrostatic force field and frictional and drag forces. The origin and nature of forces is a fundamental subject and must be conceived before their effect is dealt with in statics and dynamics. There is little scope of solving any numericals on the nature of forces at this stage. Appropriate and adequate number of examples will be taken up in the context of statics and dynamics during our study.

Example 2.1

A 50 cm × 30 cm horizontal plate ABCD is acted on by a 10 kN force at B in the plane of the plate as shown in Fig. Ex. 2.1. Determine the moment of the force about D and about A.

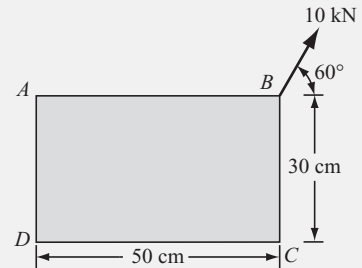


Fig. Ex. 2.1

Solution In order to determine the moment about D, fix the origin at D and let the x and y axes be along DC and DA respectively as shown in Fig. Ex. 2.1 (Solution).

The force is specified as

$$= 10 \cos 60^\circ \quad + 10 \sin 60^\circ$$

$$= 5 \quad + 8.66 \quad \text{kN}$$

and the position vector for a point B on the line of action of the force is

$$\mathbf{B} = 0.5 \quad + 0.3 \quad \text{m}$$

The moment of the force about D is, therefore,

$$M_D = \mathbf{B} \times$$

$$= (0.5 \quad + 0.3 \quad) \times (5 \quad + 8.66 \quad)$$

$$= 2.83 \quad \text{kN m}$$

It may also be noticed that the magnitude of the moment equals

$$M_D = 10 \, d$$

where d is the perpendicular distance from D to the line of action of the force. From the knowledge of this distance by a drawing to scale or by trigonometry, the moment is determined as

$$M_D = 10 \times 0.283 = 2.83 \, \text{kN m}$$

directed anticlockwise, i.e., along the positive z direction.

Similarly, the moment about point A may be evaluated by selecting the origin at A and the axes in the same directions as before

$$\mathbf{B} = 0.5$$

$$= 5 \quad + 8.66$$

$$M_A = \mathbf{B} \times = 0.5 \quad \times (5 \quad + 8.66 \quad) = 4.33 \quad \text{kN m}$$

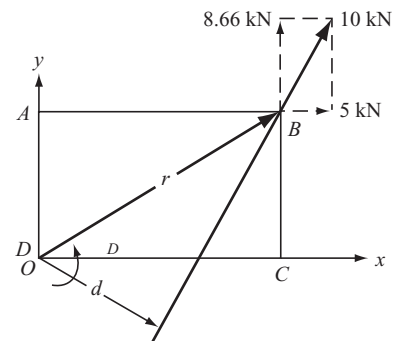


Fig. Ex. 2.1 (Solution)

2.2 FORCE FIELDS: LINEAR, PLANE AND SPATIAL

A force may be originated by the action of one body on another body in contact with it or by virtue of a force field. A force field implies the existence of a force as a function of the space coordinates and time. In general, in a force field,

$$F = F(x, y, z, t) = F(r, \theta, z, t)$$

A force field is said to be *steady* if, at any point in space, it is independent of time, i.e.,

$$F = F(x, y, z) = F(r, \theta, z)$$

Force fields are classified as linear, plane and spatial as follows:

Linear Force Field A linear force field is confined to a line only, i.e., the force at any point is a function of one dimension only.

$$F = F(x) \text{ or } F = F(s)$$

A linear spring is an example of a linear force field. As shown in Fig. 2.2(a), the force on the weight W by the spring is a function of the horizontal displacement x of the weight only.

Plane Force Field A plane force field refers to a field where the force varies with two space coordinates, i.e., x, y or r, θ , etc.

$$F = F(x, y) \text{ or } F = F(r, \theta)$$

The magnetic field generated by a current-carrying conductor with rectangular cross-section AB is an example of a plane force field. The force exerted on any magnetic particle near it is a function of its two coordinates (x, y) as shown in Fig. 2.2(b).

Spatial Force Field A spatial force field is a general force field in which the force varies with respect to the position of the point in space, i.e., with all the three coordinates x, y, z , or r, θ, z , etc.

$$F = F(x, y, z) \text{ or } F = F(r, \theta, z)$$

The gravitational force field generated by a body is an example of a spatial force field. The force exerted on any particle P towards the body is a function of the position of this particle with respect to the body in space, i.e., x, y, z as shown in Fig. 2.2(c).

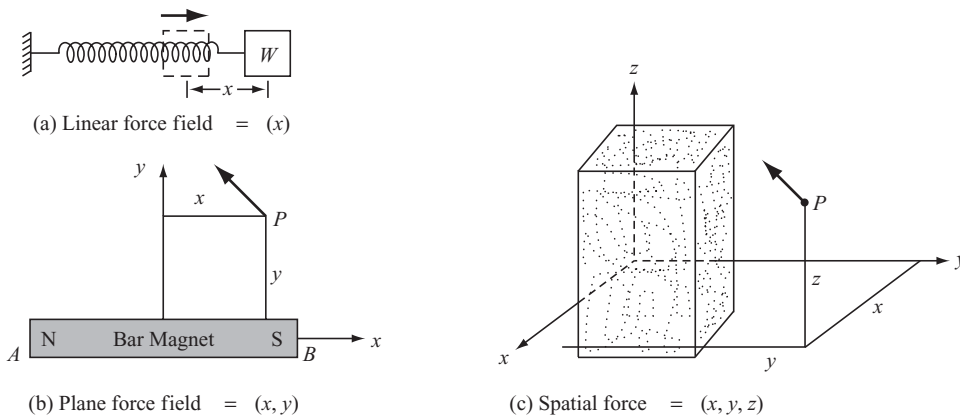


Fig. 2.2 Examples of force fields

Example 2.2

A force field represented by

$$= 6xy + 3x$$

acts on a circular plate of 2 m radius placed in the $x y$ plane with the z -axis passing through the centre of the plate. Determine the force at some salient points on the periphery of the plate.

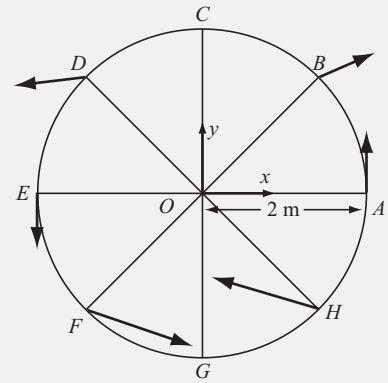


Fig. Ex. 2.2

Solution Let us consider the points A, B, C, D, E, F, G and H on the periphery as shown in Fig. Ex. 2.2. The forces at these points are determined by substituting the values of corresponding x and y . For example, at point $A(2, 0)$,

$$= 6 \times 2 \times 0 + 3 \times 2 = 6$$

| Point | x | y | Force |
|-------|--------|--------|-------------|
| A | 2 | 0 | 6 |
| B | 1.414 | 1.414 | 12 + 4.242 |
| C | 0 | 2 | 0 |
| D | -1.414 | 1.414 | -12 - 4.242 |
| E | -2 | 0 | -6 |
| F | -1.414 | -1.414 | 12 - 4.242 |
| G | 0 | -2 | 0 |
| H | 1.414 | -1.414 | -12 + 4.242 |

2.3 DISTRIBUTED FORCE FIELDS

A distributed force field is characterised by the action of a continuously distributed force. Such forces may act over a line, a surface or a volume; these are correspondingly denoted as *lineal*, *surface* and *body* forces.

Lineal Force A lineal force is one that acts along a line on the body. The force d at any small length dl is given by

$$d = w dl$$

where w is the intensity of loading at dl .

An example of the lineal distributed force is a loaded cable as shown in Fig. 2.3(a).

Surface Force A surface force acts over a surface and the force d at any area dA is given by

$$d = p dA$$

where p , the intensity of force per unit area, is generally termed as pressure.

Hydrostatic pressure acting on the surface of a cylinder immersed in water as shown in Fig. 2.3(b) is an example of a surface force.

Body Force A body force is essentially the force exerted on the mass or volume content of the body.

Thus the body force d on an element of mass dm or volume dV is given as

$$d = g dm$$

or

$$d = \gamma dV$$

where g is the force per unit mass on the small mass dm or γ is the force per unit volume on the volume dV .

An example of body force is the force exerted on a body due to gravitational attraction of the earth as shown in Fig. 2.3(c). The force is distributed throughout the volume or mass of the body.

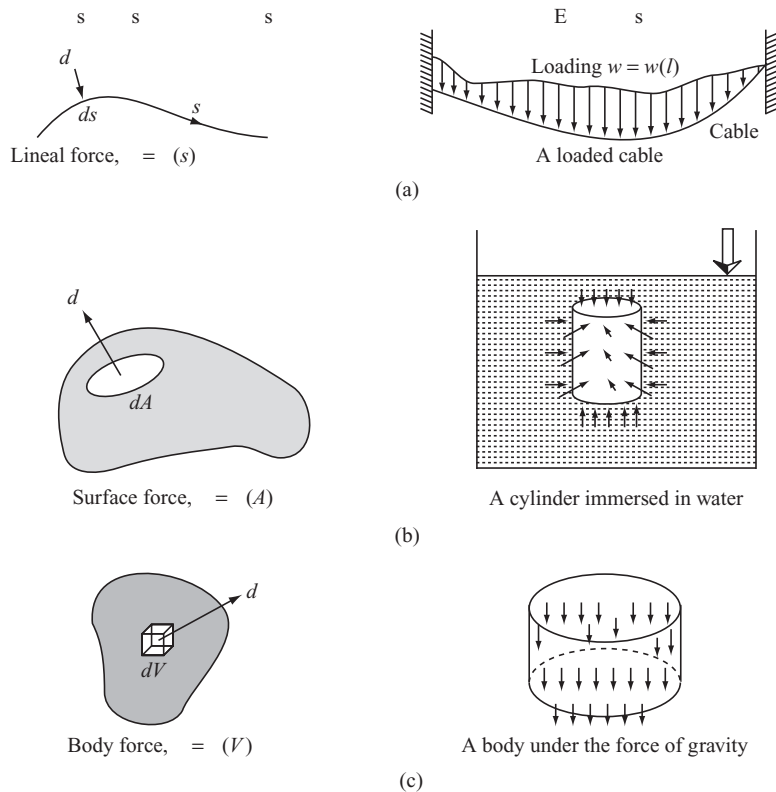


Fig. 2.3 Examples of distributed forces

2.4 FORCE SYSTEM ACTING ON A BODY

A body may be subjected to a number of forces in a specified manner

$$1, 2, 3, \dots$$

acting such that the position vectors of any points on their lines of action are

$$1, 2, 3, \dots$$

respectively referred to an arbitrary origin O .

Some of the forces may be equal and opposite and may constitute couples. The couples are sometimes bracketed separately from the other forces and are taken into account by way of couple moments. There may, in addition, be moments acting otherwise. Thus, the body may be acted upon by moments

1, 2, 3, ...

The forces acting in a specified manner and the moments acting on the body constitute a force system or a system of forces as represented in Fig. 2.4. A force system is sometimes referred to as an 'external action' on the body.

The force system acting on a body may consist of forces which may be qualified in one or more of the following classifications:

1. *Concurrent force systems*: Collinear, planar or spatial
2. *Parallel force systems*: Planar or spatial
3. *Coplanar force systems*: Concurrent and non-concurrent, parallel and non-parallel
4. *Spatial force systems*: Concurrent and non-concurrent, parallel and non-parallel.

It is obvious that a particle can only be subjected to a concurrent force system whereas a rigid body may be under the action of any force system.

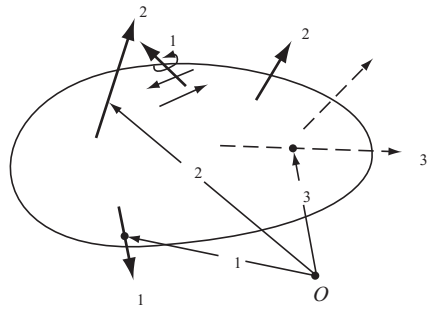


Fig. 2.4 A force system acting on a body

2.5 RESULTANT OF A FORCE SYSTEM

The resultant of a force system acting on a body implies the net external action on the body. In other words, the resultant action is a simple equivalent force system which can replace the given force system for an equivalence of effect so far as motion or tendency of motion of the body is concerned. The definition of a rigid body permits no internal dimensional or structural changes within the body; the resultant concept applied to a rigid body stands for complete equivalence of action and is therefore highly meaningful. Similarly, a particle conceived as a relatively small or point object allows the resultant concept to be used to advantage due to complete equivalence of action represented by it. We shall, therefore, confine ourselves to the resultant concept for a particle and a rigid body.

The action of a force is two-fold: first, in its own right as a translational action and second, to generate a moment or rotational action about an arbitrary point or an axis. It follows that the resultant of a force system should, in general, comprise of (a) a force and (b) a moment, as shown in Fig. 2.5.

It is necessary to qualify the point of action or line of action of the force and the direction of the moment as will be shown later. It is also likely that a force system can result in a force only or in a moment only for certain force fields and certain choices of the point of action of the resultant.

Sometimes, the resultant of a force system is referred to as the equivalent or equipollent action. It is, in general, incorrect to term the resultant as the equivalent because the equivalence of an action has a wider implication. The equivalence of action on a body may be desired with different objectives. For example, the objective may be to study the motion of the body, to analyse its internal forces or to compute its deformation or rate of deformation. Thus, by definition, the equivalent

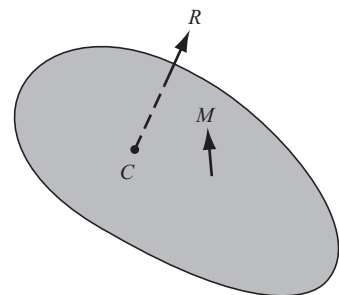


Fig. 2.5 Resultant action for the force system acting on a body

system of forces for a given system of forces is such that it produces the same desired effect. In particular, for a rigid body in motion or having a tendency of motion, the analysis of motion can be made with the resultant replacing the given system of forces. The resultant is, therefore, the equivalent action of a system of forces for the dynamic consideration of a rigid body.

It may be seen that the resultant of a plane system of forces must be a force in that plane which may be accompanied by a couple-moment in a direction normal to that plane. Similarly, the resultant of a system of parallel forces should be a force parallel to them which may be accompanied by a moment in a direction normal to the parallel forces. The resultant of a system of concurrent forces should be a single force which must pass through the point of concurrency. These statements cannot be taken for granted; let us discuss the individual cases.

2.6 PRINCIPLE OF PARALLEL TRANSFER OF A FORCE

Consider a force F acting through a point P_1 on a rigid body as shown in Fig. 2.6(a). If it is desired that the force be applied through a point P_2 on the body, then the force applied at P_2 must be accompanied by a moment with magnitude

$$M = Fd$$

perpendicular to the plane of transfer of the force for equivalence where d is the perpendicular distance of the parallel transfer of force. This is the principle of translation of a force to a parallel position. The principle can be proved by imagining a pair of equal and opposite forces F and $-F$ adding to null at point P_2 in the first instance. The perpendicular distance from P_2 to the given force is d . The system of three forces thus constituted

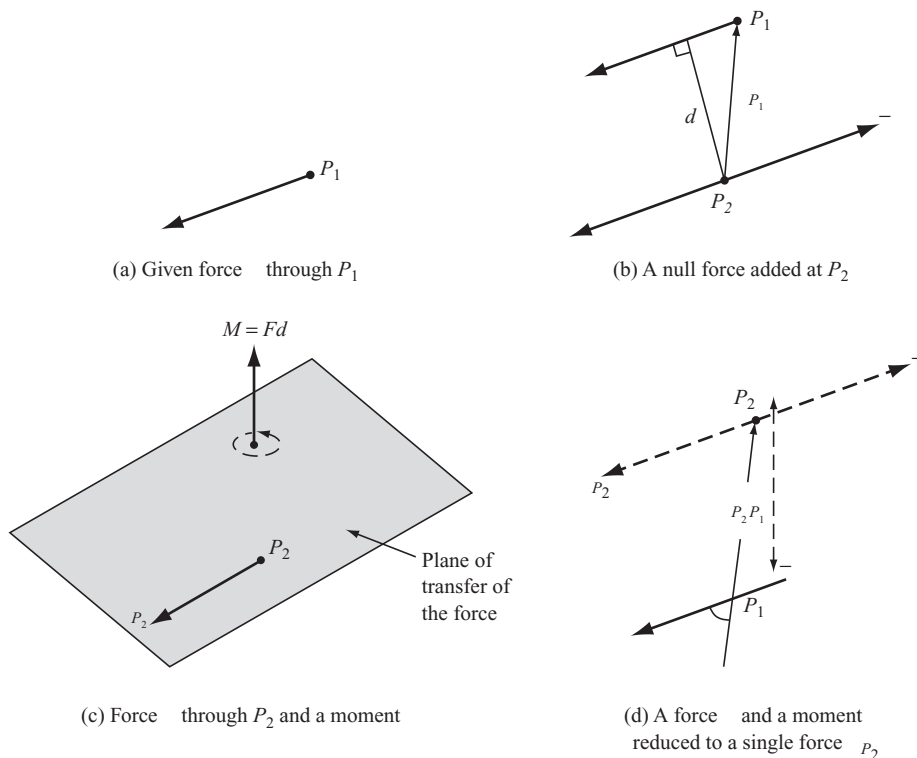


Fig. 2.6 Principle of parallel transfer of force

as shown in Fig. 2.6(b), may be visualised as a force F acting through P_2 and a couple of forces F and $-F$ acting through points P_1 and P_2 respectively. The couple of forces comprise a couple-moment M given by

$$M = Fd \quad (2.2)$$

in magnitude, perpendicular to the plane of transfer of the force F .

The sense of the accompanying couple-moment may either be visualised with the help of a neat sketch or observed vectorially as follows:

Let the position vector of a point P_2 on the new line of action of F , called r_{P_2} , with reference to a point P_1 on the initial line of action of the force, be $r_{P_2P_1}$. Then, the moment vector accompanying the transferred force is

$$M = -r_{P_2P_1} \times F$$

specified completely in magnitude and direction. If the magnitude of the displacement vector is d and it is the perpendicular distance through which the force is transferred, then the direction of the accompanying moment of magnitude

$$M = Fd$$

is equal and opposite to the moment exerted by F about a point on the initial line of action of the force as shown in Fig. 2.6(c).

Let us now discuss the reverse problem. If a force F and a moment M act on a rigid body such that the force passes through a point P_1 and the moment has no component along the direction of the force then the given system of F and M may be replaced by a single parallel force F passing through the point P_2 .

The transfer of F so as to pass through P_2 would have required as accompanying moment

$$M = -r_{P_2P_1} \times F$$

which, in this case, must nullify the existing moment M normal to the plane of transfer. Hence,

$$M = -r_{P_2P_1} \times F$$

or

$$M = r_{P_2P_1} \times F \quad (2.3)$$

The point P_2 can be located with respect to P_1 by solving Eq. (2.3) by way of decomposing it into scalar equations as shown in Fig. 2.6(d).

A force acting through a point and a moment can be replaced by a single parallel force for equivalence if the moment has no component in the direction of the force. Let us observe the difficulty when the moment has a component in the direction of the force. Clearly, the transfer of the force parallel to itself involves the application of a moment perpendicular to the plane of transfer of the force; no moment along the direction of the force may creep in. Conversely, any amount of transference of a force parallel to itself cannot involve a moment along the direction of the force so as to nullify the given component along that direction. The simplest system to which an arbitrary force acting at a point and a moment acting on the rigid body may be reduced is the force parallel to itself and a moment along the line of action of the force. The simplest equivalent form, i.e., a force and couple-moment directed along the force is called *wrench*. If the force in the wrench is displaced to any parallel position, the moment directed along the force will remain unaltered and an additional moment arising from the parallel transfer of the force will be required.

The concept of a wrench is explained with reference in Fig. 2.7(a) where an arbitrary force F passing through a point P_1 in a rigid body and an arbitrary moment M are given. The moment has a component M_f along the direction of the force and a component M_n normal to the direction of the force. The force F may be transferred to act at another point P_2 in the body such that it is equivalent to F at P_1 and the component M_n of M . There is no possibility for the component M_f in the direction of the force to be taken care of by a

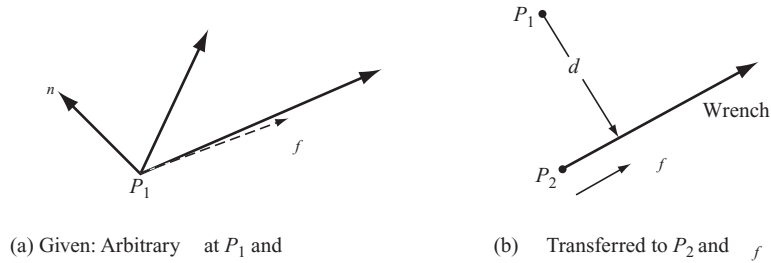


Fig. 2.7 Concept of a wrench on a rigid body

parallel transfer of f . The simplest equivalent system which remains is, therefore, a force f through P_2 and a moment M in the direction of the force. This system, called a wrench, is shown in Fig. 2.7(b).

Example 2.3

A rigid parallelepiped is made of parallel bars AB, CD, EF and OP as shown in Fig. Ex. 2.3. A force F of 10 kN acting along CD is to be replaced by an equivalent action by applying the force along any of the other parallel bars. Determine the equivalent action in each case.

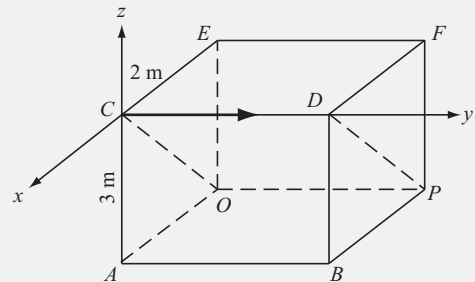


Fig. Ex. 2.3

Solution If a 10 kN force were to act along EF , it should be accompanied by a moment

$$M = 10 \times 2 = 20 \text{ kN m}$$

along the positive z -axis or by a moment vector

$$\begin{aligned} &= - EC \times E \\ &= -(-2) \times 10 = 20 \text{ kN m} \end{aligned}$$

If it were to act along AB , the accompanying moment would be

$$M = 10 \times 3 = 30 \text{ kN m}$$

along the negative x -axis or by a moment vector

$$\begin{aligned} &= - AC \times A \\ &= -(-3) \times 10 = -30 \text{ kN m} \end{aligned}$$

Similarly, if the force were transferred to the bar OP , the moment accompanying it would be

$$M = 10 \times \sqrt{(3^2 + 2^2)} = 36.06 \text{ kN}$$

along a direction normal to the plane containing CD and OP , i.e., normal to the plane $OCDP$.

Vectorially, the accompanying moment would be

$$\begin{aligned} &= - oc \times c \\ &= -(-2 \ -3) \times 10 = 20 \ -30 \text{ kN m} \end{aligned}$$

Example 2.4

A force acts at a position vector :

$$= 5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k} \text{ kN}$$

$$= -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \text{ m}$$

A couple-moment also acts:

$$= 2\mathbf{i} + 3\mathbf{j} \text{ kN m}$$

It is desired to replace the system by a wrench. Specify the equivalent wrench.

Solution The unit vector in the direction of the force is

$$= \frac{5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}}{\sqrt{5^2 + 6^2 + 4^2}} = 0.57\mathbf{i} + 0.684\mathbf{j} + 0.456\mathbf{k}$$

The given couple moment may be resolved into a component f in the direction of the force and a component n normal to it.

$$f = \mathbf{M} \times \mathbf{u}$$

$$= (2\mathbf{i} + 3\mathbf{j}) \times (0.57\mathbf{i} + 0.684\mathbf{j} + 0.456\mathbf{k})$$

$$= 1.14 + 2.05 = 3.19 \text{ kN m}$$

$$f = 3.19 (0.57\mathbf{i} + 0.684\mathbf{j} + 0.456\mathbf{k})$$

$$= 1.82\mathbf{i} + 2.18\mathbf{j} + 1.45\mathbf{k}$$

$$n = \mathbf{M} - f$$

$$= (2\mathbf{i} + 3\mathbf{j}) - (1.82\mathbf{i} + 2.18\mathbf{j} + 1.45\mathbf{k})$$

$$= 0.18\mathbf{i} + 0.82\mathbf{j} - 1.45\mathbf{k}$$

The moment component f is there to stay as a component of the wrench but the normal component n may be eliminated by way of parallel transfer of the force. Let the force be transferred to a new position P_2 defined by a position vector

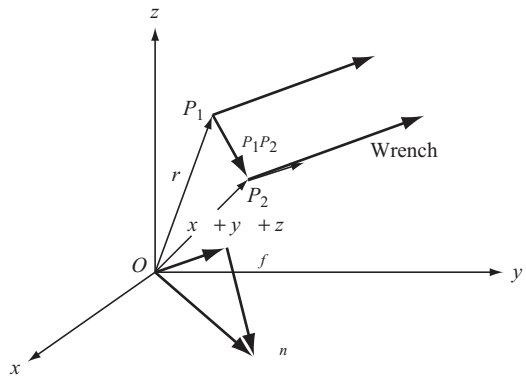


Fig. Ex. 2.4 (Solution)

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The displacement vector for the force is

$$P_2P_1 = (x - (-2))\mathbf{i} + (y - 3)\mathbf{j} + (z - 4)\mathbf{k}$$

$$= (x + 2)\mathbf{i} + (y - 3)\mathbf{j} + (z - 4)\mathbf{k}$$

The parallel transfer must be accompanied by a moment vector

$$- P_2P_1 \times P_2$$

$$= - \begin{vmatrix} x+2 & y-3 & z-4 \\ 5 & 6 & 4 \end{vmatrix}$$

which should be negative of n in order to nullify it, i.e., equal to $-0.18\mathbf{i} - 0.82\mathbf{j} + 1.45\mathbf{k}$.

The solution of this equation provides x, y and z which implies that the equivalent wrench is such that the force $5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$ passes through this point $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the accompanying moment in the direction of the force is $1.82\mathbf{i} + 2.18\mathbf{j} + 1.45\mathbf{k}$.

Example 2.5 Replace the force system consisting of three forces shown in the figure by a wrench passing through a point in the yz plane.

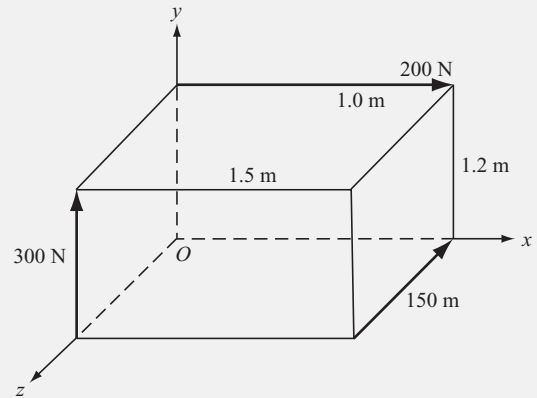


Fig. Ex. 2.5

Solution The resultant force $R = 200\mathbf{i} + 300\mathbf{j} - 150\mathbf{k}$ N. The moment of the given forces about the origin O is given by

$$\begin{aligned} &= 1.2 \times 200 + 1 \times 300 + 1.5 \times (-150) \\ &= -300 + 225 - 240 \text{ Nm.} \end{aligned}$$

Let the wrench be located at point $P(o, y, z)$ located by the position vector $y\mathbf{j} + z\mathbf{k}$. The moment must be $M\mathbf{i}$, i.e. normal to yz plane.

Then,

$$(y\mathbf{j} + z\mathbf{k}) \times (200\mathbf{i} + 300\mathbf{j} - 150\mathbf{k}) + M\mathbf{i} = -300\mathbf{i} + 225\mathbf{i} - 240\mathbf{i}$$

whence,

$$\begin{aligned} 300z + 150y &= M \\ 200z &= 225 \\ 200y &= 240 \\ y = 1.2 \text{ m, } z &= 1.125 \text{ m, } M = 217.5 \text{ Nm} \end{aligned}$$

The desired wrench, therefore, consists of a force $200\mathbf{i} + 300\mathbf{j} - 150\mathbf{k}$ N and a moment 217.5 Nm at the point $(0, 1.2, 1.125)$.

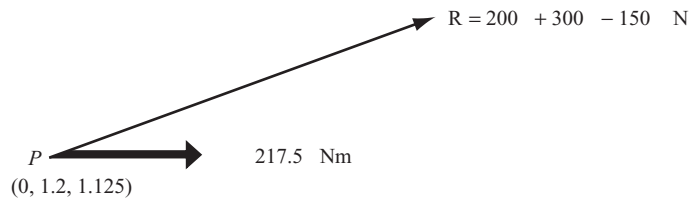
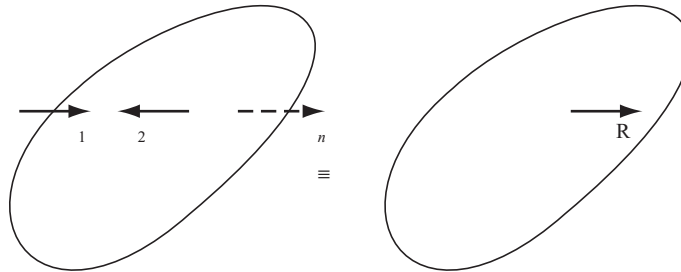


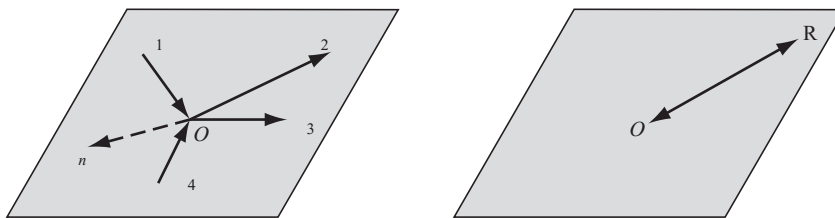
Fig. Ex. 2.5 (Solution)

2.7 RESULTANT OF A CONCURRENT FORCE SYSTEM

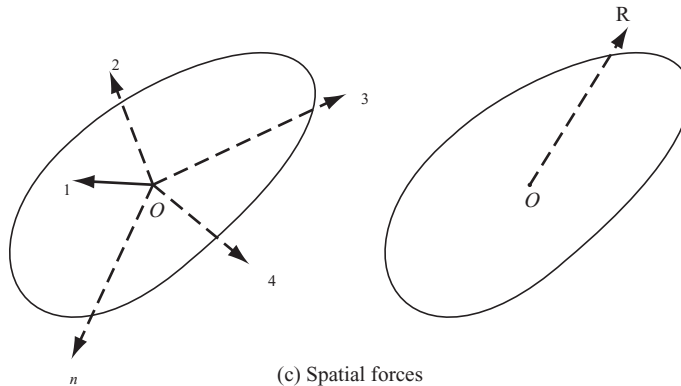
A concurrent force system may be collinear, coplanar or spatial; collinear if the forces have the same line of action, coplanar if the lines of action of the forces lie in a plane and spatial if the lines of action of the forces lie in space as shown in Fig. 2.8.



(a) Collinear forces



(b) Coplanar forces



(c) Spatial forces

Fig. 2.8 Concurrent force systems

(a) Collinear Forces

A system of collinear forces

$$1, 2, 3, \dots$$

acting on a body may be replaced by a single resultant force R acting in the same line of action as the given forces where

$$R = 1 + 2 + 3 + \dots \tag{2.4}$$

In other words, the sum of the forces in the collinear force system must provide the resultant.

(b) Coplanar Concurrent Forces

A system of coplanar concurrent forces

$$1, 2, 3, \dots$$

acting on a body may be replaced by a single resultant force R passing through the point of concurrency in that plane where

$$R = F_1 + F_2 + F_3 + \dots$$

In terms of rectangular components,

$$F_1 = F_{1x} + F_{1y}$$

$$F_2 = F_{2x} + F_{2y}$$

$$F_3 = F_{3x} + F_{3y}$$

and

$$\begin{aligned} R &= (F_{1x} + F_{2x} + \dots) + (F_{1y} + F_{2y} + \dots) \\ &= R_x + R_y \end{aligned} \tag{2.5}$$

Geometrically, a polygon of forces can be constructed to add the coplanar concurrent forces F_1, F_2, F_3, \dots to result in R .

(c) Spatial Concurrent Forces

A system of concurrent forces not confined to a plane,

$$F_1, F_2, F_3, \dots$$

acting on a body may also be replaced by a single resultant force R passing through the point of concurrency such that

$$\begin{aligned} R &= F_1 + F_2 + F_3 + \dots \\ &= (F_{1x} + F_{2x} + \dots) + (F_{1y} + F_{2y} + \dots) + (F_{1z} + F_{2z} + \dots) \\ &= R_x + R_y + R_z \end{aligned} \tag{2.6}$$

In view of the space distribution of the forces, the geometrical construction, though possible is not feasible for adding the spatial forces because of the complexity of drawing space diagrams.

Example 2.6

At a point P on a vertical mast three forces F_1, F_2 and F_3 act:

$$\begin{aligned} F_1 &= 50 \\ F_2 &= -30 \quad -15 \\ F_3 &= -25 \quad -10 \quad +5 \end{aligned}$$

Determine the resultant force at P .

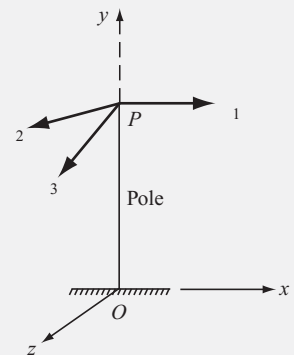


Fig. Ex. 2.6

Solution The resultant of the concurrent forces is given by

$$\begin{aligned} R &= F_1 + F_2 + F_3 \\ &= 50 \quad -30 \quad -15 \quad -25 \quad -10 \quad +5 \\ &= -5 \quad -25 \quad +5 \end{aligned}$$

This single force when applied at P results in the same dynamic action as that exerted by the given system of concurrent forces.

2.8 RESULTANT OF A PARALLEL FORCE SYSTEM

A force system consisting of parallel forces may be planar or spatial: *planar*, if the parallel forces lie in a plane and the moments applied externally or couple-moments by the applied forces are directed normal to the plane and *spatial*, if the parallel forces are in space or the moments applied externally are directed in arbitrary directions.

An analysis of parallel force systems such as that of concurrent force systems can be made under the heads of plane and spatial systems but the parallel force systems constitute an important class in engineering and will, therefore, be dealt with separately.

Plane Parallel Force System

Consider a simple case of two parallel forces. Any two parallel forces must be coplanar. Let the two parallel forces be equal in magnitude. If they are in the same sense as in Fig. 2.9(a, i), they add up to result in a single force which has twice the magnitude of each force, directed parallel to them and in the same sense as the constituent forces. The forces also produce a net moment about any point in the plane. The net moment is zero about any point midway between the forces. The net effect of the pair of forces, in this case, is to result in a resultant force R as well as an appropriate moment about the point considered. If they are in the opposite sense as in Fig. 2.9(a, ii), they add up to result in a null force. In addition, they have a net turning effect. In fact, they constitute a *couple* and the net turning effect is denoted by the moment of the couple. *The moment of the couple* is determined as

$$= \times$$

where is the position vector of a point on the line of action of one force with respect to an origin taken on the line of action of the other and is the former force.

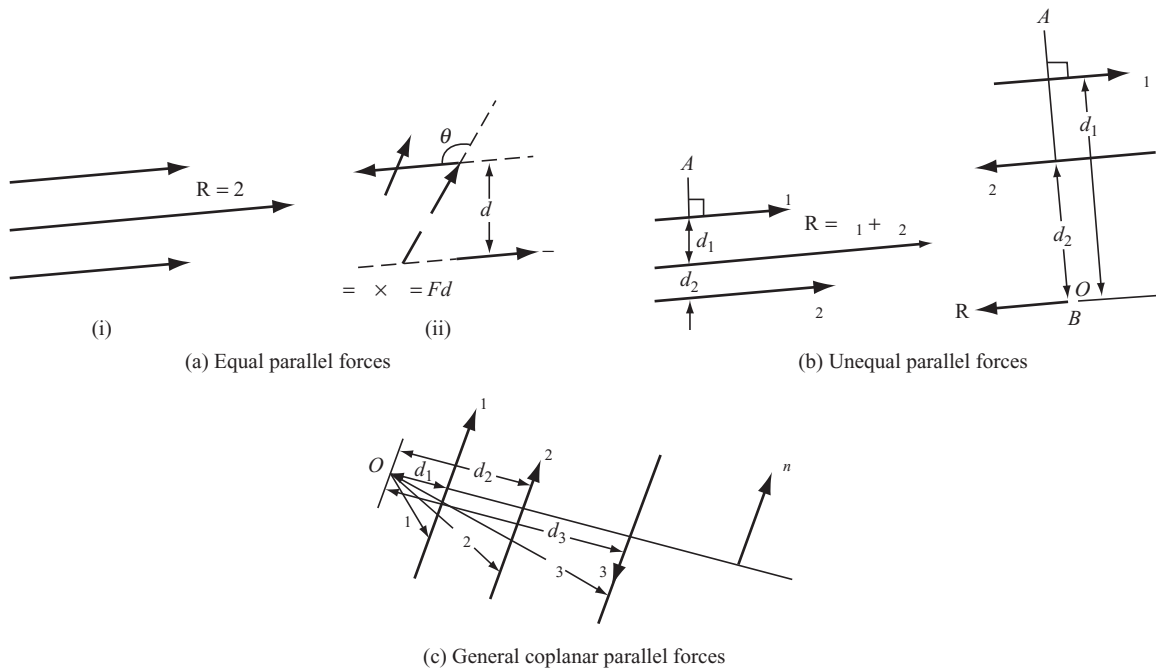


Fig. 2.9 Resultant of parallel forces

Alternatively, the magnitude

$$M = rF \sin \theta$$

$$= Fr \sin \theta = Fd$$

where d is the perpendicular distance between the forces constituting the couple and θ is the angle between r and F .

It follows that the moment of the couple is directed perpendicular to the plane containing r and F , i.e., perpendicular to the plane of the couple. It can be concluded, therefore, that a pair of equal parallel forces with opposite sense can be replaced by an equivalent moment. It must be understood that the moment is a free vector; the directions and the lines of action of the forces constituting the couple are of no consequence. Whether the couple is in a particular plane or in any parallel plane is also immaterial. The moment may well be due to an equivalent couple of smaller forces but placed further apart as illustrated in Fig. 2.10.

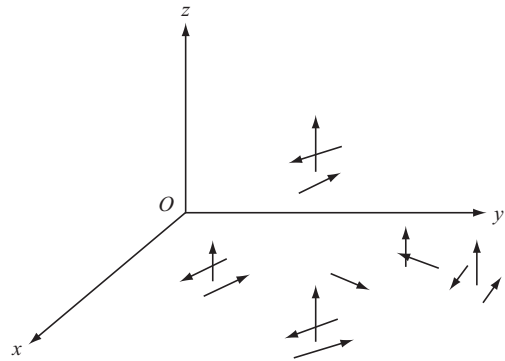


Fig. 2.10 Equivalent couple moments

Consider next the case of a pair of parallel but unequal forces, F_1 and F_2 . If they have the same sense, their resultant R is parallel to them with the same sense and its magnitude

$$R = F_1 + F_2$$

If they have opposite sense, their resultant R is parallel to them in sense of the larger force and the magnitude

$$R = F_1 - F_2 \tag{2.7}$$

The net effect of the forces is also in producing a turning effect; the moment of the forces F_1 and F_2 is different at different places. If a line AB is drawn perpendicular to the lines of action of the forces, it is easy to observe that the net moment of the force is different both in magnitude and sense about different points. In each case, a point O can be located such that the net moment about that point is zero. This point O should lie closer to the larger force; on the side of the smaller force if they are in the same sense and on the opposite side of the smaller force if they are in opposite sense. In terms of d_1 and d_2 , the distances between the forces F_1 and F_2 , and the point O ,

$$F_1 d_1 = F_2 d_2 \tag{2.8}$$

as is clear in Fig. 2.9(b).

Let us now consider a case of coplanar parallel forces as shown in Fig. 2.9(c).

$$F_1, F_2, F_3, \dots$$

The resultant force is

$$R = F_1 + F_2 - F_3 + \dots \tag{2.9}$$

and the resultant moment about a point O through which R is passed has the magnitude

$$M = F_1 d_1 + F_2 d_2 - F_3 d_3 + \dots$$

whereas, in vector notation,

$$O = F_1 \times d_1 + F_2 \times d_2 - F_3 \times d_3 + \dots \tag{2.10}$$

is the moment vector about O accompanying the resultant force R passed through this point.

The general case of coplanar parallel forces can, therefore, be represented by an equivalent system of a single force R passing through a point P and a moment about an axis perpendicular to the plane of the forces. If R is non-zero, it can be made to pass through a point such that the moment vanishes and the single resultant force R becomes the equivalent. If R is zero, the moment remains as the single equivalent. The single

resultant of a system of coplanar parallel forces may, therefore, be a single force if the force is non-zero or a single moment if the resultant force is zero.

The concept of equivalence of coplanar parallel forces may be extended to the case of spatial parallel forces. If the system of parallel forces in space consists of

$$F_1, F_2, F_3, \dots$$

specified by the position vectors

$$r_1, r_2, r_3, \dots$$

respectively as shown in Fig. 2.11, then the resultant force is given by

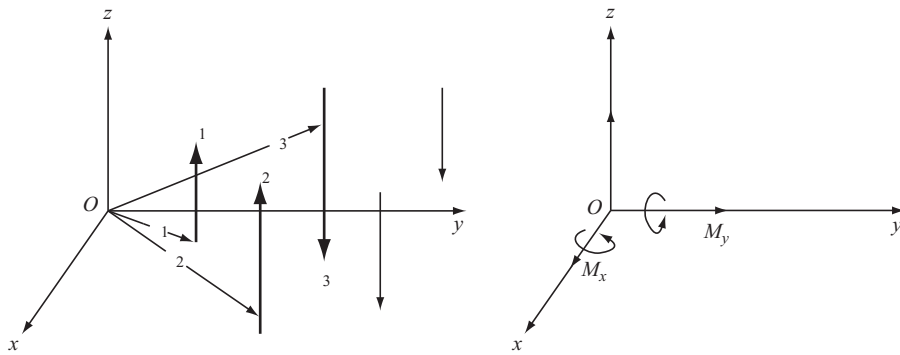
$$R = F_1 + F_2 + F_3 + \dots$$

and if this force is passing through the origin O , the moment accompanying it for equivalence should be

$$M_O = r_1 \times F_1 + r_2 \times F_2 + \dots$$

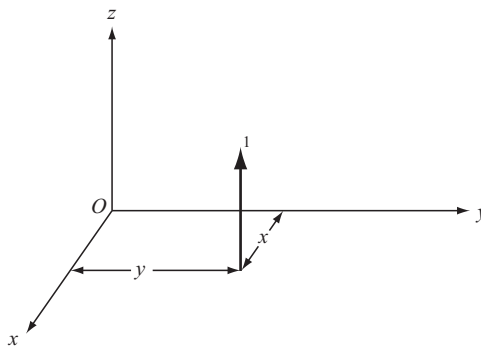
The system of parallel forces cannot generate a moment in the direction of the parallel forces; the moment M_O is wholly normal to the lines of action of the parallel forces. It is, therefore, possible to reduce the resultant to

- (i) a single equivalent non-zero force by shifting it parallel to itself so as to reduce the accompanying moment to null,
- (ii) a single equivalent moment if the resultant force happens to be zero, or
- (iii) a null force and null moment in a particular case.



(a) Specified spatial force system

(b) Equivalent system at O



(c) Single non-zero equivalent force

Fig. 2.11 Equivalence of spatial parallel forces

Example 2.7

A rigid bar AB is subjected to a system of parallel forces as shown in Fig. Ex. 2.7. Reduce the given system of forces to an equivalent (a) single resultant, (b) force-moment system at A , (c) force-moment system at D and (d) force-moment system at B .

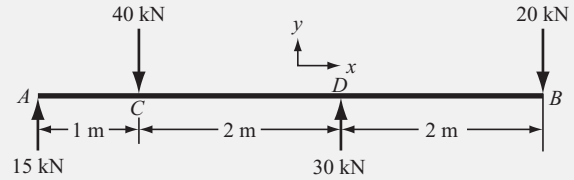


Fig. Ex. 2.7

Solution The resultant force for the coplanar parallel force system is

$$R = F_1 + F_2 + F_3 + F_4 = 15 - 40 + 30 - 20 = -15 \text{ kN}$$

which implies that it is 15 kN downward. This force may be made to act through any desired point in the bar.

(a) In order that the single force -15 kN be equivalent to the given system of forces, it should be so located that the moment exerted by it about any point should be the same as that exerted by the given system of forces. For example, let us consider the moments about A for the given system

$$M_A = 1 \times (-40) + 3 \times 30 + 5 \times (-20) = -40 + 90 - 100 = -50 \text{ (i)}$$

If the resultant is assumed to act at a distance x from A , then

$$M_A = x \times (-15) = -15x \text{ (ii)}$$

Equating (i) and (ii) as postulated before,

$$-15x = -50 \\ x = 3.33 \text{ m}$$

(b) If the force of -15 were acted at A , the moment accompanying it should be

$$M_A = -50 \text{ kN m}$$

as determined before.

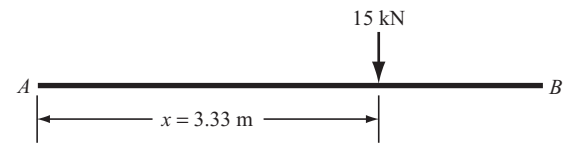
(c) If the force of -15 were acted at D , the moment accompanying it would be

$$M_D = -3 \times 15 - 2 \times (-40) + 2 \times (-20) = (-45 + 80 - 40) = -5 \text{ kN m}$$

(d) If the force of -15 were acted at B , the moment accompanying it would be

$$M_B = 5 \times 15 - 4 \times (-40) - 2 \times 30 = (-75 + 160 - 60) = 25 \text{ kN m}$$

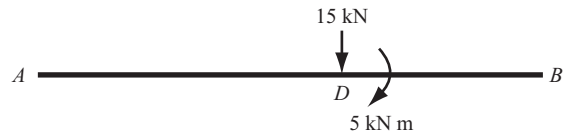
The results (a), (b), (c) and (d) are shown in Fig. Ex. 2.7 (Solution).



(a) Single resultant force



(b) Equivalent force – Moment at A



(c) Equivalent force – Moment at D



(d) Equivalent force – Moment at B

Fig. Ex. 2.7 (Solution)

Example 2.8

A 4 m × 5 m slab carries four forces normal to it as shown in Fig. Ex. 2.8. Determine the equivalent action which can be applied only at point O and determine the single resultant of the force system.

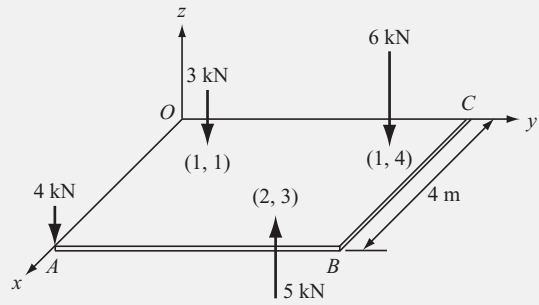


Fig. Ex. 2.8

Solution The resultant force R for the given force system must be

$$R = 4 - 3 + 5 - 6 = -8 \text{ kN}$$

which means that it should be 8 kN force acting vertically downward. If this was located to pass through O, the moment accompanying it for equivalence would be

$$\begin{aligned} M_o &= 4 \times (-4) + (1 + 1) \times (-3) + (2 + 3) \times 5 + (1 + 4) \times (-6) \\ &= -12 + 15 \text{ kN m} \end{aligned}$$

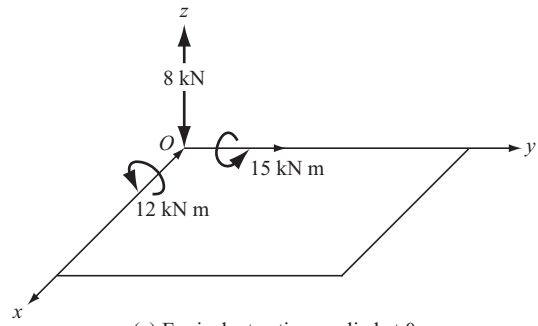
The moment accompanying the force 8 kN downward applied at O is, therefore, such that it has an x-component of -12 kN m and the y-component of 15 kN m as shown in Fig. Ex. 2.8(a) (Solution).

In order to locate the point (x, y) through which the force -8 should be passed for complete equivalence, the moment accompanying it should be null. In other words, the moment generated due to the parallel transfer of the force should just cancel the moment M_o , i.e.,

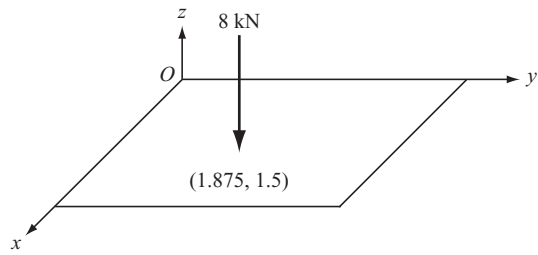
$$\begin{aligned} -M_o &= -x \times M_y + y \times M_x \\ -(-12 + 15) &= -(x + y) \times (-8) \\ &= -8x + 8y \end{aligned}$$

whence $8y = 12; \quad y = 1.5 \text{ m}$
 $-8x = -15; \quad x = 1.875 \text{ m}$

The single resultant force -8 kN can, therefore, be equivalent to the prescribed parallel force system if it is applied at a point (1.875 m, 1.5 m) as shown in Fig. Ex. 2.8(b) (Solution).



(a) Equivalent action applied at O



(b) Single equivalent force

Fig. Ex. 2.8 (Solution)

2.9 RESULTANT OF A COPLANAR FORCE SYSTEM

When the forces constituting a force system lie in a plane and the moments applied externally or generated by the applied forces are directed normal to the plane, the force system is said to be coplanar. Coplanar systems

may consist of concurrent or non-concurrent forces, parallel or non-parallel forces. The analyses of plane concurrent and plane parallel forces has been dealt with already. Attention is now focussed on the analysis of general coplanar systems.

For a coplanar force system in the x - y plane, there can be no forces in the z -direction and no moments about the x and y axes.

A system of coplanar forces

$$F_1, F_2, F_3, \dots$$

whose lines of action are specified by the position vectors

$$r_1, r_2, r_3, \dots$$

confined in the x - y plane and couple moments

$$M_1, M_2, M_3, \dots$$

directed normal to the x - y plane are reducible to a resultant force

$$R = F_1 + F_2 + F_3 + \dots \tag{2.11}$$

which can be made to pass through any point and a resultant moment M . If the resultant force R is acted through the origin O , the moment which must accompany it is given by

$$\begin{aligned} M &= F_1 \times r_1 + F_2 \times r_2 + \dots \\ &= (F_1 + F_2 + F_3 + \dots) \times r \end{aligned} \tag{2.12}$$

This is shown in Figs 2.12(a) and (b).

It is possible to shift the resultant force R parallel to itself until the moment M is cancelled. The position vector of the line of R would then be such that

$$r \times R = 0$$

as shown in Fig. 2.12(c).

In case the resultant force vanishes,

$$R = 0; \quad R_x = 0, \quad R_y = 0$$

the resultant of the coplanar force system may be a couple-moment.

$$\begin{aligned} M &= F_1 \times r_1 + F_2 \times r_2 + \dots \\ &= M_1 + M_2 + \dots \end{aligned}$$

In case the couple-moments add to zero in addition to the resultant force being zero, the resultant is a null force and a null couple-moment.

The discussion on coplanar force systems may be summed up by stating that the force system may be replaced by a non-zero single force R in that plane passing through any desired point and an appropriate moment by a non-zero single force R appropriately located in that plane, by a single moment normal to that plane if the resultant force R is zero or by a zero moment and zero force in some particular case.

A method known as *funicular polygon method* is usually applied to determine the resultant of a plane system of parallel or non-parallel forces.

The forces F_1, F_2, F_3, \dots are drawn and the areas enclosed between their lines of action are marked in capital letters. Such a figure is called *space diagram*. If the space on the left of the line of action of F_1 is called A and the other spaces as B, C, D, \dots as shown in Fig. 2.13(a), the forces are also referred to as AB, BC, CD, \dots in terms of the space notation.

The next step is to draw the *force polygon* by drawing the forces in magnitude and direction and joining them end to end. In the force polygon the forces are named in small letters thus the force between regions A and B will be named ab in the force polygon. The closing side of the polygon in opposite order gives the resultant of the forces in magnitude and direction in accordance with the polygon law of forces.

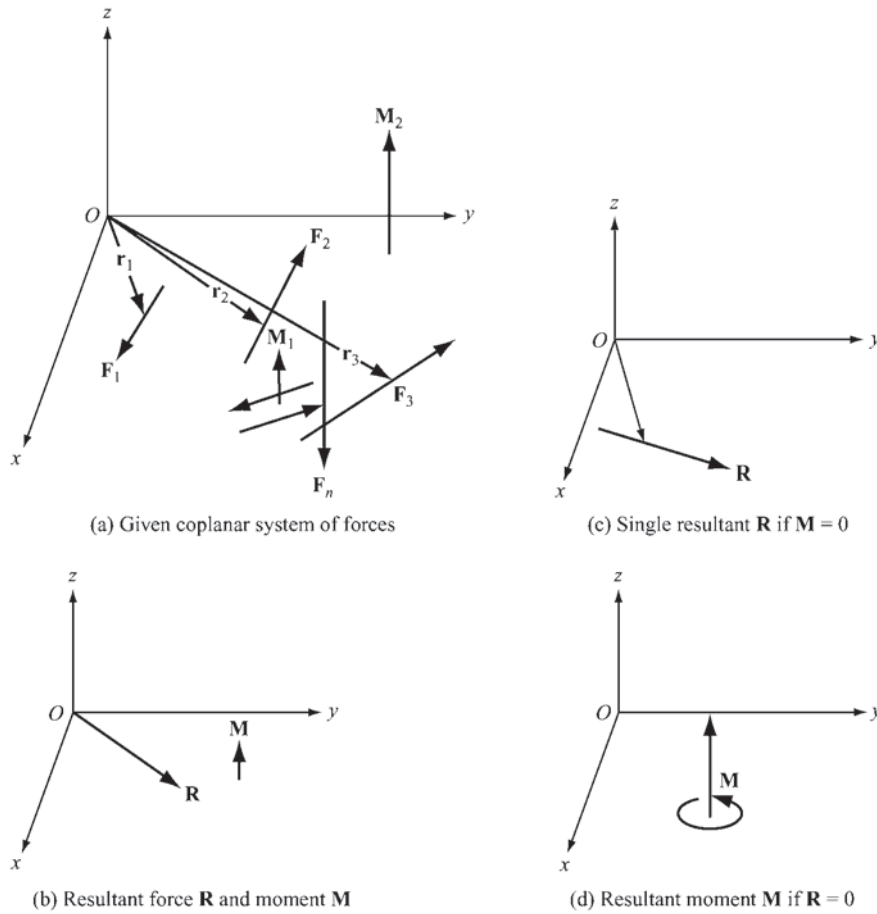


Fig. 2.12 Resultant force and moment of a given coplanar system of forces

In order to find the line of action of the resultant, the following procedure is adopted:

Take any arbitrary point p inside or outside the force polygon and join all the vertices of the force polygon with this point. This arbitrary point is called pole and the lines joining the pole to the vertices of the force polygon are called rays. These rays can be imagined as the components of the force, e.g. cp and pd are components of cd ; bp and pc are components of bc , and so on. The diagram showing the force polygon, pole and rays is called *ray diagram* as shown in Fig. 2.13(a). Starting with an arbitrary point s on the line of action of the force F_1 , a line sb is drawn parallel to the ray pb such that it intersects the line of the action of the next force F_2 as shown in Fig. 2.13(b). From that point, another line is drawn parallel to the next ray pc and produced to intersect the line of action of the next force, and so on. Finally, a line parallel to sa is drawn through the starting point s and the point of intersection O with the last line drawn as above is located. This is the point through which the resultant R of the given forces should act. The figure so drawn is called *funicular polygon*. The resultant is, therefore, completely specified by stating that it is a force denoted by ae in magnitude and direction as shown in the force polygon and acting through the point O located by drawing the funicular polygon.

It may be pointed out that, in general, the ray diagram should be such that it does not coincide with or is parallel to the forces; otherwise it may not be possible to draw the funicular polygon. This fact places a

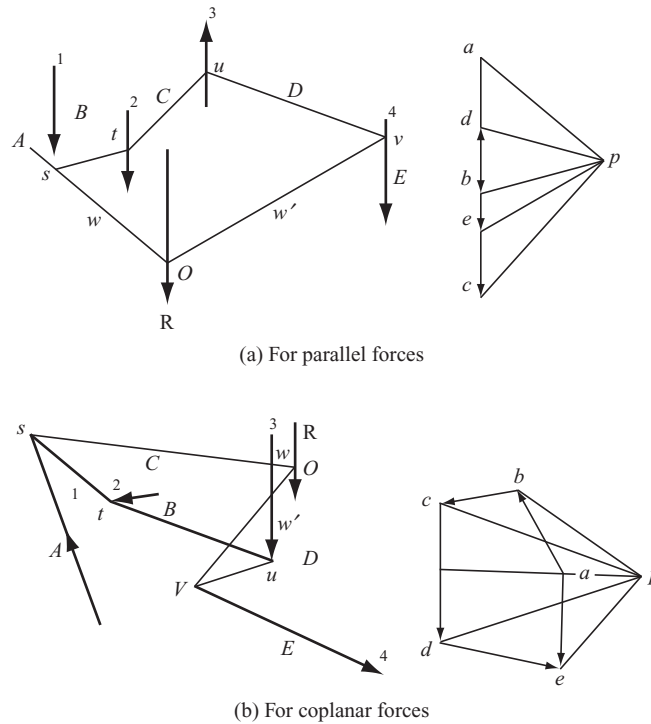


Fig. 2.13 Graphical methods

restriction on the choice of the pole p , namely, the pole should not lie on the closing side ae of the force polygon. In particular, when the given forces are parallel, the force polygon is made up of collinear lines and the pole p must not lie on the line.

The graphical procedure is a geometrical manifestation of the theoretical method. The simplicity offered by it is at the expense of accuracy of results. Moreover, the graphical procedure is restricted to the plane system of forces. An insight into the equivalence of the graphical procedure with the theoretical method is provided as follows.

The closing side ae of the force polygon $abcde$ provides the direction and magnitude of the resultant force in accordance with the polygon law of forces. The construction of the ray diagram enables us to replace each of the given forces by two components, e.g., F_1 by ap and pb , F_2 by bp and pc and F_3 by cp and pd . Summation of the forces shows that the resultant R is made up of ap and pe because the pairs of forces pb and bp , pc and cp , etc. cancel off mutually as shown by the arrows on the force diagram. The construction of the funicular diagram by drawing lines parallel to the rays enable us to locate the points on a hypothetical string such that it is in equilibrium. The force F_1 is balanced by the virtual tensions ap and pb in the string. Similarly, the force F_2 is balanced by bp and pc , and so on. A point O is located where the resultant R would be balanced by the components ap and pe , so that point must be the point of application of the resultant. It may be noted that different choices of the pole result in different rays and hence in different shapes of the funicular polygons. In each case, however, the equilibrium criteria is automatically satisfied and the point O finally located must lie on the line of action of the resultant force. It may also be added that a funicular polygon is indeed the shape of a string it would acquire under the application of the given forces at the corresponding points. For the same reason, the method of funicular polygon is also called *string analogy method*.

Example 2.9

Three forces of magnitudes 1, 2 and $\sqrt{2}$ N act along the sides of a rigid, triangular frame formed by the lines AB , BC and CA specified by

$$x + y = 1, \quad y - x = 1 \quad \text{and} \quad y = 2$$

in the same order. Find the resultant and the equation of its line of action.

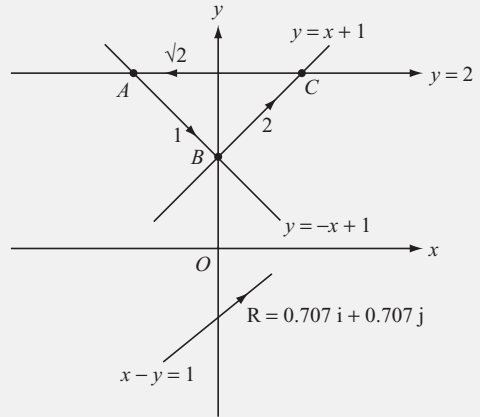


Fig. Ex. 2.9

Solution The sides of the triangle are given by

$$y = -x + 1, \quad y = x + 1 \quad \text{and} \quad y = 2$$

indicating that their slopes are -1 , $+1$ and 0 respectively. The unit vectors along the sides are, therefore,

$$0.707 \mathbf{i} - 0.707 \mathbf{j}, \quad 0.707 \mathbf{i} + 0.707 \mathbf{j} \quad \text{and} \quad - \mathbf{j} \quad \text{respectively.}$$

The three forces are, in turn, represented as

$$\begin{aligned} F_1 &= 0.707 \mathbf{i} - 0.707 \mathbf{j} && \text{passing through } (0, 1) \\ F_2 &= 1.414 \mathbf{i} + 1.414 \mathbf{j} && \text{passing through } (0, 1) \\ F_3 &= -1.414 \mathbf{j} && \text{passing through } (0, 2) \end{aligned}$$

The resultant R is given by their sum

$$R = 0.707 \mathbf{i} + 0.707 \mathbf{j}$$

The line of action of the resultant is located by considering the equivalence of the given system with the resultant in regard to the moments about the origin O ;

$$(x + y) \times (0.707 \mathbf{i} + 0.707 \mathbf{j}) = 1 \times (0.707 \mathbf{i} - 0.707 \mathbf{j}) + 1 \times (1.414 \mathbf{i} + 1.414 \mathbf{j}) + 2 \times (-1.414 \mathbf{j})$$

whence $0.707x - 0.707y = 0.707$

or $x - y = 1$

which is the line of action of the resultant force.

Example 2.10

The moments of a given plane system of forces about three points $(1, 0)$, $(0, 1)$ and $(1, 2)$ are $+4$, $+25$, $+22$ units respectively. Find the resultant force and prove that it acts along the line $12x - 9y = 16$.

Solution The plane system of forces may be replaced by a resultant force

$$R_x \mathbf{i} + R_y \mathbf{j}$$

in that plane acting at a point (x, y) in accordance with the concept of equivalence.

The moments of the resultant force are expressed as follows:

$$\begin{aligned} ((x-1) + (y-0)) \times (R_x + R_y) &= 4 \\ ((x-0) + (y-1)) \times (R_x + R_y) &= 25 \\ ((x-1) + (y-2)) \times (R_x + R_y) &= 22 \end{aligned}$$

whence

$$(x-1)R_y - yR_x = 4 \tag{i}$$

$$xR_y - (y-1)R_x = 25 \tag{ii}$$

$$(x-1)R_y - (y-2)R_x = 22 \tag{iii}$$

Subtracting (i) from (iii) and (ii) respectively,

$$R_x = 9, \quad R_y = 12$$

Hence, the resultant of the force system is

$$R = 9 + 12$$

From (i),

$$12(x-1) - 9y = 4$$

or

$$12x - 9y = 16$$

is the relationship between x and y . The point of application (x, y) cannot be determined uniquely because Eqs. (i), (ii) and (iii) cannot provide any additional information, they being a set of three equations for four unknowns.

Since, the point (x, y) can lie on a straight line

$$12x - 9y = 16$$

this equation is the locus of the points on the line of application of the resultant force. The resultant force is, therefore, directed along this line.

Example 2.11

A wrench of 20 cm size is used to tighten a pipe assembly ABC at socket A , as shown. Determine the moment the vertically downwards force of 100 N, exerts on the socket along x -axis when both the pipe assembly and the wrench lie in the x - y plane.

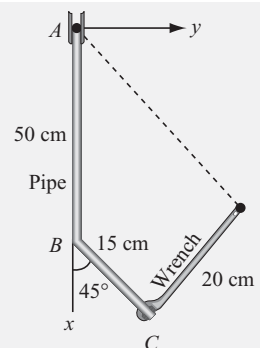


Fig. Ex. 2.11

Solution In terms of vector notation, and SI units

$$\begin{aligned} &= -100 \text{ kN} \\ &= 0.5 + 0.15 \cos 45^\circ - 0.2 \sin 45^\circ + 0.15 \sin 45^\circ + 0.2 \cos 45^\circ \\ &= (0.5 + 0.106 - 0.141) + (0.106 + 0.141) \\ &= 0.445 + 0.247 \text{ m} \end{aligned}$$

Moment of the force about the point A is given by

$$= x \times = (0.445 + 0.247) \times (-100) \text{ Nm} = 44.5 - 24.7$$

The moment which acts about the x -axis in tightening the socket is given by

$$M_{xx} = \dots = (44.5 - 24.7) \cdot = -24.7 \text{ Nm.}$$

Reflection on the Solution

We could have determined the moment of the applied force about B instead of A and simplified the solution. In that case,

$$= B = -0.055 + 0.247 \text{ m}$$

and the moment acting about the x -axis would have resulted to be the same!

It is also interesting to observe that a scalar solution to this plane problem would have been easier.

The perpendicular distance from to the x -axis is

$$d = 0.15 \sin 45 + 0.2 \cos 45 = 0.247 \text{ m}$$

The moment of the 100 N downward force about the x -axis would then be $100 \times 0.247 = 24.7 \text{ Nm}$ clockwise, i.e., -ve.

2.10 RESULTANT OF A SPATIAL FORCE SYSTEM

Let us now arrive at some conclusions regarding the equivalence of a system of forces which are arbitrarily placed on a rigid body. The forces

$$F_1, F_2, F_3, \dots$$

act through points specified by the position vectors

$$r_1, r_2, r_3, \dots$$

respectively as shown in Fig. 2.14(a).

The forces which constitute couples and produce couple-moments together with any external moment are

$$C_1, C_2, C_3, \dots$$

The resultant force R must be given by

$$R = F_1 + F_2 + F_3 + \dots \tag{2.13}$$

no matter which point it is made to pass through. If it is made to pass through point O as shown in Fig. 2.14(b), then the moment which must accompany the force for equivalence is given by

$$M = F_1 \times r_1 + F_2 \times r_2 + \dots + C_1 + C_2 + C_3 + \dots \tag{2.14}$$

The moment M may have a component M_R in the direction of R and M_N in a plane normal to R . The force R may now be transferred parallel to itself to a point P such that the M_N component of moment is nullified and the simplest equivalent system is a wrench with force R passing through P and moment M_R in the direction of R as shown in Fig. 2.14(c).

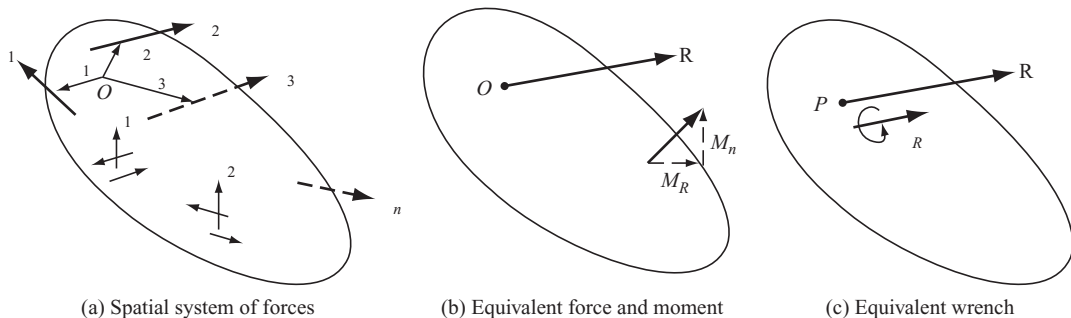


Fig. 2.14 Resultant of a spatial force system

Example 2.12 A horizontal plate, 3 m × 3 m is acted upon by three forces F_A , F_B and F_C at A , B and C respectively, as shown.

- (a) Can the forces be replaced by a single force ?
- (b) Can the forces be replaced by a wrench? If so, locate the force and moment components of the wrench, i.e., point P where its line of action intersects the plate.

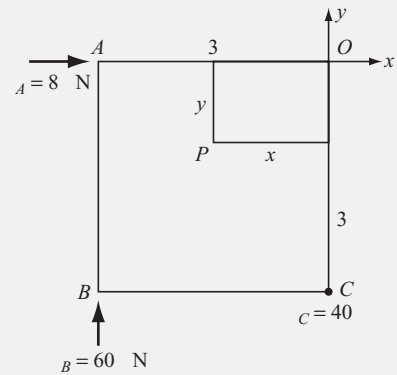


Fig. Ex. 2.12

Solution

- (a) The force system consists of F_A , F_B and F_C which add to a single force $80 + 60 + 40$. In order to replace the given forces, it must be positioned such that the moment due to this force equals the moments due to the given forces at the same point. Let the point be $P(x, y)$. Moments due to F_A and F_B would be along the z-direction, which could balance each other but the moment due to F_C would be in x and y direction; there is nothing to balance it. Hence, there is no point possible where the single force could be applied to result in zero moment; hence not possible to be replaced by a single force !
- (b) Any given force system can, of course, be reduced to a wrench, i.e., a force and a collinear moment M at a point. Let such a point be $P(x, y)$ through which the force may be made to pass.

The force component of the wrench must be

$$= 80 + 60 + 40, \text{ which has the magnitude,}$$

$$|F| = \sqrt{80^2 + 60^2 + 40^2} = 108 \text{ N}$$

If this has to pass through $P(x, y)$, the accompanying moment

$$\begin{aligned} M_P &= y \times 80 + (x-3) \times 60 + (x-(3-y)) \times 40 \\ &= -80y + (x-3)60 + (-40x) - (3-y)40 \\ &= -120 + 40y - 40x + (60(x-3) - 80y) \\ &= 40(y-3) - 40x + (60x - 180 - 80y) \end{aligned}$$

This must be collinear with $80 + 60 + 40$, which means that

$$\frac{40(y-3)}{80} = \frac{-40x}{60} = \frac{60x-180-80y}{40}$$

or

$$\frac{y-3}{2} = \frac{-2x}{3} = \frac{3x-9-4y}{2}$$

or

$$3y - 9 = -4x; -4x = 9x - 27 - 12y$$

or

$$4x + 3y = 9; 13x - 12y = 27$$

whence $x = 2.2$ m and $y = 0.07$ m

The moment M_P can now be determined by substituting x and y in the expression above:

$$\begin{aligned} M_P &= 40(0.07-3) - 40 \times 2.2 + (60 \times 2.2 - 180 - 80 \times 0.07) \\ &= 117.2 - 88 - 53.6 \end{aligned}$$

and

$$|M_P| = \sqrt{117.2^2 + 88^2 + 53.6^2} = 156 \text{ N}$$

2.11 GRAVITATIONAL FORCE FIELD

Newton's law of universal gravitation states that

Any two particles or point-masses are attracted towards each other along a line connecting them with a mutual force whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance between the particles.

Mathematically, the magnitude of the gravitational force is

$$F = G \frac{m_1 m_2}{r^2} \quad (2.15)$$

where G is the universal constant of gravitation, its value being $6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ or $\text{m}^3/\text{kg s}^2$, m_1 and m_2 are the masses of the bodies and r is the distance between them.

The gravitational force field of the earth is due to its mass M and acts on a body of mass m placed at a height h above the surface of the earth of radius R

$$F = GM \frac{m}{(R + h)^2} \quad (2.16)$$

Denoting the acceleration due to gravity g , and recognising that the weight of the body is given by

$$W = F = m g \quad (2.17)$$

$$g = \frac{GM}{(R + h)^2} \quad (2.18)$$

It may, however, be pointed out here that the earth has been assumed as a spherical and stationary body. Since the earth is not a perfect sphere, the radial distance of the surface from the centre of the earth varies with the latitude as well as the altitude. The variation of g has been investigated and expressed as

$$g = 9.806\ 16 - 0.025\ 928 \cos^2 \lambda + 0.000\ 069 \cos^2 2\lambda - 0.000\ 003\ h \text{ m/s}^2 \quad (2.19)$$

where λ is the latitude of the place and h is the height in metres above the mean sea level.

It is usual in engineering to consider g , the acceleration due to gravity as constant and the weight force directed perpendicular to the surface of the earth for most earthbound objects. This assumption is known as *assumption of a flat earth*. It is indeed incorrect to allow such an assumption in principle but it is acceptable to admit this assumption for most practical purposes, particularly when the analysis is confined to a region close to the earth.

Another noteworthy fact is that the gravitational force between two bodies of finite sizes is not necessarily along the line joining their centres, nor is it necessary that the magnitude of the force be given by Eq. (2.15). A careful reading of the statement of Newton's law of gravitation would reveal that the law relates to the force between two particles or point bodies. A body of finite size may be thought of as a distributed mass or a distribution of mass elements over the domain of the body. According to the law of gravitation, each element on a body would experience a force from each element of the other body and the total force on a body would require double-volume integration. Only in the special case of two homogeneous solid or thin hollow spheres does the law hold in its stated form because the force of gravitation turns out to be along the line joining their centres and the magnitude is given by Eq. (2.15). It is for this reason that the law holds fairly well for the nearly spherical celestial bodies. In fact, Kepler's laws of planetary motion, formulated before Newton was born, contain Newton's law of gravitation for spherical planetary bodies; Newton's original contribution was to state the laws applicable to particles or infinitesimal elements which may belong to any distributed masses of which spheres are only particular cases.

Example 2.13 Determine the gravitational force of attraction of a thin uniform rod of length a and mass M on a concentrated mass m outside the rod but on the same line as the rod and at a distance b from the nearer end.

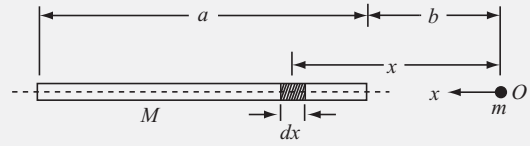


Fig. Ex. 2.13

Solution The force on the concentrated mass m can be determined by writing the force due to an element of length dx situated at a distance x from it and integrating the same for the entire length of the uniform rod. Let the mass density of the rod be ρ per unit length of the rod. The infinitesimal element of length dx has a mass ρdx and the force of attraction on the mass m due to it is

$$dF = \frac{Gm\rho dx}{x^2} \quad (i)$$

The total force of attraction due to the entire length of the rod must be

$$\begin{aligned} F &= \int dF = \int_b^{a+b} \frac{Gm\rho dx}{x^2} \\ &= Gm\rho \left[-\frac{1}{x} \right]_b^{a+b} \\ &= \frac{Gm\rho a}{b(a+b)} = \frac{GMm}{b(a+b)} \end{aligned} \quad (ii)$$

where the mass of the rod is taken as

$$M = \rho a$$

The distance between the concentrated mass and centre of the rod is

$$x_c = b + a/2$$

If the mass of the rod was concentrated at its mid-point, the force of attraction would have been

$$F_c = \frac{GMm}{(b + a/2)^2} \quad (iii)$$

It may be appreciated that F and F_c are of different magnitudes; they are closer if the length of the rod is less,

$$a \rightarrow 0, \quad x_c \rightarrow x_s$$

Equation (iii) enables us to interpret that the force of attraction of the rod would be the same if the entire mass of the rod was concentrated at x_s

where

$$F = \frac{GMm}{b(a+b)} = \frac{GMm}{x_s^2}$$

whence

$$x_s = \sqrt{b(a+b)}$$

Again, x_s coincides with x_c if a tends to zero, as expected.

2.12 HYDROSTATIC FORCE FIELD

In a static fluid, the intensity of pressure at a point is the same in all directions. This statement is due to Pascal and can be proved by considering the equilibrium of a fluid element at a point.

The rate of change of pressure in a vertically downward direction must equal the local specific weight of the fluid. This statement comprises the hydrostatic law, mathematically

$$\frac{dp}{dz} = \rho g \quad (2.20)$$

where dp/dz is the rate of change of pressure in the downward direction and ρg is the specific weight or weight density, ρ being the mass density and g , the acceleration due to gravity.

Alternatively, Eq. (2.20) is written as

$$dp = \rho g dz \quad (2.21)$$

implying that the increment of pressure dp in the vertically-downward direction over a distance of dz equals the product of mass density ρ , gravitational acceleration g and the distance dz .

In a fluid of constant density, i.e., incompressible fluid,

$$\int dp = \int \rho g dz = \rho g \int dz$$

Assuming the atmospheric pressure at the free surface of a liquid, as the reference, the pressure at a point with the atmospheric pressure as the reference is

$$p = \rho g z \quad (2.22)$$

In other words, the pressure varies linearly as the depth in an incompressible fluid at rest.

The pressure on a surface acts normal to that surface, by

$$= p \Delta A \text{ normal to the area } \Delta A. \quad (2.23)$$

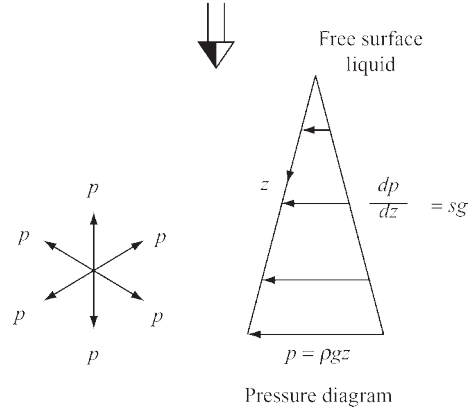


Fig. 2.15 Pascal's law and hydrostatic law

Concept Review Questions

Classify the following statements as true or false and state the reasons:

- The simplest equivalent of a force system is either a force or a couple-moment.
- The force in a force-field must vary monotonically only.
- Force is a transmissible vector or a free vector.
- Equivalent forces are defined on the basis of providing an identical action in a particular capacity.
- Resultant and equivalent are identical concepts.
- A particle may be subjected only to a concurrent force system; collinear, coplanar or spatial.
- A force is a bound vector; its line of action on a rigid body must be specified.

(h) A moment is a free vector; its line of action on a rigid body need not be specified.

- Can a system of forces acting on a rigid body be replaced by a wrench at any desired point on it?
- If a rigid slender bar is subjected to a number of forces, which factors would decide whether the resultant will be a single force or not? Is it necessary that the system of forces should not be distributed in space?

A number of plane forces act on a simply supported rigid body at two points. Determine the resultant of these forces and appreciate the fact that it is also the equivalent force so far as the determination of the reactions from the supports is concerned but is not the

equivalent force if the body is analysed either for deflection or for the resisting moments and forces developed in the body.

Which of the following system of forces may be represented by a single resultant non-zero force

- (a) Concurrent force system
- (b) Parallel force system
- (c) Coplanar force system
- (d) Spatial force system.

State Newton's law of gravitation for two point-objects of masses m_1 and m_2 placed a distance r apart. How would you proceed to determine the gravitational force between two bodies which have an arbitrary but specified distribution of mass, e.g., a cricket bat and ball.

A plate AB of mass m and dimensions $2\text{ m} \times 3\text{ m}$ is to be lifted by a string tied to it and going over a set of pulleys with a suspended mass M at the other end as shown in Fig. CRQ 6. Recognise the sources of force in the system.

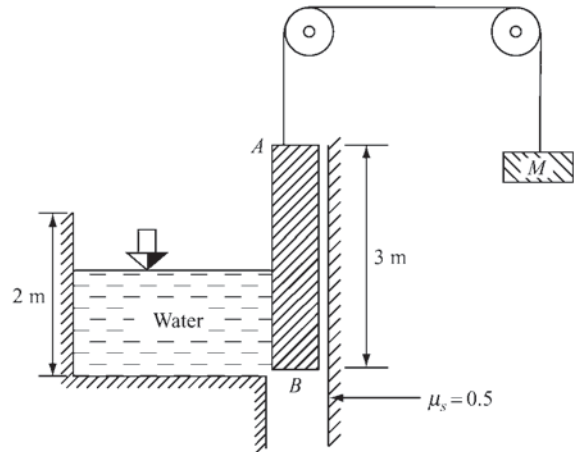


Fig. CRQ 6

If the sum of couple-moments and moments of all the forces of a force system about any three noncollinear points is zero, show that the force system results in a null.

Tutorial Problems

It is desired to transfer a 50 kN force parallel to itself from a point (2, 1) to a point (1, 2) as shown in Fig. Prob. 2.1. Determine the additional moment, if any, required to maintain equivalence. (s 70.7 kN m; 50 + 50)

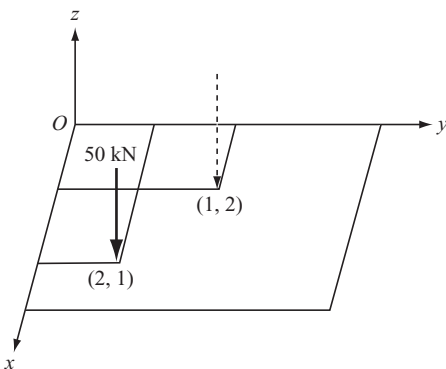


Fig. Prob. 2.1

A force of 100 N acting tangential to a drum at A must be transferred parallel to itself to its centre O or to a diametrically opposite point B . Determine the moments which should accompany it for equivalent effect.

(s 25 N m and 50 N m)

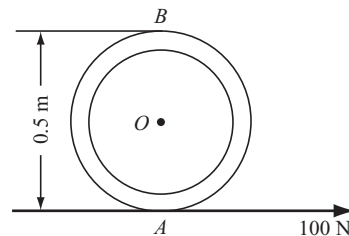


Fig. Prob. 2.2

Determine the resultant of the coplanar concurrent force system shown in Fig. Prob. 2.3

(s 49 N, $\theta = -26^\circ$)

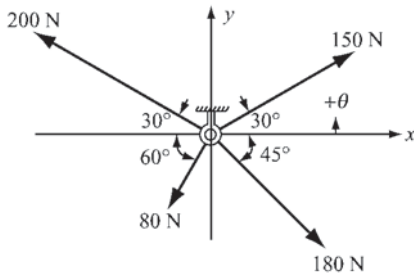


Fig. Prob. 2.3

Determine the resultant of the concurrent force system acting at a point O as shown in Fig. Prob. 2.4.

(s 19.7 N; $\alpha = 43$; $\beta = 56$ and $\gamma = 66$)

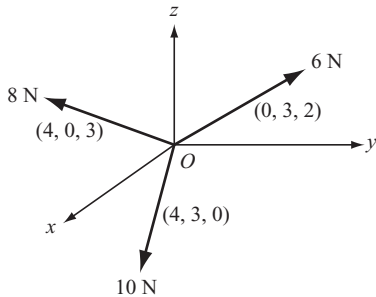


Fig. Prob. 2.4

Two unlike parallel forces, each 10 N, act 0.3 m apart as shown in Fig. Prob. 2.5(a). What is their resultant action? If one force was 11 N instead of 10 N as shown in Fig. Prob. 2.5 (b), what would the resultant action be? Can these systems be replaced by single resultant forces for equivalence? (s 3 Nm; 1N, 3.15 Nm)

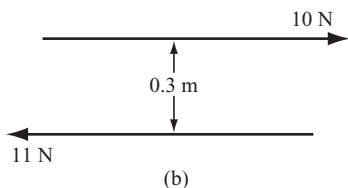
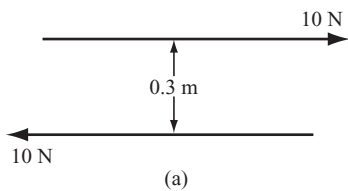


Fig. Prob. 2.5

Determine the resultant action of a coplanar parallel force system in Fig. Prob. 2.6.

(s $M = -30$ N m)

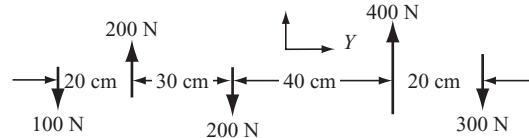


Fig. Prob. 2.6

The resultant of four vertical forces is a couple-moment 30 N m acting counterclockwise. Three of the four forces are shown in Fig. Prob. 2.7. Determine the fourth force.

(s 33 N upward at 44.5 cm to the right of A on AB)

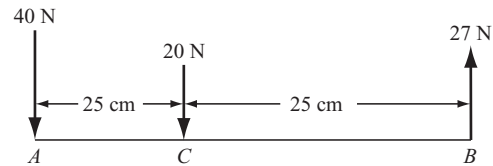


Fig. Prob. 2.7

8 The following forces act parallel to the z -axis. Their respective points of application in the x y plane are also given. Determine the single resultant and locate it in each of the following cases:

Case 1

| | | | | |
|-------------|--------|---------|--------|----------|
| Force in | 3 | -4 | 2 | -5 |
| kN | | | | |
| x, y in m | (2, 5) | (1, -5) | (3, 3) | (-4, -4) |

Case 2

| | | | |
|--------------|----------|-----------|-----------|
| Force in N | 100 | 200 | -300 |
| x, y in cm | (10, 10) | (20, -50) | (30, -40) |

(s Case 1: $R = -4$ kN at $(-7.0, -15.3)$ m

Case 2: $M_x = 30$ N m, $M_y = 40$ N m)

A coplanar parallel force system consisting of three forces acts on a rigid bar AB as shown in Fig. Prob. 2.9. Determine the simplest equivalent action for the force system. If an additional force of 10 kN acts along the bar A to B , what would be the simplest equivalent action?

(s -10 kN along 40 kN force; 14.14 kN, -45)

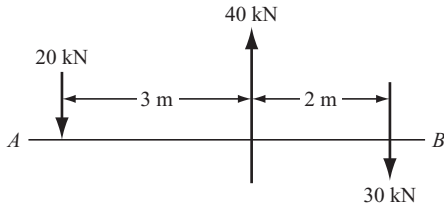


Fig. Prob. 2.9

A pulley of 1 m diameter is subjected to 2 kN and 1 kN force at A and B respectively as shown in Fig. Prob. 2.10. Its own weight of 0.5 kN acts at the centre O . Determine the resultant force and its line of action with respect to AOB .

(s 3.04 kN making an angle of 80.5° with AOB)

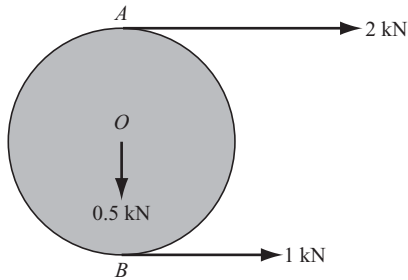


Fig. Prob. 2.10

A bell-crank lever AOB is subjected to a horizontal force 10 N at A while a weight of 7.5 N is attached at B . Determine the resultant action on the lever. If the 10 N force were transferred to point P , what would be the change in the resultant action?

(s In equilibrium; 1 N m moment and lever turning counterclockwise)

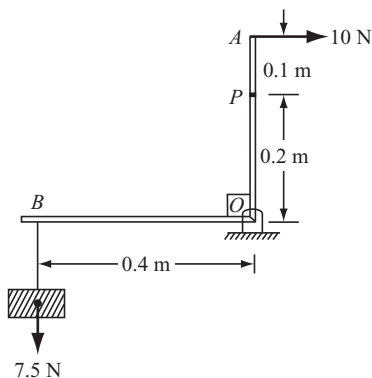


Fig. Prob. 2.11

A dam is subjected to three forces; 50 kN force on the upstream vertical face AB , 30 kN force on the downstream inclined face and its own weight 120 kN. Determine the single equivalent force and locate its point of intersection with the base AC , assuming all the forces to lie in the same plane.

(s 137 kN; $y + 5.6x = 0$)

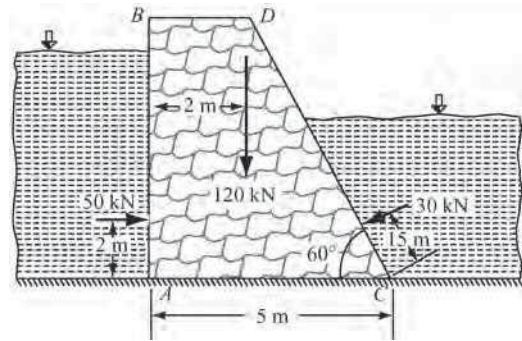


Fig. Prob. 2.12

A $2\text{ m} \times 4\text{ m}$ plate is subjected to a system of three coplanar forces as shown in Fig. Prob. 2.13. Determine the equivalent action at O which may replace the force system.

(s 5.9 kN and 1.7 kN m)

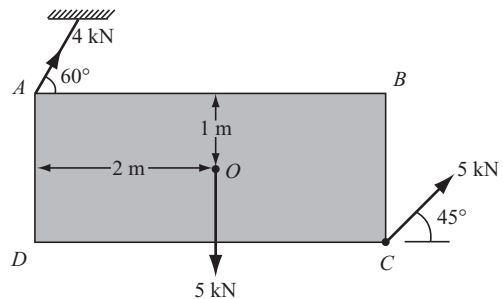


Fig. Prob. 2.13

A symmetrical truss is loaded by five forces as shown in Fig. Prob. 2.14. Obtain the resultant load and its line of action.

(s 15.5 kN along $0.29 - 0.95$)

Determine the resultant of the forces acting on a bell-crank lever as shown in Fig. Prob. 2.15.

(s 4.25 Nm clockwise)

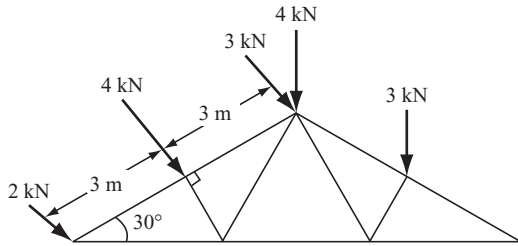


Fig. Prob. 2.14

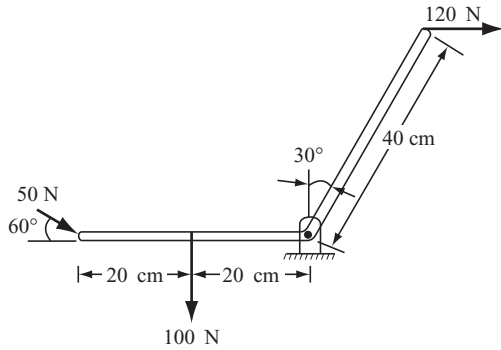


Fig. Prob. 2.15

A bracket is subjected to a coplanar force system shown in Fig. Prob. 2.16. Determine the magnitude and the line of action of the single resultant of the system.

$$(\quad s \quad -200 \quad -154 \quad ; \quad 35 \text{ cm from } A)$$

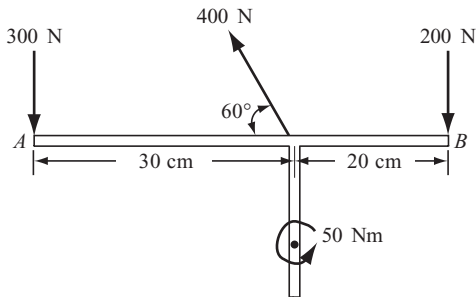


Fig. Prob. 2.16

Reduce the wrench along the x -axis as shown in Fig. Prob. 2.17 to a system of two forces perpendicular to the y -axis acting at A and B on the y -axis. It is given that $OA = 0.2 \text{ m}$ and $OB = 0.6 \text{ m}$.

$$(\quad s \quad 100 \quad -7.5 \quad ; \quad -50 \quad +7.5 \quad)$$

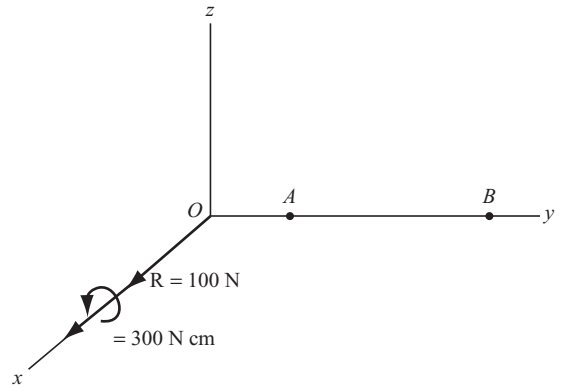


Fig. Prob. 2.17

- 8 A system of three forces acts on a parallelepiped as shown in Fig. Prob. 2.18. Replace the forces by a wrench and specify its point of action on a face.

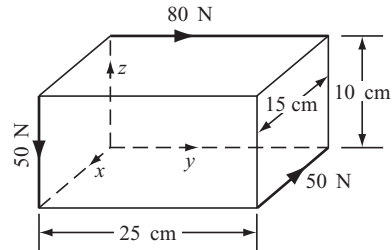


Fig. Prob. 2.18

Reduce the given force system to an equivalent force plus couple-moment system at corner O as shown in Fig. Prob. 2.19.

$$(\quad s \quad -56.5 \quad +17.4 \quad -17.4 \quad ; \quad -3.5 \quad -28.7 \quad +30 \quad)$$

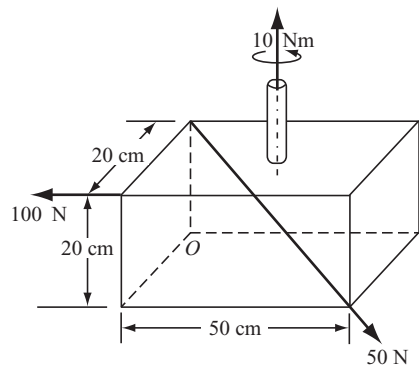


Fig. Prob. 2.19

Find the force of attraction between two homogeneous solid spheres of radii r_1 and r_2 and masses m_1 and m_2 placed at a distance r between their centres.

$$\left(\begin{array}{l} \text{s. } \frac{Gm_1 m_2}{r^2} \text{ along the line joining their} \\ \text{centres} \end{array} \right)$$

Find the force of attraction of a thin uniform rod of length $2a$ and mass M on a particle of mass m placed at a distance b from its mid-point such that the particle is equidistant from its ends.

$$\left(\begin{array}{l} \text{s. } \frac{Gm_1 m_2}{r^2} \text{ along the line joining the} \\ \text{mid-point of the rod} \end{array} \right)$$

Determine the maximum possible friction force which may be developed between a pair of sliding surfaces if the normal reaction between them is 20 kN and the coefficient of static friction is 0.30. What happens to the frictional force if the applied tangential force is increased further and the bodies acquire relative motion? (s 6 kN)

A stack of plates of different materials are placed one above the other and a horizontal force is applied to one of them in the middle of the stack. Discuss the circumstances under which (a) the plate on which force is applied and all the plates above it slide together (b) the plate on which force is applied and some more plates below it slide together and (c) only the plate on which force is applied slides out.

Three forces act along the three sides of a triangle in the same order with their magnitudes proportional to the sides along which they act. Prove that their simplest resultant is a single moment whose magnitude is proportional to twice the area of the triangle. Hence, extend it for a polygon of any number of sides.

Replace the three forces shown in Fig. Prob. 2.25 by a resultant force R passing through the point O and a couple for equivalent effect.

$$\left(\begin{array}{l} \text{s } R = 200 \quad -400 \quad -500 \text{ N} \\ \quad \quad \quad = 15 \quad -24 \text{ N m} \end{array} \right)$$

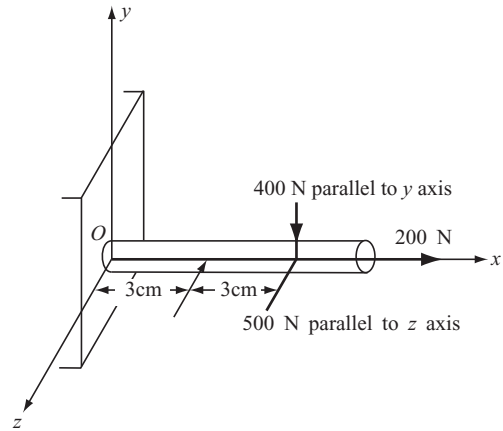


Fig. Prob. 2.25

Determine the tension in cable BC (Fig. Prob. 2.26). Neglect the weight of AB . (s 5.0 kN)

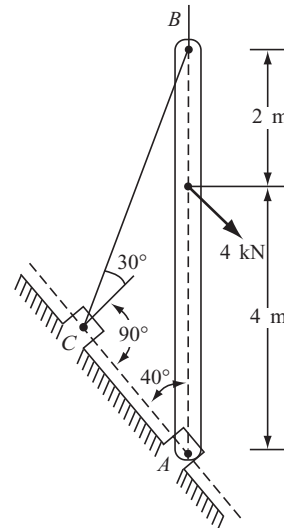


Fig. Prob. 2.26

Determine the point on the line joining the centres of the earth and moon (Fig. Prob. 2.27) at which the gravitational forces of the earth and moon are equal. It is given that the mass of moon is .0123 times that of earth and distance from earth to moon is 3.8×10^5 km.
 (s. 3.45×10^5 km from earth)

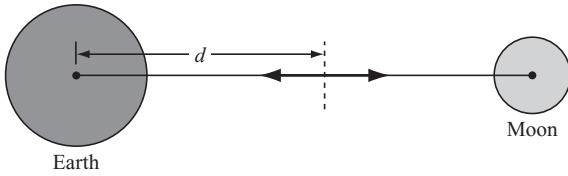


Fig. Prob. 2.27

In the punch shown in Fig. Prob. 2.28 links OA and AB have negligible mass and all friction may be neglected. The punch has a mass of 2 kg and moves in a vertical direction only. Find the moment which must be applied to OA to maintain its speed constant at 10 rad/s clockwise, with the mechanism in the position shown in Fig. Prob. 2.28 and with zero.

(s. 4.53 Nm)

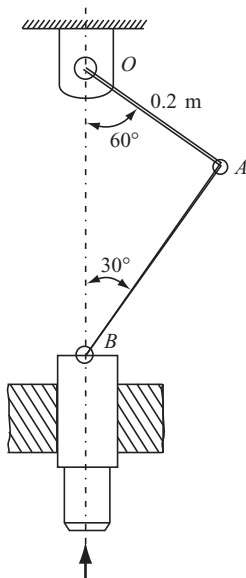


Fig. Prob. 2.28

Determine the moment of a force 10 kN acting as shown in Fig. Prob. 2.29, (a) about the point C (b) about the point H and (c) about the axis CF .

(s (a) and (b) $3.71 + 5.94$ Nm;
 (c) 2 Nm)

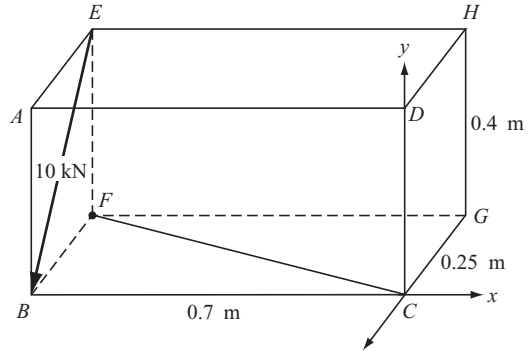


Fig. Prob. 2.29

(a) Find the resultant of the force system shown in Fig. Prob. 2.30 and locate the position of its line of action on the x -axis.
 (b) Determine the magnitude and sense of a single vertical force to be applied at point C so as to make the resultant of the entire system pass through A . Also, find that resultant.

(s (a) $300 - 280$ N at 107 mm from O
 (b) 72 N; $300 - 208$ N at A)

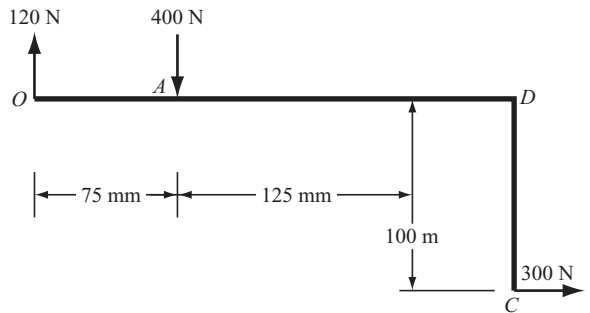


Fig. Prob. 2.30

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

A rigid body is acted upon by a force system. It can in general be brought to equilibrium by the application of a force acting

- (a) on a suitable point on the body
- (b) anywhere along a suitable line
- (c) along a suitable line and a moment along the direction of the force
- (d) along a suitable line and a moment in the direction perpendicular to the direction of force

The simplest resultant of a spatial parallel force system is always

- (a) a wrench
- (b) a force
- (c) a moment
- (d) a force and moment

The force of gravitation between two bodies will be inversely proportional to the square of

the distance between their centres of masses if the two are

- (a) of constant densities
- (b) spherical
- (c) of any arbitrary shape
- (d) of the same shape and size

A force acting on a rigid body at a point P can be replaced by a force of equal magnitude and in the same direction at a point Q on the body, together with a moment

- (a) equal in magnitude to PQ times F , acting normal to the plane of P and Q
- (b) equal in magnitude to F times the distance moved in the lines of actions of the force, acting in the plane of P and Q
- (c) given by $F \times PQ$
- (d) given by $F \times PQ$

Answers to Multiple-Choice Questions

- 1 (c) 2 (c) 3 (b) 4 (c)

Equilibrium of a Particle and a Rigid Body

3



3.1 EQUILIBRIUM CONCEPT IN MECHANICS

A body is said to be in a state of equilibrium if the body is either at rest or is moving at a constant velocity. The phrase ‘constant velocity’ implies motion along a straight line at a constant speed. The state of equilibrium, in other words, implies that the body must be at rest with respect to some inertial frame. Equilibrium is a kinematic state of the body; a special state when there is no motion or when the motion is at a constant speed along a straight line. Clearly, all other possible states of motion do not qualify to be categorised as states of equilibrium. Such states are characterised by the presence of the rate of change of momentum, linear and angular. Examples of non-equilibrium are: a particle accelerating along a straight line, a particle going round a curved path and a body rotating about any axis within or outside it.

Let us now try and relate the concept of equilibrium to the action of forces and moments acting on a body. The laws of mechanics governing the motion of a body are

$$\text{Newton's Law: } \mathbf{F} = \dot{\mathbf{p}} = -(\quad)$$

$$\text{Euler's Law: } \mathbf{H} = \dot{\mathbf{L}} = -(\quad \times \quad)$$

For a body in equilibrium, there should be no rate of change of momentum, linear or angular, i.e.:

$$\dot{\mathbf{p}} = -(\quad) = 0 \quad \text{and} \quad \dot{\mathbf{H}} = -(\quad \times \quad) = 0$$

and hence, the necessary conditions are

$$\mathbf{F} = 0 \quad \text{and} \quad \mathbf{H} = 0$$

The resultant external force and resultant external moment about any point should, therefore, vanish for a body to be in equilibrium.

Let the forces acting on a body be

$$F_1, F_2, F_3, \dots$$

and the position vectors of any points on the lines of action of the forces be

$$r_1, r_2, r_3, \dots$$

respectively with reference to an arbitrary origin O as shown in Fig. 3.1. The forces will then tend to compel the body to change its state of rest or of constant velocity in two ways:

1. By adding to comprise a resultant force

$$R = \Sigma F = F_1 + F_2 + F_3 + \dots \tag{3.1}$$

tending to destroy the equilibrium by bringing about a translational acceleration.

2. By generating moment vectors about the reference point O ,

$$\begin{aligned} M_1 &= r_1 \times F_1 \\ M_2 &= r_2 \times F_2 \\ M_3 &= r_3 \times F_3 \end{aligned}$$

and the sum of the moments

$$\Sigma M = M_1 + M_2 + M_3 + \dots$$

tending to destroy the equilibrium by bringing about angular acceleration.

It is possible that the force system acting on a body can as well include some couples with couple-moments

$$C_1, C_2, C_3, \dots$$

in addition to the forces. In that case, the sum of the moments should include the couples provided the forces comprising the couples are not counted in their own right for producing moments.

$$\Sigma M = M_1 + M_2 + \dots + C_1 + C_2 + \dots \tag{3.2}$$

It follows, therefore, that the necessary conditions of equilibrium for a body are:

$$\begin{aligned} \Sigma F &= 0 \\ \Sigma M &= 0 \end{aligned} \tag{3.3}$$

These are, however, not the sufficient conditions for the equilibrium of a body.

In words, the sum of all the forces acting on the body should be zero and the sum of all the moments produced due to the force system about any point should also be zero for equilibrium. It may also be seen that each of these vector equations represents three scalar equations, i.e., one along each coordinate axis. In the rectangular system of coordinates, the equivalent scalar equations are:

$$\begin{aligned} \Sigma F_x &= 0 & \Sigma M_x &= 0 \\ \Sigma F_y &= 0 & \Sigma M_y &= 0 \\ \Sigma F_z &= 0 & \Sigma M_z &= 0 \end{aligned} \tag{3.3a}$$

If, in a given situation, a body which is acted upon by a system of forces is to be brought to a state of equilibrium it is achieved in two steps:

1. The system of forces is reduced to an equivalent force, its point of application as desired and the accompanying moment.
2. An equilibrant action equal and opposite to the corresponding equivalent action is applied in the desired manner:

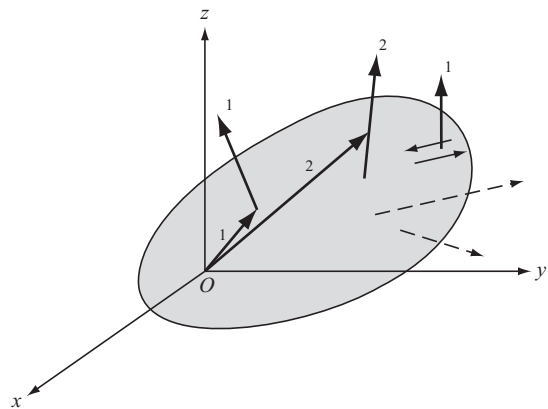


Fig. 3.1 Equilibrium under the action of a system of forces

$$E_f = -R = -\Sigma$$

$$E_m = -\Sigma$$

If the simplest equivalent for a particular case can be reduced to a single force R suitably applied, then the simplest equilibrant for that case

$$E = -R$$

may also be applied along the same line of action.

Example 3.1

The following forces as shown in Fig. Ex. 3.1 are applied to a rigid body initially at rest:

- $F_1 = 2 \mathbf{i} + 3 \mathbf{j} + 3 \mathbf{k}$ at $(7, 2, 3)$,
- $F_2 = -2 \mathbf{i} - 4 \mathbf{j}$ at $(5, 1, 0)$,
- $F_3 = -2 \mathbf{i} + 2 \mathbf{j} + 2 \mathbf{k}$ at $(4, 0, -1)$,
- $F_4 = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ at $(2, 2, 1)$

Show that the body is in equilibrium.

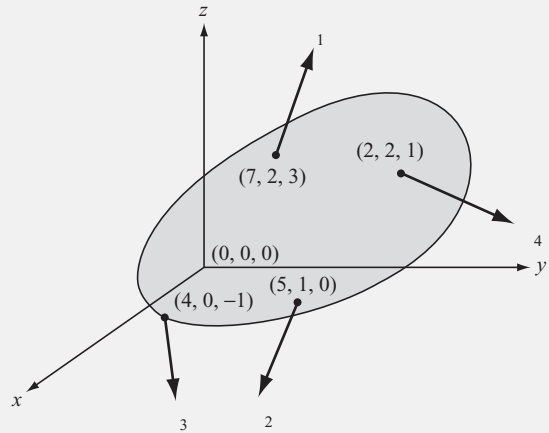


Fig. Ex. 3.1

Solution The necessary conditions of equilibrium of a body are

$$\Sigma F_x = 0 \text{ and } \Sigma F_y = 0. \text{ In this case,}$$

$$\Sigma F_x = (2 + -2 - 2) + (-2 + 2 - 2) + (3 - 4 + 2 - 1) = \text{zero, identically.}$$

Taking moments about the origin $(0, 0, 0)$,

$$\Sigma M_o = (7 \mathbf{j} + 2 \mathbf{k} + 3 \mathbf{i}) \times (2 \mathbf{i} + 3 \mathbf{j} + 3 \mathbf{k}) + (5 \mathbf{i} + 1 \mathbf{j} + 0 \mathbf{k}) \times (-2 \mathbf{i} - 4 \mathbf{j})$$

$$+ (4 \mathbf{i} + 0 \mathbf{j} - 1 \mathbf{k}) \times (-2 \mathbf{i} + 2 \mathbf{j} + 2 \mathbf{k}) + (2 \mathbf{i} + 2 \mathbf{j} + 1 \mathbf{k}) \times (-\mathbf{i} - \mathbf{j} - \mathbf{k}) = 0,$$

identically, which shows that the necessary conditions of equilibrium are identically satisfied. The body should, therefore, be in equilibrium.

It is interesting to observe that for a body in equilibrium, the summation of the moments about any arbitrary point must vanish. Let us take moments about the point $(2, 2, 1)$:

$$\Sigma M_{(2,2,1)} = ((7-2) \mathbf{j} + (2-2) \mathbf{k} + (3-1) \mathbf{i}) \times (2 \mathbf{i} + 3 \mathbf{j} + 3 \mathbf{k}) + ((5-2) \mathbf{i} + (1-2) \mathbf{j} + (0-1) \mathbf{k})$$

$$\times (-2 \mathbf{i} - 4 \mathbf{j}) + ((4-2) \mathbf{i} + (0-2) \mathbf{j} + (-1-1) \mathbf{k}) \times (-2 \mathbf{i} + 2 \mathbf{j} + 2 \mathbf{k})$$

$$= (-2 + 2 + 0) \mathbf{i} + (-11 + 11 + 0) \mathbf{j} + (5 - 5 + 0) \mathbf{k}$$

$$= \text{zero, identically, as expected}$$

3.2 FREE-BODY DIAGRAMS

The first step to solving a problem in mechanics is to draw the free-body diagram (*fbd*) of the body under study. It is pointed out in Chapter 1 that a free-body diagram results in showing all the external forces on the body, when isolated from its surroundings. Typically, an *fbd* consists of

- the weight of the body shown acting at a point
- the external applied forces and moments
- the reactions of the surroundings, from which it is isolated
- the reactions of the supports, from which it is also isolated; tension in the string, if any

What is the idea of drawing an fbd?

It is the *fbd* which shows all the forces (and moments) that a body is subjected to, whether it is in equilibrium (i.e., at rest or in uniform motion) or in a state of acceleration. Hence, the *fbd* shows the action of all the forces (and moments) on a body, which is the starting point of all analysis.

Let us consider a case of two cylinders 1 and 2 resting in a V-groove, as shown in Fig. 3.2(a).

The *fbd* of the upper cylinder 1 is shown in Fig. 3.2(b). It shows the weight of the cylinder and the reactions from the groove at *A* and from the other cylinder 2 because these are the external forces acting on it.

The *fbd* of the lower cylinder 2, shown in Fig. 3.2(c) consists of its weight and reactions from the surroundings.

Now, the *fbd* of the two cylinders, taken together as one subsystem, would be as shown in Fig 3.2(d). Notice that the actions and reactions between the two cylinders do not come up as the external forces as they mutually cancel!

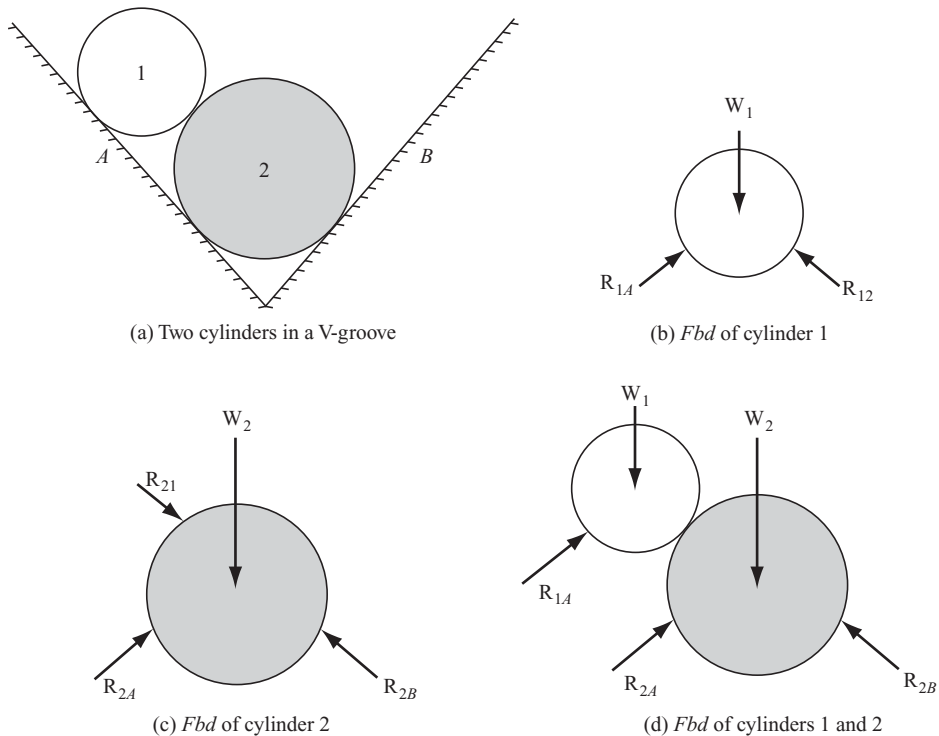


Fig. 3.2 Two cylinders resting in a V-groove

Some further examples of *fbd* are illustrated in Fig. 3.3. The *fbd* of a ladder on a rough floor but smooth wall is shown with a component of frictional force at the floor, tending to resist slippage. The *fbd* of the beam is drawn to show the vertical reactions at the knife-edge support and the roller support. The *fbd* of the submerged body is shown in the hydrostatic forces.

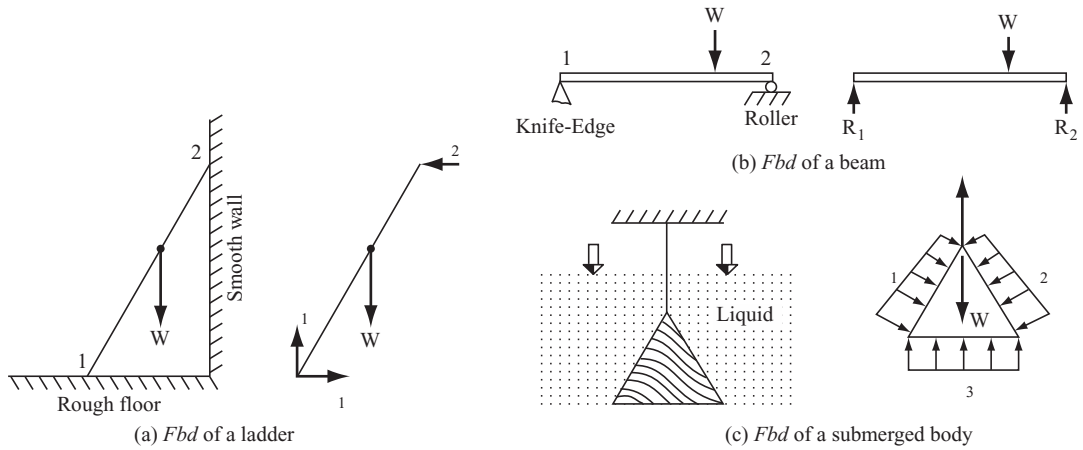


Fig. 3.3 Examples of free-body diagrams of objects

3.3 REACTIONS BY SUPPORTS

Different types of supports are employed to hold structural members and components in motion. The purpose of a support is to provide a desirable reaction. Let us see the nature of reactions offered by different types of supports as shown in Fig. 3.4.

| Type of support | Reaction |
|--|---|
| <p>Ball or roller Rough or Smooth</p> <p>(a) Normal to the surface at that point</p> | <p>$R =$ 90° O</p> |
| <p>Rough O</p> <p>(b) With normal and tangential components</p> | <p>O</p> |
| <p>Pin joint Hinge</p> <p>(c) Vertical and horizontal components</p> | <p>O</p> |
| <p>Ball and socket</p> <p>(d) Along the member in space (three components)</p> | <p>O</p> |

Fig. 3.4 (Continued)

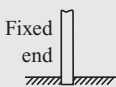

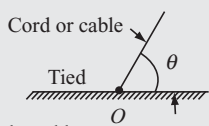
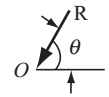
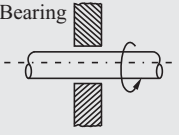
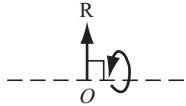
| Type of support | Reaction |
|--|--|
|  <p>(e) A moment and a force depending upon loading</p> |  |
|  <p>(f) Tension in the cable</p> |  |
|  <p>(g) A normal force and a twisting moment</p> |  |

Fig. 3.4 Nature of Reactions by some Typical Supports

A ball or roller support on a rough surface can roll and hence not provide a tangential reaction. The reaction at a ball or roller support is, therefore, normal to the surface at that point. Similarly, the reaction by a smooth surface to a member in contact with it must be normal to the surface.

Reaction by a rough surface to an element in contact with it can be non-normal, i.e., at an angle θ to it even if the member is inclined at any angle α or normal to it. This is because both normal and tangential components of reaction are possible.

A pin-joint or a hinge gives rise to reaction along the axis of the member.

A ball-and-socket joint provides reaction along the axis of the member. Since a ball-and-socket joint is a universal joint in space, the reaction is also in space.

A fixed end of a member is capable of providing a general reaction R and a general moment M .

A cord or a cable tied to a surface can be under tension only, the reaction must be along the cord or the cable.

A bearing of a rotating shaft can provide a reaction R and a frictional moment M .

Example 3.2

Draw the free-body diagrams of all the members of a simple-loaded system sketched in Fig. Ex. 3.2 and comment on their usefulness in the analysis of the system.

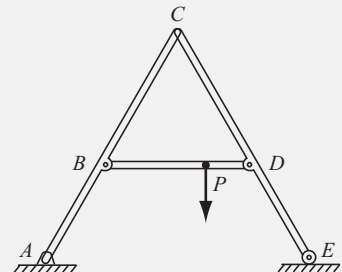


Fig. Ex. 3.2

Solution First, the free-body diagram of the centre system is drawn in Fig. Ex. 3.2(a) Solution. This is necessary to estimate the reactions at the supports A and E . The unknowns R_A and R_E can be determined by considering the equilibrium of the system.

Free-body diagrams of the members are drawn in Fig. Ex. 3.2 (Solution) (b) and (c).

In Fig. Ex. 3.2 (Solution) (b) all the reactions are taken in the direction of the positive coordinate axes. By doing so we get 15 reaction unknowns, viz. $x, y, B_x, B'_x, B_y, B'_y$, etc. These 15 unknowns can be solved by using three equilibrium equations for each member and 2 action-reaction equations at each joint of two members. (Total number of equations = $3 \times 3 + 2 \times 3 = 15$.)

Though such a procedure leads to the correct solution of the structure, it is not a normal practice to assume all the reactions to be initially positive in a free-body diagram.

While drawing the direction of these forces in free-body diagram, the following points are considered:

1. The action-reaction equations are applied directly while drawing the free-body diagram itself. For example in Fig. (c), the action-reaction principle gives that at joint C, $x = -x', y = -y'$. These two equations are eliminated from the calculations and x', y' are shown equal and opposite to x, y respectively in the free-body diagram itself. By doing this in the present case at every point we are left to analyse the structure for 9 equilibrium equations only.
2. It is a usual practice to assume the direction of the reactions intuitively rather than taking them arbitrarily. For example, in Fig. (c) on the member BD , B_y and y should act in upward direction in order to balance y . Similarly intuitively we can see that y, E_y should act upward and x should act to the left on CA . However, a later check by the equilibrium equations decides the validity of these assumptions.

It must also be noted that free-body diagrams of the members are not mutually independent and hence it is, in general, not possible to determine the unknowns on a member by considering the equilibrium of that member alone. For example, suppose after finding R_A and R_E , we wish to determine the reactions at B, C and D, i.e., B_x, B_y, x, y, x and y . There are four unknowns on the members ABC considered alone and also four unknowns on the member CDE considered alone, but taken together as $ABCDE$, there are again only four unknowns. Similarly, on member BD alone, there are four unknowns. It may not be possible to solve for the unknowns by considering the equilibrium of each member one by one, but it must be possible to determine all the unknowns by writing all the equations of equilibrium for all the members.

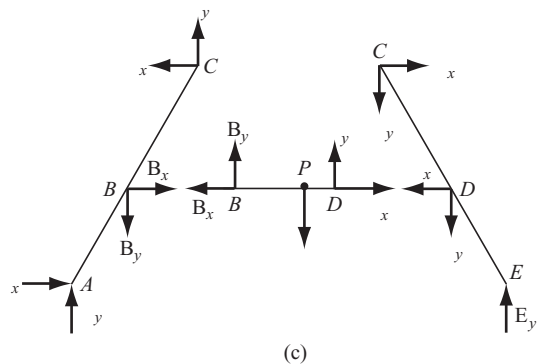
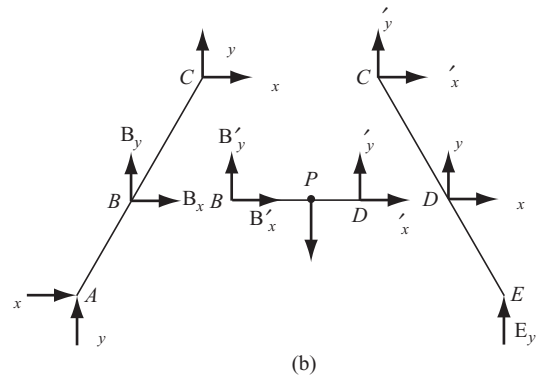
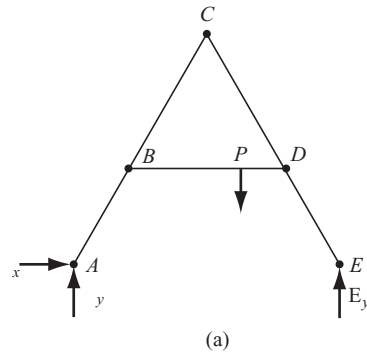


Fig. Ex. 3.2 (Solution)

3.4 EQUILIBRIUM OF A PARTICLE

A particle, by definition, has negligible dimension in comparison with the coordinates describing its motion. It is an idealisation of a real body when its mass can be considered to be concentrated at a central point. A particle can, therefore, experience a system of forces which must be concurrent at the particle. The concurrent system of forces can act along a common line to constitute collinear forces or they can belong to a plane or a spatial system as shown in Fig. 3.5. In each case, the condition of equilibrium must be

$$\Sigma = 0$$

Since there is a point through which all the forces pass, it follows that the summation of the moments of all the forces about that point vanishes. The summation of the moments of the concurrent forces about any other point in space must also be zero. The other condition

$$\Sigma = 0$$

is, therefore, automatically satisfied by the state of concurrency of the forces. As a matter of fact, the conditions of equilibrium, namely

$$= 0 \quad \text{and} \quad = 0$$

are alternative conditions for a concurrent force system; one implies the other also. Hence, it is a matter of choice whether to use one or the other.

For equilibrium under the action of coplanar concurrent forces,

$$\Sigma = 0$$

which implies that

$${}_{1x} + {}_{2x} + \dots = 0; \quad {}_{1y} + {}_{2y} + \dots = 0$$

In words, the sums of the components of the forces along any two mutually perpendicular directions in the plane of the forces must be zero for the equilibrium of a particle under the action of coplanar concurrent forces. The polygon of forces should automatically close to provide the zero resultant force for equilibrium.

It is interesting to observe that *two concurrent forces must be coplanar* since a plane can always be passed through two intersecting straight lines. Further, *three concurrent forces in equilibrium must be coplanar*, for if the third force was not in the plane of the other two it would result in a component normal to the plane and upset the equilibrium. However, four concurrent forces in equilibrium may be coplanar or spatial. Three concurrent forces F_1, F_2 and F_3 which maintain the particle in equilibrium as shown in Fig. 3.5(c) are related by the sine law.

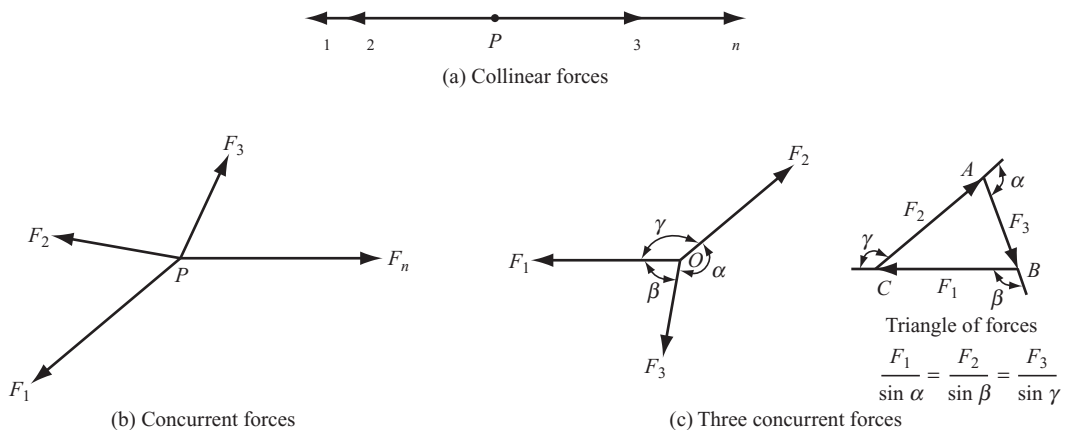


Fig. 3.5 Equilibrium of a particle

Lami's Theorem

Lami's theorem states that, if three forces acting on a particle keep it in equilibrium, then each force is proportional to the sine of the angle between the other two and the constant of proportionality is the same. Symbolically,

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma} = k \tag{3.4}$$

where α, β and γ are the angles between F_2, F_3 ; F_1, F_3 and F_1, F_2 respectively. The triangle of forces, in this case, should close to provide the resultant force zero for equilibrium and by applying the sine-rule for a triangle. Lami's theorem may be obtained as follows:

For the triangle ABC shown in Fig. 3.5(c) corresponding to the forces F_1, F_2 and F_3 , acting at a point O ,

$$\begin{aligned} \angle CAB &= 180^\circ - \alpha \\ \angle ABC &= 180^\circ - \beta \\ \angle BCA &= 180^\circ - \gamma \end{aligned}$$

From the sine rule for the triangle,

$$\frac{F_1}{\sin (180^\circ - \alpha)} = \frac{F_2}{\sin (180^\circ - \beta)} = \frac{F_3}{\sin (180^\circ - \gamma)}$$

and from the fact that $\sin (180^\circ - \alpha) = \sin \alpha$, etc., it reduces to the Lami's theorem

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

For equilibrium, under the action of spatial concurrent forces,

$$\Sigma \mathbf{F} = 0$$

which implies that

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned}$$

In words, the sums of the components of the forces along the three coordinate directions must be zero for the equilibrium of a particle under the action of spatial concurrent forces.

Let us discuss another aspect of the state of a particle. If a particle is subjected to a system of concurrent forces

$$F_1, F_2, F_3, \dots$$

then it may or may not be in equilibrium. If it is in equilibrium, then

$$\Sigma \mathbf{F} = F_1 + F_2 + F_3 + \dots = 0$$

If it is not in equilibrium, then the system reduces to a resultant force

$$R = F_1 + F_2 + F_3 + \dots$$

which represents the net external force on the particle. In case it is desired to bring the particle in equilibrium, provision must be made to apply a force equal and opposite to R . The force which, when applied on it brings it to a state of equilibrium, is called the *equilibrant force* denoted by E

$$E = -R \tag{3.5}$$

Example 3.3

A body weighing 800 N is hung from a horizontal ring 6 m in diameter by means of three cords, each 5 m long. On the ring, two of the cords are placed 90° apart and the point of attachment of the third cord bisects the remaining arc of the ring. Find the tension in each cord.

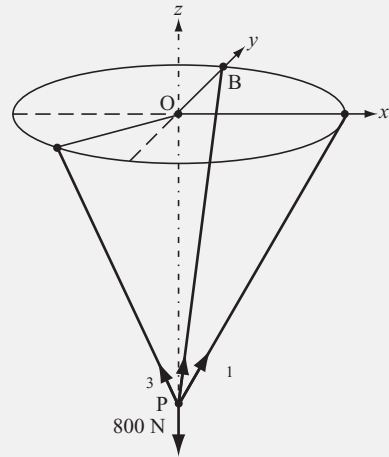


Fig. Ex. 3.3

Solution With reference to Fig. Ex. 3.3 where the body is represented by a particle P,

$$A = 3, \quad B = 3, \quad C = -3/\sqrt{2} - 3/\sqrt{2}, \quad P = -5$$

From these vectors

$$PA = 3i + 5j, \quad PB = 3j + 5k, \quad PC = -2.12i - 2.12j + 5k$$

and the unit vectors along these cords are given by

$$u_{PA} = (3i + 5j) / \sqrt{3^2 + 5^2} = 0.51i + 0.86j$$

$$u_{PB} = 0.51j + 0.86k$$

$$u_{PC} = -0.36i - 0.36j + 0.86k$$

Let the tension in these cords be F_1, F_2 and F_3 respectively in magnitude. Vectorially,

$$F_1 = F_1(0.51i + 0.86j)$$

$$F_2 = F_2(0.51j + 0.86k)$$

$$F_3 = F_3(-0.36i - 0.36j + 0.86k)$$

For equilibrium of the particle P,

$$F_1 + F_2 + F_3 - 800j = 0 \text{ is fine.}$$

whence

$$0.51F_1 - 0.36F_3 = 0$$

$$0.51F_2 - 0.36F_3 = 0$$

$$F_1 + F_2 + F_3 = 800/0.86 = 930$$

and solving these equations,

$$F_1 = 272 \text{ N}, \quad F_2 = 272 \text{ N} \text{ and } F_3 = 386 \text{ N}$$

EXPERIMENT E1

EQUILIBRIUM UNDER COPLANAR FORCES

OBJECTIVE TO STUDY THE EQUILIBRIUM OF A PARTICLE UNDER THE ACTION OF FORCES IN A PLANE.

Apparatus A horizontal circular force table, also called a vertical rectangular force table, also called standard weights and metre rod.

, as shown in Fig. E 1.1(a), or , as shown in Fig. E1.1(b),

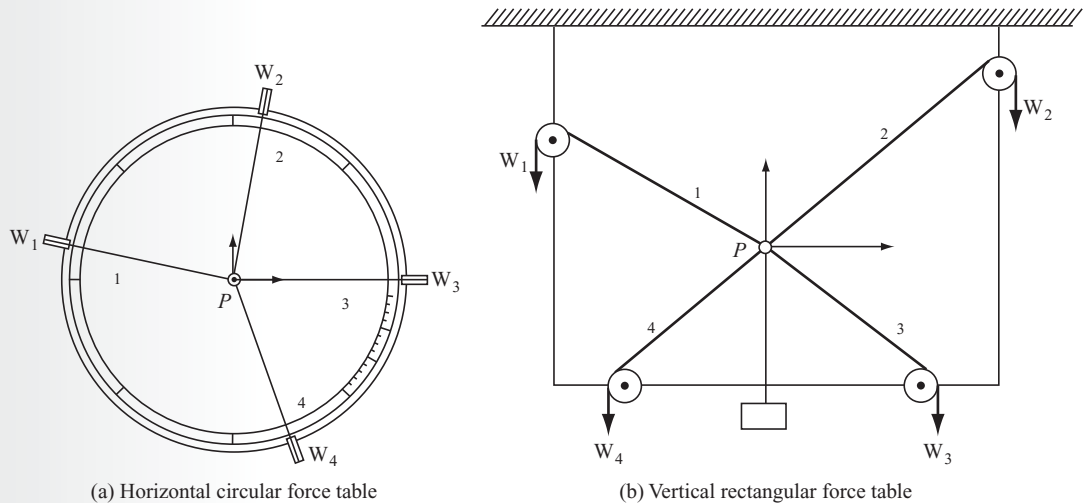


Fig.E1.1 Force table

Background Information The state of equilibrium of a particle refers to a state of uniform velocity of rest. In the present case, it is intended to study the forces acting on a particle when it is at rest.

A particle cannot be in equilibrium when a single force is applied on it. It would be in equilibrium under the action of two or more forces if the vectorial summation of forces is zero

$$\Sigma \vec{F} = 0 \tag{E1.1}$$

In particular, if a particle is subjected to only two forces, the forces must be equal and opposite, i.e., equal in magnitude, in the same line of action but opposite in sense in order to keep it in equilibrium. If it is subjected to only three forces, the three forces must be coplanar for equilibrium. Conversely, three non-coplanar forces cannot keep a particle in equilibrium.

This fact can be appreciated by recognising that the resultant of any two must be equal and opposite to the third force for equilibrium, and this cannot happen unless the three forces lie in one and the same plane. If a particle is in equilibrium under the action of four or more forces, the forces may be spatial, i.e., not necessarily confined to act along the same line or in the same plane.

When a particle is in equilibrium under the action of three forces F_1 , F_2 and F_3 , as shown in Fig. E1.2, the condition of equilibrium, i.e.,

$$\Sigma \vec{F} = 0; \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

may alternatively be expressed as Lami's theorem or Triangle Law of Equilibrium.

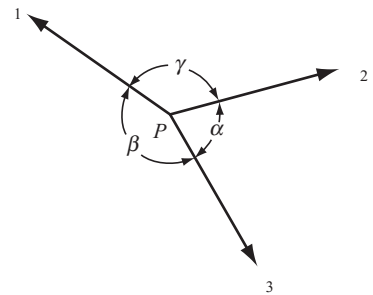


Fig.E1.2 A particle in equilibrium

Lami's Theorem If a particle is in equilibrium under the action of three forces, each force must bear the same proportionality with the sine of the angle between the other two forces.

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

Triangle Law of Equilibrium If a particle is in equilibrium under the action of three forces, the forces must be represented in magnitude, direction and sense by the sides of a triangle, taken in order, in the same sense.

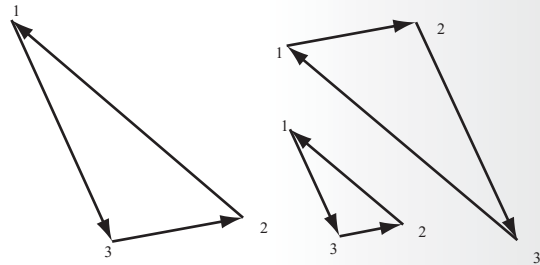


Fig. E1.3 Possible triangles to represent force F_1 , F_2 and F_3

When a particle is in equilibrium under the action of more than three coplanar forces, the condition of equilibrium i.e.,

$$\Sigma F = 0; F_1 + F_2 + F_3 + F_4 + \dots = 0$$

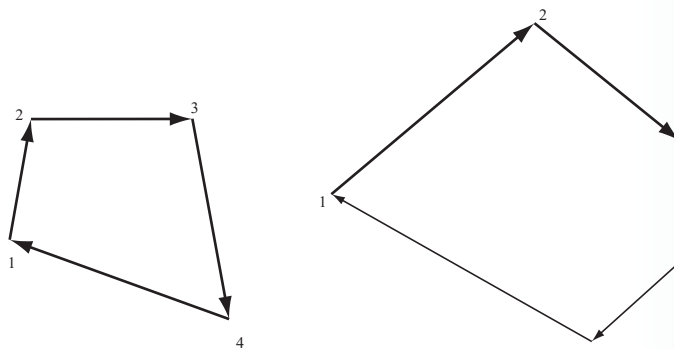
may alternatively be stated in terms of the *polygon law of equilibrium*.

If a particle is in equilibrium under the action of n coplanar forces, the forces must be represented in magnitude, direction and sense by the n sides of a polygon, taken in order, in the same sense.

Observations and Calculations The table of observations and calculations depends precisely on the motivation of the experiment which may be one or more of the following:

(a) To demonstrate the triangle law or polygon law of equilibrium

In this case, a particle may be subjected to as many forces as desired on either force table and the positions of the strings together with the loads suspended at the corresponding endpoints are noted. The forces, in magnitude and direction, are drawn end to end, in order to verify whether a closed triangle or polygon is formed. If so, within the allowable limits of error, the corresponding law is verified; if not, the sources of error must be recognised and minimised to allow a closer prediction by the law.



(a) Polygon of forces for Fig. E.1.1(a)

(b) Polygon of forces for Fig. E.1.1(b)

Fig. E1.4 Polygon law of equilibrium

(b) To demonstrate the application of Lami's theorem

In this case, only three forces must be made to act on the particle on either force table. The three forces must automatically be coplanar for maintaining equilibrium.

Measurements must be made for the directions of the three strings and the loads applied at the ends. The included angles between every pair of forces are calculated and, as per Fig. E3.2, the following constants are calculated:

$$k_1 = \frac{F_1}{\sin \alpha}; \quad k_2 = \frac{F_2}{\sin \beta}; \quad k_3 = \frac{F_3}{\sin \gamma}$$

If k_1 , k_2 and k_3 are equal within the allowable limits of error, the validity of the Lami's theorem is upheld; if not efforts must be made to recognise the sources of error and to improve upon the result.

(c) *To determine the two unknown loads hanging at the ends of two strings*

In this case, Lami's or triangle law/polygon law of equilibrium is taken for granted to be valid and, from the knowledge of the other forces, the unknown forces are determined. The procedure is to record all the directions of the strings and all the known loads. Then apply Lami's theorem or use the triangle law if the number of total forces is three or use the polygon law if the number of forces exceed three. It will be seen that, with all the directions known and with magnitudes for two less than all the forces known, it is possible to complete the polygon. The unknown forces are then read off from the corresponding sides of the closed polygon to the same scale as the other forces.

(d) *To demonstrate the validity of the condition $\Sigma = 0$ for a given case of equilibrium*

In this case, the measurements of the inclinations of the strings may be made with reference to x - and y -axis selected arbitrarily. The corresponding forces in the strings are known from the loads applied at the ends. The forces are then resolved into x and y components. Summation of the x components and of the y components are made separately and observed whether the results are close to zero or not, i.e.,

$$\begin{aligned} \Sigma x &= x_1 + x_2 + x_3 + \dots \\ \Sigma y &= y_1 + y_2 + y_3 + \dots \end{aligned}$$

If the summations Σx and Σy are each close to zero, the experimental values obey the condition of equilibrium $\Sigma = 0$, if not the sources of error must be minimised.

Result The results, whether or not a theorem or law holds for the situations examined, should be recorded. Discrepancies, if any, may be mentioned.

Points for Discussion

State the sources of error in the experiment. Notice if the pulleys have friction, if the threads have knots, if the central ring and the strings are coplanar, if the strings are tied to the ring radially, if the weights are standard, if the graduations are uniform, etc.

The condition of equilibrium $\Sigma = 0$ is valid in general. The graphical conditions, i.e., the triangle law and polygon law of equilibrium apply only when the forces are coplanar. Can you suggest how the graphical method may be extended to be employed for spatial forces?

When the experimental values differ from the theoretical and graphical values, which of them must be in error and why?

Should the condition $\Sigma = 0$ hold good for the equilibrium of a rigid body if it is subjected to a system of forces which are

- concurrent
- non-concurrent
- coplanar and
- spatial

Discuss why the answer is yes in all these cases.

What happens to the point P if a weight, say W_1 , is increased by 10%? What needs to be done to bring it back to equilibrium?

Example 3.4

Four pieces of string knotted at A support two equal masses in equilibrium in a vertical plane as shown from pulleys at D and E . Determine the tensions in the strings AB and AC and the angle θ between AB and AE for minimum tension in AB .

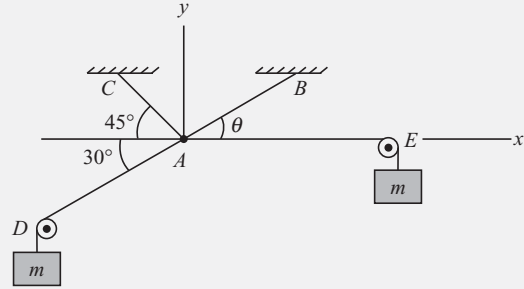


Fig. Ex. 3.4

Solution As shown in Fig. Ex. 3.4 the tension in AE must be mg ; so also the tension in AD must be mg , since the pulleys are assumed to be frictionless.

Let the x -axis be along AE and y -axis perpendicular to it. Then,

$$\begin{aligned} F_{AE} &= mg \\ F_{AD} &= -mg \cos 30^\circ - mg \sin 30^\circ \\ &= -\frac{\sqrt{3}}{2} mg - mg/2 \\ F_{AB} &= F_{AB} (\cos \theta + \sin \theta) \\ F_{AC} &= F_{AC} (-\cos 45^\circ + \sin 45^\circ) = -\frac{F_{AC}}{\sqrt{2}} + \frac{F_{AC}}{\sqrt{2}} \end{aligned}$$

For equilibrium at A , $\Sigma F_x = 0$

or

$$F_{AB} + F_{AC} + F_{AD} + F_{AE} = 0$$

$$mg - \frac{\sqrt{3}}{2} mg + F_{AB} \cos \theta - \frac{F_{AC}}{\sqrt{2}} = 0 \tag{i}$$

$$-mg/2 + F_{AB} \sin \theta + \frac{F_{AC}}{\sqrt{2}} = 0 \tag{ii}$$

These are 2 equations for the three unknowns F_{AB} , F_{AC} and θ . The third equation comes up from the physical constraint, i.e., the force in AB must be the minimum.

From (i),

$$F_{AB} \cos \theta - \frac{F_{AC}}{\sqrt{2}} = \left(\frac{\sqrt{3}}{2} - 1 \right) mg$$

and from (ii),

$$F_{AB} \sin \theta + \frac{F_{AC}}{\sqrt{2}} = \frac{mg}{2}$$

Adding the two,

$$F_{AB} (\cos \theta + \sin \theta) = \frac{(\sqrt{3} - 1)}{2} mg$$

Minimum F_{AB} corresponds to maximum $(\cos \theta + \sin \theta)$

i.e.,

$$\begin{aligned} \frac{d}{d\theta} (\cos \theta + \sin \theta) &= 0 \\ -\sin \theta + \cos \theta &= 0 \end{aligned}$$

or $\sin \theta = \cos \theta$, i.e., $\theta = 45^\circ$

Now,
$$F_{AB} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{(\sqrt{3} - 1)}{2} mg$$

whence,
$$F_{AB} = \frac{(\sqrt{3} - 1)}{2\sqrt{2}} mg \cong \underline{0.26 mg}$$

Now,
$$0.26 mg \cdot \frac{1}{\sqrt{2}} + \frac{F_{AC}}{\sqrt{2}} = \frac{mg}{2}$$

$$F_{AC} = \underline{0.45 mg}$$

3.5 EQUILIBRIUM OF A RIGID BODY

A rigid body may be subjected to one of the force systems classified as follows:

- (a) Concurrent force system: Collinear, plane or spatial
- (b) Parallel force system: plane or spatial
- (c) Coplanar force system: concurrent and non-concurrent, parallel and non-parallel
- (d) Spatial force system: concurrent and non-concurrent, parallel and non-parallel.

(a) Concurrent Force System

The analysis of the static equilibrium of a rigid body under the action of concurrent forces is quite similar to that of a particle. Concurrent forces may be collinear, coplanar or spatial and the vector method or the algebraic method can be employed with advantage.

If there are two forces acting at a point in equilibrium, the forces must be collinear. If there are three forces acting at a point in equilibrium, the forces must lie in a plane, i.e., the force system must be coplanar. This follows from the fact that any two lines of forces acting at a point must constitute a plane and the third force cannot have a component normal to that plane; otherwise that unbalanced component would upset the equilibrium. If there are four or more forces acting at a point in equilibrium, these may be coplanar or spatial.

The condition of equilibrium for a rigid body under the action of concurrent forces

$$1, 2, 3, \dots$$

is that their resultant

$$R = \Sigma = 1 + 2 + 3 + \dots$$

must vanish, i.e.,

$$R = \Sigma = 0$$

This vector equation for equilibrium stands for three scalar equations for a general concurrent force system:

$$\begin{aligned} R_x = \Sigma x = 0 &= 1_x + 2_x + 3_x + \dots \\ R_y = \Sigma y = 0 &= 1_y + 2_y + 3_y + \dots \\ R_z = \Sigma z = 0 &= 1_z + 2_z + 3_z + \dots \end{aligned}$$

where x , y and z are the coordinate axes arbitrarily drawn through the point of concurrency as shown in Fig. 3.6(a).

If, on the other hand, there is a rigid body subjected to a system of concurrent forces resulting in R , then an equilibrant force E given by

$$E = -R$$

must be applied passing through the point of concurrency in order to bring the body to equilibrium.

(b) Parallel Force System

If a rigid body is subjected to a parallel force system, the resultant of the forces may be a non-zero force or a zero force, unaccompanied or accompanied by a couple-moment. Equilibrium of the rigid body demands that the resultant force and resultant moment must vanish:

$$\Sigma F = 0; \quad \Sigma M = 0 \quad (3.6)$$

It may be understood that the moment about any arbitrary point O as shown in Fig. 3.6(b) may be considered and should be equated to zero.

For a plane parallel-force system, the moment of the forces about any point in the plane of the forces must be perpendicular to the plane of the forces. It is interesting to note that moment summation about different points provide different equations, such as

$$\Sigma M_1 = 0, \quad \Sigma M_2 = 0, \quad \Sigma M_3 = 0, \quad \dots$$

In fact, there can be only two of these equations mutually independent; one in its own right and the other in lieu of

$$\Sigma F = 0$$

It follows that for a system of plane parallel forces acting on a body, the conditions of equilibrium may alternatively be stated as

$$\Sigma M_1 = 0 \quad \text{and} \quad \Sigma M_2 = 0 \quad (3.7)$$

where 1 and 2 are two suitably chosen points.

(c) Coplanar Force System

If the forces applied on a rigid body are such that their lines of action lie in the same plane and the moments due to the couples or otherwise are directed perpendicular to the plane, the body is said to be subjected to a coplanar force system. It is usual to choose the x - y plane in the plane of the coplanar force system and the x - and y -axes are chosen conveniently in regard to the directions of the forces as shown in Fig. 3.6(c). The necessary conditions of equilibrium reduce to a set of three equations:

$$\begin{aligned} \Sigma F_x &= 0 \quad \text{and} \quad \Sigma M = 0 \\ \Sigma F_y &= 0 \end{aligned} \quad (3.8)$$

The moment summation referred to above is about any point in the plane of the forces; it is about the z -axis through the chosen point.

An interesting and extremely useful point is to express the force summations of equilibrium in terms of equivalent moment summations. The advantage of doing so is that the moments can be taken about the line of action of a force which is unknown and needs to be eliminated at least temporarily. For example, if there are three unknown coplanar forces in a system, then moments about the line of action of each force, in turn, will yield three moment equations.

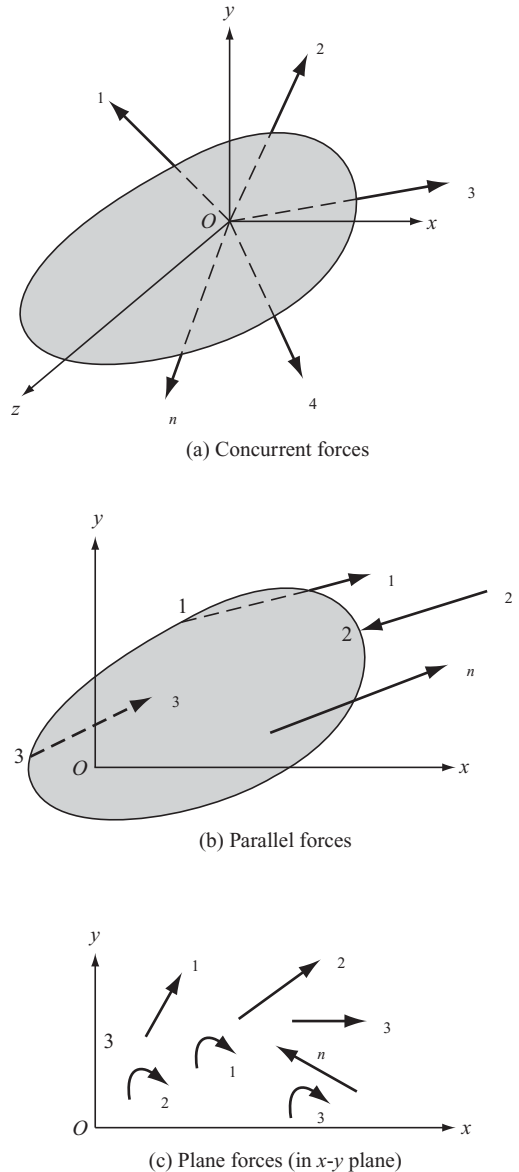


Fig. 3.6 Different force systems acting on a rigid body

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned} \tag{3.9}$$

which comprise a set equivalent to Eq. (3.8).

It may as well be decided to consider the equivalent equations of equilibrium as

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned} \tag{3.10}$$

In each case, there are, in essence, two force equations and one moment equation; the apparent difference is only in the embodiment of these equations. It may so happen that the three equations set up in a particular case may not form an independent or a complete set, e.g., when a moment is taken about the point of intersection of two or more lines of forces. In such cases further equations for moment summations would provide an answer. It may also be noted that it is often advisable to set up a redundant equation to provide a check on the solution of the problem.

(d) Spatial Force System

The necessary conditions of equilibrium for a rigid body subjected to a general force system are those specified earlier by general equilibrium considerations

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \end{aligned}$$

These two vector equations are equivalent to a set of six scalar equations:

$$\begin{aligned} \Sigma F_x &= 0 & \Sigma M_x &= 0 \\ \Sigma F_y &= 0 & \Sigma M_y &= 0 \\ \Sigma F_z &= 0 & \Sigma M_z &= 0 \end{aligned} \tag{3.11}$$

where the x , y and z axes are chosen arbitrarily but with due regard to convenience of handling the force system. For example, it may be preferable to have an axis in a direction in which a number of forces act and it may be advantageous to choose the x - y plane as the plane in which a number of forces lie.

3.6 EQUILIBRIUM OF A SYSTEM OF PARTICLES

Consider a system consisting of three particles P_1 , P_2 and P_3 as shown in Fig. 3.7. The system is subjected to net external forces F_1 at P_1 , F_2 at P_2 and F_3 at P_3 as shown.

The resultant action on the system of particles consists of a single force R equal to the sum of the external forces.

$$R = F_1 + F_2 + F_3$$

The system will be in equilibrium if the resultant of the external forces vanishes, i.e.,

$$R = F_1 + F_2 + F_3 = 0$$

or F_1 , F_2 and F_3 constitute a closed triangle.

It is essential that each constituent particle of a system must also be in equilibrium. Take, for example, particle P_1 . It is subjected to the external force F_1 as well as the internal forces, F_{13} due to particle P_3 and F_{12} due to particle P_2 . Obviously, F_1 , F_{12} and F_{13} must keep the particle in equilibrium.

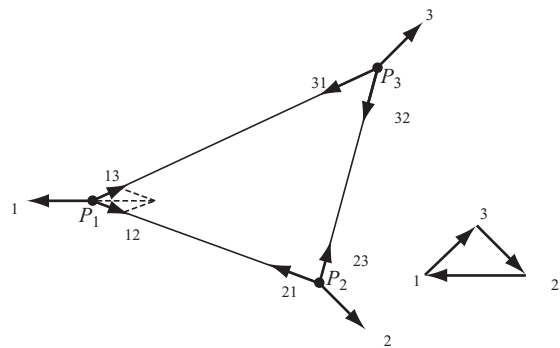


Fig. 3.7 A system of three particles

Since action and reaction must be equal and opposite

$$\begin{aligned} 12 &= -21; & 12 + 21 &= 0 \\ 13 &= -31; & 13 + 31 &= 0 \\ 23 &= -32; & 23 + 32 &= 0 \end{aligned}$$

It follows that the sum of all internal forces in a system must be zero.

Extending the argument to a system of a number of particles, say n , the equilibrium demands that

1. The vector sum of all external forces is zero

$$\Sigma e = 0$$

2. The vector sum of all external plus all internal forces must be zero

$$\Sigma e + \Sigma i = 0$$

in order that each particle be in equilibrium separately.

It is interesting to note that a system of three particles must lie in a plane and the forces on the particles must constitute a coplanar force system for equilibrium. A system of four or more particles may not be coplanar and may constitute spatial force systems for equilibrium.

General Comments

A number of rigid bodies may be interconnected to comprise a system. In such a case, if the total system of rigid bodies is in equilibrium, every subsystem and every component of the system must also be in equilibrium.

It follows, therefore, that the necessary conditions for equilibrium may be written for the total system as well as for any desired subsystem or a component of the system. For example, if two rigid bodies are connected by an inextensible string which is kept taut, then the net external forces and moments should satisfy the conditions of equilibrium as well as the net external forces and moment, as observed from the free-body diagram of each body, should also satisfy the conditions of equilibrium separately.

Example 3.5

Three homogeneous cylinders of the same material having masses m , $2m$ and m are placed in a container with a curved base as shown in Fig. Ex. 3.5. Determine the reaction on cylinder P from the left wall, upper cylinder and curved base. Assume that the cylinders are of equal length and that they are placed such that their centres of gravity lie in the same vertical plane. The radius of the curved base is three times the radius of cylinder P .

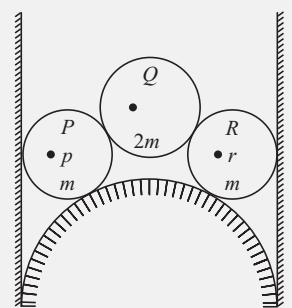


Fig. Ex. 3.5

Solution By geometry of the system,

$$op = 3r + r = 4r$$

$$pc = 3r - r = 2r$$

$$oc = \sqrt{(4r)^2 - (2r)^2} = 3.464r$$

$$p = r + \sqrt{2} r = 2.414r$$

$$c = \sqrt{2.414^2 - 2^2} r = 1.35r$$

and

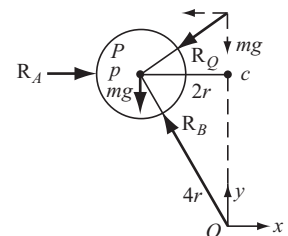


Fig. Ex. 3.5 (Solution)

because the radius of the top cylinder is $\sqrt{2} r$ from the ratio of their masses, i.e., given by the expression

$$\sqrt{\frac{2 \cdot \pi r^2 l \rho}{\pi l \rho}}$$

The free-body diagram of the cylinder P shows the following forces acting on it:

Reaction R_A by the left vertical wall; R_A must be horizontal.

Reaction R_B from the supporting curved surface; R_B must be directed along op .

Reaction R_Q by the top cylinder Q ; R_Q is directed along p .

Weight of the cylinder itself; acting downwards.

From the equilibrium consideration of the cylinder P ,

$$R_A + R_B + R_Q + mg = 0$$

or
$$R_A \mathbf{i} - R_B \frac{2r}{4r} \mathbf{i} + R_B \frac{3.464r}{4r} \mathbf{j} - R_Q \frac{2r}{2.414r} \mathbf{i} - R_Q \frac{1.35r}{2.414r} \mathbf{j} - mg \mathbf{j} = 0$$

whence,

$$R_A - 0.5 R_B - 0.829 R_Q = 0 \tag{i}$$

and

$$0.866 R_B - 0.56 R_Q - mg = 0 \tag{ii}$$

It is also known that the vertical component of R_Q must be equal to half the weight of the top cylinder;

$$0.56 R_Q = mg$$

or

$$R_Q = 1.79 mg \tag{iii}$$

Substituting R_Q from (iii) in (ii),

$$R_B = \frac{2 mg}{0.866} = 2.31 mg$$

and substituting these values in (i),

$$R_A = (0.5 \times 2.31 + 0.829 \times 1.79) mg = 2.64 mg$$

The reactions R_A , R_B and R_Q are, therefore, given by

$$R_A = 2.64 mg$$

$$R_B = (-0.5 + 0.866) \times 2.31 mg = -1.155 mg + 2 mg$$

$$R_Q = (-0.829 - 0.56) \times 1.79 mg = -1.484 mg - mg$$

Example 3.6

Three identical cylinders, each weighing W , are stacked, as shown in Fig. Ex. 3.6, on smooth inclined surfaces, each inclined at an angle θ with the horizontal. Determine the smallest angle θ to prevent the stack from collapsing.

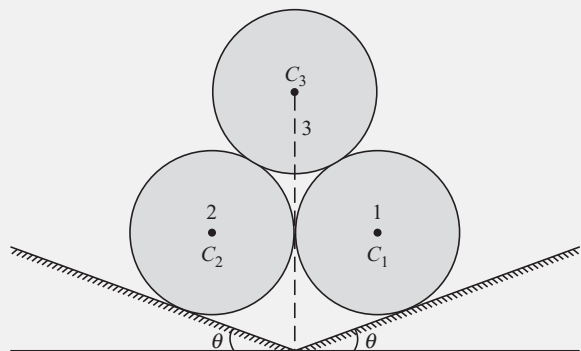


Fig. Ex. 3.6

Solution The free-body diagrams are drawn (Fig. Ex. 3.6 (Solution)) for cylinders 1 and 3 in order to understand the forces acting on each of them. The limiting case of collapse of the stack implies that the reaction between cylinder 2 and cylinder 1 becomes zero

$$R_{21} = 0$$

From the equilibrium of the cylinder 3 with regard to its free-body diagram,

$$\Sigma_y = 0; \quad R_{13} \cos 30^\circ + R_{23} \cos 30^\circ - W = 0$$

$$\Sigma_x = 0; \quad R_{23} \sin 30^\circ - R_{13} \sin 30^\circ = 0$$

whence
$$R_{13} = R_{23} \frac{W}{2 \cos 30^\circ}$$

From the equilibrium of cylinder 1 with regard to its free-body diagram,

$$\Sigma_y = 0; \quad R_s \cos \theta - W - R_{31} \cos 30^\circ = 0$$

$$\Sigma_x = 0; \quad R_{31} \sin 30^\circ - R_s \sin \theta = 0$$

Substituting

$$R_{31} = R_{13} = \frac{W}{2 \cos 30^\circ}$$

Simplifying and solving for θ ,

$$\tan \theta = \frac{\tan 30^\circ}{3} = \frac{0.577}{3} = 0.1924$$

and $\theta = 10.9^\circ$

It shows that if the inclined surfaces are frictionless, the angles between the surfaces and the horizontal line should not be less than 10.9° . If it is less, the reaction R_s will not be able to provide a horizontal resisting component to balance the horizontal component of R_{31} and the cylinders will fall apart. On the other hand, if the angle exceeds 10.9° , R_s will provide a horizontal resisting component in excess of that required by R_{31} and then the reaction from cylinder 2 will also act to guarantee equilibrium.

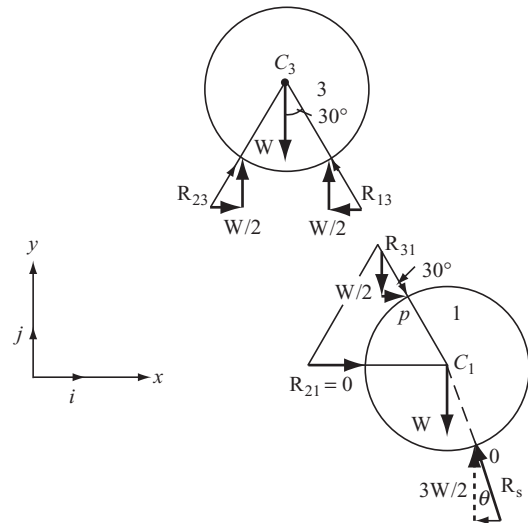


Fig. Ex. 3.6 (Solution)

Example 3.7

A painter's scaffold 10 m long and weighing 0.75 kN is supported in a horizontal position by vertical ropes attached at equal distances from the ends of the scaffold as shown in Fig. Ex. 3.7. Find the greatest distance from the ends that the ropes may be attached to permit a 1 kN painter to stand safely at one end of the scaffold.

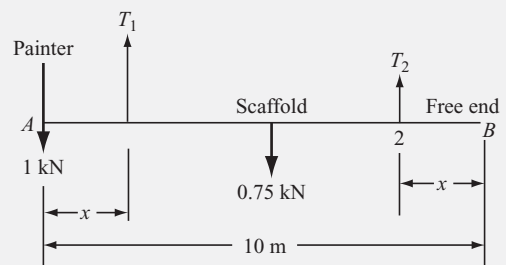


Fig. Ex. 3.7

Solution If the painter stands at A then the free-body diagram of the scaffold is as shown in Fig. Ex. 3.7. The tensions in the strings are T_1 and T_2 . For equilibrium of the scaffold,

$$\begin{aligned} \Sigma \quad &= 0; T_1 + T_2 = 1 + 0.75 = 1.75 \text{ kN} & (i) \\ \Sigma \quad &_A = 0; T_1 x + T_2 (10 - x) - 0.75 \times 5 = 0 & (ii) \end{aligned}$$

For x to be as large as possible, the tension T_2 in the string 2 reduces to zero. Then,

$$T_1 = 1.75 \text{ kN}$$

and

$$x = \frac{0.75 \times 5}{1.75} = 2.14 \text{ m}$$

If x is more than 2.14 m, i.e., if the ropes are attached closer to each other, then the solution of (i) and (ii) would show negative tension T_2 ; a state of compression which is not possible in a rope. On the other hand, if x is less than 2.14 m, i.e., if the ropes are attached farther, T_2 remains positive, permitting safe operation of the painter.

As a digression, let us demonstrate what we meant by the equivalence of the set of conditions, Eq. (3.7) with the set of conditions, Eq. (3.6) for equilibrium. In this example, the conditions chosen to provide Eqs. (i) and (ii) came from Eq. (3.6). Instead, Eq. (3.7) would provide, say,

$$\begin{aligned} \Sigma \quad &_1 = 0; \quad 1x - 0.75(5 - x) + T_2 (10 - 2x) = 0 \\ \Sigma \quad &_2 = 0; \quad 1(10 - x) - T_1 (10 - 2x) + 0.75(5 - x) = 0 \end{aligned}$$

or

$$1.75x + 10T_2 - 2T_2x = 3.75 \quad (iii)$$

$$-1.75x - 10T_1 + 2T_1x = -13.75 \quad (iv)$$

which are equivalent to Eqs. (i) and (ii). For x to be as large as possible $T_2 = 0$ which when substituted in Eq. (iii), provides

$$x = \frac{3.75}{1.75} = 2.14 \text{ m}$$

and from (iv)

$$T_1 = 1.75 \text{ kN, as before}$$

Example 3.8

The boom of a crane is shown in Fig. Ex. 3.8. If the weight of the boom is negligible compared with the load $W = 60 \text{ kN}$, find the compression in the boom and also the limiting value of the tension when the boom approaches the vertical position.

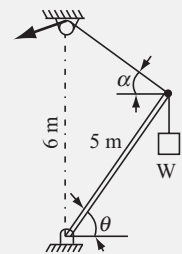


Fig. Ex. 3.8

Solution By drawing the free-body diagram of the boom, we observe that it is under the action of three forces:

- (i) 60 kN acting downwards at B
- (ii) Tension along the string inclined at an angle α with the horizontal. It can be resolved into $T \sin \alpha$ and $T \cos \alpha$ components acting vertically and horizontally respectively at B.
- (iii) Reaction at A acting along AB. It can be resolved into $A \sin \theta$ and $A \cos \theta$ components as shown in Fig. Ex. 3.8 (Solution).

For equilibrium of the boom,

$$\Sigma = 0; \Sigma_x = 0, \Sigma_y = 0$$

$$A \cos \theta - T \cos \alpha = 0; \cos \alpha = \frac{A \cos \theta}{T}$$

$$T \sin \alpha + A \sin \theta - 60 = 0; \sin \alpha = \frac{60 - A \sin \theta}{T}$$

whence
$$\tan \alpha = \frac{60 - A \sin \theta}{A \cos \theta}$$

By the geometrical configuration,

$$\tan \alpha = \frac{CD}{BD} = \frac{AC - AD}{BD} = \frac{6 - 5 \sin \theta}{5 \cos \theta}$$

From Eqs (i) and (ii), by comparison, A must be 50 kN

$$\text{and } T^2 (\cos^2 \alpha + \sin^2 \alpha) = A^2 \cos^2 \theta + (60 - A \sin \theta)^2$$

$$\text{or } T^2 = A^2 + 3600 - 120 A \sin \theta$$

For vertical position of the boom,

$$\theta = 90^\circ, \sin \theta = 1$$

$$T^2 = A^2 + 3600 - 120A = 2500 + 3600 - 6000 = 100$$

and
$$T = 10 \text{ kN}$$

It is interesting to observe the implication of this answer. The reaction at A remains 50 kN for all values of θ , i.e., for all inclinations of the boom. The load of 60 kN, therefore, requires only 10 kN to be shared by the string.

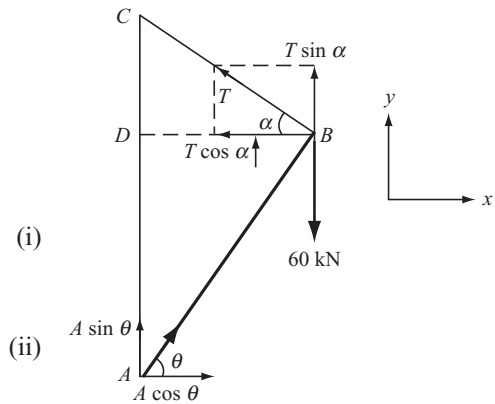


Fig. Ex. 3.8 (Solution)

Example 3.9

A three-bar pendulum $ABCD$ has three bars each 2 m in length and weighing 2.5 kN as shown in Fig. Ex. 3.9. It is held in equilibrium by applying a horizontal force of 3 kN at the free end. Determine the angles θ_1 , θ_2 and θ_3 made with the vertical.

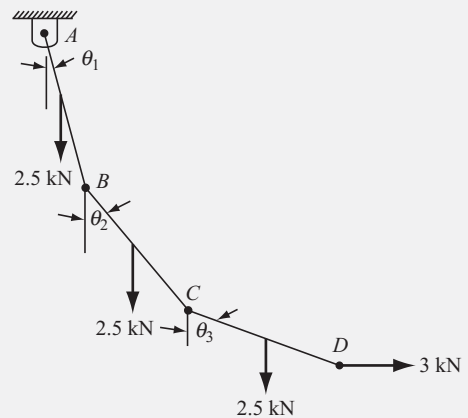


Fig. Ex. 3.9

Solution

$$AB = BC = CD = 2 \text{ m}$$

Weight of each bar assumed to act at their respective midpoints = 2.5 kN

Consider the equilibrium of bar CD :

$$\Sigma_c = 0; 3 CD \cos \theta_3 - 2.5 CD/2 \sin \theta_3 = 0$$

whence $\tan \theta_3 = \frac{6}{2.5} = 2.4$

and $\theta_3 = 67.4^\circ$

Also, by the equilibrium of bar CD,

$$\Sigma = 0; R_1 + 3 - 2.5 = 0$$

whence $R_{1x} = -3 \text{ kN}$ and $R_{1y} = 2.5 \text{ kN}$

Consider now the equilibrium of bar BC:

$$\Sigma_{B} = 0; -2.5 \times 2/2 \sin \theta_2 + 2 (\sin \theta_2 - \cos \theta_2) \times (3 - 2.5) = 0$$

or $-2.5 \sin \theta_2 - 5 \sin \theta_2 + 6 \cos \theta_2 = 0$

$$\tan \theta_2 = \frac{6}{7.5} = 0.8$$

$$\theta_2 = 38.66^\circ$$

$$\Sigma = 0; R_2 + 3 - 2.5 - 2.5 = 0$$

and $R_2 = 5 - 3$

and the free-body of bar AB is drawn. For the equilibrium of AB,

$$\Sigma_{A} = 0; -2.5 \times 2/2 \sin \theta_1 + 2 (\sin \theta_1 - \cos \theta_1) \times (3 - 5) = 0$$

or $-2.5 \sin \theta_1 - 10 \sin \theta_1 + 6 \cos \theta_1 = 0$

$$\tan \theta_1 = \frac{6}{12.5} = 0.48$$

$$\theta_1 = 25.64^\circ$$

It may be checked that

$$R_3 = -3 + 7.5$$

by applying the condition of $\Sigma = 0$ for this bar. This value of R_3 was expected from the equilibrium consideration of the entire system:

$$R_3 + 3 - 2.5 - 2.5 - 2.5 = 0$$

$$R_3 = -3 + 7.5$$

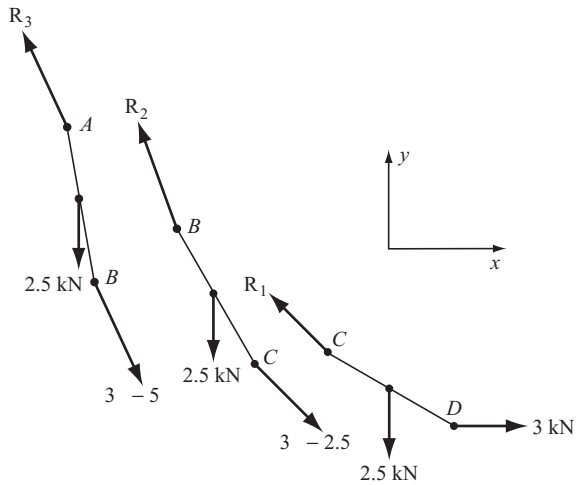


Fig. Ex. 3.9 (Solution)

Example 3.10

A uniform bar AB hinged at A, is kept horizontal by supporting and settling a 40 kN weight with the help of a string tied at B and passing over a smooth peg C as shown in Fig. Ex. 3.10. The bar weighs 20 kN. Determine the reactions at the supports A and C as well as the tension in the string.

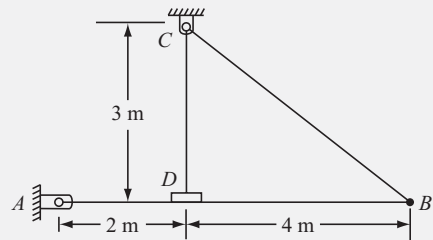


Fig. Ex. 3.10

Solution Consider the equilibrium of the bar AB with reference to its free-body diagram as shown in Fig. Ex. 3.10 (Solution). It may be noted that the weight of 40 kN is partly resting on the bar and is partly supported by the string to the extent of tension T .

$$\sum M_A = 0$$

From this relation

$$-(40 - T) \times 2 - 20 \times 3 + 6 \times T(-\cos \theta + \sin \theta) = 0$$

or $-140 + 2T + 6T \sin \theta = 0$

Taking $\sin \theta = \frac{3}{5} = 0.6$,

$$T = 25 \text{ kN}$$

$$\sum F_x = 0; A_x - T \cos \theta = 0$$

$$A_x = 25 \times \frac{4}{5} = 20 \text{ kN}$$

$$\sum F_y = 0; A_y + T \sin \theta - (40 - T) - 20 = 0$$

$$A_y = 20 \text{ kN}$$

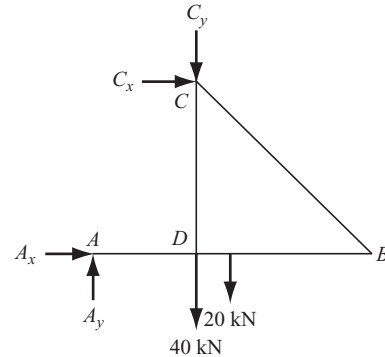
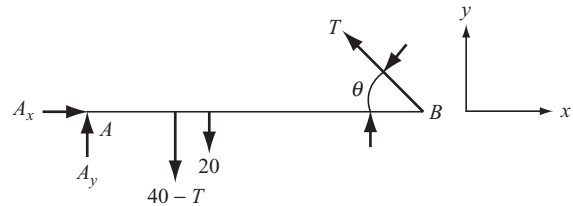


Fig. Ex. 3.10 (Solution)

Hence the tension in the string is 25 kN and the reaction at the hinge A is $(20 \quad +20)$ kN which is 28.28 kN inclined at 45° upwards with the bar. Consider now the equilibrium of the entire system with reference to the free-body diagram:

$$\sum F_x = 0; A_x + C_x = 0$$

$$\sum F_y = 0; A_y - 40 - 20 - C_y = 0$$

whence

$$C_x = -A_x = -20 \text{ kN}$$

$$C_y = A_y - 60 = 20 - 60 = -40 \text{ kN}$$

which shows that the reaction at C must be $(-20 \quad -40)$ kN

Example 3.11

A rectangular table $1 \text{ m} \times 2 \text{ m}$ is mounted on three equal supports. The table weighs 2 kN which acts through its centre of gravity c . If two vertical loads 1 kN and 4 kN are applied on the surface of the table as shown in Fig. Ex. 3.11, calculate the reaction at the supports.

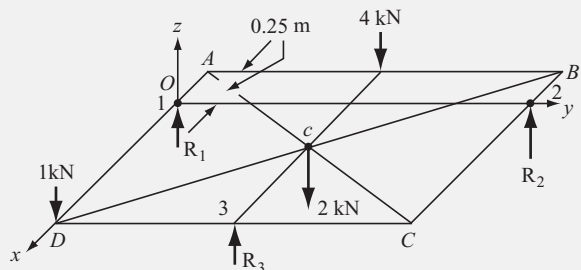


Fig. Ex. 3.11

Solution Choosing the plane of the table as the x - y plane, the origin O at 1 and coinciding the y -axis with the line joining the reactions 1 and 2 as shown, the loads and reactions are specified as follows:

| Description | Load | Position vector |
|----------------|-------|-----------------|
| 1 kN downward | -1 | 0.75 |
| 2 kN downward | -2 | 0.25 + 1 |
| 4 kN downward | -4 | -0.25 + 1 |
| Reaction R_1 | R_1 | 0 + 0 |
| Reaction R_2 | R_2 | 2 |
| Reaction R_3 | R_3 | 0.75 + 1 |

For equilibrium, the necessary conditions are:

$$\Sigma F_x = 0 \quad \text{and} \quad \Sigma F_y = 0$$

From the former,

$$(-1 - 2 - 4 + R_1 + R_2 + R_3) = 0$$

or

$$R_1 + R_2 + R_3 = 7$$

(i)

From the latter,

$$0.75 \times (-1) + (0.25 + 1) \times (-2) + (-0.25 + 1) \times (-4) + 0 + 2 \times R_2 + (0.75 + 1) \times R_3 = 0$$

or

$$0.75 - 2.5 - 1.00 - 4 + 2R_2 - 0.75R_3 + R_3 = 0$$

whence

$$(-6 + 2R_2 + R_3) = 0$$

or

$$2R_2 + R_3 = 6$$

(ii)

and

$$(0.25 + 0.75R_3) = 0$$

$$0.75R_3 = 0.25$$

(iii)

or

From (iii),

$$R_3 = 0.33 \text{ kN}$$

From (ii),

$$R_2 = 2.835 \text{ kN}$$

and

From (i),

$$R_1 = 3.835 \text{ kN}$$

Example 3.12

A horizontal rigid bar AB weighing 1 kN/m carries a load of 2 kN at its free end A . It is supported by the ball-and-socket joint at B and the cables PQ and RS are taut. Determine the tensions in the cables and the reaction at B .

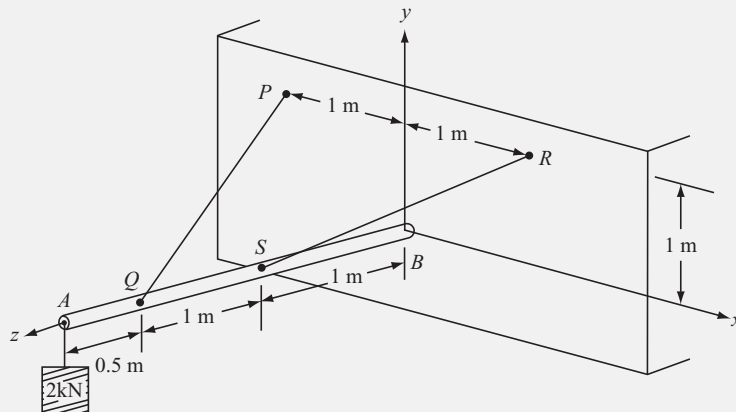


Fig. Ex. 3.12

Solution With reference to the x , y and z -axes, as shown in Fig. Ex. 3.12, the forces on the bar AB are shown and tabulated below. The unit vector $_{QP}$ along QP may be determined by locating P and Q ;

$$P(-1, 1, 0) \quad \text{and} \quad Q(0, 0, 2)$$

$$_{QP} = \frac{(-1-0) \mathbf{i} + (1-0) \mathbf{j} + (0-2) \mathbf{k}}{\sqrt{(-1)^2 + 1^2 + (-2)^2}} = -0.41 \mathbf{i} + 0.41 \mathbf{j} - 0.82 \mathbf{k}$$

Similarly, the unit vector $_{SR}$ along SR is expressed as

$$_{SR} = \frac{(1-0) \mathbf{i} + (1-0) \mathbf{j} + (0-1) \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = 0.58 \mathbf{i} + 0.58 \mathbf{j} - 0.58 \mathbf{k}$$

| Description | Force | Position vector |
|-------------------------------------|---|-----------------|
| 1. Weight at the free end | -2 | 2.5 |
| 2. $_{Q}$, the force in cable QP | $F_Q(-0.41 \mathbf{i} + 0.41 \mathbf{j} - 0.82 \mathbf{k})$ | 2 |
| 3. $_{s}$, the force in cable SR | $F_s(0.58 \mathbf{i} + 0.58 \mathbf{j} - 0.58 \mathbf{k})$ | 1 |
| 4. Weight of the bar | -1 | 1.25 |
| 5. Reaction at B | $B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$ | 0 |

For equilibrium, $\Sigma \mathbf{i} = 0$ and $\Sigma \mathbf{j} = 0$.

From the summation of the forces on AB ,

$$-0.41 F_Q + 0.58 F_s + B_x = 0 \tag{i}$$

$$-2 + 0.41 F_Q + 0.58 F_s - 1 + B_y = 0 \tag{ii}$$

$$-0.82 F_Q - 0.58 F_s + B_z = 0 \tag{iii}$$

The moments of the forces about the origin at B are calculated as follows:

$$1. \quad 2.5 \times (-2 \mathbf{j}) = 5 \mathbf{k}$$

$$2. \quad 2 \times F_Q(-0.41 \mathbf{i} + 0.41 \mathbf{j} - 0.82 \mathbf{k}) = (-0.82 \mathbf{i} - 0.82 \mathbf{k}) F_Q$$

$$3. \quad 1 \times F_s(0.58 \mathbf{i} + 0.58 \mathbf{j} - 0.58 \mathbf{k}) = (0.58 \mathbf{i} - 0.58 \mathbf{k}) F_s$$

$$4. \quad 1.25 \times (-1 \mathbf{j}) = 1.25 \mathbf{k}$$

$$5. \quad 0$$

Summation of the moments about the origin yields

$$\Sigma \mathbf{k} = 5 + (-0.82 \mathbf{i} - 0.82 \mathbf{k}) F_Q + (0.58 \mathbf{i} - 0.58 \mathbf{k}) F_s + 1.25 \mathbf{k} = 0$$

whence,

$$6.25 - 0.82 F_Q - 0.58 F_s = 0 \tag{iv}$$

$$-0.82 F_Q + 0.58 F_s = 0 \tag{v}$$

Now, we have a set of five Eqs (i) to (v) for the five unknowns

$$F_Q, F_s, B_x, B_y \text{ and } B_z$$

which may be solved to provide

$$F_Q = 3.80 \text{ kN along } QP$$

$$F_s = 5.37 \text{ kN along } SR$$

$$B_x = 1.56 \text{ kN}$$

$$B_y = -1.67 \text{ kN}$$

$$B_z = 6.23 \text{ kN}$$

Example 3.13

- (a) A parachuter of mass 60 kg descends to the ground in a parachute of mass 20 kg. If the drag on the parachute varies as $3V^2$ N where V is the speed of the decent, will the parachute be in equilibrium before it reaches the ground?
- (b) Comment on the falling of the raindrops from the cloud to the ground; are they in equilibrium before reaching the ground?

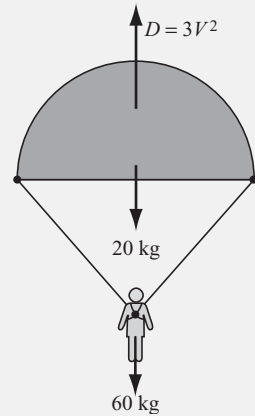


Fig. Ex. 3.13

Solution

(a) The parachute commences with almost zero vertical speed but it soon acquires a speed V .

For equilibrium, $20 \times 9.81 + 60 \times 9.81 = 3V^2$

whence $V = 16$ m/s.

It may be noted that a body moving at a constant velocity, such as this, is in equilibrium!

Just to get an estimate how fast it happens, imagine the descent without drag

$$V = u + gt$$

$$16 = 0 + 9.81t; t \approx 1.6\text{s}$$

In reality, it may take a few seconds, to attain a speed of 16 m/s but it will happen well before it reaches the ground.

(b) Raindrops are tiny masses, subjected to wind drag while falling from the cloud. They also attain equilibrium at a constant speed, well before reaching the ground.

Example 3.14

Three identical wooden blocks, each with mass m and length a are stacked on top of each other as shown in Fig. Ex. 3.14 with overhangs on one side. Determine the maximum possible overhang the top block can have from the bottom of the stack in order to stay in equilibrium.

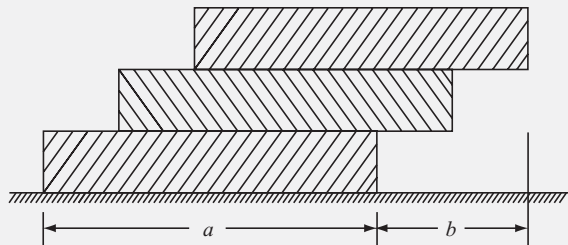


Fig. Ex. 3.14

Solution While the base block is stable, the other two blocks 1 and 2 are likely to topple over at points P_1 and P_2 respectively.

Let the overhang of block 1 be b_1 and the further overhang of block 2 be b_2 , as shown in Fig. 3.14 (Solution). First, we must ensure ‘no toppling’ of the top block 2 over the point P_2 . That requires b_2 to be a maximum of $a/2$. In order to determine the maximum b_1 , let us redraw the figure and ensure ‘no toppling’ over point P_1 .

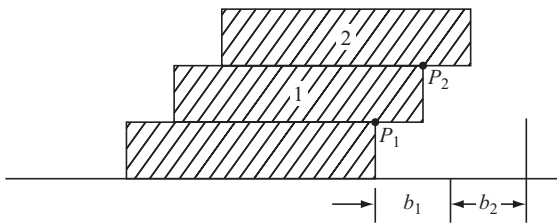


Fig. Ex. 3.14(a) (Solution)

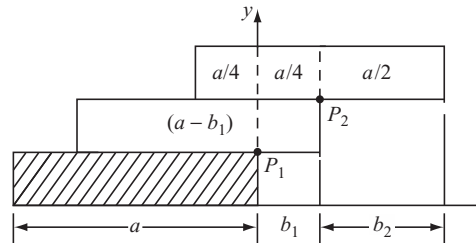


Fig. Ex. 3.14(b) (Solution)

Taking moments of the components of the weights of the blocks about the point P_1 and equating to zero,

$$\frac{a}{4} \cdot mg + (a - b_1)mg - \left(\frac{a}{4} + \frac{a}{2}\right)mg - b_1 mg = 0$$

or
$$\frac{a}{4} + a - b_1 - \frac{a}{4} - \frac{a}{2} - b_1 = 0$$

or
$$\frac{a}{2} - 2b_1 = 0; \quad b_1 = \frac{a}{4}.$$

The maximum possible overhang of the top block from the bottom of the stack is, therefore,

$$b = b_1 + b_2 = \frac{a}{2} + \frac{a}{4} = \frac{3a}{4}.$$

Reflection The problem does not state that the overhangs b_1 and b_2 are equal; that would not be the case for the maximum possible overhang of $(b_1 + b_2)$. Nor can one start by finding out b_1 before finding out b_2 . It means that one has to start from the top block and proceed downwards. For example, if these were four blocks, the maximum overhang possible would have been $a/2 + a/4 + a/8$, i.e., $7/8a$. What does it mean? Larger the number of blocks, greater the maximum possible overhang, tending to be equal to a itself!

Example 3.15

A bar $ABCD$ is hinged at A and supported by a cable, at BC , passing over a frictionless pulley at P above it. Determine the tension in the cable and the reaction at A for a load of 500 N hanging at D .

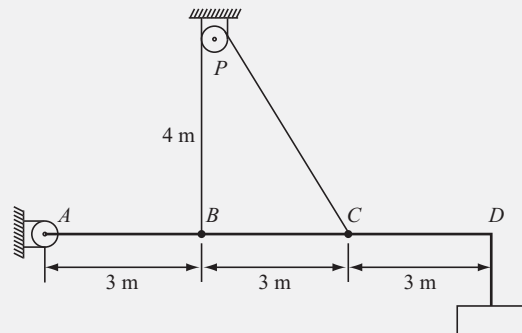


Fig. Ex. 3.15

Solution The cable passes over a frictionless pulley; the tension the part BP must be the same in the part CP . A free-body diagram of the bar is drawn, as shown.

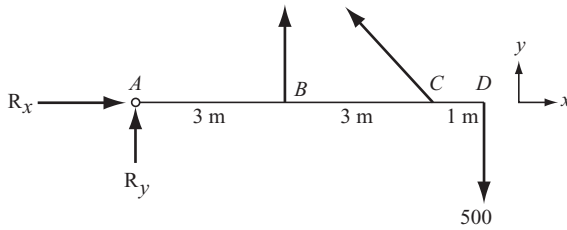


Fig. Ex. 3.15(a) (Solution)

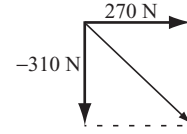


Fig. Ex. 3.15(b) (Solution)

For equilibrium of the bar,

$$\Sigma x = R_x - \frac{3}{5}T = 0 \quad (i)$$

$$\Sigma y = R_y + T - 500 + \frac{4}{5}T = 0 \quad (ii)$$

$$\Sigma A = 3T + \frac{4}{5}T \times 6 - 500 \times 7 = 0 \quad (iii)$$

From (iii),

$$7.8 T = 3500$$

$$T = 450 \text{ N}$$

By substitution in (i) and (ii)

$$R_x = 270 \text{ N}$$

$$R_y = -310 \text{ N}$$

which shows that the reaction at A consists of a vertically down component.

Example 3.16

A rigid weightless diving board, supported on two springs 1 and 2, each of stiffness 20 kN/m is in the horizontal position. Determine the angle of tilt θ the board will make when a person of weight 500 N stands at the edge of the board, before diving.

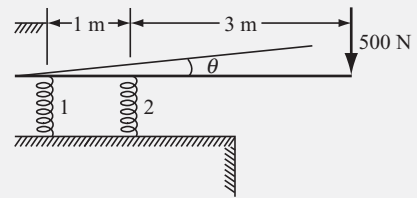


Fig. Ex. 3.16

Solution Let the compressions of the springs 1 and 2 be x_1 and x_2 when the diving board is tilted at an angle θ . Assuming that the board tilts by turning over a point a metre to the left of spring 1,

$$x_1 = a\theta; \quad x_2 = (a + 1)\theta \quad (i)$$

whence

$$x_2 = x_1 + \theta$$

Considering the *fbd* of the diving board, as shown in Fig. Ex. 3.16, for equilibrium,

$$\Sigma = 0; \quad kx_1 + kx_2 - 500 = 0 \quad (ii)$$

$$\Sigma _1 = 0; \quad kx_2 \times 1 - 500 \times 4 = 0; \quad x_2 = \frac{2000}{20 \times 1000} = 0.1 \text{ m}$$

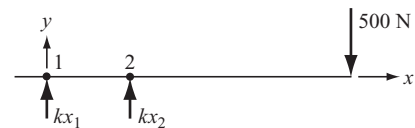


Fig. Ex. 3.16 (Solution) *fbd* of the diving board

By substituting in (i), $x_1 = 0.1 - \theta$
 and by substituting in (ii), $k(0.1 - \theta) + k \times 0.1 - 500 = 0$
 whence, $\theta = \frac{500 - 2 \times 20 \times 1000 \times 0.1}{20 \times 1000}$
 $= 0.175$ radian
 $= 0.175 \times 180/\pi = 10.0^\circ$

Example 3.17 An exercise machine is designed to be a lightweight cart on an inclined ramp on which a lady may lie and hold herself with two parallel strings, which are attached to her cart through frictionless pulleys. If the lady weighing 700 N is able to pull and keep the strings horizontal while the ramp is inclined at 15° , determine the force she is exerting by each hand. Find also the reaction of the ramp on the cart.

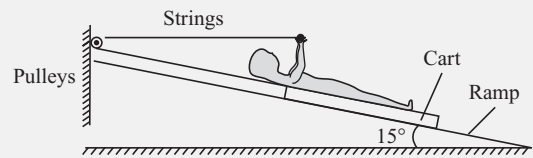


Fig. Ex. 3.17

Solution Let the pull exerted by each hand be P , then the total pull $2P$ exerted horizontally is also the tension in the strings pulling her up the incline. Referring to the *fbd* of the lady,

$$\Sigma x = 0; -2P - 2P \cos 15^\circ + 700 \sin 15^\circ = 0$$

whence, $3.93 P = 181.2; P = 46$ N

$$\Sigma y = 0; R - 700 \cos 15^\circ + 2P \sin 15^\circ = 0$$

whence, $R = 676.1 - 23.8 = 652.3$ N

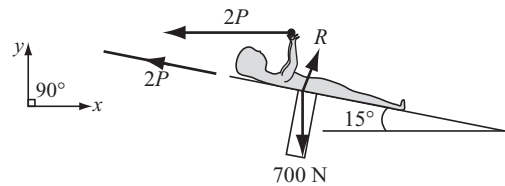


Fig. Ex. 3.17 (Solution)

Example 3.18 A person on a wheelchair is trying to roll up a small step of 10 cm. If the diameter of the wheel is 60 cm and the combined weight of the person and the wheelchair is 1500 N, how much force will he need to apply at the periphery of the wheel? What should be a reasonable incline of a ramp if the person can exert up to 100 N force?

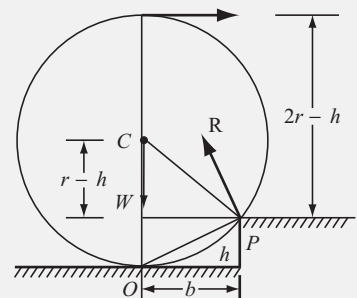


Fig. Ex. 3.18(a)

Solution For the given position, the free body diagram shows the horizontal force F applied by the person, weight W acting at the centre of the wheel and reaction R at the corner point P . At the instant of the wheel being lifted up, the reaction at O must be zero.

From the geometry,

$$b^2 = r^2 - (r-h)^2 = 2rh - h^2 = 2 \times 0.3 \times 0.1 - 0.1^2$$

and $b = 0.224$ m

Taking moments about the point P and equating to zero for equilibrium,

$$-F(2r - h) + Wb = 0$$

$$F = \left(\frac{1500}{2} \times 0.224 \right) / (0.6 - 0.1) = 335 \text{ N.}$$

It is assumed that each wheel takes half the total load and the person applies an equal force F by each hand.

The force of 335 N is indeed too much for a wheelchair-ridden person.

In order to roll over a ramp of inclination θ , the person must apply a force F , as shown, along the incline such that the sum of moments about a point O is zero. In doing so, the force (due to reaction and friction) at O does enter the calculations and need not be known.

$$W \frac{d}{2} \sin \theta - Fd = 0$$

whence

$$F = \frac{W}{2} \sin \theta$$

If the total weight of the wheelchair and person, 1500 kg is assumed to be shared equally by two wheels, $W = 750$ kg.

$$F = \frac{750}{2} \sin \theta = 375 \sin \theta$$

For an inclination of 30° , $F = 187.5$ N

For, 20° , $F = 128$ N

For, 15° , $F = 97$ N

For, 10° , $F = 65$ N

A value of 10° to 15° requiring a force less than 100 N appears to be a reasonable incline.

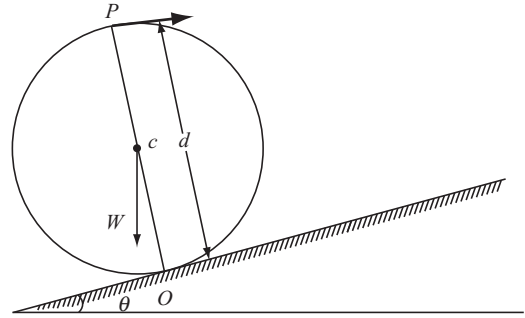


Fig. Ex. 3.18(b)

Example 3.19

A jib crane as shown in Fig. Ex. 3.19 consists of a boom anchored by two coplanar strings above it. At an instant when the boom and the strings make angles of 30° and 15° with the horizontal respectively and the load carried at the hook is 100 kN, determine the forces in the boom and each of the two strings making an angle of 30° between them.

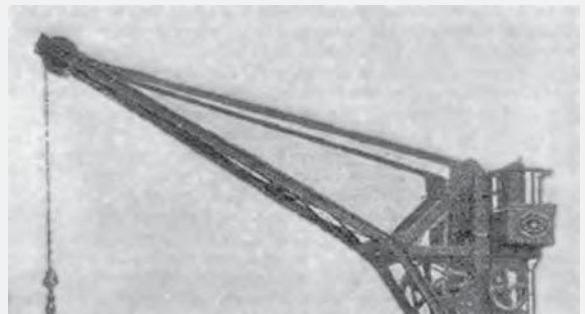


Fig. Ex. 3.19

Solution Let us consider the equilibrium at the tip O . Assuming the forces (tensile) as shown; OC being the force in the boom on OB the resultant of forces F in each of the two strings. Notice that each of the string forces F is inclined OB on either side of it at $30/2$, i.e., 15° . Such that

$$OB = 2F \cos 15^\circ$$

Let us solve this problem by two different methods, i.e. by using vectors and graphically:

By Vector Method

$$\begin{aligned} \Sigma &= 0 \\ \Sigma_x = 0 &= OB \cos 15^\circ + OC \cos 30^\circ \\ &= 2F \cos 15^\circ \cos 15^\circ + OC \cos 30^\circ \\ &= 2 \times 0.966 \times 0.966 F + 0.866 OC \\ &= 1.87 F + 0.866 OC \end{aligned}$$

$$\begin{aligned} \Sigma_y = 0 &= -100 - OB \sin 15^\circ - OC \sin 30^\circ \\ &= -100 - 2F \cos 15^\circ \sin 15^\circ - OC \sin 30^\circ \\ &= -100 - 2 \times 0.966 \times 0.259 F - 0.5 OC \\ &= -100 - 0.5 F - 0.5 OC \end{aligned}$$

which reduces to

$$\begin{aligned} 1.87 F + 0.866 OC &= 0 \\ 0.5 F + 0.5 OC &= -100 \end{aligned}$$

whence, $F = 172 \text{ kN}$ and $OC = -371 \text{ kN}$

which means that,

- (a) each of the two strings is under tension of 172 kN and
- (b) the boom OC is under compression of 371 kN.

By Graphical Method

The graphical method can be used to draw the polygon of forces only if the forces are coplanar. In this case, the three forces F , F and OC acting at point o are coplanar; hence a triangle of forces can be drawn by drawing lines parallel to the three forces, respectively.

The triangle drawn must be closed because the three forces F , F and OC keep the point o in equilibrium. The force OC was drawn to a chosen scale of $1 \text{ cm} = 50 \text{ kN}$; 2 cm in length. Measuring the lengths of OB and OC and using the same scale results in the following:

$$\begin{aligned} &= 6.6 \text{ cm} \rightarrow 330 \text{ kN} \\ &= 7.4 \text{ cm} \rightarrow 370 \text{ kN} \end{aligned}$$

Also, it is seen that the direction of OB is the same as was assumed; so the strings are in tension. The direction of OC is reversed which means that the boom is under compression of 370 kN force.

Now, the force of 330 kN is, in fact, the resultant of forces F, F in the two strings anchoring the boom. We can use the graphical method to draw the triangle or parallelogram of forces, as follows, to evaluate F .

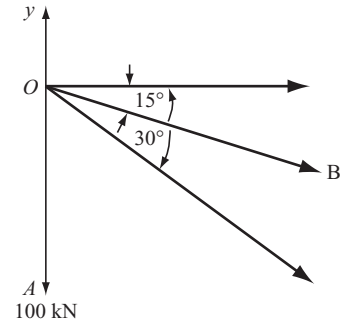


Fig. Ex. 3.19(a) (Solution)

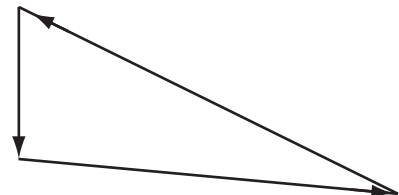


Fig. Ex. 3.19(b) (Solution)

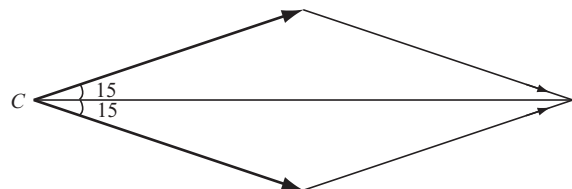


Fig. Ex. 3.19(c) (Solution)

Employing the same scale, 1 cm = 50 kN, in drawing the parallelogram of forces, we measure

$$F = 3.4 \text{ cm} \rightarrow 170 \text{ kN}$$

which shows that each string is in tension with 170 kN force!

Comparing the two methods

It is seen that the vector method is more accurate and applicable to all situations whereas the graphical method applies only to plane forces and it is inherently not so accurate; it also requires drawing work.

EXPERIMENT E2 EQUILIBRIUM UNDER SPATIAL FORCES

OBJECTIVE TO STUDY THE EQUILIBRIUM OF A PARTICLE UNDER THE ACTION OF FORCES IN SPACE.

Apparatus A skeleton space frame consisting of bars with provisions to pass the strings over frictionless pulleys at the desired points. Standard weights and metre rod.

Background Information The equilibrium of a particle under the action of forces in space is essentially governed by the condition

$$\Sigma = 0$$

i.e., the vector sum of all the forces must vanish.

In the space frame shown in figure a load W is supported by three strings AP , BP and CP in order to keep the knot P in equilibrium at rest. Taking the origin at a corner point O , the coordinates of A , B , C and P are measured and are used to find the unit vectors and along AP , BP and CP respectively.

For example,

$$A = x_A + z_A$$

$$P = x_P + y_P + z_P$$

$$AP = (x_A - x_P) + y_P + (z_A - z_P)$$

whence,
$$AP = \sqrt{(x_A - x_P)^2 + y_P^2 + (z_A - z_P)^2}$$

and
$$= \frac{(x_A - x_P)}{AP} + \frac{y_P}{AP} + \frac{(z_A - z_P)}{AP}$$

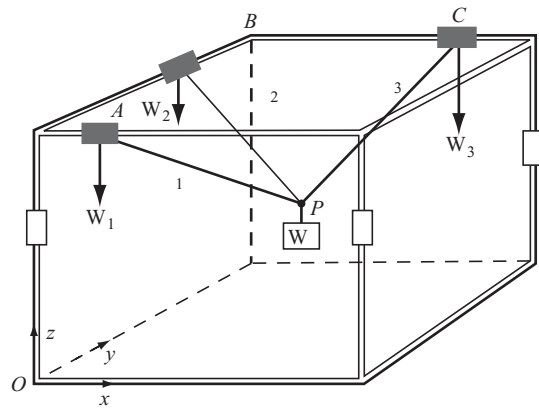
The forces F_1 , F_2 and F_3 in AP , BP and CP respectively are such that

$$\Sigma = 0; F_1 + F_2 + F_3 - W = 0$$

or
$$F_1 + F_2 + F_3 - W = 0$$

From a knowledge of the unit vectors and in terms of , and , this vector equation is rewritten as three scalar equations and the three unknowns F_1 , F_2 and F_3 are obtained.

In the apparatus the strings are kept taut under the application of hanging loads at A , B and C . Neglecting friction at the pulleys, the tensions in the strings must equal the corresponding applied loads.



Skeleton space Frame

Observations and Calculations It is advisable to record the coordinates and to formulate the equations in a tabular form. For example, the table of force $_1$ is as follows:

| POINT P | POINT A | FOR AP |
|---------------|---------------|---------------------------------------|
| $x_P y_P z_P$ | $x_A y_A z_A$ | $(x_A - x_P) (y_A - y_P) (z_A - z_P)$ |

$AP =$

Table of Results

| Member | Forces in members | | Difference |
|--------|-------------------|------------------|------------|
| | Theoretical (N) | Experimental (N) | Abs. % |
| AP | | | |
| BP | | | |
| CP | | | |

Points for Discussion

- State the necessary and sufficient conditions of equilibrium for a rigid body. In particular, is the condition $\Sigma =$ just necessary or necessary and sufficient as far as the equilibrium of a particle is concerned?
- If a load W hangs from P maintained in equilibrium under the action of $_1$, $_2$ and $_3$ as done in the experiment and the load W is increased by 10%, what will happen to the position of the strings and why? Would the strings stay unaltered if the tension in each string were increased by 10%?
- Enumerate the sources of error in the experiment and suggest how each can be minimised.
- Would you suggest some other design of the skeleton frame to be able to perform experiments on equilibrium under the action of spatial and coplanar forces?

Concept Review Questions

- (a) If a system is in equilibrium, is it necessary that it should be static? Relate the concept of equilibrium with the laws of motion.
 - (b) If a rigid body is rotating at a constant angular velocity about some axis, is the body said to be in equilibrium or not?
- Draw the free-body diagrams for the following:
- (a) a nut-cracker in action
 - (b) a ladder leaning against a wall
 - (c) a bullock-cart in motion
 - (d) an aeroplane in flight
 - (e) a body floating in a liquid.
- (a) State the Lami's theorem for the equilibrium of a body under the action of three coplanar forces.
 - (b) Prove that a body must be in equilibrium if Lami's theorem is obeyed.
- Examine the truth in the following statements and rewrite them after necessary corrections.
- (a) A rigid body must be in equilibrium if the resultant of the force system acting on it vanishes.
 - (b) A rigid body subjected to a couple may be brought to equilibrium by a force placed suitably in the plane of the couple.
 - (c) A pin joint and a hinge are identical supports so far as the reaction is concerned.
 - (d) If three concurrent forces keep a particle or a rigid-body in equilibrium, then the forces must be coplanar.

- (e) The equilibrant action required to bring a rigid-body into equilibrium should be equal and opposite to any equivalent force system acting on the body.

Illustrate why the necessary conditions of equilibrium

$$\Sigma \dots = 0 \quad \text{and} \quad \Sigma \dots = 0$$

can be replaced by the equivalent moment equations

$$\Sigma \dots_1 = 0 \quad \text{and} \quad \Sigma \dots_2 = 0$$

for a plane force system acting on a body.

Tutorial Problems

A small boat is held static in a river by means of three inextensible taut ropes OA , OB and OC . The water in the river exerts a force on the boat in the direction of the flow. If the tensions in OA and OB are 1 kN and 0.6 kN respectively, as shown in Fig. Prob. 3.1, determine the force exerted by the flow on the boat and the tension in rope OC .

- (a) Will the boat remain in equilibrium if rope OC breaks?
 (b) What would be the tension in OA after OC breaks?
 (c s 0.116, 0.808 kN; yes, 0.067, 0.133 kN)

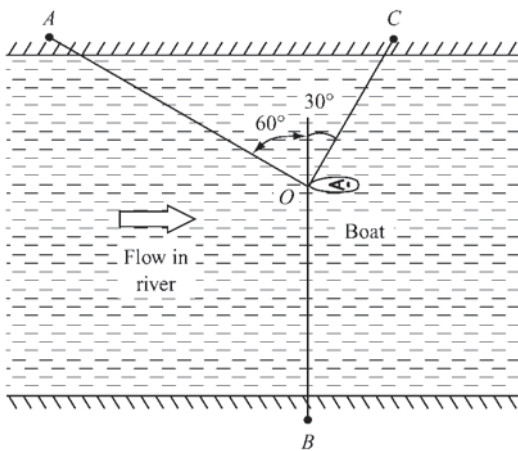


Fig. Prob. 3.1

A force applied to a stretched elastic string at O stretches it to a position AOB as shown in Fig. Prob. 3.2. If the tension in each part of the string is 50 N, determine the magnitude and direction of the force applied. (c s 61 N, 7.5)

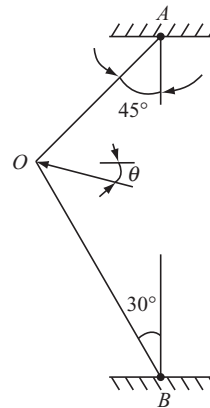


Fig. Prob. 3.2

In an old drawing, four forces are shown acting at a point but the line of action of the fifth force has been disfigured. However, it is known that the moments exerted by the forces about some origin in space are: $2 + 2 - 3$, $3 - 4 + 7$, $2 + 3 - 4 - 8 + -2$ and $1 - 2 + 2$ respectively. Determine whether the fifth force passes through the same point or not.

(c s Yes, since $\Sigma M = 0$)

A $4 \text{ m} \times 5 \text{ m}$ slab carries four forces normal to it as shown in Fig. Prob. 3.4.

- (a) Determine the magnitude and point of application on it of a single force equivalent to the given system of forces.
 (b) If the slab is to be in equilibrium, determine the magnitude and location of the equivalent.

- (c) If the slab must be brought into equilibrium by holding it at the origin O , determine the reaction necessary from the device to hold it.
- (d) Is it possible to hold the slab in equilibrium by applying suitable forces at the three free corners A , B and C ?
 (\quad s -8 kN; at 1.9 m, 1.5 m point)

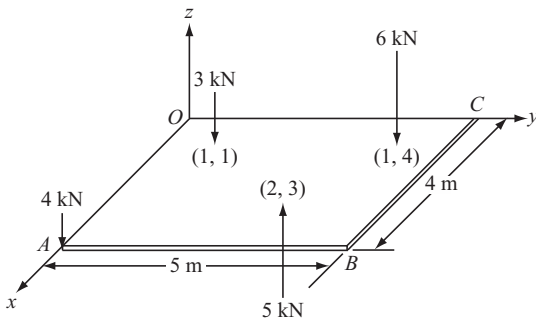


Fig. Prob. 3.4

A simple stone-crushing mechanism consists of a piston on which a force of 15 kN acts and three rigid weightless links OA , OB and OC hinged at O , A , B and C as shown in Fig. Prob. 3.5. At the given orientation, what is the force exerted on the stone S trapped between the jaw and the fixed wall.
 (\quad s 28 kN)

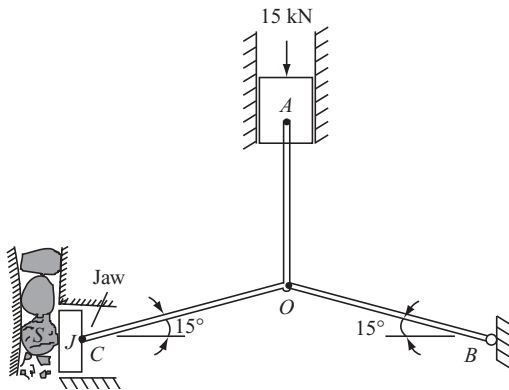


Fig. Prob. 3.5

A cylinder having a diameter of 0.5 m and mass 50 kg is supported on a uniform rigid link AO 2 m long and of mass 10 kg by means of a string OB held taut as shown in Fig. Prob. 3.6. Assuming the surfaces of contact of the cylinder

as frictionless, calculate the tension in the string and the reaction at the hinge A .

$$\left(\quad \text{s } T = 335 \text{ N}, R_A = 424 \text{ N at } 89.4^\circ \text{ to horizontal} \right)$$

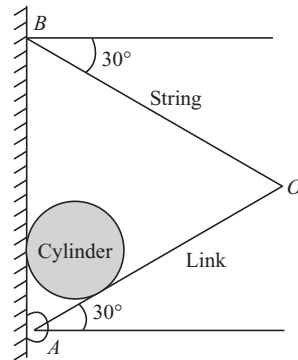


Fig. Prob. 3.6

- (a) A weight W tied to the lower end of a suspended cord of length l is pulled by a horizontal force so as to displace it by a distance d away from its vertical position. Express the force and tension T in the cord as a function of the horizontal displacement of the weight.
- (b) Determine the horizontal distance to which a 10 m long inextensible nylon thread holding a mass of 1000 N can be drawn before it breaks. The thread can withstand a maximum tension of 10 kN.

$$\left(\quad \text{s } = W \cot(\cos^{-1} d/l), T = W \operatorname{cosec}(\cos^{-1} d/l), d = 10 \text{ m} \right)$$

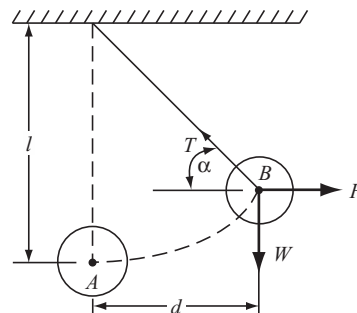


Fig. Prob. 3.7

- 8 A uniform metre rod AB , assumed rigid, of mass 0.5 kg is suspended from its ends in an inclined position and a mass of 1 kg is suspended from a point D . Determine the tension in each string.

Where should the suspended mass be placed in order to get equal tension in the strings?

(s $T_1 = 4.9 \text{ N}$ and $T_2 = 9.81 \text{ N}$; at C)

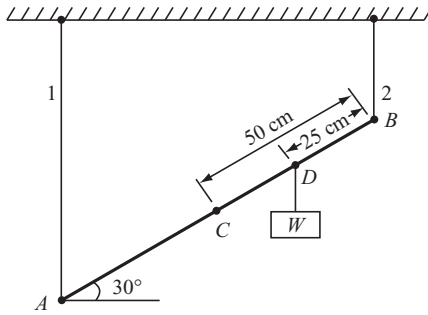


Fig. Prob. 3.8

A crane is idealised by a uniform rigid arm of weight W_a supported on a knife edge of the rest and held by a cable together with a counter weight of 5 kN as shown in Fig. Prob. 3.9. The 10 kN load held by it is moved outward on the arm with a constant velocity v of 0.2 m/s. Assuming the system to be in equilibrium when $\theta = 30^\circ$ find the rate of change of the tension T in the cable.

(s $\frac{dT}{dt} = 5 \frac{V}{\cos \theta} = 1.155 \text{ kN/s}$)

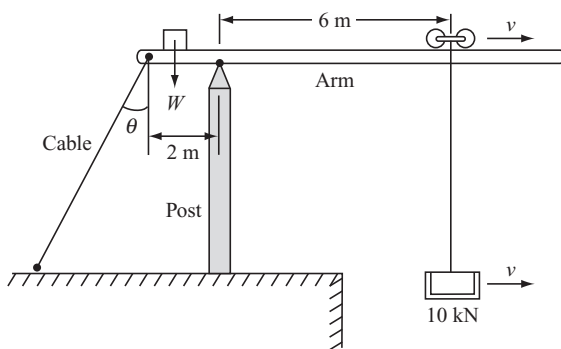


Fig. Prob. 3.9

Two cylinders 1 and 2 are connected by a rigid bar of negligible weight hinged at each

cylinder and are left to rest in equilibrium in the position shown under the application of a force P applied at the centre of cylinder 2. Determine the magnitude of force P if the masses of the cylinders are $m_1 = 100 \text{ kg}$ and $m_2 = 50 \text{ kg}$.

(s 263 N)

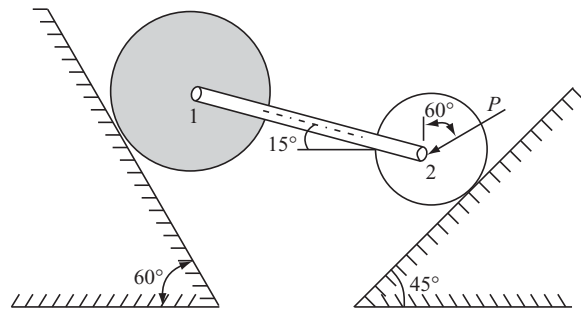


Fig. Prob. 3.10

Three identical balls rest on a smooth horizontal surface touching one another as shown in Fig. Prob. 3.11. A fourth ball of the same size and weight is placed on top of these three to form a pyramid and the three lower balls are now held together by an encircling string as shown in the plane view. Determine the tension in the string if the mass of each ball is 900 kg. Assume all surfaces of contact to be smooth.

(s 1200 N)

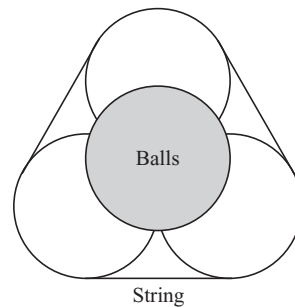


Fig. Prob. 3.11

Figure Prob. 3.12 shows a weight W tied to the end of a cord of length L . Find the magnitude

of the force required to pull the weight to an angle α from its vertical position and the tension in the cord.

(s $W \cot \alpha, W \operatorname{cosec} \alpha$)

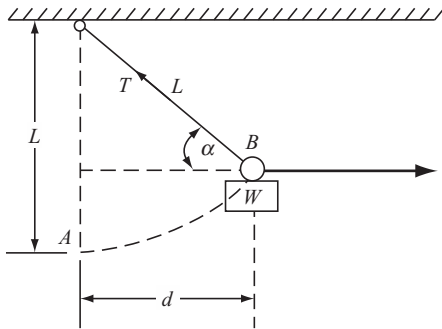


Fig. Prob. 3.12

A three-wheeler scooter rickshaw with weight 2 kN acting at its centre of gravity C is shown schematically in Fig. Prob. 3.13. The driver D weighing 0.5 kN and the passenger P weighing 0.8 kN are located as shown. Calculate the reactions at the wheels 1, 2 and 3 for equilibrium on a horizontal road.

(s 1.00, 1.23, 1.07 kN)

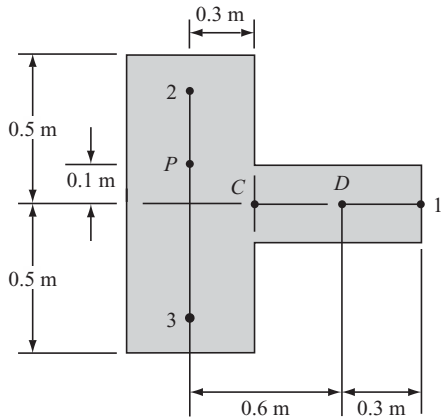


Fig. Prob. 3.13

A vertical tower of height h is subjected to a horizontal force at its top and it is

anchored by two equal guy wires symmetrically as shown in Fig. Prob. 3.14. Determine the tension T in the guy wires if $h = 20$ m, $a = 3$ m, $b = 4$ m and (b) the horizontal force = 10 kN.

(s 34.5 kN)

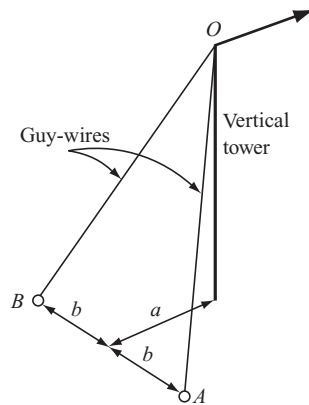


Fig. Prob. 3.14

A vertical mast AB is supported in a ball-and-socket joint at A and by cables BC and DE as shown in Fig. Prob. 3.15. A force

= 500 + 400 - 300

is applied at B . Calculate the reaction provided by the ground at A .

(s 5100 N)

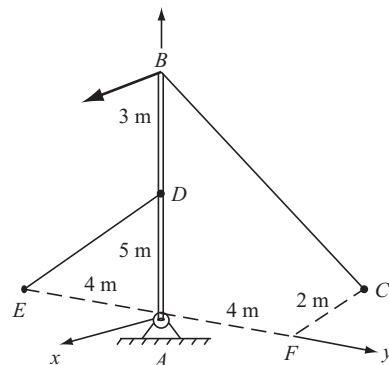


Fig. Prob. 3.15

Two cables BG and BH are attached to hold the boom ABC 1.4 m long, hinged at A , horizontally as shown in Fig. Prob. 3.16. Determine

the tensions in the cables if a load of 1 kN acts at C .

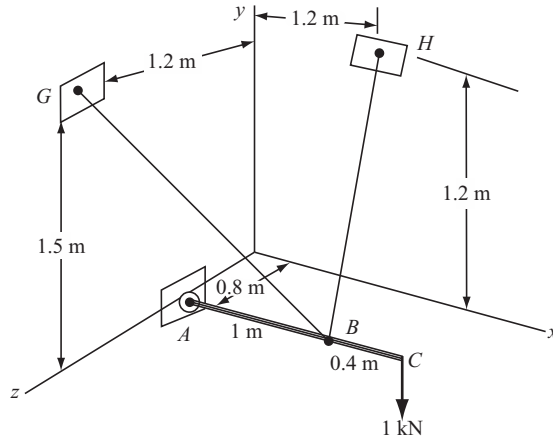


Fig. Prob. 3.16

(Answers: 1230 N and 486 N)

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

A rigid body is in equilibrium under the action of three forces. It implies that the forces must

- (a) be concurrent
- (b) be coplanar
- (c) either be concurrent or coplanar
- (d) pass through the centre of mass

A rigid body is in equilibrium. Given that the moment of all the forces acting on the body about some axis is zero and also given that forces are concurrent, implies that

- (a) the resultant force is zero
- (b) the forces have a line of action passing through the axis

- (c) the resultant forces have a line of action parallel to the axis
- (d) any of (a), (b), (c) can be true

A body is acted upon by a force system. It can in general be brought to equilibrium by the application of

- (a) a force acting on a suitable point on the body
- (b) a force acting anywhere along a suitable line
- (c) a force acting along a suitable line and a moment along the direction of the force
- (d) a wrench acting anywhere on the body.

Lami's theorem

- (a) relates the forces with the sines of angles
- (b) state that, for equilibrium under the action of three concurrent forces, there is a unique constant of proportionality between a force and the angle between the other two forces

- (c) may be applied to consider a relationship between forces and angles of a polygon representation of forces
- (d) may be applied for a body which may or may not be in equilibrium
- (a) must be in equilibrium
- (b) cannot be in equilibrium
- (c) may be in equilibrium provided the forces are concurrent
- (d) may be in equilibrium provided the forces are parallel

If the sum of all the forces acting on a body is zero, it may be concluded that the body

Answers to Multiple-Choice Questions

1 (b) 2 (d) 3 (c) 4 (b) 5 (c)

Applications in Statics

This section consists of some salient applications in Statics under the following three chapters:

-
- **S1 SIMPLE STRUCTURES**
 - **S2 THIN RIGID BEAMS**
 - **S3 APPLICATION WITH FRICTION**
-



Simple Structures

S1

S1.1 TYPES OF STRUCTURES

A structure may consist of a truss or a frame—pin-connected or rigidly secured. A truss is an assemblage of slender bars fastened together at their ends by smooth bolts or ball-and-socket joints acting as hinges. A truss, by definition, is a pin-connected structure. The bar members, therefore, act as two-force members which can either be in tension or in compression; there can be no transverse force in a member of a truss. A frame structure, on the other hand, consists of members which may be subjected to a transverse load in addition to the axial load. We shall again limit our discussion to pin-connected frames. The reason for leaving out rigidly-secured structures, such as welded trusses is that the members may then be subjected to initial loads, axial or transverse, the estimation of which is a task by itself.

A simple structure is thus a pin-connected frame or truss. A truss consists of slender-bar members which can carry no transverse loads. It follows that the loading in a truss must be at the joints only. A truss consisting of members which lie in a plane and are loaded in the same plane is called a *plane truss*. If a truss is made of non-coplanar members, it is referred to as a *space truss*.

Similarly, a frame may be a *plane frame* or a *space frame* depending upon its structure.

Let us now examine trusses with regard to their rigidity. Trusses are classified as *statically determinate* and *statically indeterminate*. If the members are allowed any relative movement, then the assemblage of members is called a *non-rigid truss* or *mechanism* and if the members are not allowed any relative movement, then it is called a *rigid truss*. A *just-rigid truss* is that which, on the removal of any single member, becomes non-rigid. An *over-rigid truss* is the one that has redundant members which may be removed to render the truss just-rigid. Examples of such trusses are shown in Fig. S1.1. We shall confine our study to the just-rigid simple trusses and frames.

The number of joints in a truss is related to the number of members. A necessary relationship between the number of joints and the number of members for a just-rigid plane truss is

$$m = 2j - 3$$

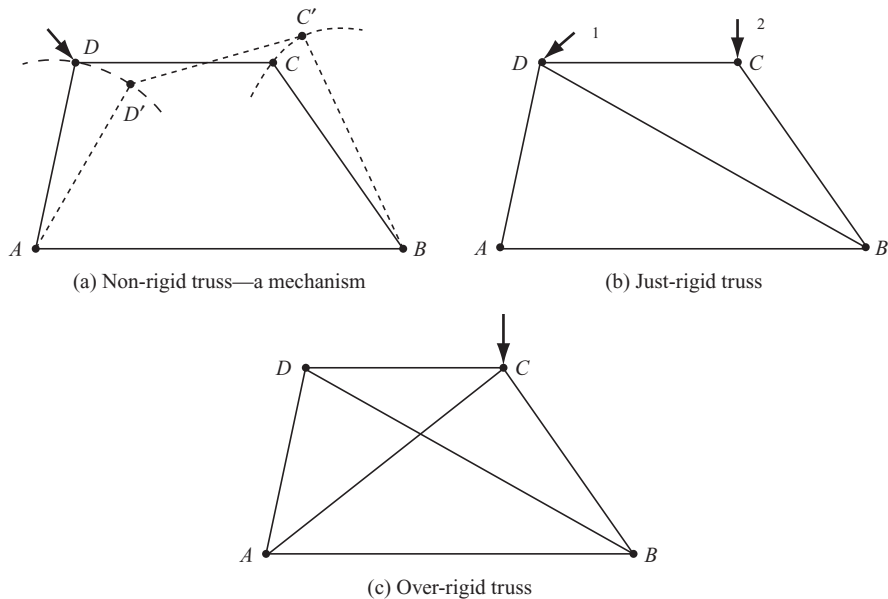


Fig. S1.1 Types of trusses

This is, however, not a sufficient relationship. In other words, a truss cannot be just-rigid if m and j are related otherwise but a truss may not be just-rigid even if m and j are related as before.

Consider, for example a simple truss $ABCDEF$ as shown in Fig. S1.2. A just-rigid truss requires the number of members to be given by

$$m = 2j - 3$$

According to the law, the number of members must be $(2j - 3)$ for a truss to be just-rigid but if the number is different from $(2j - 3)$ the following may happen:

$m < (2j - 3)$ the truss cannot be just-rigid; parts of which must be under-rigid and a part may be just-rigid or over-rigid as shown in Fig. S1.2(b)

$m > (2j - 3)$ the truss cannot be just-rigid; parts of which must be over-rigid and a part may be under-rigid or just-rigid as shown in Fig. S1.2(c).

It may be appreciated that $m = (2j - 3)$ is no guarantee for a truss to be just-rigid. Some parts of such a truss may be over-rigid and some other parts under-rigid. For example, in Fig. S1.2(d), parts $ABCF$ are under-rigid and $FCDE$ over-rigid.

It may be appreciated that simple or just-rigid trusses are generated from the basic triangular truss by successively adding a pair of new members to the existing joints and by generating a new joint by connecting the new members. Now, for a basic triangular truss which is just-rigid, the number of joints $j = 3$. For each additional joint, two members must be added to keep it just-rigid. If we wish to visualise a truss of j joints, then $(j - 3)$ joints must be added to the basic triangular truss. The number of members which will be added are $2(j - 3)$ and the total number of members become

$$m = 2(j - 3) + 3$$

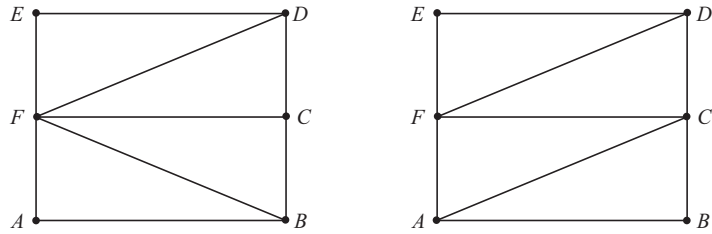
whence

$$m = 2j - 3$$

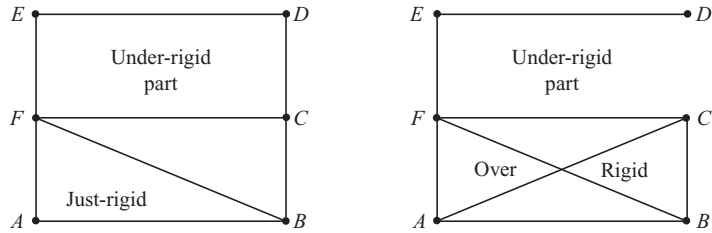
It may also be observed from the equation that the number of members in a simple just-rigid truss must be odd.

A space truss (or frame) consists of members which do not lie in a single plane. If the non-coplanar members are pin jointed, it is called a *simple space truss*. A necessary relationship between the number of joints j and the number of members m for a just rigid simple space truss is

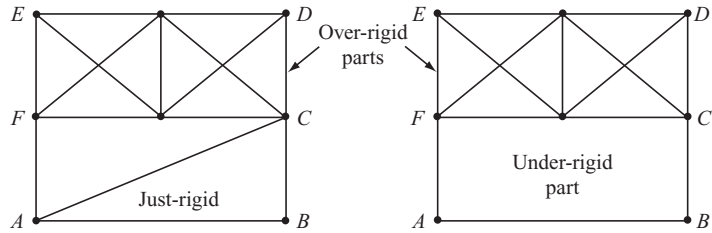
$$m = 3j - 6$$



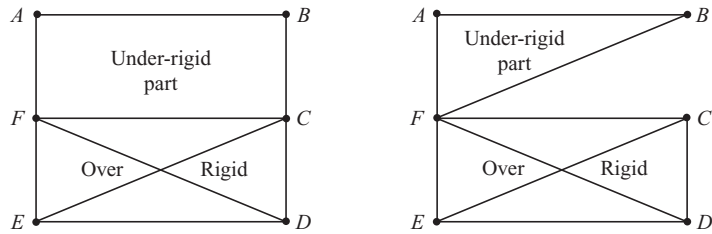
(a) Just-rigid trusses $m = (2j - 3)$



(b) $m < (2j - 3)$, non-rigid trusses



(c) $m > (2j - 3)$, part over-rigid

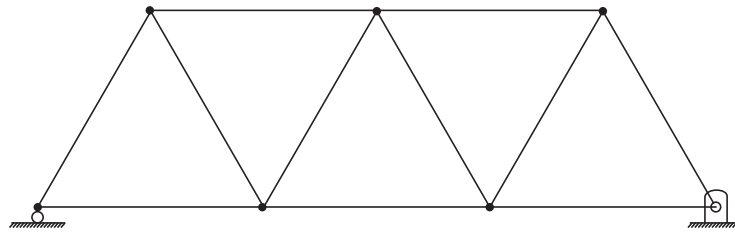


(d) $m = (2j - 3)$, non-just-rigid trusses

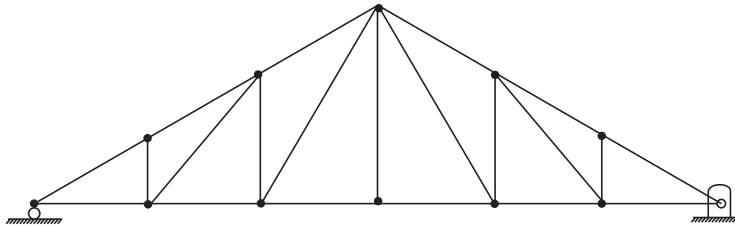
Fig. S1.2 Conditions of rigidity of a truss

Table S1.1 Typical just-rigid trusses

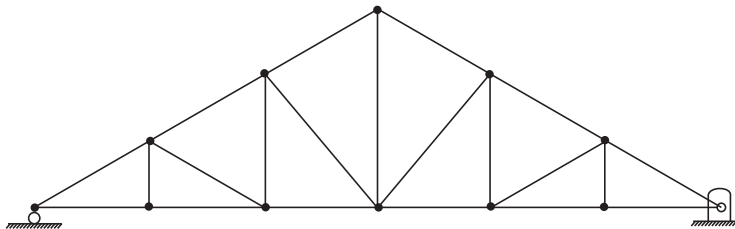
| Types of truss | Joints j | Members m | Condition $m = 2j - 3$ |
|------------------|------------|-------------|------------------------|
| (a) Warren Truss | 7 | 11 | Satisfied |
| (b) Pratt Truss | 12 | 21 | Satisfied |
| (c) Howe Truss | 12 | 21 | Satisfied |
| (d) K-Truss | 16 | 29 | Satisfied |



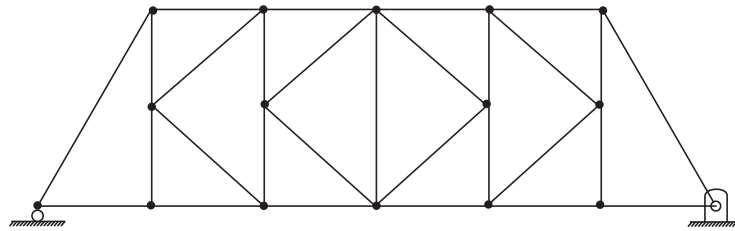
(a) Warren truss



(b) Pratt truss



(c) Howe truss



(d) K-truss

Fig.S1.3 Some standard types of trusses

There are some standard types of trusses known after the names of the originators or their shape. Some of them, *Warren truss*, *Pratt truss*, *Howe truss* and *K-truss* are shown in Figs. S1.3(a), (b), (c) and (d). Let us check the just-rigidity of the trusses:

Just-rigid trusses are *statically determinate* and the over-rigid trusses are statically indeterminate. This statement follows from the fact that the number of members in a just-rigid truss are in accordance with the necessary conditions of equilibrium for the number of joints. We shall, therefore, confine ourselves to the analysis of just-rigid trusses. The task of determining the reactions at the supports and the forces in the

members of simple plane trusses is achieved by three standard techniques known as *method of joints*, *method of sections* and *graphical method*.

This is because the basic space truss is a tetrahedron consisting of 6 members and 4 joints, i.e., $m = 6, j = 4$. It can be extended to a bigger space truss by adding 3 non-coplanar members from any 3 previously existing joints and creating one additional joint, i.e., $m = 9, j = 5$ and $m = 12, j = 6$, etc. which leads to the equation $m = 3j - 6$. Like plane truss relationship, this relationship is necessary but not a sufficient condition for a space truss to be just-rigid. For example, a truss may be partly over-rigid and partly non-rigid even if $m = 3j - 6$.

S1.2 INTERNAL FORCES: TENSION AND COMPRESSION

In a truss members are interconnected by pin joints. If a member is in tension as shown in Fig. S1.4(a), the member is pulled towards its ends on either side by external forces. Consequently, the internal forces in the member tend to resist it and hence act in a sense away from the end.

Likewise, if a member is in compression as shown in Fig. S1.4(b) the external forces compress it at its ends and the internal forces tend to resist it.

External and internal forces on a member are indeed in accordance with the action and reaction principle.

Members in a frame may be subjected only to forces along them. A member in a truss cannot be subjected to a transverse force. That is why members are said to be 2-force (or two-force) members, i.e., subjected to a pair of tensile forces or a pair of compressive forces.

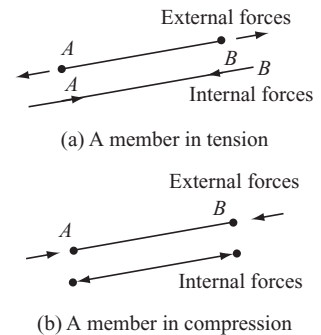


Fig. S1.4 Internal forces in members

S1.3 ANALYSIS BY THE METHOD OF JOINTS

A plane truss or frame can be subjected only to a coplanar force system. Any joint of a plane truss or frame may be subjected only to a coplanar and concurrent force system. A space truss or frame can be subjected to a spatial force system. Any joint of a space truss may be subjected to a spatial concurrent force system. The condition of concurrency of a force system at a joint in a truss follows from the equilibrium of the forces at that joint which is also the point of concurrency.

The method of joints consists of taking up one joint at a time and analysing it for equilibrium. At every joint in a truss the forces must be along the members at that joint. The forces acting at every joint must satisfy the necessary condition of equilibrium:

$$\Sigma = 0$$

which implies that

$$\Sigma_x = 0, \quad \Sigma_y = 0, \quad \Sigma_z = 0$$

for spatial forces at a joint
and

$$\Sigma_x = 0, \quad \Sigma_y = 0$$

for plane forces at a joint.

In addition to the equations of equilibrium at each joint, the overall equilibrium of a truss provides additional equations which can be used to determine the reactions of the supports. In fact, one of the first tasks in the method of joints is to evaluate the reactions from the supports. The equations of overall equilibrium are:

$$\begin{aligned} \Sigma &= 0 \\ \Sigma &= 0 \end{aligned}$$

which, for a plane truss reduce to only three equations:

$$\Sigma x = 0 \quad \Sigma y = 0$$

and

$$\Sigma = 0$$

or an alternative set of equations

$$\Sigma y = 0$$

and

$$\Sigma_1 = 0, \quad \Sigma_2 = 0$$

as has been shown earlier.

The points about which moments are taken may be the points of application of the support reactions for convenience and for simplicity in analysis.

Once the reactions are known, there must be a pin-joint in a simple just-rigid plane truss where there are only two or less unknown forces in magnitude. These values are determined by analysing the joint for equilibrium. The force in any member at a joint being known, the force at the joint at the other extremity of that member is known by the *action-reaction principle*. We then look for another joint where there are a maximum of two unknown forces in magnitudes and analyse it for equilibrium. In this manner, a chain process is set up to proceed from one joint to another and analyse the forces in the members. When the objective is to determine the force in a particular member, it is necessary to search a joint nearest to the member where to start with and proceed towards a joint where the member is connected.

It is a useful convention in the method of joints to indicate the forces at a joint such that the known forces are taken in the correct directions and the unknown forces are assumed positive, away from the joint. On evaluation, if a force turns out to be positive, the member must be in tension and if a force comes out to be negative, the member must be in compression. It is also a universal practice to designate the force at a joint due to a member by the name of the member itself. For example, the force exerted by a member BC at B or C is referred to as B .

It may also be noted that bold-face notation does not necessarily mean 'vector' but 'magnitude of vector'. Also, it may happen to use bold-face and nonbold-face interchangeably for the sake of convenience!

Some remarks on the loading conditions of some joints are also in order:

1. If there are only two members and no external force at a joint, the two members must be collinear in order that any force is taken by them. The reason for this is that the two forces maintaining a point in equilibrium must act along the same line of action and their magnitudes must be equal. The two members at the joint should, therefore, not only be collinear but also have equal forces and their nature must be the same, i.e., either both tensile or both compressive.

It follows from this that, in the special case when the two members do not have any forces, they may not be collinear. Conversely, if in a truss, there exists a joint of two members which are non-collinear and there is no external force, the forces in the members must be zero. This is shown in Fig. S1.5(a).

It also follows that if an external force acts on a two-member joint then the members cannot be collinear.

2. If there are only three members and no external force at a joint, the members will carry forces in accordance with the condition of equilibrium. If two of the three members are collinear, these two members should have equal forces and the third must be a zero-force member, otherwise equilibrium will not be maintained.

If an external force is applied at a joint of three members, the forces in the members can again be determined in the light of equilibrium of the joint. If two of the three members are collinear and the force acts in line with the third member, then the force in the third member must be equal and opposite to the external force. These facts are illustrated in Fig. S1.5(b).

3. If there are four members at a joint, the members will carry forces in accordance with equilibrium. If, however, there are two pairs of collinear members at a joint and there is no external force then the forces in the collinear members must be equal. One such example is shown in Fig. S1.5(c) where the joint B is in equilibrium under the action of two pairs of collinear forces.

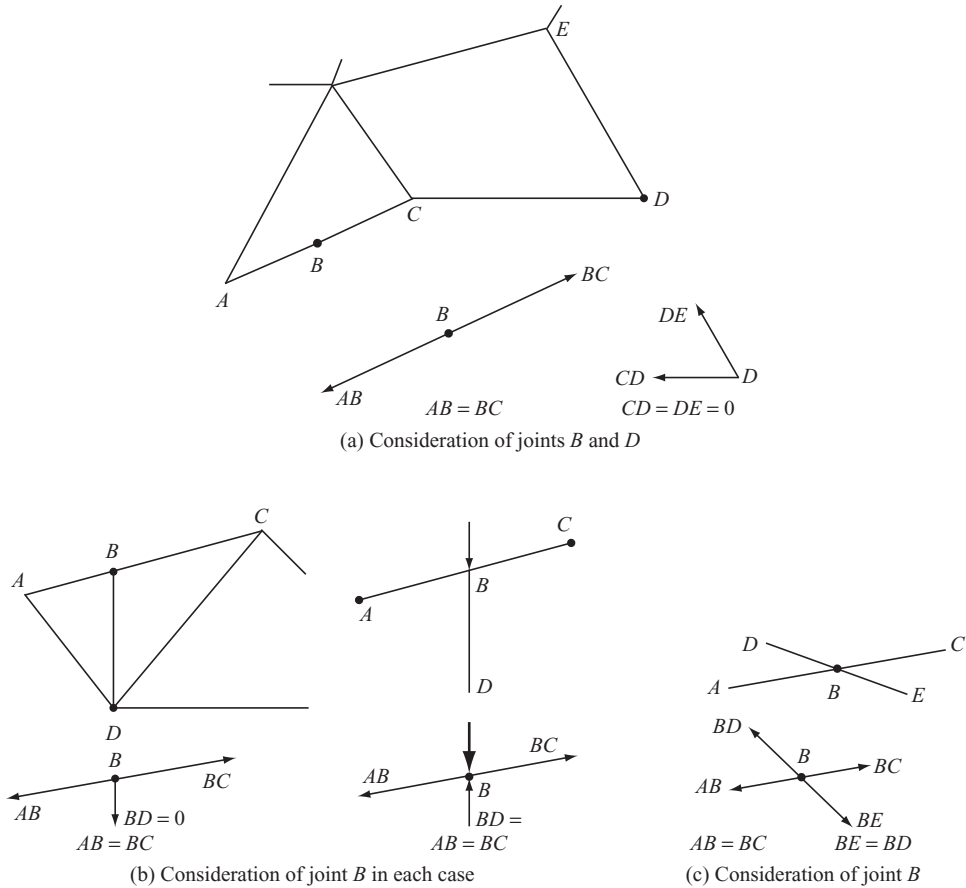


Fig.S1.5 Consideration of joints

Example S1.1

A pin-jointed frame $ABCO$ is supported and loaded as shown in Fig. Ex. S1.1. The members AB and BC are each 3 m long. Find the magnitude and nature of force in each of the members due to a load of 10 kN at the apex.

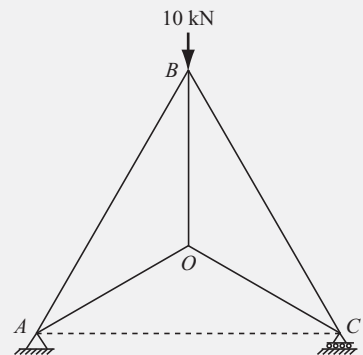


Fig.Ex. S1.1

Solution Let us first draw the *fbd* of the frame, as shown in Fig S1.1 (Solution). Considering the equilibrium of the structure as a whole

$$\begin{aligned} \Sigma_y = 0; \quad R_{Ay} + R_c - 10 &= 0 \\ \Sigma_x = 0; \quad R_{Ax} &= 0 \end{aligned}$$

and $\Sigma_A = 0; 3 R_c - 10 \times 3/2 = 0$
whence $R_c = 5 \text{ kN}, R_{Ay} = 5 \text{ kN}, R_{Ax} = 0$

For equilibrium of the joint *A* (Fig. Ex. S1.1 (Solution)),

$$\begin{aligned} \Sigma_y = 0; 5 + B \sin 30 + B \sin 60 &= 0 \\ \Sigma_x = 0; B \cos 60 + B \cos 30 &= 0 \\ &= -0.577 B \end{aligned}$$

and $5 + 0.866 B - 0.577/2 B = 0$
whence $B = -8.66 \text{ kN};$ compression
and $= 5.00 \text{ kN};$ tension

For equilibrium of the joint *B*,

$$\begin{aligned} \Sigma_x = 0; B \sin 30 - B \sin 30 &= 0 \\ \Sigma_y = 0; -10 - B - B \cos 30 - B \cos 30 &= 0 \end{aligned}$$

whence, $B = -8.66 \text{ kN};$ compression
and $B + 2 \times 0.866 B = -10$
or $B = -10 + 2 \times 8.66 \times 0.866$
or $B = 5 \text{ kN};$ tension
and $= 5 \text{ kN};$ tension

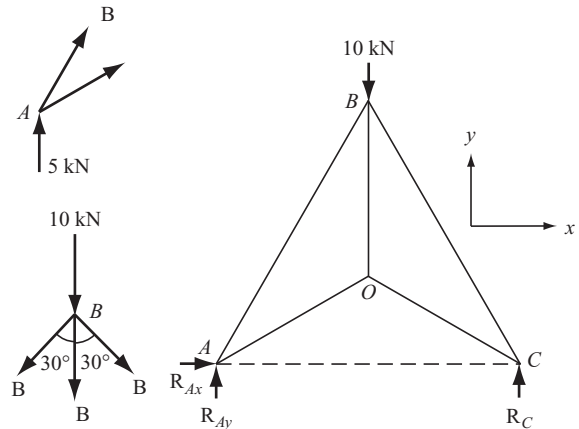


Fig. Ex. S1.1 (Solution)

Example S1.2 A frame *PQRSTU* is hinged to a rigid support at *P* and is simply supported at *T*. It is loaded as indicated in the Fig. Ex. S1.2. Estimate the magnitude and nature of the force in the members *PQ* and *UT*.

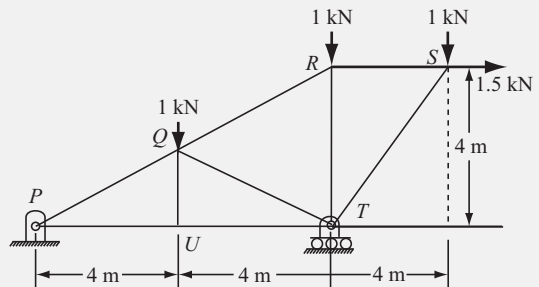


Fig. Ex. S1.2

Solution The free-body diagram of the structure is shown in Fig. Ex. S1.2. For equilibrium,

$$\begin{aligned} \Sigma_x &= 0 \\ \Sigma_y = 0; P_y + &- 1 - 1 - 1 = 0 \\ P_y + &= 3 \text{ kN} \end{aligned}$$

$$\begin{aligned} \Sigma x = 0; 1.5 - P_x &= 0 \\ P_x &= 1.5 \text{ kN} \\ \Sigma p &= 0 \\ 8T - 1 \times 4 - 1 \times 8 - 1 \times 12 - 1.5 \times 4 &= 0 \\ \text{whence} \quad &= 3.75 \text{ kN} \\ \text{and hence} \quad P_y &= 3 - 3.75 = -0.75 \text{ kN} \end{aligned}$$

Consider the equilibrium at joint P :

$$\begin{aligned} \Sigma x = 0; -1.5 + P \cos \theta &= 0 \\ \Sigma y = 0; P \sin \theta - 0.75 &= 0 \\ \text{Taking} \quad \sin \theta &= \frac{2}{\sqrt{20}} = 0.447 \\ \text{and} \quad \cos \theta &= \frac{4}{\sqrt{20}} = 0.895 \end{aligned}$$

$$\begin{aligned} P &= 0.75/0.447 = 1.68 \text{ kN; tension} \\ P &= 1.5 - 1.68 \times 0.895 = 0 \end{aligned}$$

Since UQ is perpendicular to PU and UT , UQ can transmit no force and the force in UT must be the same as in PU which is zero.

Hence $\quad \quad \quad = 0$

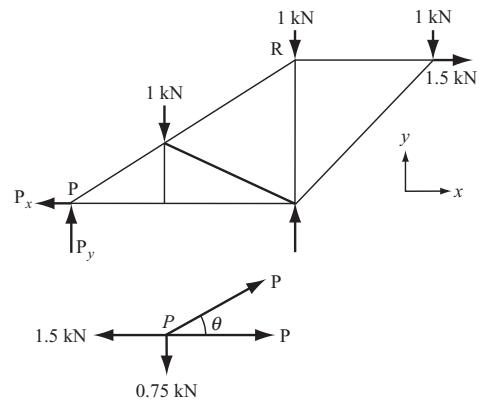


Fig. Ex. S1.2 (Solution)

Example S1.3

A simple structure $ABCDE$ is supported on a hinge at A and on rollers at B while it carries a horizontal force of 1000 kN at E as shown in Fig. Ex. S1.3. Determine the force in member AC , using the method of joints.

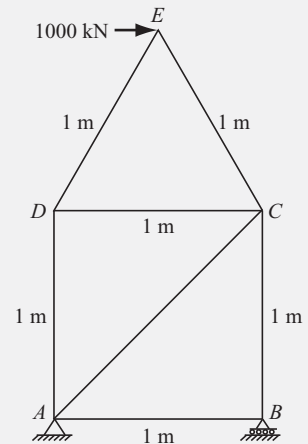


Fig. Ex. S1.3

Solution:

Method of Joints Starting with E

For joint E ,

$$\begin{aligned} \Sigma F_x = 0; 1000 + EC \sin 30 - ED \sin 30 &= 0 \\ \Sigma F_y = 0; -EC \cos 30 - ED \cos 30 &= 0 \end{aligned}$$

whence $-EC = ED = 1000 \text{ kN}$

Next, for joint D ,

$$\Sigma F_x = 0; \quad DE \cos 60 + CD = 0$$

$$\Sigma F_y = 0; \quad DE \sin 60 - DA = 0$$

whence, $CD = -500 \text{ kN}$

$$DA = 866 \text{ kN}$$

Finally, for joint C ,

$$\Sigma F_x = 0; \quad -DC - EC \cos 60 - AC \cos 45 = 0$$

$$\Sigma F_y = 0; \quad -BC - AC \sin 45 + EC \sin 60 = 0$$

and from the former

$$500 + 1000/2 - AC \times 0.707 = 0$$

whence $AC = \frac{1000}{0.707} = 1414 \text{ kN}$

which implies that the force in AC is 1414 kN tensile. The internal forces in AC are shown in Fig. Ex. S1.3 (Solution) and it is clear that the external force on AC must be tensile.

Method of Joints Starting with B

For this method, we should first determine the reactions at A and B .

Let the reaction at hinge A be

$$A_x + A_y$$

and the reaction at the roller support B be vertical, i.e., B .

For the equilibrium of the structure,

$$\Sigma F_x = 0; \quad A_x + 1000 = 0$$

$$\Sigma F_y = 0; \quad A_y + B = 0$$

$$\Sigma M_A = 0; \quad B \times 1 - 1000 \times 1.866 = 0$$

whence $B = 1866, A_x = -1000$ and $A_y = -1866 \text{ kN}$

At B , the reaction is vertical and there are two members BA and BC ; one horizontal and the other vertical. The horizontal member can carry no force because if it did it would not be balanced.

For joint A ,

$$\Sigma F_x = 0; \quad A_x + AC \cos 45 = 0$$

$$\Sigma F_y = 0; \quad A_y + AD + AC \sin 45 = 0$$

whence $AC = \frac{1000}{0.707} = 1414 \text{ kN}$

The internal or resistive forces in AC are outwards which, as before, imply that the member AC carries a tensile force of 1414 kN.

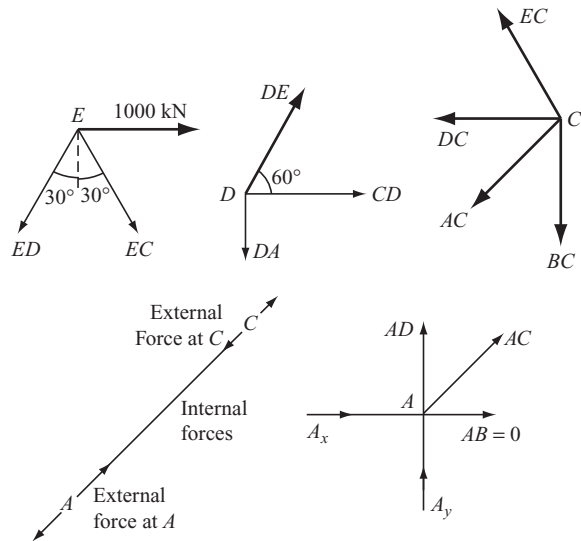


Fig. Ex. S1.3 (Solution)

EXPERIMENT E3 FORCES IN A PLANE TRUSS

OBJECTIVE TO DETERMINE THE FORCES IN THE MEMBERS OF A STATICALLY DETERMINATE PLANE TRUSS.

Apparatus A pin-jointed, simply supported plane determinate truss, some members of which have spring balances installed in them. Standard weights and a metre rod.

Background Information The forces in the members of a truss may be computed analytically by the method of joints and by the method of sections and also graphically based upon the concept of equilibrium of the whole or of a part of the truss. Experimentally, the spring balances installed within the members are read off without loading and with loading in order to estimate the forces developed due to the loading of the truss. A typical truss *ABCDEFGHJIJ* hinge-supported at *A* and roller-supported at *G* as shown in Fig. E3.1, may be subjected to loads at *C*, *D* and *E*, for example. If it is required to determine the force in member *AC*, the procedure would be to determine the reactions R_A and R_G at *A* and *G* respectively in the first instance. This may be done by considering the equilibrium of the entire truss. For the vertical loading as given, R_A and R_G are directed upwards.

$$\Sigma \quad = 0; \quad R_A + R_G + W_C + W_D + W_E = 0 \tag{i}$$

$$\Sigma \quad = 0; \quad W_C 3l + W_D 2l + W_E l - R_A 4l = 0 \tag{ii}$$

whence R_A and R_G are determined.

By the method of joints, one would first consider the simplest joint *B* where only two members *AB* and *BC* are pinned at right angles. From the free-body diagram of joint *B*, it is seen that it can be in equilibrium only if each of *BA* and *BC* is zero, because if either is non-zero it will leave an unbalanced force at *B*. Next, consider the joint *A*. The free-body diagram of joint *A* shows that there are two forces *AJ* and *AC* unknown in magnitude as shown in Fig. E3.2. From the equilibrium of *A*,

$$\Sigma F_x = 0; \quad AJ + AC \sin \theta = 0 \tag{iii}$$

$$\Sigma F_y = 0; \quad R_A - AC \cos \theta = 0 \tag{iv}$$

The solution of Eqs (iii) and (iv) provide the desired force in the member *AC*. Whether the member is in tension or compression may be sorted out by observing the sign of *AC*. It would come out to be positive in this

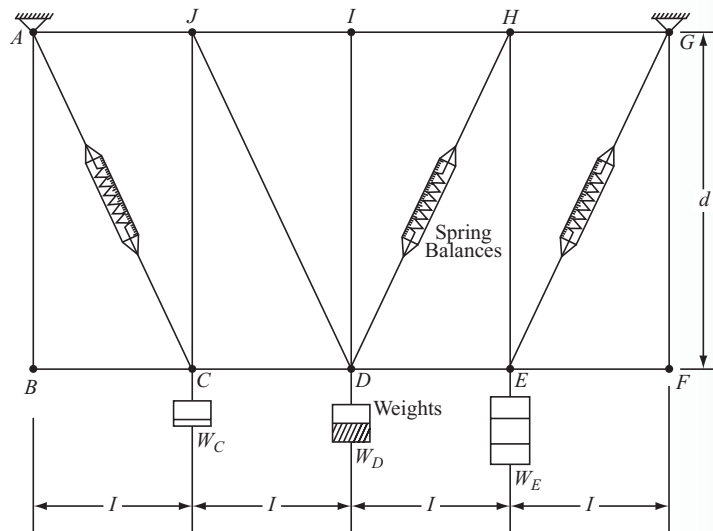


Fig. E3.1 Free-body diagram of a loaded plane truss

case which may be interpreted as tension, following the text; otherwise, a free-body diagram of the member AC may be drawn as also shown in Fig. E3.3. The internal forces being inward, the external forces at A and C must be outward which suggest that the member AC must bear a tensile force.

By the method of sections, one would cut a section through members AJ , AC and BC . Consequently external forces AJ , AC and BC are shown acted upon the left section in its free-body diagram drawn in Fig. E3.3. Consider the section for equilibrium:

$$\Sigma M_A = 0; \quad BC \cdot d = 0$$

whence the force BC is seen to be zero.

$$\Sigma F_x = 0; \quad AJ + AC \sin \theta + BC = 0$$

$$\Sigma F_y = 0; \quad R_A - AC \cos \theta = 0$$

The desired force AC is thus obtained from these equations. The sign of AC would decide whether it is tensile or compressive. It would come out to be positive in this case which may be interpreted as tension; otherwise, as noted from the fact that the external force AC on the member AC is outward, the member must be in tension.

Experimentally, the spring balance installed in the member AC would show a tensile force acting on the member when the truss is loaded as shown.

Table of Results Forces in the members with spring balances for the prescribed loading should be reported.

| Forces in members | AC | DH | EG |
|-------------------|------|------|------|
| Experimentally | | | |
| Analytically | | | |
| Difference | | | |

Results on the linearity of response of the system should be reported in the form of a curve plotted between the load at C and the force in AC .

Points for Discussion

Recognise the sources of error in the experiment. In particular, observe the play in the spring balances and joints. The use of a spring balance which operates on the principle of elongation of the member is inherently prone to error.

Suggest some means of eliminating the sources of error. One of the ways of minimising the error in the experiment would be to use larger loads for the first part of the experiment and to change the load in larger steps in the second part of the experiment.

Explain the truth in the statement: The method of joints is a special case of the method of sections .

Do the forces in members of a truss respond linearly to loading? How?

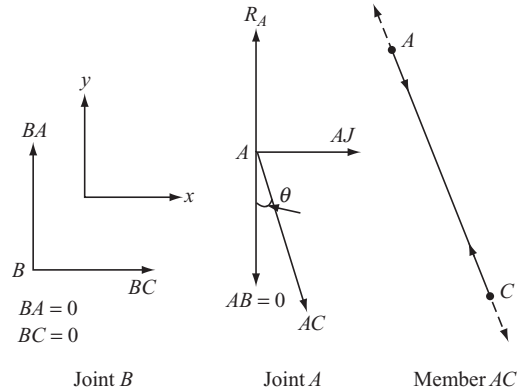


Fig. E3.2 Free-body diagram

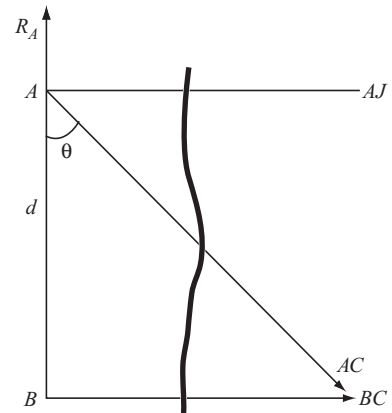


Fig. E3.3 Free-body diagram of the section

S1.4 ANALYSIS BY THE METHOD OF SECTIONS

The method of sections consists of hypothetically cutting a section of the given truss and analysing it for equilibrium. Equilibrium of the entire truss guarantees the equilibrium of every part of the truss. In the method of joints, equilibrium of every joint was considered. In the method of sections, equilibrium of any selected section of the truss is considered. The section of the truss is selected in such a way as to 'cut' the desired member only once. The free-body diagram of the section will thus include the unknown force in the member. The analysis of the section for equilibrium requires the application of

$$\Sigma = 0 \quad \text{and} \quad \Sigma = 0$$

For a plane section, the necessary conditions of equilibrium reduce to

$$\Sigma F_x = 0, \Sigma F_y = 0 \quad \text{and} \quad \Sigma M_z = 0$$

or an equivalent set of three equations.

The force system acting on a section of a plane truss can only be a coplaner force system. The three necessary equations as above are also sufficient conditions of equilibrium.

The free-body diagram of the desired section of a truss may have one, two, three or more unknown forces. The equations of equilibrium written for a section may or may not be adequate to determine all the unknowns. In case a selected section of the truss is not amenable to solution, an attempt must be made to locate a section which includes some of the unknown forces of the desired section that can be determined. The knowledge gained from the analysis of simpler sections must enable us to determine all the unknowns. It may be appreciated that the method of joints is a special case of the method of sections. When a section is chosen in the vicinity of a joint so as to enclose the joint, the section in question reduces to the joint only. Equilibrium of the section implies equilibrium at the joint. The method of joints is thus the method of sections applied to enclose one joint at a time.

It is a useful convention in the method of sections to indicate the forces on a section such that the known forces are taken in the correct directions and the unknown forces are assumed acting away from the section under consideration. On evaluation, if a force turns out to be positive, the member must be in tension and if a force comes out to be negative, the member must be in compression. The forces are named after the names of the members for convenience in recognition.

Example S1.4

A simple structure $ABCDE$ is supported on a hinge at A and on rollers at B while it carries a horizontal force of 1000 kN at E as shown in Fig. Ex. S1.4. Determine the force in member AC , using the method of sections.

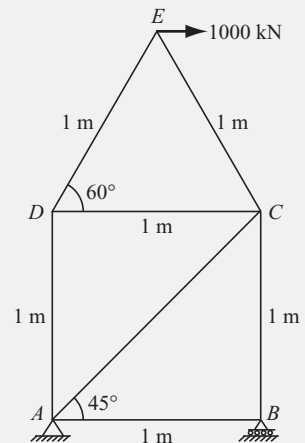


Fig. Ex. S1.4

Solution

Method of Sections (a)

A section *SS* can be cut through members *AD*, *AC* and *BC* as shown in Fig. Ex. S1.4 (Solution), and a consideration of equilibrium of either side of the section should lead to the force in *AC*. Consider the upper part of the structure. The external forces on the part are

- (a) 1000 kN at *E*, acting horizontally
- (b) Forces along *AD*, *AC* and *BC* due to the lower part of the structure; tensile if away from the section under consideration.

For equilibrium,

$$\begin{aligned} \Sigma F_x &= 0; 1000 - AC \cos 45^\circ = 0 \\ \Sigma F_y &= 0; -AD - BC - AC \sin 45^\circ = 0 \end{aligned}$$

and from the former,

$$\begin{aligned} AC &= \frac{1000}{0.707} \\ &= 1414 \text{ kN} \end{aligned}$$

which shows that the member *AC* is subjected to 1414 kN tensile force.

Method of Sections (b)

It is instructive to consider the free-body diagram and the equilibrium of the lower part of the section *SS* for the desired purpose and to show that it involves more work. The additional work is required to calculate the reactions by considering the equilibrium of the entire structure first. After obtaining that

$$\begin{aligned} B_y &= 1866 \text{ kN} \\ A_x &= -1000 \text{ kN} \\ A_y &= -1866 \text{ kN} \end{aligned}$$

and

The equilibrium of the lower part requires that

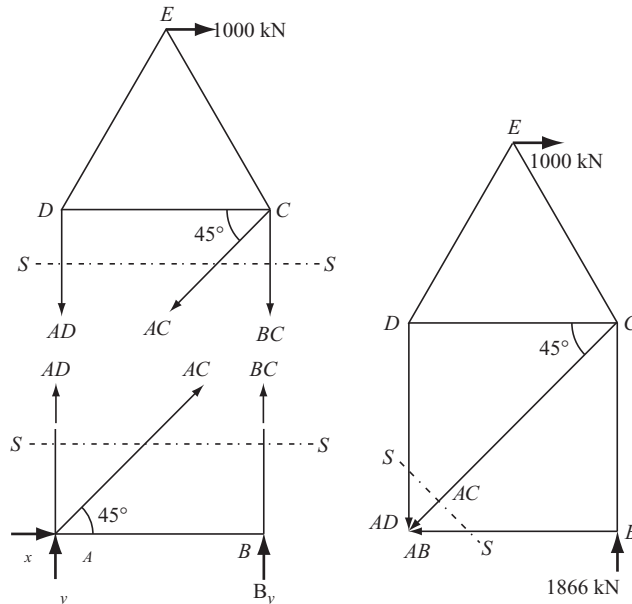


Fig. Ex. S1.4 (Solution)

$$\begin{aligned} \Sigma \quad &= 0; \quad + \quad \cos 45 = 0 \\ \Sigma \quad &= 0; \quad + \quad + \quad + \quad + \quad \sin 45 = 0 \end{aligned}$$

whence

$$= \frac{1000}{0.707} = 1414 \text{ kN}$$

which shows that the member is subjected to 1414 kN tensile force.

The section may instead be cut through members , and and the part of the structure on its right can be considered for equilibrium

$$\Sigma \quad = 0; \quad 1000 - \quad - \quad \cos 45 = 0 \tag{i}$$

$$\Sigma \quad = 0; \quad 1866 - \quad - \quad \sin 45 = 0 \tag{ii}$$

$$\Sigma \quad = 0; \quad -1000 \times 1.866 + 1 \times \quad + \quad \times 0.707 = 0 \tag{iii}$$

whence,

$$+ 0.707 \quad = 1000 \tag{i}$$

$$+ 0.707 \quad = 1866 \tag{ii}$$

$$+ 0.707 \quad = 1866 \tag{iii}$$

and it can be noticed that the Eqs (ii) and (iii) are identical. The three unknown forces , and cannot be evaluated by only two independent equations.

Another look on the section reveals that the three forces under question are concurrent. Such sections which render three concurrent forces as unknowns are not plausible.

The way out of this difficulty can be found by evaluating one of the three unknowns by considering some other section or joint for equilibrium. In this case, we notice from the consideration of equilibrium at , that the force in is zero.

Hence, from Eq. (i),

$$\begin{aligned} 0.707 \quad &= 1000 \\ &= 1414 \text{ kN} \end{aligned}$$

which shows that must carry a tensile force of 1414 kN.

Example S1.5

A derrick crane has ball-and-socket joints at , , and as shown in Fig. Ex. S1.5. Determine the forces in links , and when it is supporting a dead load of 1000 kN.

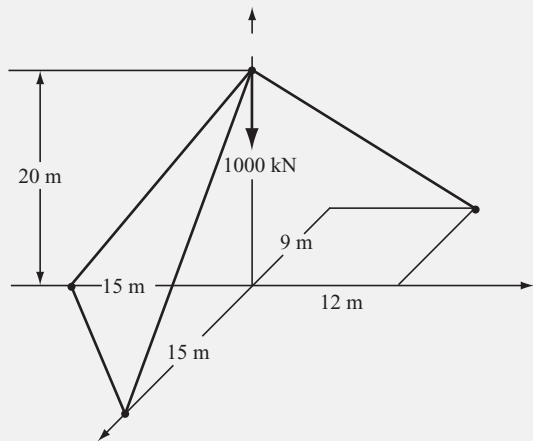


Fig. Ex. S1.5

Solution With respect to the given origin O , position vectors of the salient points are as follows:

$$\begin{aligned} &= -15 \\ \mathbf{B} &= 15 \\ &= 12 \quad -9 \\ &= 20 \end{aligned}$$

It follows that the unit vectors along the three members all pointing to D are:

$$\mathbf{e}_{AD} = \frac{\mathbf{OD} - \mathbf{OA}}{\sqrt{\mathbf{OD}^2 + \mathbf{OA}^2}} = \frac{20 \mathbf{j} + 15 \mathbf{i}}{25} = 0.8 \mathbf{j} + 0.6 \mathbf{i}$$

$$\mathbf{e}_{BD} = \frac{20 \mathbf{j} - 15 \mathbf{k}}{25} = 0.8 \mathbf{j} - 0.6 \mathbf{k}$$

$$\mathbf{e}_{CD} = \frac{20 \mathbf{j} - 12 \mathbf{i} + 9 \mathbf{k}}{25} = -0.48 \mathbf{i} + 0.8 \mathbf{j} + 0.36 \mathbf{k}$$

Let the magnitudes of forces in members be denoted by AD , BD and CD , then for equilibrium,

$$AD (0.8 \quad +0.6 \quad) + BD (0.8 \quad -0.6 \quad) + CD (-0.48 \quad +0.8 \quad +0.36 \quad) - 1000 = 0$$

This vector equation is written as three component scalar equations:

$$\begin{aligned} 0.6 AD - 0.48 CD &= 0, && \text{for } x\text{-direction} \\ 0.8 AD + 0.8 BD + 0.8 CD - 1000 &= 0, && \text{for } y\text{-direction} \\ -0.6 BD + 0.36 CD &= 0, && \text{for } z\text{-direction} \end{aligned}$$

and

Solving these equations,

$$AD = 416.7 \text{ kN}, \quad BD = 312.5 \text{ kN}, \quad CD = 520.83 \text{ kN}.$$

The unit vectors were taken pointing to the common point D . Now that AD , BD and CD all turn out positive the members exert forces pointing to D as shown in Fig. Ex. S1.5 (Solution) which means the members must be under compression.

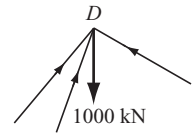


Fig. Ex. S1.5 (Solution)

Example S1.6

A simple crane rests on a ball-and-socket joint O and it is supported by two strings AB and AC as shown in Fig. Ex. S1.6. Determine the forces in the cables and the reaction at O when it holds a load of 1280 kN.

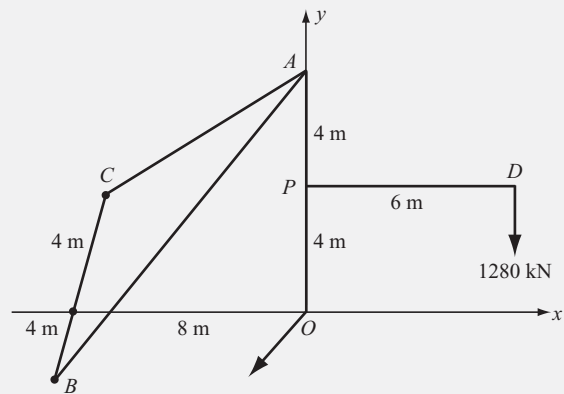


Fig. Ex. S1.6

Solution Position vectors of A , B , C and D are as follows:

$$\begin{aligned} &= 8 \\ \mathbf{B} &= -8 \mathbf{i} + 4 \\ &= -8 \mathbf{i} - 4 \\ &= 4 \mathbf{j} + 6 \end{aligned}$$

and the load applied at D is -1280 kN .

The unit vectors along the cables, pointing towards A are

$$\begin{aligned} \mathbf{e}_{CA} &= \frac{8\mathbf{j} + 8\mathbf{i} + 4\mathbf{k}}{12} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \\ \mathbf{e}_{BA} &= \frac{8\mathbf{j} + 8\mathbf{i} - 4\mathbf{k}}{12} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \end{aligned}$$

The negative sign implies that the forces exerted by the strings are not pointing towards A but away from A , which means that they must be in tension. This is indeed true for strings and cables.

The ball-and-socket joint at O cannot resist any moment. It can provide only force reactions. Therefore, using

$$\Sigma = 0$$

for equilibrium of the crane structure,

$$\begin{aligned} 720 \left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) + 720 \left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right) \\ -1280 \mathbf{j} + R_x \mathbf{i} + R_y \mathbf{j} + R_z \mathbf{k} = 0 \end{aligned}$$

whence

$$\begin{aligned} R_x &= 960 \text{ kN} \\ R_y &= 2240 \text{ kN} \\ R_z &= 0 \end{aligned}$$

Taking moments about the joint O ,

$$\begin{aligned} (-8 \mathbf{i} + 4 \mathbf{j}) \times AB + (-8 \mathbf{i} - 4 \mathbf{j}) \times AC + (4 \mathbf{j} + 6 \mathbf{k}) \times (-1280 \mathbf{j}) \\ = (-8\mathbf{i} + 4\mathbf{j}) \times \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right) AB + (-8\mathbf{i} - 4\mathbf{j}) \times \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \right) AC \\ + (6 \mathbf{j} + 4 \mathbf{k}) \times (-1280 \mathbf{j}) \\ = \left(-\frac{16}{3}\mathbf{k} - \frac{8}{3}\mathbf{j} + \frac{8}{3}\mathbf{j} - \frac{8}{3}\mathbf{i} \right) AB - \left(\frac{16}{3}\mathbf{k} + \frac{8}{3}\mathbf{j} - \frac{8}{3}\mathbf{j} + \frac{8}{3}\mathbf{i} \right) AC - 7680 \mathbf{k} \\ = -3000 \mathbf{k} - \frac{8}{3}(AB - AC)\mathbf{i} - \frac{16}{3}(AB + AC)\mathbf{k} \end{aligned}$$

Now

$$\mathbf{e}_{OB} = \frac{-8\mathbf{i} + 4\mathbf{k}}{\sqrt{80}} = \frac{-2\mathbf{i} + \mathbf{k}}{\sqrt{5}}$$

and

$$\mathbf{e}_{OC} = \frac{-2\mathbf{i} - \mathbf{k}}{\sqrt{5}}$$

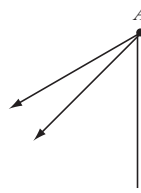


Fig. Ex. S1.6(a) (Solution)

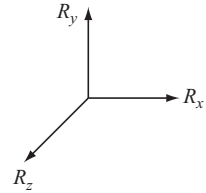


Fig. Ex. S1.6(b) (Solution)

The moment about axis OB , i.e.,

$M_{OB} = \mathbf{O} \cdot \mathbf{O}B$ must be zero for equilibrium.

$$\left[-\frac{8}{3}(AB - AC)\mathbf{i} - \frac{16}{3}(AB + AC + 1440)\mathbf{k} \right] \cdot \left[\frac{-2\mathbf{i} + \mathbf{k}}{\sqrt{5}} \right] = 0$$

$$\frac{16}{3\sqrt{5}}(AB - AC) - \frac{16}{3\sqrt{5}}(AB + AC + 1440) = 0$$

or
$$\frac{-32}{3\sqrt{5}} AC = \frac{16}{3\sqrt{5}} \times 1440$$

or
$$AC = -720 \text{ kN.}$$

By symmetry (or by considering the moment about axis OC), $AB = -720 \text{ kN.}$

Example S1.7

A vertical pole CD is supported on a hinge D at the base and two guy wires tied at its top as shown in Fig. Ex. S1.7. The pole is supporting a horizontal pull of 1 kN along x -axis. Determine the reactions at the support A , B and C .

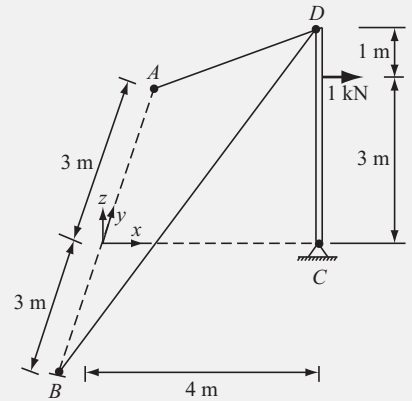


Fig. Ex. S1.7

Solution Reactions at A and B should be along the guy wires; Reaction at C consists of R_x , R_y and R_z .

From the coordinates of the points $A(0, 3, 1)$, $B(0, -3, 1)$, $C(4, 0, 0)$ and $D(4, 0, 4)$, unit vectors along the wires are

$$\mathbf{e}_{DA} = \frac{(-4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})}{\sqrt{34}} \quad \mathbf{F}_{DA} = F\mathbf{e}_{DA}$$

$$\mathbf{e}_{DB} = \frac{(-4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})}{\sqrt{34}} \quad \mathbf{F}_{DB} = F\mathbf{e}_{DB}$$

By symmetry forces in guy wires must be equal. As shown in Fig. Ex. S1.7 (Solution) the force exerted by the wires on the pole at D is

$$\mathbf{F}_{DA} + \mathbf{F}_{DB} = \frac{(-8\mathbf{i} - 6\mathbf{j})}{\sqrt{34}} F' = (-8\mathbf{i} - 6\mathbf{j}) F'$$

For equilibrium of the pole,

$$(-8\mathbf{i} - 6\mathbf{j}) F' + 1000\mathbf{i} + R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k} = 0$$

The isolated component along y ; $R_y = 0$.

$$-8F' + 1000 + R_x = 0$$

$$-6F' + R_z = 0$$

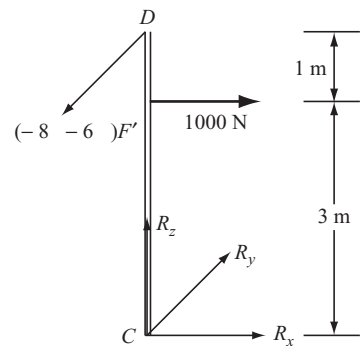


Fig. Ex. S1.7 (Solution)

and $\Sigma c = 0; 4 \times (-8 - 6) F' + 3 \times 1000 = 0$

whence $(-32 F' + 3000) = 0; F' = 93.75$

and $F = \sqrt{34} \times 93.75 = 547 \text{ N}$

Also, $R_x = 8 \times 93.75 - 1000 = -250 \text{ N}$

$R_z = 6 \times 93.75 = 562.5 \text{ N}$

Let us check the solution. Taking moments about D , for example,

$-1000 \times 1 - 4 R_x = 0; R_x = -250 \text{ N}$

which is the same as before.

Example S1.8

A simple smoothly jointed frame work hinged to vertical wall at A and B , as shown, carries a load of 400 N at D . Find the forces in members EC and BC .

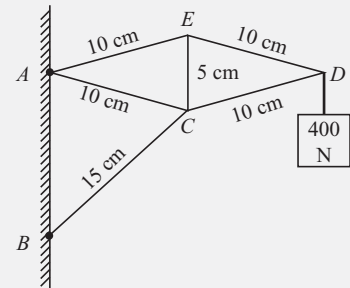


Fig. Ex. S1.8

Solution The first step is to determine the reactions at A and B . From the *fbd* of the entire framework, Fig. Ex. S1.8(a) (Solution).

From geometry,

$AP = \sqrt{10^2 - 2.5^2} = 9.7 \text{ cm} = KC$

$AD = 2 AP = 19.4 \text{ cm}$

$AK = 2.5 \text{ cm}$

$KB = \sqrt{15^2 - 9.7^2} = 11.4 \text{ cm}$

$AB = 2.5 + 11.4 = 13.9 \text{ cm}$.

For equilibrium,

$\Sigma F_x = 0 = \Sigma F_y$

$R_{Ax} + R_{Bx} = 0$

$R_{Ay} + R_{By} - 400 = 0$

and $M_A = R_{Bx} AB - 400 AD = 0$

whence $R_{Bx} = 400 \cdot \frac{AD}{AB} = 400 \cdot \frac{19.4}{13.9} = 644 \text{ N}$

Therefore, $R_{Ax} = R_{Bx} = -644 \text{ N}$.

It is not possible to determine R_{Ay} and R_{By} from the single equation between them!

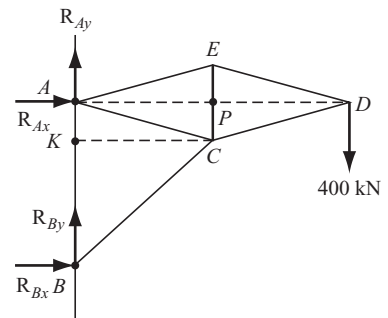


Fig. Ex. S1.8(a) (Solution)

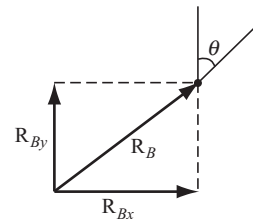


Fig. Ex. S1.8(b) (Solution)

But we know that there is a hinge at B and just one member BC which can only bear a tensile or compressive force; the reaction at B must, therefore, be along BC . Hence

$$\frac{R_{By}}{R_{Bx}} = \tan \theta = \frac{KC}{KB} = \frac{9.7}{11.4} = 0.85$$

and $R_{By} = 0.85 \times 644 = 547 \text{ N}$

$$R_{Ay} = 400 - 547 = -147 \text{ N}$$

Now, Let us proceed to determine the required forces in the members:

For equilibrium at joint D ,

$$DE = 400 \times \frac{10}{5} = 800 \text{ N (tension)}$$

$$DC = 400 \times \frac{10}{5} = 800 \text{ N (compression)}$$

For equilibrium at E ,

$$EC = 400 \text{ N (compression)}$$

$$EA = 800 \text{ N (tension)}$$

For equilibrium at C , by measuring to the scale,

$$CA = 230 \text{ N (compression)}$$

$$CB = 860 \text{ N (compression)}$$

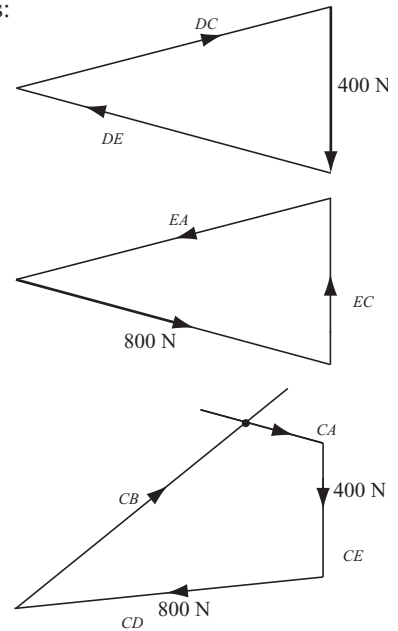


Fig. Ex. S1.8(c) (Solution)

S1.5 ANALYSIS BY THE METHOD OF TENSION COEFFICIENTS

This method is another version of the method of joints. It is a powerful method to analyse pin-jointed 3-dimensional space frames and it can be adapted for computer-based analysis.

In this method, for each joint, we consider the length of the members and the coordinates of the joints. All the members are initially assumed to be in tension in order to commence the analysis.

What is a tension coefficient?

If the tension in a member is assumed as T and the length of the member is L , then

$$t = \frac{T}{L} \text{ is the tension coefficient for that member.}$$

Let us consider a joint A with members AB , AC , AD , etc. pin-jointed at A . Let us concentrate on a member AB with tension (assumed) T_{AB} and the coordinates of A be (x_A, y_A) and B be (x_B, y_B) .

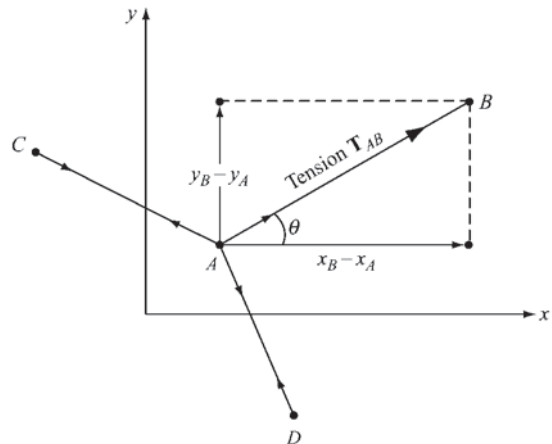


Fig. S1.6

With reference to the chosen coordinates,

$$\cos \theta = \frac{x_B - x_A}{AB}; \quad \sin \theta = \frac{y_B - y_A}{AB}$$

The tension T_{AB} in AB can be resolved into the component along x -axis as

$$T_{AB} \cos \theta = \frac{AB}{AB} (x_B - x_A) = T_{AB} (x_B - x_A)$$

Likewise, the component of T_{AB} along the y -axis is

$$T_{AB} \sin \theta = \frac{AB}{AB} (y_B - y_A) = T_{AB} (y_B - y_A)$$

where $T_{AB} = \frac{T_{AB}}{AB}$, the tension coefficient for bar AB .

One can write similar equations for each of the other members AC, AD , etc., and then apply the conditions of equilibrium to determine the magnitude and nature of forces in the members. 'Positive' forces imply 'tension' and 'negative' compression.

The method of tension coefficients may simply be extended to space frames, merely by writing 3 equations, i.e., in x, y and z directions (in place of 2 for plane frames) for each joint. If a space frame has j joints, then we shall get $3j$ equations.

Example S1.9

A load of 7.2 kN is supported by two ropes, suspended from two points 5 m apart. Determine the forces in the ropes.

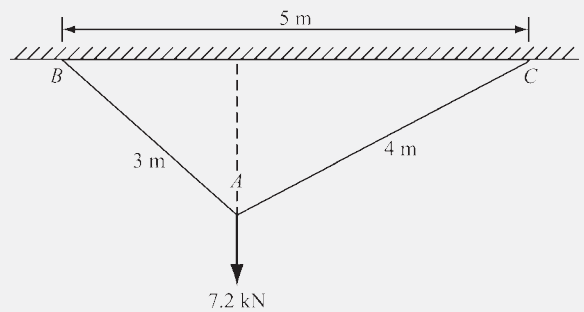


Fig. Ex. S1.9

Solution Let us solve the problem by two different methods.

By the Method of Joints

There is first one joint A . We consider the equilibrium of the joint A . Noting that the ropes subtend a right angle at A ; also the angles BPA and APC are right angles, we can get $BP = 1.8$ m, $PC = 3.2$ m and $AP = 2.4$ m.

$$\begin{aligned} \Sigma F_x = 0; \quad T_{AC} \cos \theta - T_{AB} \cos \phi &= 0 \\ \frac{3.2}{4} T_{AC} - \frac{1.8}{3} T_{AB} &= 0 \\ \text{or, } 0.8 T_{AC} - 0.6 T_{AB} &= 0 \end{aligned}$$

$$\begin{aligned} \Sigma F_y = 0; \quad T_{AC} \sin \theta + T_{AB} \sin \phi - 7.2 &= 0 \\ \frac{2.4}{4} T_{AC} + \frac{2.4}{3} T_{AB} - 7.2 &= 0 \\ \text{or, } 0.6 T_{AC} + 0.8 T_{AB} - 7.2 &= 0 \end{aligned}$$

from (i) and (ii),

$$T_{AB} = 5.76 \text{ kN}; \quad T_{AC} = 4.32 \text{ kN.}$$

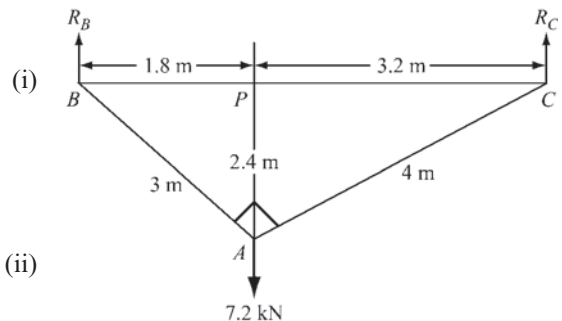


Fig. Ex. S1.9(a) (Solution)

By the Method of Tension Coefficients

Now, we write the equations for the joint *A*, in *x* and *y* directions:

Along *x*-axis, $-1.8 t_{AB} + 3.2 t_{AC} = 0$

Along *y*-axis, $2.4 t_{AB} + 2.4 t_{AC} - 7.2 = 0$

where t_{AB} and t_{AC} are the tension coefficients for the members *AB* and *AC* respectively.

From these equations, $t_{AB} = 1.92$, $t_{AC} = 1.08$.

We know, by definition of tension coefficients,

$$t_{AB} = \frac{AB}{AB} \quad \text{or} \quad t_{AC} = \frac{AC}{AC}$$

whence,

$$AB = 1.92 \times 3 = 5.76 \text{ kN}$$

and

$$AC = 1.08 \times 4 = 4.32 \text{ kN}$$

Positive values of both t_{AB} and t_{AC} show that these are indeed tensions!

Comment

On the first look, the method of tension coefficients may not appear to be ‘significantly better’ than the method of joints but it is not true. The method of tension coefficients proves to be more useful and much easier for 3-dimensional or space frames!

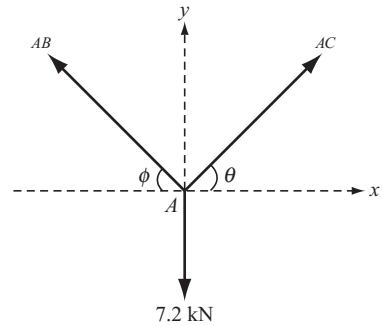


Fig. Ex. S1.9(b) (Solution)

Example S1.10

A derrick crane has ball-and-socket joints at *A*, *B*, *C* and *D* as shown. Determine the forces in links *AB*, *BD* and *CD* when it is supporting a dead load of 1000 kN.

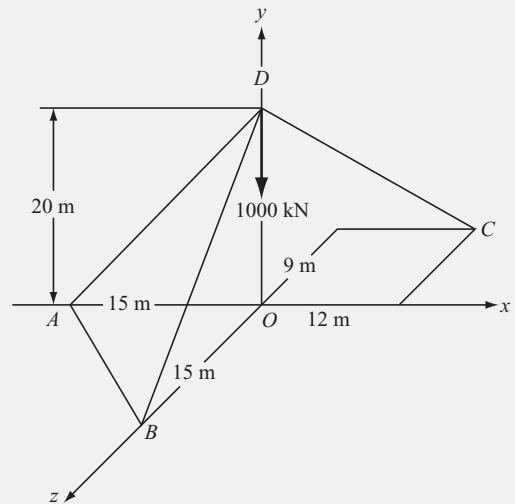


Fig. Ex. S1.10

Solution Let us write 3 equations along the 3 axes at the joint *D*, considering the origin at *D*.

Along *x*-axis $12 t_{DC} + 0 t_{DB} - 15 t_{DA} = 0$ (i)

Along *y*-axis $-9 t_{DC} + 15 t_{DB} + 0 t_{DA} = 0$ (ii)

Along *z*-axis $-(20 t_{DC} + 20 t_{DB} + 20 t_{DA}) - 1000 = 0$ (iii)

From (iii),

$$t_{DC} + t_{DB} + t_{DA} = -50$$

Solving the above equations,

$$t_{DC} = -20.83, t_{DB} = -12.5, t_{DA} = -16.66$$

Lengths of the links are given by

$$CD = \sqrt{12^2 + 9^2 + 20^2} = 25 \text{ m}$$

$$BD = \sqrt{15^2 + 20^2} = 25 \text{ m}$$

and

$$AD = \sqrt{15^2 + 20^2} = 25 \text{ m}$$

Hence

$$F_{DC} = -20.83 \times 25 = -520.8 \text{ kN (compressive)}$$

$$F_{DB} = -12.5 \times 25 = -312.5 \text{ kN (compressive)}$$

and

$$F_{DA} = -16.66 \times 25 = -416.5 \text{ kN (compressive)}$$

Example S1.11

A simple space frame (also called shear-legs frame) as shown carries a load of 21 kN. Determine the forces in the three members.

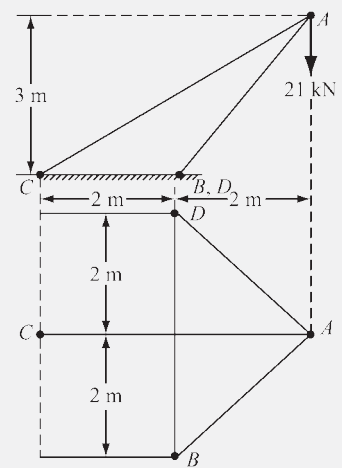


Fig. Ex. S1.11

Solution Let us solve this problem by the method of tension coefficients. Let the origin be at A and the x, y, z axes as shown to follow the right-hand screw rule.

Writing the 3 equations along the axes:

Along x -axis, $2 t_{AB} - 2 t_{AD} = 0$

Along y -axis, $-2 t_{AD} - 2 t_{AB} - 4 t_{AC} = 0$

Along z -axis, $-3 t_{AB} - 3 t_{AD} - 3 t_{AC} + 21 = 0$

From (iii), $t_{AB} + t_{AD} + t_{AC} = -7$

Using other equations, $t_{AB} = -7, t_{AD} = -7$ and $t_{AC} = +7$

These are the tension coefficients for the members AB, AD and AC respectively.

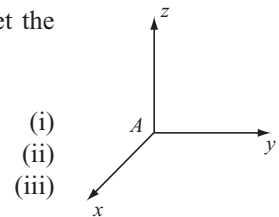
Now,

$$AC = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

$$AB = AD = \sqrt{2^2 + 2^2 + 3^2} = 4.12 \text{ m}$$

Therefore,

$$F_{AC} = 7 \times 5 = 35 \text{ kN (tension)}$$



$$t_{AB} = t_{AD} = -7 \times 4.12 = -28.86 \text{ kN (compression)}$$

If we were to use a computer to solve the equations, the three equations for tension coefficient would have to be written in a tabular form. For the present case, the table would appear as follows:

| Joint | Direction | Equation | Tension coeff. |
|-------|-----------|--|--------------------|
| A | x | $2 t_{AB} - 2 t_{AD} = 0$ | $t_{AB} = t_{AD}$ |
| | y | $-2 t_{AB} - 2 t_{AD} - 4 t_{AC} = 0$ | $t_{AB} = -t_{AC}$ |
| | z | $-3 t_{AB} - 3 t_{AD} - 3 t_{AC} - 21 = 0$ | $t_{AC} = +7$ |

Computation would involve solving 3 simultaneous equations at each pin-joint of the space frame.

EXPERIMENT E4 FORCES IN A SPACE FRAME

OBJECTIVE TO DETERMINE THE FORCES IN THE MEMBERS OF A LOADED SHEAR-LEGS SPACE FRAME EXPERIMENTALLY, VECTORIALLY AND GRAPHICALLY.

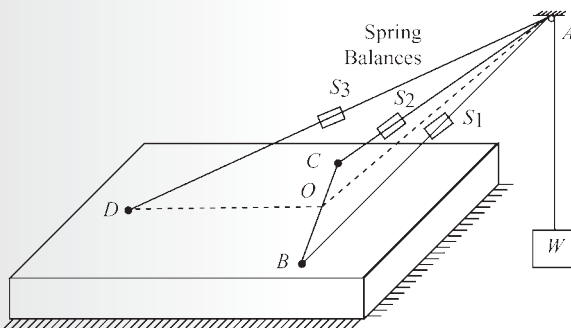


Fig. E4.1 Shear legs apparatus

Apparatus Shear-legs apparatus consisting of two rigid bars AB and AC and a tie-bar AD together with a provision for loading at A as shown in Fig. E4.1. Metre rod, spring balances and standard weights.

Background Information Since the frame is in equilibrium, every part and sub-part of the frame must also be in equilibrium. Consider, for example, the joint A . The forces that act at A to keep it in equilibrium are the known loads acting vertically downward and the forces along AB , AC and AD . The directions of AB ,

AC and AD are determined from coordinate geometry; only the magnitudes of forces are unknown which may be obtained experimentally by reading the spring balances installed in the members. Theoretically, the method of vector analysis or a graphical construction may be employed to estimate the forces.

Vectorially, let the unit vectors along AB , AC and AD be \hat{u}_1 , \hat{u}_2 and \hat{u}_3 respectively. These are obtained by measuring the coordinates of the end points. For example, if the origin is chosen at O ,

$$\begin{aligned} \hat{u}_1 &= \frac{x_A}{AB} \hat{i} + \frac{z_A}{AB} \hat{k} \\ \hat{u}_2 &= \frac{y_B}{AB} \hat{j} \\ \hat{u}_3 &= \frac{x_A}{AB} \hat{i} + \frac{y_B}{AB} \hat{j} + \frac{z_A}{AB} \hat{k} \end{aligned}$$

whence

$$AB = \sqrt{(x_A^2 + y_B^2 + z_A^2)}$$

and

$$\hat{u}_1 = \frac{x_A}{AB} \hat{i} + \frac{z_A}{AB} \hat{k}$$

Let the unknown forces be F_1 , F_2 and F_3 in the members AB , AC and AD respectively. Then, for equilibrium of the joint A

$$\begin{aligned} \Sigma F_x &= 0 \\ F_1 \frac{x_A}{AB} + F_3 \frac{x_A}{AB} - W &= 0 \end{aligned}$$

This equation may be written as three equations, one along each coordinate direction to obtain the three unknowns.

Graphically, the forces in the members may be determined in two steps. First, a fictitious member AO is assumed to replace the bars AB and AC . The point A may then be considered for equilibrium under the action of the load and forces in AO and AD . A triangle of forces is drawn whence, by measurement to appropriate scale, the forces in AO and AD are determined, as shown in Fig. E4.2.

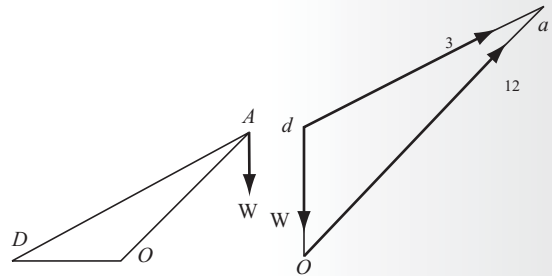


Fig. E4.2 Construction of triangle of forces

It is advisable to draw a scale diagram ADO of the frame with the fictitious member AO before drawing the triangle of forces. It may be seen that the force in AD must be F_3 and the force in AO in F_{12} . The nature of forces in AD and AO is observed by drawing the arrows of the internal forces and of the external forces on the members.

Since the force exerted by the member AD on the joint is F_3 , directed A to D , the external force acting at A on AD must be equal and opposite to it as shown by the dotted lines in Fig. E4.3. Outward external forces on AD imply that it must be in tension. Similarly, the fictitious member AO is observed to be in compression.

The second step is to resolve the force F_{12} along AO into forces F_1 and F_2 along AB and AC respectively. After drawing the force F_{12} along AO , the directions of AB and AC are drawn and the force F_{12} is resolved into F_1 along AB and F_2 along AC by completing the parallelogram of forces with AO as the resultant diagonal as shown in Fig. E 4.4.

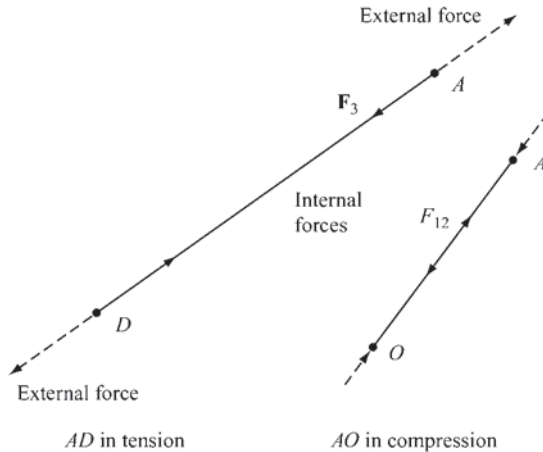


Fig. E4.3 Internal forces in members

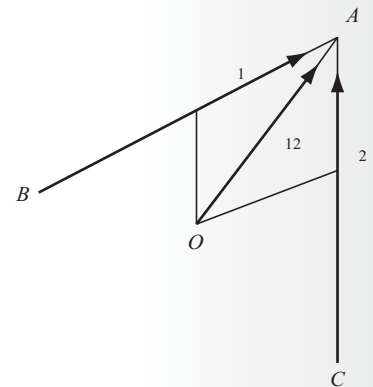


Fig. E4.4 Construction of parallelogram of forces

Table of Results

| Force in member | Experimental values | | | Vectorial analysis | Graphical procedure |
|-----------------|---------------------|-------------|-----------------------|--------------------|---------------------|
| | Initial reading | When loaded | Force (by difference) | | |
| AB | | | | | |
| AC | | | | | |
| AD | | | | | |

Points for Discussion

What are the assumptions made in the vectorial and graphical analyses in respect of the rigidity and mass of the members?

Examine the joints and supports of the frame and comment on the validity of your analysis in view of the same.

Is it possible to calibrate one of the spring-balances to determine the weight of a given body?

Comment on the validity of the assumption of a fictitious member AO lying in the plane of AB and AC .

Concept Review Questions

- (a) What is meant by a simple structure?
- (b) Differentiate between a structure and mechanism.
- (c) Can a simple structure be a space structure?
- (a) Draw a just-rigid structure with five members and another just-rigid structure with five joints. Draw one additional member in each of them to render them over-rigid.
- (b) Is it possible to have a just-rigid structure with an even number of members? Why or why not?

In the analysis by the method of joints, should one proceed from a joint on the extreme left to the joint on the extreme right or the reverse or are there some other important considerations?

Draw the free-body diagram of a member subjected to tension. What would be the internal forces at the two joints?

Recalling the comments on loading conditions, fill in the blanks:

- (a) If there are only two members and no external force at a joint,.....
- (b) If there are three members, two of which are collinear and there is no external force at a joint,.....

Illustrate the implication of the Bow's notation. What are the space diagram and rays diagram?

Comment on the graphical method of analysing a simple structure. Under what conditions is the graphical method preferred and for what conditions does it fail?

Tutorial Problems

Determine the forces in the members of the pin-jointed truss shown in Fig. Prob. S1.1.

$$\begin{aligned}
 & (s_x = 3W) \\
 & s_x = 2W, \quad s_y = -3W, \quad B = -W \text{ (Comp)}, \\
 & B = -W \text{ (Comp)}, \quad AC = \sqrt{2}W \text{ (tens)}, \\
 & \quad = W \text{ (tens)}, \quad BD = 2\sqrt{2}W \text{ (tens)}, \\
 & \quad BE = -3W \text{ (comp)}, \quad E = 0)
 \end{aligned}$$

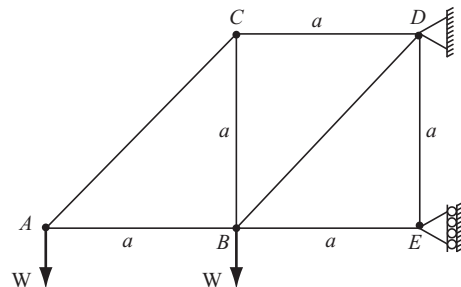


Fig. Prob. S1.1

A coplanar simple truss $ABCDE$ is loaded with a force of 50 kN at A as shown in Fig. Prob. S1.2. Determine

- (a) the forces exerted in the members of the truss
- (b) the force exerted on the pins at the joints.

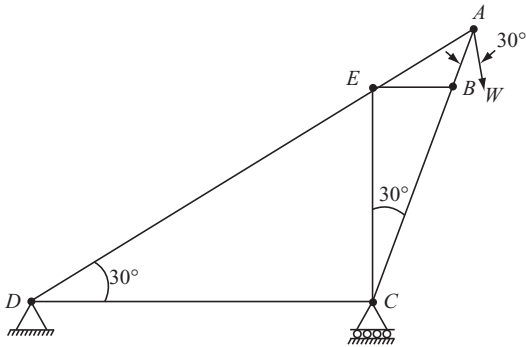


Fig. Prob. S1.2

Using the method of joints or the method of sections, calculate the force in each member of the trusses shown in Fig. Prob. S1.3. State whether the members are in tension or in compression

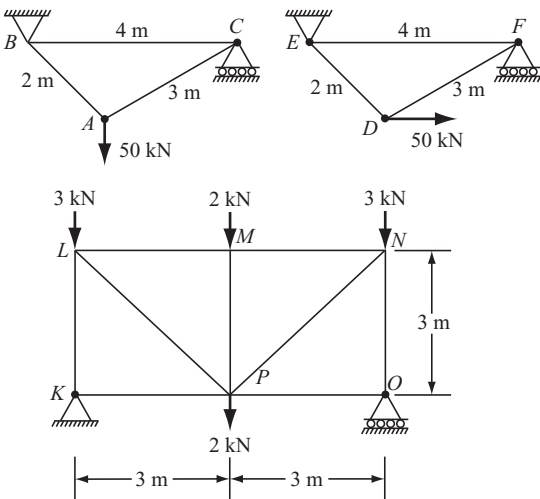


Fig. Prob. S1.3

- (s $B = 40.6$, $= 34.6$, $B = 27.95$ kN
- $PN = 2\sqrt{2}$, $= 2$

- $E = 29.8$, $= -36.6$, $E = 29.5$ kN
- $P = 2$, $= 2$
- $= 5$, $P = 0 = P$
- $LP = 2\sqrt{2}$, $= 5$)

Determine the forces in the members of the given simple truss. (Fig. Prob. S1.4)

- (s $B = 100\sqrt{2}/3$, $= -100\sqrt{2}/3$,
- $E = 100/3 = E$
- $B = -100/3 = E =$ etc.)

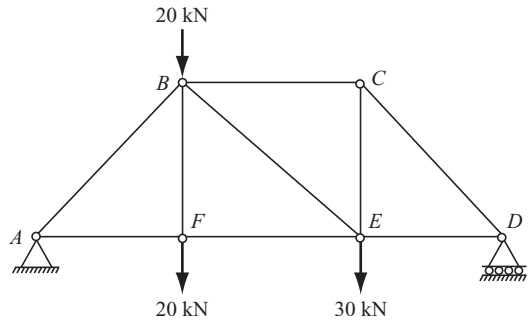


Fig. Prob. S1.4

A simple truss is loaded as shown in Fig. Prob. S1.5. Calculate the forces in the members.

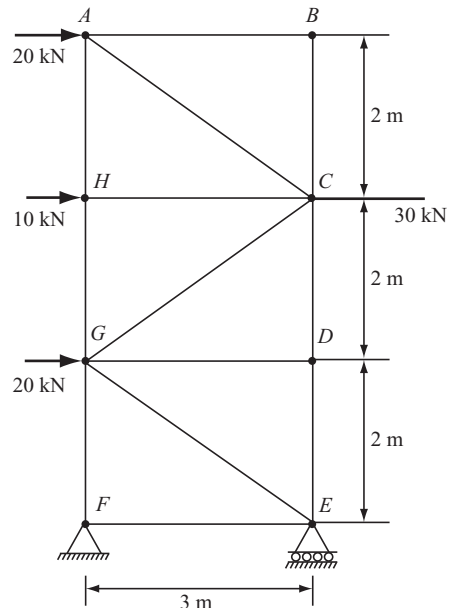


Fig. Prob. S1.5

Determine the forces produced in the bars of the system due to the horizontal force P applied at the hinge B (Fig. Prob. S1.10)

$$\begin{aligned} F_{AB} &= P \text{ (tension);} \\ F_{AC} &= 1.414 P \text{ (comp)} \end{aligned}$$

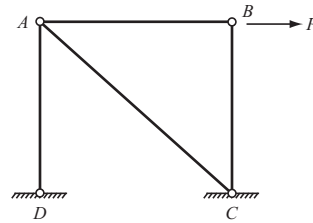


Fig. Prob. S1.10

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

- If, for a plane pin-jointed truss, $m = 2j - 3$ with the usual notion,
- (a) it must be a just-rigid truss
 - (b) it cannot be over-rigid over any part of the truss
 - (c) it may or may not be a just-rigid truss
 - (d) it cannot be non-rigid over any part of the truss

The method of joints for the analysis of forces in the members of a pin-jointed truss

- (a) is a special case of method of sections
- (b) does not need the determination of reactions at the supports

- (c) works equally well, irrespective of starting point for the analysis
- (d) fails when there are only two members at a joint and no external load is applied there

In the method of sections for the analysis of forces in the members of a pin-jointed truss,

- (a) the section can be cut through any set of members for equal ease of analysis
- (b) the sections must be cut so that the number of unknowns is limited and determined by employing the conditions of equilibrium.
- (c) care must be taken to ensure that the section being cut is in equilibrium.
- (d) the sections to be cut are as small as possible

Answers to the Multiple-Choice Questions

- 1 (c) 2 (a) 3 (b)

Simply Supported Beams: Shear Forces and Bending Moments

S2



S2.1 TYPES OF BEAMS

A beam is a structural member, generally horizontal, which can take transverse load in addition to other loading. By ‘transverse load’, we mean ‘perpendicular to the length’ of the beam. A beam may also take other loads, e.g., load in the axial direction, i.e., tension or compression and torsion about the axis!

A beam is a very important structural member and is often used in engineering structures, bridges, etc., wherever a transverse load is encountered. In contrast, members of pin-jointed frames can take only compressive or tensile load. Shafts typically take only twisting loads. Struts take only compressive loads. Beams can take it all! A two-force frame member becomes a beam if it also takes transverse load. Similarly, a shaft transmitting power becomes a beam if it also takes transverse load! Beams may carry different types of transverse loads, e.g., point loads, uniformly distributed loads or varying loads or even time dependent loads along its length. We shall consider the beams as ‘rigid beams’, which implies that the size and shape of the beam must remain the same. A rigid beam cannot deflect under-applied loads; there may, however, be a tendency to deflect or curve under the action of shear forces and bending moments, which we shall work out.

Beams are classified depending upon the types of supports. The simplest of all beams is called a ‘simply



A structure showing a horizontal beam in front

supported beam', which means a beam supported on a hinge at one end and a roller support at another. Such a beam can carry only a transverse load, as shown in Fig. S2.1. A simply supported beam could as well be an overhanging beam at one or both ends, as shown in Fig. S2.2.

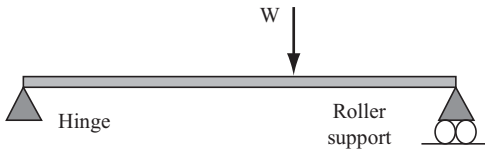


Fig. S2.1 A simply supported beam

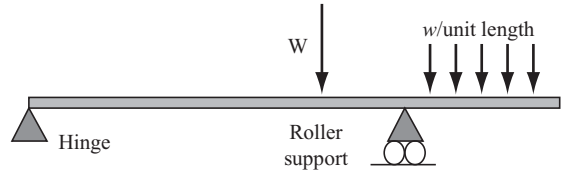


Fig. S2.2 A simply supported overhanging beam

A beam fixed at one end and free at the other is called a *cantilever beam* or just a *cantilever*. Cantilevers are useful because they can support transverse loads on overhangs and at such places where a second support cannot be placed. Examples of cantilevers are fixed-end wall brackets, commonplace balconies and half-bridges from either side of the over. Look at the Cantilever Tower Bridge over river Thames in London. It opens up for ships to pass by and comes down flat to make the road!

A cantilever may be loaded with point loads or distributed loads as shown in Fig. S2.3.

A beam could also be supported by fixed supports, such as by nails, bolts, rivets, welding or be cast in concrete. Such beams are called fixed-end beams. A continuous beam is one which has more than two supports. Some such beams as shown in Figs. S2.4 (a, b and c) are also called 'statically indeterminate' beams because the reactions cannot be obtained by employing only the equations of equilibrium.



Double cantilever tower bridge, London

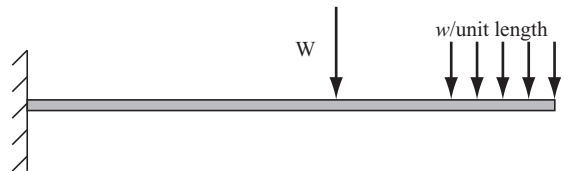


Fig. S2.3 A cantilever beam

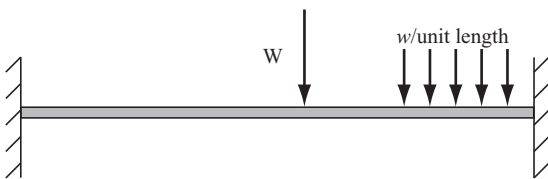


Fig. S2.4 (a) A fixed-ends beam

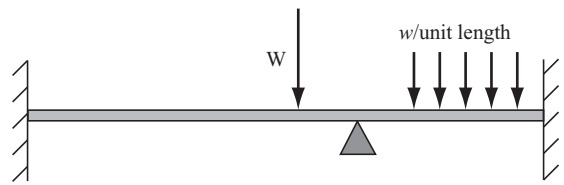


Fig. S2.4 (b) A continuous beam: with an intermediate support

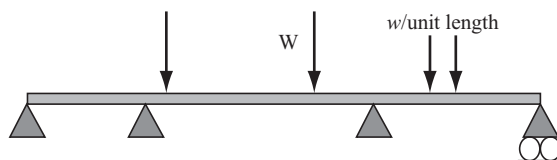


Fig. S2.4 (c) A continuous beam: with two intermediate supports

S2.2 SHEAR FORCE AND BENDING MOMENT AT A CROSS-SECTION IN A BEAM

By virtue of the fact a beam takes transverse load, i.e., across the longitudinal axis, it is bound to experience resisting forces and moments over the cross-sections. The internal resisting forces over a cross section are called shear forces and the internal resisting moments are called bending moments.

Shear force acting at a cross section tend to shear the beam across that cross section. The bending moment at a cross section tends to curve the beam concave upwards or convex upwards. But the beam is considered to be rigid; resisting shear and bending. It may also be pointed out that the cross section of a beam is either a rectangle with more depth than width, or it is I or T shaped to resist shear and bending stresses.

Consider the free-body diagram of simply supported beam, as shown in Fig. S2.5. It shows the external load W and the reactions at the supports. The roller support at B can only provide a vertical reaction, whereas the hinge can, in principle, provide a vertical and a horizontal reaction depending upon the need. In this case, however, there is not going to be a horizontal reaction because there is no other horizontal force! The vertical reactions at A and B can be easily worked out by considering the equilibrium of the beam and applying summation of 'forces', and 'moments about any point' to be zero!

How can we work out the shear force and bending moment at a given cross section?

Simple! Consider a part of the beam, say the left part, until that cross section and consider its equilibrium, as shown in Fig. S2.5. At that cross section, the equilibrant force would be the shear force and the equilibrant moement called the bending moment.

Clearly, the shear force at P would be equal to R_A acting downwards

$$= R_A$$

and the bending moment at P would be

$$= R_A \cdot AP = R_A \cdot x$$

acting in anticlockwise direction

What would have happened if we considered that part of the beam to the right of P ?

Let us draw the *fbd* and proceed to determine the shear force and bending moment at P .

The shear stress comes out to be

$$= W - R_B$$

which is the same, in magnitude as R_A because

$$W = R_A + R_B$$

But is now acting upward.

The two answers for the shear force at the same point must be the same in magnitude and sense. How do we resolve the issue of sign difference? Notice that the downward was acting on a positive face of the cross section upward is now acting on the negative face.

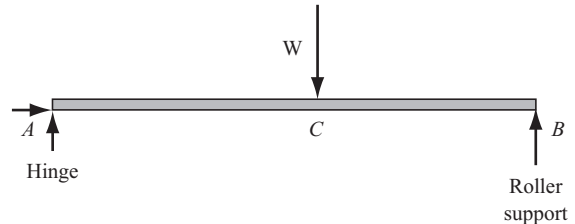


Fig. S2.5 Free-body diagram of a simply supported beam

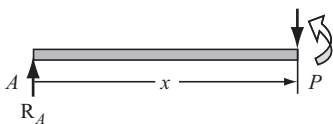


Fig. S2.5 (a) Free-body diagram of part to the left of the cross section

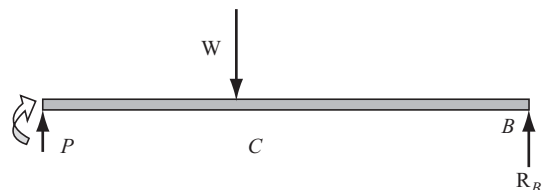


Fig. S2.5 (b) Free-body diagram of part to the right of the cross section

Sign Convention for Shear Force V

A shear force is said to be positive if it acts downwards on a positive face such that the right part of the beam tends to shear down compared to the left part.

Now, the shear force at P in the case is $+$ because it is downward on the left part; it was tending to shear the right part downwards! The shear force obtained by considering the part of the beam on the right of the section is also $+$ since it acts on a negative face.

The bending moment at the same point P can be obtained but it would appear to be negative of the obtained earlier. How do we resolve the issue of sign difference?

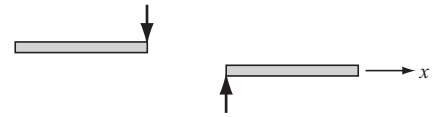


Fig. S2.6 Sign convention: positive shear force V downward on a positive face and upward on a negative face

Sign Convention for Bending Moment M

A bending moment is said to be positive if it acts anticlockwise on a positive face such that the left part of the beam tends to sag down compared to the right part.

We should always assume and show the bending moment to be positive in a *fbd* and decide on it actually being positive or negative depending upon the sign that emerges by equilibrium analysis.

Let us determine the shear force at any cross-section between C and B by choosing a point P in that part and considering the equilibrium of the part on the left or right of that cross-section as before. The shear force at a point P in that part, Fig. S2.8 as given by



Fig. S2.7 Sign convention: positive bending moment M anticlockwise on a positive face and clockwise on a negative face

$$= W - R_A$$

acting upwards. That means it is negative, by using the sign convention, i.e., $-$.

But, that is not how we should proceed. It is recommended to draw the *fbd* by showing the positive shear force (any positive bending moment) and then by working out whether it is indeed positive or negative, depending upon what sign we get from equilibrium consideration. So, we should draw an *fbd* by assuming as positive downward, as shown in Fig. S2.9.

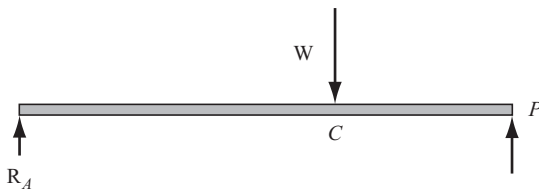


Fig. S2.8 Free-body diagram of the part on the left of cross section

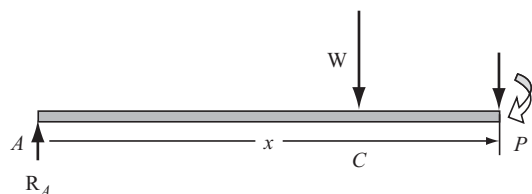


Fig. S2.9 Free-body diagram of the part on the left of cross section with V and M assumed positive

The analysis shows that, for equilibrium,

$$- + R_A - W = 0$$

whence,

$$= R_A - W$$

and that would be negative in numerical terms, W being greater than R_A .

The bending moment at this section is given by

$$- R_A \cdot AP + W \cdot AC = 0$$

whence,

$$- R_A \cdot AP - W \cdot AC = R_A x - W \cdot (x - AC)$$

It results in , both in magnitude and sign.

Now that we understand the shear force and bending moment are vectors, we also understand that we are generally dealing with vertical loading, reactions and forces, and we are dealing with MAGNITUDES, so we may as well not use vector notation. Let us write the loading as W and w , reactions as R , shear force as V and bending moment as M .

Let us now consider the shear force and bending moment at some sections in the cantilever shown in Fig. S2.3 by drawing its *fbd* as shown in Fig. S2.10.

By equilibrium considerations,

$$R_A - W - (w \cdot DB) = 0$$

and
$$M_A - W \cdot AC - (w \cdot DB \cdot AK) = 0$$

where K is the midpoint of DB

whence,
$$R_A = W + (w \cdot DB)$$

and
$$M_A = W \cdot AC + (w \cdot DB \cdot AK)$$

Once the reactions at the support are known, it is easy to draw the *fbd* of any required part of the cantilever to determine the shear force at any cross section. In order to obtain the shear force at a section between A and C , the *fbd* of the left hand part of the cantilever is drawn in Fig. S2.11.

By equilibrium of the part with V and M shown positive according to the sign convention and the moment also positive, according to sign convention,

$$R_A - V = 0; \quad V = R_A$$

and
$$M_A + M - R_A \cdot AP = 0$$

or
$$M = R_A \cdot x - M_A$$

which would provide M both in magnitude and direction.

Likewise, the bending moment at another section on the right of the load W can be obtained by looking at the part of the cantilever. By equilibrium of the part with V and M both shown positive according to the sign conventions in Fig. S2.12,

$$M_A + M - R_A \cdot x + W \cdot (x - AC) = 0$$

whence,
$$M = R_A \cdot x - M_A - W \cdot (x - AC)$$

both in magnitude and direction.

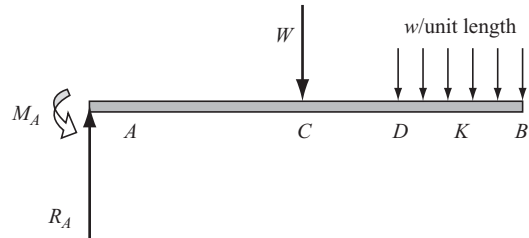


Fig. S2.10 Free-body diagram of the cantilever beam

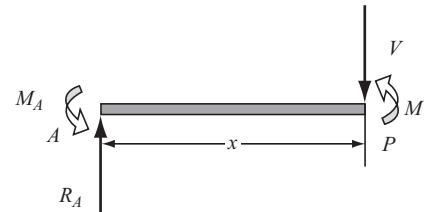


Fig. S2.11 Free-body diagram of the cantilever part

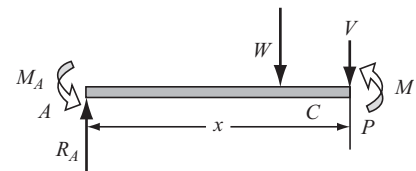


Fig. S2.12 Free-body diagram of the cantilever part AP

S2.3 SHEAR FORCE DIAGRAM FOR A BEAM

A diagram showing the variation of shear force at different cross sections along the beam is called Shear Force Diagram (often referred as SF Diagram).

In order to draw a shear force diagram, one needs to find out shear forces at some salient points, i.e., cross sections of a beam and then draw a continuous graph. Now that we have to determine shear forces at several salient points, we should proceed by writing a general expression, by inspection of a given beam and loading, which can result in shear forces without drawing so many free-body diagrams and taking too many steps. It is possible to do so with some practice. Let us demonstrate how to write general expressions for some cases:

For the simply supported beam example,

$$V = +R_A \quad | \quad -W$$

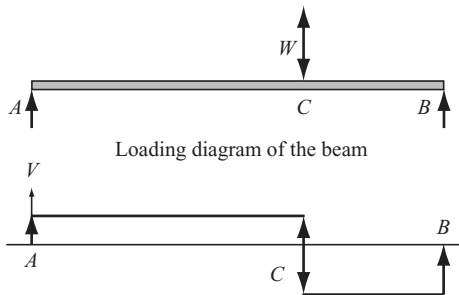


Fig. S2.13 Shear force diagram for the beam

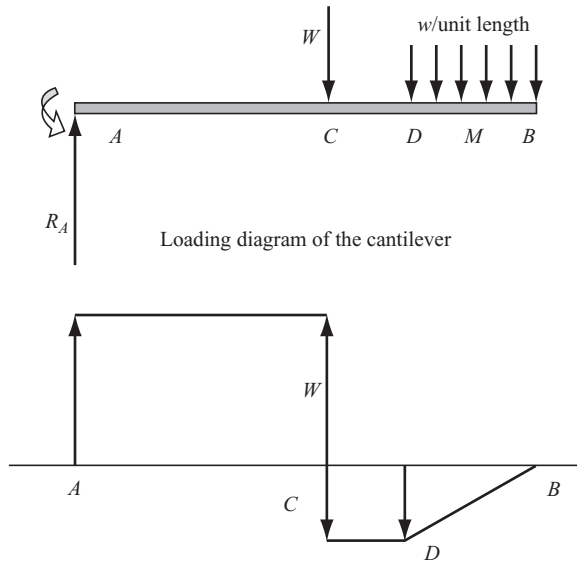


Fig. S2.14 Shear force diagram for the cantilever

which means that the shear stress is $+R_A$ 'until the next load' and $+R_A - W$, thereafter!

The shear force diagram may now be drawn. It is usually drawn right under the loading diagram to show point by point correspondence as in Fig. S2.13:

It may be noted that the diagram shown the shear force V to be positive and equal to R_A . From A to C and then it shows it to be negative and equal to $W - R_B$ from C to B.

Let us now draw the SF diagram for the cantilever shown loaded in Fig. S2.13. We write an expression for shear force V in the cantilever:

$$V = R_A - W - wx(x - AD)$$

which means that the shear force is R_A from A to C; $R_A - W$ from C to D it varies linearly from D to B.

The SF diagram is now drawn in Fig. S2.14 under the loading diagram.

S2.4 BENDING MOMENT DIAGRAM FOR A BEAM

A diagram showing the variation of bending moment at different cross sections along the beam is called Bending Moment Diagram (often referred as BM Diagram).

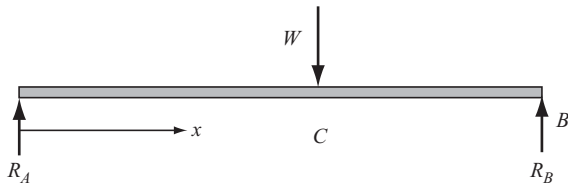
In order to draw a bending moment diagram, one needs to find out bending moment at some salient points, i.e., cross sections of a beam and then draw a continuous graph. Now that we have to determine bending moment at several salient points, we should proceed by writing a general expression, by inspection of a given beam and loading, which can result in bending moments without drawing so many free-body diagrams and taking too many steps. It is possible to do so with some practice. Let us demonstrate how to write general expressions for some cases:

For the simply supported beam, the general expression for the bending moment is

$$M - R_A \cdot x + W \cdot (x - AC) = 0$$

whence,

$$M - R_A \cdot x - W \cdot (x - AC)$$



Loading diagram of the beam

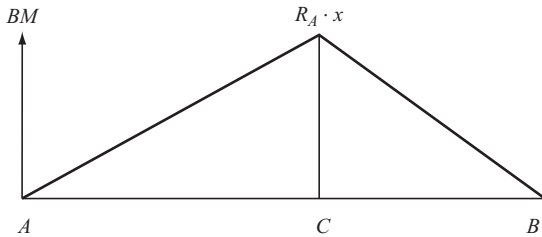
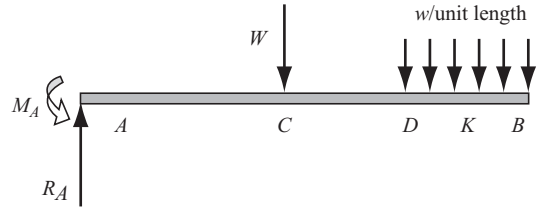


Fig. S2.15 Bending moment diagram for the beam



Loading diagram of the cantilever beam

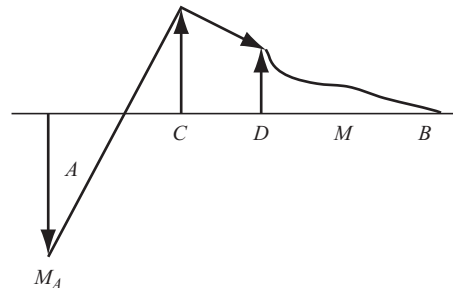


Fig. S2.16 Bending moment diagram for the cantilever

Implying that it is $R_A \cdot x$ until the next load and the whole expression thereafter. The bending moment diagram is shown plotted in Fig. S2.15. It may be noted that the bending moment is the maximum at C, where shear force changed sign and passed through zero!

Let us now turn to the cantilever. The general expression for the bending moment at any cross section can be written by equilibrium:

$$M + M_A - R_A \cdot x + W \cdot (x - AC) + w(x - AD) \cdot (x - AD)/2 = 0$$

whence,

$$M = -M_A + R_A \cdot x - W \cdot (x - AC) - w(x - AD) \cdot (x - AD)/2$$

with usual meanings. The bending moment diagram is shown plotted under the loading diagram in Fig. S2.16. It may also be noted that the bending moment has maximum values at A and C, where the shear force passes through zero!

S2.5 GENERAL RELATION BETWEEN LOADING, SHEAR AND BENDING MOMENT

It is possible to relate the loading with shear and bending moment at the same cross section of a beam with distributed loads. Consider a beam with loading w per unit length, considered positive downwards, as shown in Fig. S2.17.

The shear force V and moment M are shown positive, by convention, on both faces of the element, isolated from the beam. Distances are measured from an origin O . Equilibrium of the element requires that

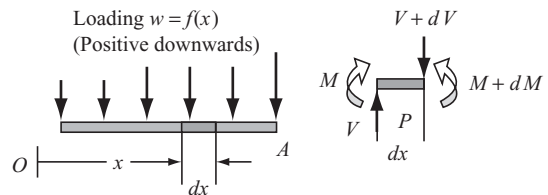


Fig. S2.17 An element of a beam

$$V - w \cdot dx - (V + dV) = 0$$

and

$$M + w \cdot dx \cdot dx/2 + (V + dV) \cdot dx - (M + dM) = 0$$

From the former,

$$w = -dV/dx$$

It shows that the slope of the shear force diagram must be equal to the negative of the loading at that cross section. This equation also holds good on either side of a concentrated load but not at the point of the load.

Integrating the above equation w.r.t. x ,

$$\int dV = -\int w \cdot dx$$

which means that the area under the loading curve must be equal to the shear force at a cross section.

The moment equation for the equilibrium of the element simplifies to provide

$$V = dM/dx$$

which expresses that the slope of the bending moment curve equals the shear force at that point. Likewise, by integration of the above

$$\int dM = -\int V \cdot dx$$

shows that the area under the area under the shear force diagram equals the bending moment at that point.

It may also be noted that the second derivative of M w.r.t. x must be equal to $-w$. It also shows that if w is a given function of x , the shear force and bending moment diagrams can be drawn very easily!

The following observations are also helpful:

1. That the degree of the curve for shear force is one higher than that for loading. And the degree for curve of bending moment is one higher than that for the shear force.
2. That the shear force must have a step change at a point of concentrated load; it usually changes sign!
3. That a point of zero shear force must correspond to a point of maximum or minimum bending moment.

Some conclusions are summed up in Table S2.1.

Table S2.1 Shapes of loading, shear force and bending moment diagrams

| Type of loading on a beam | Shape of loading diagram | Shape of shear force diagram | Shape of bending moment diagram |
|---------------------------|--------------------------|------------------------------|---------------------------------|
| Zero; No Load | Zero Line | Rectangular | Triangular Line |
| Uniform Loading | Rectangular | Triangular Line | Parabolic Curve |
| Linearly Varying Loading | Triangular Line | Parabolic Curve | Cubic Curve |

Example S2.1

A simply supported beam 4 m long carries a 5 kN load at 1 m from one end, as shown. Calculate the shear force and bending moments at a point on either side of the load.

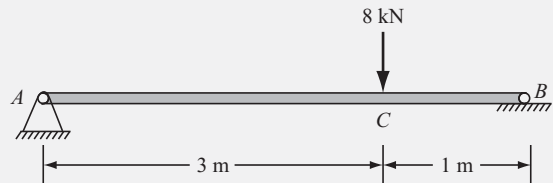


Fig. Ex. S2.1

Solution We need to determine the reactions by considering the equilibrium of the beam AB . Reaction R_B at the roller end must be vertical; hence the reaction at the hinge must also be vertical since the external load is only vertical!

$$\sum F_y = 0 \quad R_A + R_B - 8 = 0; \quad R_A + R_B = 8 \text{ kN}$$

$$\sum M_A = 0 \quad R_B \times 4 - 3 \times 8 = 0; \quad R_B = 6 \text{ kN}$$

and, by difference, $R_A = 2 \text{ kN}$.

Now, consider the equilibrium of a part AP of the beam from the left end, with shear force V shown positive downward on a positive face.

By equilibrium,

$$2 - V = 0; \quad V = 2 \text{ kN}$$

The shear force at P is $+2 \text{ kN}$. Shear force V at all sections from A to C on this part of the beam is $+2 \text{ kN}$.

Now, consider the equilibrium of a part AP' where P' is on the right side of the applied load, again assuming V to be $+ve$, acting downwards.

By equilibrium,

$$+2 - 8 - V = 0; \quad V = -6 \text{ kN}$$

It is seen that, at point C , the V changes from $+2 \text{ kN}$ to -6 kN , i.e., by 8 kN which due to the applied load of 8 kN .

In order to determine the bending moment at a cross section, we again need to consider the equilibrium of a part of the beam until that cross section.

Considering the part AP , of length x , metre, the equilibrium moment at P (exerted by the rest of the beam on part AP) must be $+2x \text{ kN/m}$. It shows that it varies linearly from 0 at A to 2×3 or 6 kN/m at C .

Now, consider the equilibrium of part AP' as before and the equilibrant moment at P' should be $+2x - 8(x - 3)$, i.e., $-6x + 24 \text{ kN/m}$. This again varies linearly between C and B .

$$M \text{ at } C = 6 \text{ kN/m}$$

$$M \text{ at } B = 0$$

Reflection

There are two useful observations:

1. We need to verify the answers as we proceed and 2. We may decide to determine the V and M at a cross section by considering the part of the beam from the left or from the right of convenient to us. In order to verify the V and M at P' , we could have considered the same cross section by choosing the right part of the beam as follows:

The equilibrant force at P' would be -6 kN but it is at a $-ve$ face; meaning a negative -6 kN , as before.

Likewise, the equilibrant moment at P' would be $-6x'$ but it is at a $-ve$ face; meaning a positive BM .

Since x' is measured from B ,

$$M \text{ at } B = 0$$

$$M \text{ at } C = 6 \times 1 = 6 \text{ kN/m.}$$

We have verified our results by considering the part of the beam from the right end.

We could have written expressions for V and M at all the sections from one end to the other:

$$V = +2 \text{ ; } -8 \text{ kN}$$

$$M = 2x \text{ ; } +8(3 - x)$$

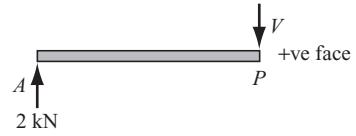


Fig. Ex. S2.1 (a) (Solution)

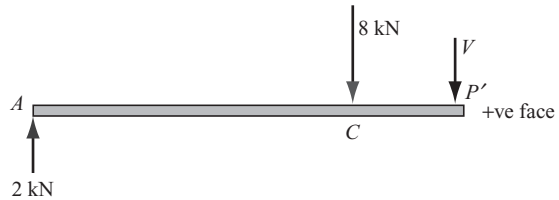


Fig. Ex. S2.1 (b) (Solution)

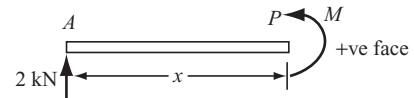


Fig. Ex. S2.1 (c) (Solution)

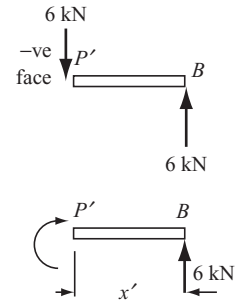


Fig. Ex. S2.1 (d) (Solution)

where the first term applies from A to C and the total expression for the part C to B .

The advantage of doing so is to be able to write such expressions for all different sections of a beam with several different types and points of loading.

Example S2.2 Calculate the shear and bending moment at a point on either side of the concentrated load for the cantilever AB carrying a concentrated load and a uniformly distributed load as shown.

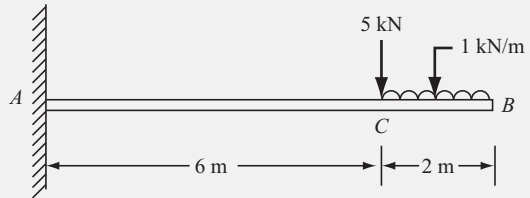


Fig. Ex. S2.2

Solution We draw a fbd for the cantilever and determine the reaction at A :

$$\sum F_y = 0; \quad R_A - 5 - 1 \times 2 = 0; \quad R_A = 7 \text{ kN}$$

$$\sum M_A = 0; \quad M_A - 5 \times 6 - 2 \times 1 \times 6(6+1) = 0$$

$$M_A = 30 + 14 = 44 \text{ kN/m.}$$

Let us draw the fbd of a part of the beam starting from left side A , and assume both V and M to be +ve.

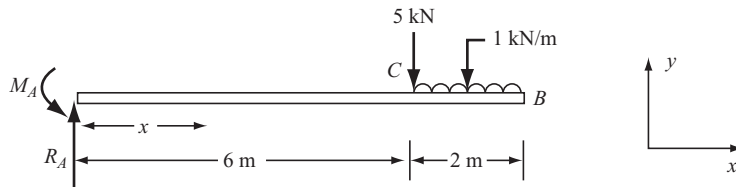


Fig. Ex. S2.2(a) (Solution)

For the cross section at P : by equilibrium

$$R_A - V = 0; \quad V = R_A \Rightarrow 7 \text{ kN}$$

$$M + M_A - R_A \cdot x = 0; \quad M = R_A \cdot x - M_A$$

Likewise, just imagine (or draw) the fbd of a part equal distance of the beam until a point P on the right of the 5 kN load.

By equilibrium

$$R_A - V - 5 - 1(x - 6) = 0; \quad V = 7 - 5 - 1(x - 6)$$

$$M + M_A - R_A x + 5(x - 6) + 1(x - 6)^2 / 2 = 0$$

whence
$$M = -M_A - R_A x + 5(x - 6) - 1(x - 6)^2 / 2$$

We could have written down expressions for V and M at all the sections from A to the other end by a little imagination:

$$V = +7 \dot{-} 5 - 1(x - 6)$$

$$M = -M_A + R_A x \dot{-} 5(x - 6) - 1(x - 6)^2 / 2 = -44 + 7x \dot{-} 5(x - 6) - (x - 6)^2 / 2$$

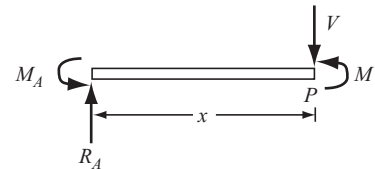


Fig. Ex. S2.2(b) (Solution)

where the parts before the vertical divider apply from A to C and the whole of it applies from C to B . From the above,

at $x = 3 \text{ m}$ $V = +7 \text{ kN}$ $M = -23 \text{ kN/m}$

at $x \leq 6 \text{ m}$ $V = +7 \text{ kN}$ $M = -2 \text{ kN/m}$

at $x \geq 6 \text{ m}$ $V = +2 \text{ kN}$ $M = -2 \text{ kN/m}$

at $x = 7 \text{ m}$ $V = +1 \text{ kN}$ $M = -\frac{1}{2} \text{ kN/m}$

at $x = 8 \text{ m}$ $V = 0$ $M = 0$

Shear force and bending moment are both zero at the free end B , indeed!

Example S2.3 Draw the loading diagram for the simply supported beam shown in Fig. Ex. S2.3. Calculate the shear force and bending moment at some salient points and hence sketch the shear force and bending moment diagrams.

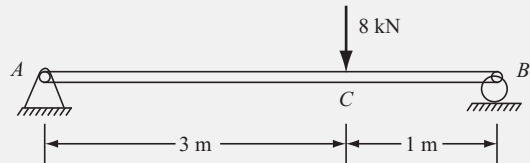


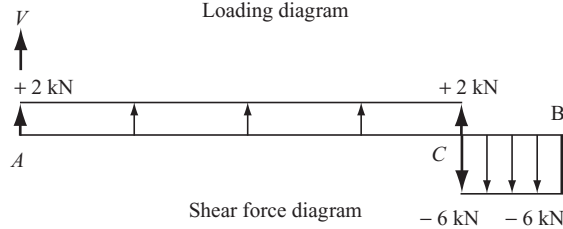
Fig. Ex. S2.3

Solution The reactions at A and B are calculated in solving Ex. S2.1. The shear forces and bending moments at some representative sections are also calculated there.

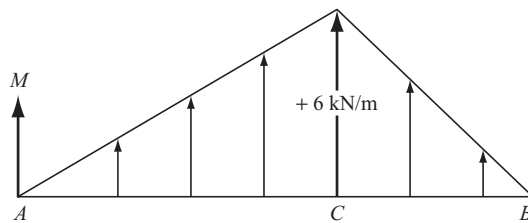
Taking into account those values, let us draw the loading, shear force and bending moment diagram as follows: Notice that the loading diagram is the free-body diagram for the beam.



Loading diagram



Shear force diagram



Bending moment diagram

Fig. Ex. S2.3 (Solution)

Example S2.4 Draw the loading diagram for the cantilever shown in Fig. Ex. S2.4. Calculate the shear force and bending moment at some salient points and hence sketch the shear force and bending moment diagrams.

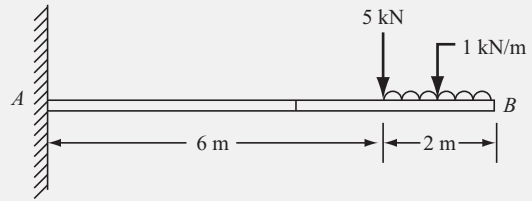


Fig. Ex. S2.4

Solution The reaction at A is calculated in the solution of Ex. S2.2. The shear forces and bending moments at some representative sections are also calculated there.

Taking into account those values, let us draw the loading (free-body) diagram, shear force and bending moment diagrams as follows.

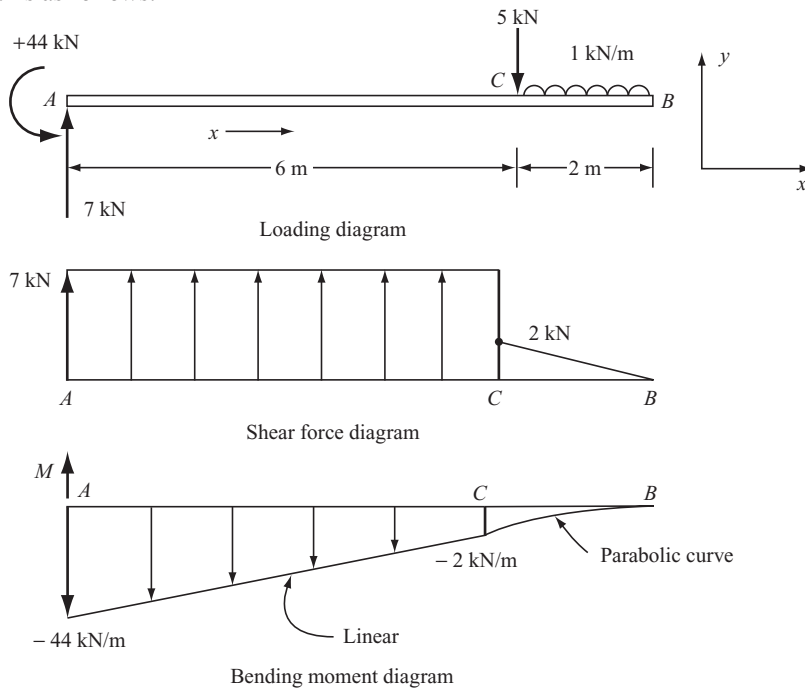


Fig. Ex. S2.4 (Solution)

Example S2.5 Draw the SF and BM diagrams for the beam loaded as shown in Fig. Ex. S2.5. Also locate the points of contraflexure.

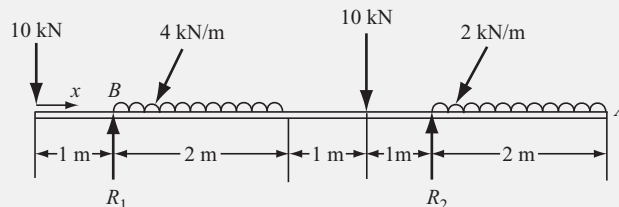


Fig. Ex. S2.5

Solution From the conditions of equilibrium of the beam

$$\Sigma = 0; -10 + R_1 - 4 \times 2 - 10 + R_2 - 2 \times 2 = 0$$

$$R_1 + R_2 = 32$$

$$\Sigma_B = 0; -10 \times 1 - 8 \times 1 - 10 \times 3$$

$$+ R_2 \times 4 - 4 \times 5 = 0$$

$$R_2 = 12 \text{ kN} \quad \text{and} \quad R_1 = 32 - 12 = 20 \text{ kN}$$

The shear force V at a point on the left of the concentrated load may be obtained by drag the *fbd* of the left hand part as shown in Fig. Ex. S2.5(c) (Solution) where V has been assumed to be positive.

For equilibrium,

$$-10 - 4 \times 2 - V = 0; \quad V = -18$$

Likewise, imaging or cross section any where in the beam, the general expression for shear force is

$$V = -10; +20 - 4(x-1); -10; +12; -2(x-5) \text{ upto } 3$$

Let us check for $x = 7 \text{ m}$,

$$V = -10 + 20 - 4(3-1); -10; +12; -2(2)$$

$$= -10 + 20 - 8 - 10 + 12 - 4 = 0; \quad \text{true!}$$

$$1 < x < 3 \quad M + 10x - 20(x-1) + 4(x-1)^2/2 = 0$$

$$M = -2(x-1)^2 + 10x - 20$$

$$\text{At } x = 1 \quad M = -10; \quad \text{at } x = 2 \quad BM = -2$$

$$\text{At } x = 3 \quad M = 2$$

$$3 < x < 4 \quad M + 10x - 20(x-1) + 8(x-2) = 0$$

$$M = 2x - 4$$

$$\text{At } x = 3 \quad M = 2; \quad \text{at } x = 4 \quad BM = 4$$

$$4 < x < 5 \quad M + 10x - 20(x-1) + 8(x-2) + 10(x-4) = 0$$

$$M = -8x + 36$$

$$\text{At } x = 4 \quad M = 4; \quad \text{at } x = 5 \quad BM = -4$$

$$5 < x < 7 \quad M + 10x - 20(x-1) + 8(x-2) + 10(x-4) - 12(x-5) + 2(x-5)^2/2 = 0$$

$$M = -(x-5)^2 + 4x - 24$$

$$\text{At } x = 5 \quad M = -4; \quad \text{at } x = 6 \quad BM = -1$$

$$\text{At } x = 7 \quad M = 0, \text{ as expected at the end } F.$$

$$M = -(x-5)^2 + 4x - 24$$

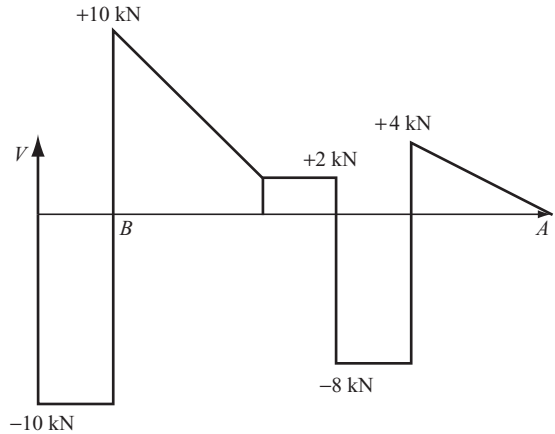
$$\text{At } x = 5 \quad M = -4; \quad \text{at } x = 6 \quad BM = -1$$

$$\text{At } x = 7 \quad M = 0, \text{ as expected at the end } F.$$

The points of contraflexure can be located by observing the change of sign of BM

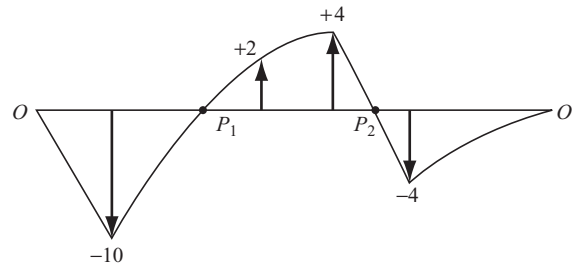
$$\text{For } P_1, -2(x-1)^2 + 10x - 20 = 0 \quad x = 2.38 \text{ m}$$

$$\text{For } P_2, -8x + 36 = 0 \quad x = 4.50 \text{ m}$$



Shear force diagram

Fig. Ex. S2.5(a) (Solution)



BM diagram

Fig. Ex. S2.5(b) (Solution)

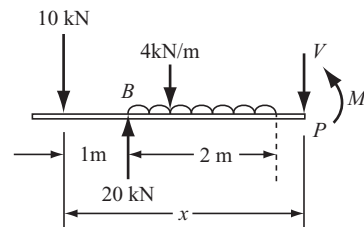


Fig. Ex. S2.5(c) (Solution)

Example S2.6

A beam carrying a uniformly-distributed load rests on two supports b m apart with an equal overhang of a m at each end. Determine the ratio b/a for the maximum bending moment to be as small as possible. Use this result to determine the most economical length for a railway sleeper if the rail centres are 1.6 m apart and also for the metre-gauge rails.

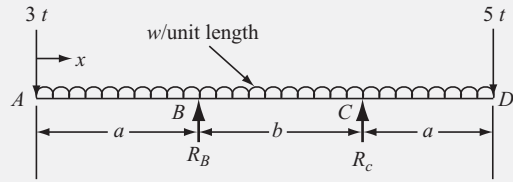


Fig. Ex. S2.6

Solution Consider the variation of the bending moment along the length of the beam

For $0 < x < a$ $M + wx \cdot x/2 = 0$; $M = -wx^2/2$

For $x = a$ $M = -wa^2/2$

The reactions are symmetrically located; these are each

$$w(2a + b)/2$$

From the above expression,

- at $x = 2$ m, $V = -10 + 20 - 4 = +6$ kN
- at $x = 3$ m, $V = -10 + 20 - 8 = +2$ kN
- at $x < 4$ m, $V = -10 + 20 - 8 = +2$ kN
- $x > 4$ m, $V = 2 - 10 = -8$ kN
- at $x = 5$ m, $V = -8 + 12 = +4$ kN
- at $x = 6$ m, $V = +4 - 2 = 2$ kN
- at $x = 7$ m, $V = 2 - 2 = 0$, or expected!

This data enables us to draw the shear force diagram as shown in Fig. Ex. S2.6 (Solution).

In order to determine the bending moment M at any point, one may also proceed to consider the equilibrium of the same part of the beam as for shear forces, by assuming M to be positive at the section P .

By equilibrium,

$$M + 10x - 20(x - 1) + 4 \times 2 \times (x - 2) = 0$$

where $M = -10x + 20x - 20 - 8x + 16 = 2x - 4$

at $x = 3$ m, $M = +2$ kN/m.

and at $x < 4$ m, $M = +4$ kN/m.

We could continue by writing the general expression for M until the other end A but it may be easier to proceed bit by bit and as follows:

Considering the equilibrium of different parts of the beam, starting from the left end, (by drawing the *fbd* or just imagining it),

$$0 < x < 1 \quad M + 10x = 0, \quad M = -10x$$

$$\text{At } x = 1, \quad M = -10 \text{ kN/m}$$

For $a < x < (a + b)$ $M + wx \cdot x/2 - w(2a + b)/2 \cdot (x - a) = 0$

$$M = -wx^2/2 + wax - wa^2 + wbx/2 - wab/2$$

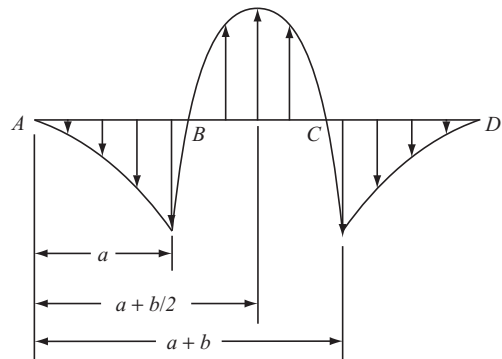


Fig. Ex. S2.6 (Solution)

As is clear from the nature of the *BM* diagram (Fig. Ex. S2.6 (Solution)) the bending moment can be maximum either at $x = a$ and $x = a + b$ or at $x = a + b/2$. The maximum bending is the least when the numerical values at these places are equal:

$$wa^2/2 = -w(a + b/2)^2/2 + wa(a + b/2) - wa^2 + wb(a + b/2)/2 - w ab/2$$

whence, cancelling w and simplifying,

$$a = b/\sqrt{8}$$

and

$$a/b = 1/\sqrt{8}$$

It is clear that, for $a > b/\sqrt{8}$ the *BM* at $x = a$, i.e., at the supports is maximum and for $a < b/\sqrt{8}$ the *BM* at $x = a + b/2$, i.e., at the centre is maximum.

The railway sleepers rest uniformly on the ground and are subjected to two equal point loads as a first approximation. For 1.6 m gauge rails,

$$b = 1.60; \quad a = 1.6/\sqrt{8} = 0.57 \text{ m}$$

The most economical length of the railway sleeper is, therefore,

$$l = 0.57 + 1.60 + 0.57 = 2.74 \text{ m}$$

For the metre gauge rails,

$$b = 1.00; \quad a = 1/\sqrt{8} = 0.36 \text{ m}$$

and the most economical length of the sleeper is

$$l = 0.36 + 1 + 0.36 = 1.72 \text{ m}$$

Example S2.7

Figure Ex. S2.7 shows the *SF* diagram for a beam supported at two points and loaded in some manner. Determine the position of the supports and details of loading over the entire length.

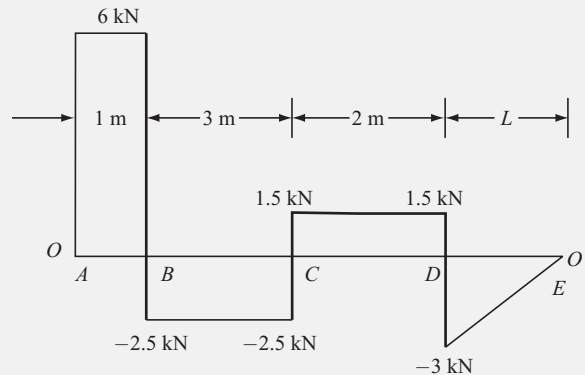


Fig. Ex. S2.7

Solution Conclusions in respect of the relationship between the loading, shear force and *BM* diagrams can be drawn from the fact that

$$\frac{d(V)}{dx} = -w$$

and

$$\frac{d(M)}{dx} = V$$

Between A and B , the shear force is constant; the loading must be zero. At A and at B , the shear force changes abruptly indicating

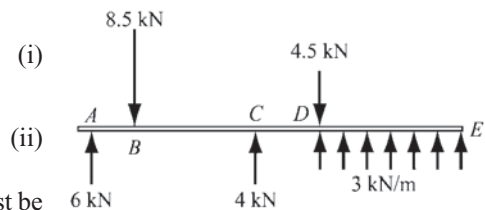


Fig. Ex. S2.7 (Solution)

the presence of concentrated loads at A and B . Positive shear force at A and from A to B indicates +ve, i.e., upward force of 6 kN at A . A change of sign of SF at B requires a downward concentrated force of 8.5 kN at B . Between B and C , there is no loading. Again, at C , there must be an upward force of 4 kN raising the V to 1.5 kN. Again, no loading between C and D and a downward force of 4.5 kN at D is observed from the SF diagram. From D to E the V increases linearly showing that the loading must be uniform and upward. This force is, by difference, 3 kN.

It appears that the supports are placed at A and C which provide upward reactions. There is additional support between D and E , which is uniform perhaps by way of floatation or some other way.

Example S2.8

A log of wood, specific gravity 0.78, 3 m long and 25 cm \times 25 cm in cross-section floats in water. Determine the load that should be placed centrally on the log so that the log is just completely immersed in water. Draw the SF and BM diagrams of the log.

Solution The log of wood is subjected to an upward hydrostatic force from below, as shown in Fig. Ex. S2.8 (Solution) the hydrostatic pressure being

$$p = \rho gh = 1000 \times 9.81 \times 0.25 = 2452.5 \text{ N/m}^2$$

which comes to an upward uniform loading

$$w_1 = 2452.5 \times 0.25 = 613 \text{ N/m}$$

The weight of the log appears as downward uniform loading

$$w_2 = 0.78 \times 1000 \times 9.81 \times 0.25 \times 0.25 = 478 \text{ N/m}$$

The effective uniform loading is

$$w = w_1 - w_2 = 613 - 478 = 135 \text{ N/m upward}$$

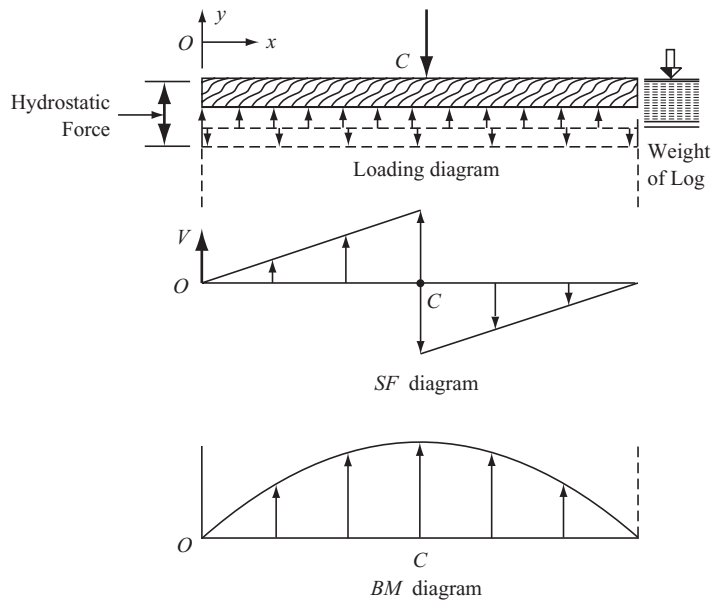


Fig. Ex. S2.8 (Solution)

The centrally placed weight should, therefore, be

$$W = 135 \times 3 = 405 \text{ N}$$

in order to keep the log in equilibrium.

The shear-force distribution is calculated thus:

| | | | |
|-----|----------------|------------------------|---|
| For | $0 < x < 1.5,$ | $-V + 135x = 0;$ | $V = +135x$ |
| At | $x = 0,$ | | $V = 0$ |
| At | $x = 1.5,$ | | V changes from $+202.5 \text{ N}$ to -202.5 N |
| For | $1.5 < x < 3,$ | $-V + 135x - 405 = 0;$ | $V = -405 + 135x$ |
| At | $x = 1.5,$ | | $V = -202.5 \text{ N}$ |
| At | $x = 3,$ | | $V = 0,$ as expected at the free end |

The *BM* diagram can be plotted from the following:

| | | | |
|-----|----------------|---|--|
| For | $0 < x < 1.5,$ | $M - 135x \cdot x/2 = 0;$ | $BM = 67.5x^2$ |
| At | $x = 0,$ | | $M = 0;$ at $x = 1.5$ $BM = 152 \text{ N/m}$ |
| For | $1.5 < x < 3,$ | $M - 135x \cdot x/2 + 405(x - 1.5) = 0$ | |
| | | | $M = 67.5x^2 - 405(x - 1.5)$ |
| At | $x = 1.5,$ | | $M = 152 \text{ N/m}$ |
| At | $x = 3,$ | | $M = 0,$ as expected at the free end |

Example S2.9

Draw the *SF* and *BM* diagrams

of the beam shown in Fig. Ex. S2.9.

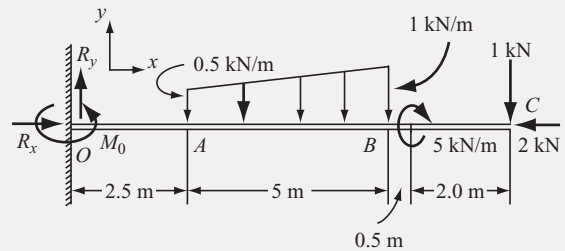


Fig. Ex. S2.9

Solution From the equilibrium of the beam,

$$\Sigma F_y = 0; R_y - (1/2 + 1)/2 \times 5 - 1 = 0;$$

$$R_y = 4.75 \text{ kN}$$

$$R_x - 2 = 0; R_x = 2 \text{ kN}$$

$$\Sigma M_O = 0; M_0 - \int_{2.5}^{7.5} wx \, dx - 5 - 1 \times 10 = 0$$

where the loading *w* is determined as a function of *x* as follows:

$$w = ax + b$$

$$\left. \begin{aligned} 0.5 &= 2.5a + b \\ 1.0 &= 7.5a + b \end{aligned} \right\} a = 0.1, b = 0.25$$

or

$$w = 0.1x + 0.25$$

and

$$M_0 = 34.7 \text{ kN/m}$$

Consideration of the shear force at different sections shows that

For

$$0 < x < 2.5, \quad -V + 4.75 = 0; \quad V = +4.75 \text{ kN}$$

For

$$2.5 < x < 7.5,$$

$$-V + 4.75 - \int_{2.5}^x (0.25 + 0.1x) dx = 0$$

$$= -0.05x^2 - 0.25x + 5.67$$

At $x = 2.5$, $V = +4.75 \text{ kN}$

At $x = 5.0$, $V = +3.17 \text{ kN}$

At $x = 7.5$, $V = +1.00 \text{ kN}$

For $7.5 < x < 10$,

$$-V + 4.75 - \int_{2.5}^{7.5} (0.25 + 0.1x) dx - \int_{7.5}^x (0.25 + 0.1x) dx = 0$$

$$= +1.00 \text{ kN}$$

The bending moment at different sections is determined as follows:

For $0 < x < 2.5$, $-M + 4.75x = 0; \quad M = 4.75x - 34.7$

At $x = 0$, $M = -34.7 \text{ kN/m}$

At $x = 2.5$, $M = -22.8 \text{ kN/m}$

For $2.5 < x < 7.5$, $M = 4.75x + 34.7 + (0.25 + 0.1x)(-2.5)$
 $= 4.75x - 34.7 - 0.125(-2.5)^2$
 $- 0.05(-2.5)^2 + 0.033(-2.5)^3$

At $x = 2.5$, $M = -22.8 \text{ kN/m}$

At $x = 5.0$, $M = -11.77 \text{ kN/m}$

At $x = 7.5$, $M = -7.5 \text{ kN/m}$

For $7.5 < x < 8$, varies linearly from -7.5 kN/m to -7 kN/m

At $x = 8$, changes abruptly from -7 kN/m to -2 kN/m

For $8 < x < 10$, varies linearly from -2 kN/m to 0 .

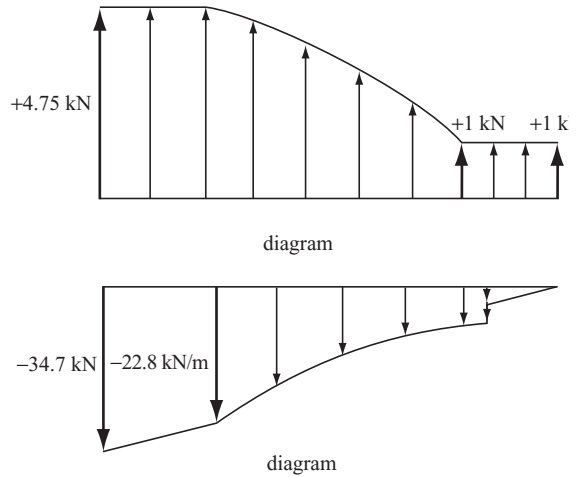


Fig. Ex. S2.9 (Solution)

Example S2.10 Compute the shear force, axial force and bending moment over the length of the bent beam shown in Fig. Ex. S2.10.

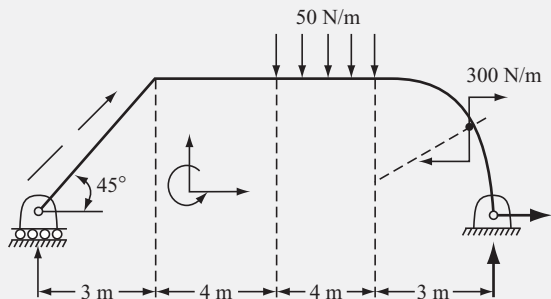


Fig. Ex. S2.10

Solution From the equilibrium of the beam,

$$\Sigma F_x = 0; \quad -300 - 14 R_A + 50 \times 4 \times 5 = 0; \quad R_A = 50 \text{ N}$$

$$\Sigma F_y = 0; \quad -50 \times 4 + 50 + R_{FY} = 0; \quad R_{FY} = 150 \text{ N}; \quad R_F = 0$$

Consider a section at a distance s from A along AB . From the free-body diagram of the beam segment

$$F_a \sin 45^\circ - V \cos 45^\circ + 50 = 0$$

$$F_a \cos 45^\circ + V \sin 45^\circ = 0$$

whence,

$$F_a = -70.7 \text{ N}; \quad V = +70.7 \text{ N}$$

$$M = 50 s \cos 45^\circ$$

$$= 35.35 s$$

At $s = 4.24$, $M = 150 \text{ N/m}$

Consider now a section between B and C . From the free-body diagram,

$$-V + 50 = 0; \quad V = +50 \text{ N} \quad \text{and} \quad AF = 0$$

$$M - 50(s - 4.24 + 3) = 0$$

or $M = 50s - 62.0$

At $s = 4.24$, $M = 150 \text{ N/m}$; at $s = 8.24$ $M = 350 \text{ N/m}$

From the free body of a section between C and D of the beam,

$$-V + 50 - 50(s - 4.24 - 4) = 0; \quad V = -50s + 462$$

$$AF = 0$$

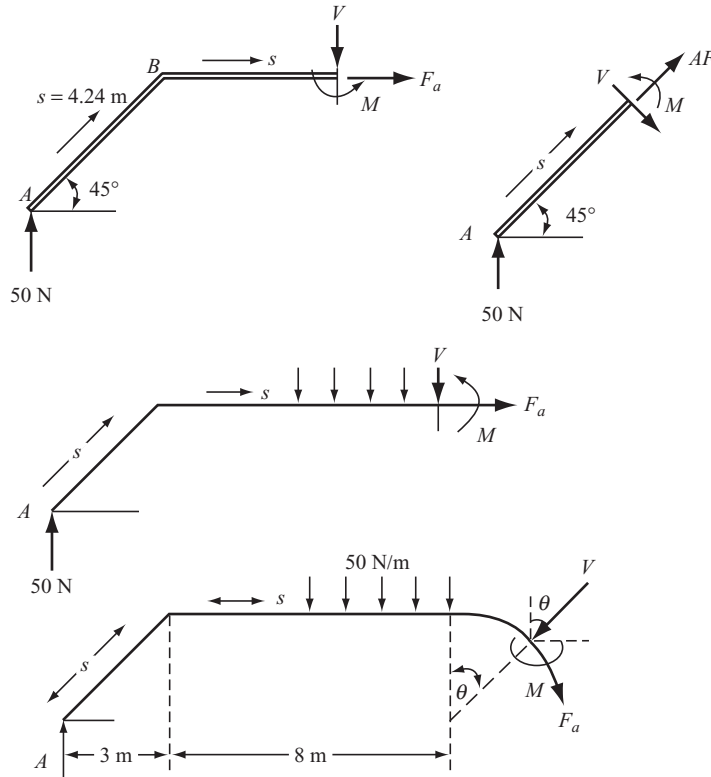


Fig. Ex. S2.10 (Solution)

At $s = 12.24, V = 150 \text{ N}$
 $M - 50(s - 4.24 + 3) + 50(s - 4.24 - 4)^2/2 = 0$
 $M = 50s - 62 - 25(s - 8.24)^2$

At $s = 8.24, M = 350 \text{ N/m}$
 At $s = 12.24, M = 150 \text{ N/m}$

For a section between D and E , from the free-body diagram of the part on the left,

$$-V \cos \theta - F_a \sin \theta + 50 - 200 = 0$$

$$-V \sin \theta + F_a \cos \theta = 0$$

whence

$$V = -150 \cos \theta \quad \text{and} \quad F_a = -150 \sin \theta$$

$$M - 50(s - 4.24 + 3 + 3 \sin \theta) + 200(2 + 3 \sin \theta) = 0$$

$$M = 50s - 450 \sin \theta - 462$$

At $s = 12.24, M = 150 \text{ N/m}$ and $\theta = 0$
 At $s = 12.24, M = -168 \text{ N/m}$ and $\theta = \pi/4$

At E , the bending moment changes abruptly from -168 N/m to 132 N m and varies linearly down to zero between E and F . The shear force and the axial force are given by

$$V = -150 \cos \theta; \quad F_a = -150 \sin \theta$$

until F where these values drop down to zero.

Example S2.11 An overhanging beam AB 20 m long simply supported at A and D carries a uniformly-distributed load and two concentrated loads as shown in Fig. Ex. S2.11. Determine the location of the supports placed 12 m apart sharing the load equally. Draw the BM diagram and locate the points of zero shear force in the beam.

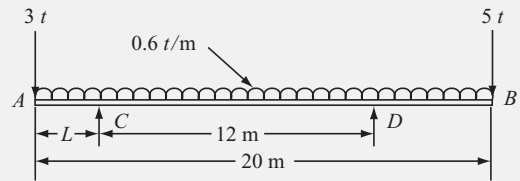


Fig. Ex. S2.11

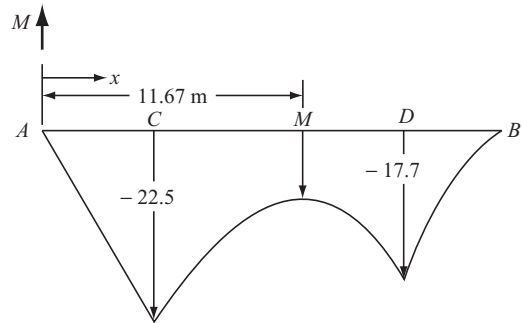
Solution From the equilibrium of the beam,
 $\Sigma = 0; \quad R_C + R_D - 3 - 0.6 \times 20 - 5 = 0$
 $R_C + R_D = 20$

and using the condition $R_C = R_D$,
 $R_C = 10 = R_D$

$\Sigma_A = 0;$
 $10L + 10(L + 12) - 5 \times 20 - 20 \times 0.6 \times 10 = 0$
 whence $L = 5 \text{ m}$ the distance of C from A .

For evaluating the bending moment in tonne metres along the beam, taking the origin at A and proceeding for the equilibrium of different sections:

$0 < x < 5 \quad M + 3x + 0.6x^2/2 = 0; \quad M = -3x - 0.3x^2$
 At $x = 0, \quad M = 0$, as expected at the end
 At $x = 2.5, \quad M = -9.375$
 At $x = 5, \quad M = -22.5$



Bending-moment diagram

Fig. Ex. S2.11 (Solution)

$$\begin{array}{ll}
 5 < x < 17 & M + 3x + 0.6x^2/2 - 10(x - 5) = 0 \\
 & M = 7x - 0.3x^2 - 50 \\
 \text{At } x = 5, & M = -22.5, \text{ providing a check} \\
 \text{At } x = 11, & M = -9.3 \\
 \text{At } x = 17, & M = -17.7 \\
 17 < x < 20 & M + 3x + 0.6x^2/2 - 10(x - 5) - 10(x - 17) = 0 \\
 & M = +17x - 0.3x^2 - 220 \\
 \text{At } x = 17, & M = -17.7, \text{ providing a check} \\
 \text{At } x = 20, & M = 0, \text{ as expected at the end}
 \end{array}$$

The *BM* diagram is shown in Fig. Ex. S2.11 (Solution). There is no point of contraflexure in the beam because the bending moment does not change sign at any point.

The point of zero shear in the beam corresponds to the location of the minimum bending moment:

$$\begin{aligned}
 \frac{d}{dx} (7x - 0.3x^2 - 50) &= 0 \\
 7 - 0.6x &= 0
 \end{aligned}$$

or at $x = 11.67$ m; at point *M* as shown.

It may also be noted that the shear force changes sign, i.e., passes through zero at points *C* and *D* but the value at these points is not said to be zero.

Example S2.12 A flat plate (4 m × 1 m) (Fig. Ex. S2.12) hinged at the top 1 m edge serves as a wide wall of a tank containing oil of specific gravity 0.9 to a depth of 3 m above the lower edge of the plate. Determine the horizontal force required at the lower edge of the plate to keep it in equilibrium. Draw the *SF* and *BM* diagrams of the plate.

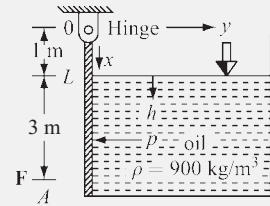


Fig. Ex. S2.12

Solution The pressure at a depth *h* in oil is given by

$$\begin{aligned}
 p &= \rho gh \\
 &= 900 \times 9.81 \times h = 8829h \text{ N/m}^2 \\
 &= 8.83h \text{ kN/m}^2
 \end{aligned}$$

Total hydrostatic force acting on the plate is

$$\begin{aligned}
 P &= \int p \, dA = \int_0^3 8.83h \times (1 \times dh) \\
 &= \left[8.83 \frac{h^2}{2} \right]_0^3 = 8.83 \times 9/2 \\
 &= 39.74 \text{ kN}
 \end{aligned}$$

which acts at 1 m from the base of the plate, i.e., at 3 m below the hinge.

For equilibrium of the plate

$$\begin{aligned}
 \Sigma \tau_0 &= 0; \\
 4F &= 39.74 \times 3 \\
 F &= 29.80 \text{ kN}
 \end{aligned}$$

whence

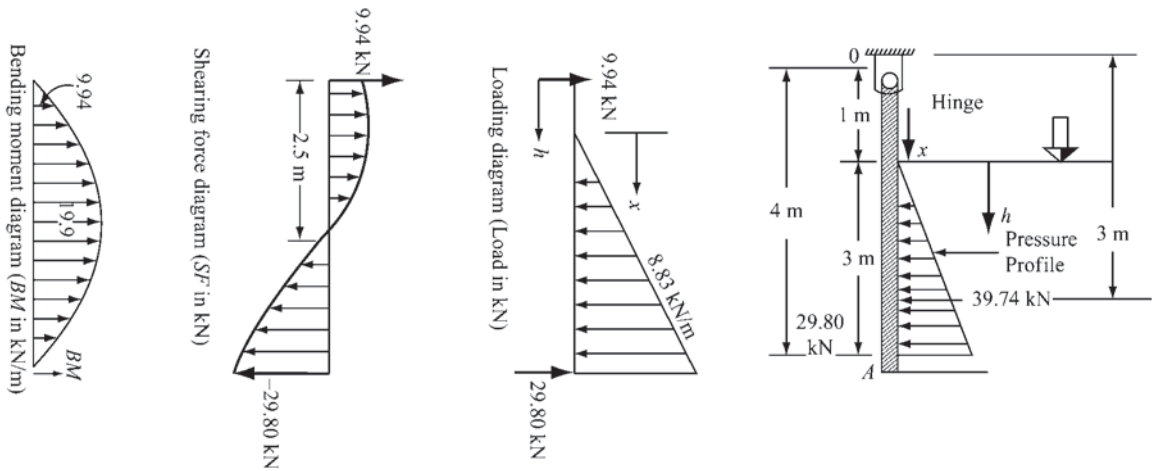


Fig. Ex. S2.12 (Solution)

The reaction at the hinge 0 must be $(39.74 - 29.80) = 9.94$ kN towards right in the horizontal direction.

The shearing force is estimated as follows:

For $0 < x < 1$, $V = +9.94$ kN

For $1 < x < 4$,

$$\text{or } 0 < h < 3, \quad V = +9.94 - \int_0^h 8.83h \cdot (1 \times dh)$$

$$= +9.94 - 4.41h^2 \text{ kN}$$

which is zero at $h = \sqrt{\frac{9.94}{4.41}} = 1.50$ m

or at $x = 1.50 + 1 = 2.5$ m

and maximum being

$$+9.94 - 4.41 \times 3^2 \text{ or } -29.80 \text{ kN}$$

At $h = 3$ m or $x = 4$ m

which was expected because the reaction of 29.80 kN would bring it to zero at the end.

The bending moment is calculated as follows:

For $0 < x < 1$, $M = 9.94x$ kN/m

At $x = 1$, $M = 9.94$ kN/m

For $1 < x < 4$,

$$\text{or } 1 < h < 3, \quad M = 9.94x - 4.41 h^2 \cdot h/3$$

$$= 9.94(h + 1) - 1.47h^3$$

$$= 9.94 + 9 + 94h - 1.47h^3 \text{ kN/m}$$

It may be seen that the bending moment is maximum at $h = 1.5$ m.
 Maximum $M = 9.94 + 9.94 \times 1.5 - 1.47 \times 1.5^3$
 $= 19.9$ kN m

It should also be appreciated that, at $h = 3$ m, the M reduces to
 $9.94 + 9.94 \times 3 - 1.47 \times 3^3 = 0$

as is expected at the lower end of the plate.

The SF and BM diagrams are consequently as shown in Fig. Ex. S2.12 (Solution).

EXPERIMENT E5 SHEAR FORCE AND BENDING MOMENT IN A BEAM

OBJECTIVE TO DETERMINE THE SHEAR FORCE AND BENDING MOMENT AT A CROSS SECTION OF A BEAM AND TO COMPARE THE SAME WITH THE CORRESPONDING THEORETICAL VALUES.

Apparatus A simply supported level beam with provision for loading at the desired points and spring balances provided to enable the measurement of shear force and bending moment at a cross section.

Background Information The shear force at a cross section of a beam is defined as the transverse force tending to cause shear across the section. The bending moment in a beam at any cross-section is the transverse moment tending to cause bending of the beam in the plane of the loading.

Experimental determination of the shear force and bending moment at a cross-section is made by improvising a beam in two parts and by measuring the reaction of one part on the other.

Initially, the unloaded beam is positioned to be horizontal by adjusting the wing nuts provide on the spring and the initial readings on the spring are noted. The beam is then loaded as desired. This may be achieved by suspending masses m_1, m_2 and m_3 at three places at distances of x_1, x_2 and x_3 from the left end A as shown in Fig. E5.1. The loads acting at these locations are, therefore, m_1g, m_2g and m_3g . The two parts of the beam tend to get disturbed from the horizontal position on the application of the loads. The wing nuts are turned suitably to bring them in the horizontal position.

The magnitude of the shear force at the cross section equals the corrected reading on the spring balance S_1

$$V = S_1 \tag{E5.1}$$

The theoretical value may be obtained by computing the reactions R_A and R_B and then by noting that, for the equilibrium of the left-hand part,

$$R_A - m_1g - m_2g + V = 0$$

whence

$$V = -m_1g - m_2g + R_A \tag{E5.2}$$

The magnitude of the bending moment at the cross-section equals the moment exerted by the force observed as the corrected reading S_2 on the spring balance placed at an arm of length a , i.e.,

$$M = S_2 \cdot a \tag{E5.3}$$

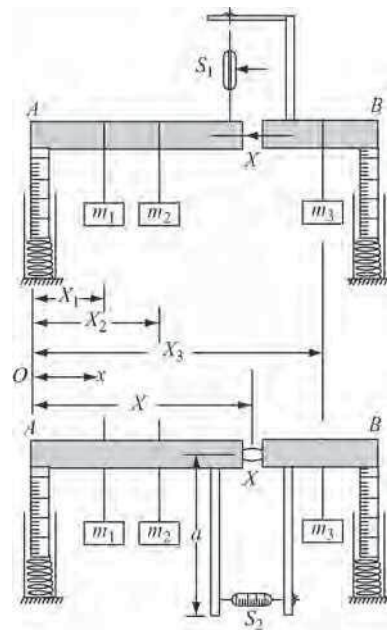


Fig. E5.1 Measurement of shear force and bending moment

The theoretical value may be obtained by observing that, for the equilibrium of the left-hand part,

$$-R_A \cdot x + m_1 g \cdot (x - x_1) + m_2 g \cdot (x - x_2) + M = 0$$

whence
$$M = R_A \cdot x - m_1 g (x - x_1) - m_2 g (x - x_2) \tag{E5.4}$$

Observations and Calculations The initial and final readings of the springs for the unloaded and loaded beams respectively are recorded:

A set of observations and calculations for a prescribed loading may be arranged as follows:

Length of the Beam:

Loading on the Beam:

Distances of Loads:

Distance a:

| | <i>Initial</i> | <i>Final</i> | <i>Difference</i> |
|--------------------------------------|----------------|--------------|-------------------|
| Spring balance <i>A</i> | | | |
| Spring balance <i>B</i> | | | |
| Spring balance <i>S</i> ₁ | | | |
| Spring balance <i>S</i> ₂ | | | |

$SF = S_1 =$

$BM = S_2 a =$

Results The measured values of the shear force and bending moment at the section may be recorded and compared with the theoretical values.

Points for Discussion

Comment on the difference between the experimentally measured and the theoretical values of shear force and bending moment.

Compute the percentage error for each system of loading and observe whether the percentage error increases as the value of theoretical shear force and bending moment decrease. If so, why? If not, why not?

Have you taken the weight of the beam into account? Explain how?

Can the method of measuring the shear force and the bending moment be employed on a cantilever? Explain how with the help of a sketch. Can the method be used on a beam with the fixed end supported and with other end conditions?

Is it possible to state from the observations *S*₁ and *S*₂ taken from the spring balances whether the shear force and bending moment are positive or negative?

Concept Review Questions

- (a) What is the difference between a beam and a member of a simple truss?
- (b) What are the implications of a beam referred to as thin and rigid?

State the different means of supporting a beam and differentiate between an overhanging beam, a cantilever beam and a continuous beam.

- (a) Define the terms shear force and bending moment at a cross-section in a beam. How are the *SF* and *BM* diagrams drawn and what useful purpose is achieved by drawing them?
 - (b) Comment on the sign conventions for the shear force and bending moment at a section.
 - (c) Can a structural member have shear force and bending moment at a cross-section?
- (a) How are the distributions of the loading, the shear force and bending moment related to each other? Are there any pre-conditions for the relationship?

- (b) Sketch a simply supported beam with some transverse and inclined loading. Draw the *SF* and *BM* diagrams for the same alongside the sketch.

What are the implications of sudden changes of

- (a) loading
- (b) shear force
- (c) bending moment

for a simply supported beam?

Is it possible to predict the location and mode of failure of a beam subjected to a given loading? Is it necessary to draw the *SF* and *BM* diagrams before predicting the same?

Tutorial Problems

Draw the *SF* and *BM* diagrams for the beams and loading shown in Fig. Prob. S2.1(a), (b), (c).

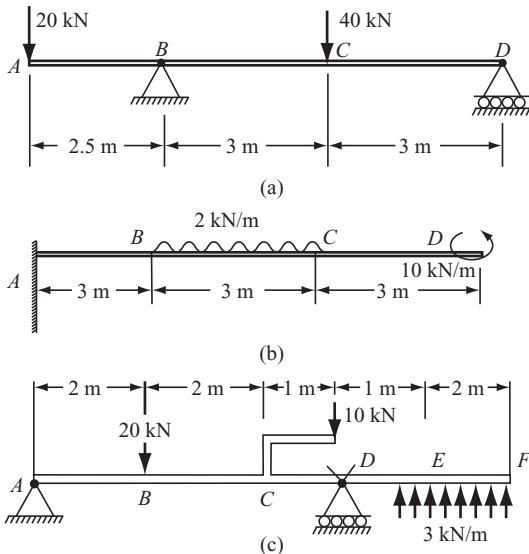


Fig. Prob. S2.1(a), (b), (c)

A beam of length L is loaded by an external moment M as shown in Fig. Prob. S2.2. Draw the *SF* and *BM* diagrams.

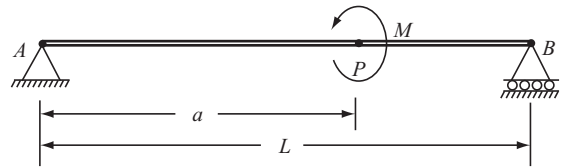


Fig. Prob. S2.2

A simply supported overhanging beam 4 m long is uniformly loaded at 2 kN/m over the entire span and carries a triangularly distributed load over the left half of the span as shown in Fig. Prob. S2.3. Draw the *SF* and *BM* diagrams for the beam.

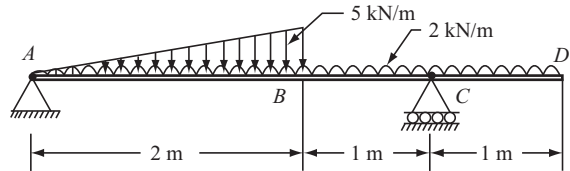


Fig. Prob. S2.3

A uniform beam AB of weight W rests horizontally on two supports C and D at one third of the span from either end respectively. There are two loads, each $2W$ at one third the distance between C and D measured from

C and D respectively. Draw the SF and BM diagrams for the beam.

A train of weight W and length L is in the centre of a bridge whose span is twice the length of the train as shown in Fig. Prob. S2.5. Assuming that the weight of the train is uniformly distributed throughout its length, calculate the bending moment at the centre of the bridge. Compare it with the value when one end of the train just reaches the pier-end of the bridge.

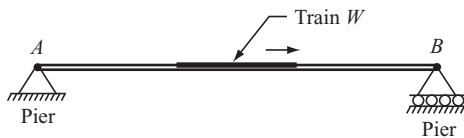


Fig. Prob. S2.5

Draw the SF and BM diagrams for a light horizontal cantilever 4 m long carrying concentrated loads each of 2 kN at its centre and its free end.

The SF diagram for a part of a loaded beam is shown in Fig. Prob. S2.7. All the SF values are

in kN. Draw the loading diagram and the BM diagram for the beam.

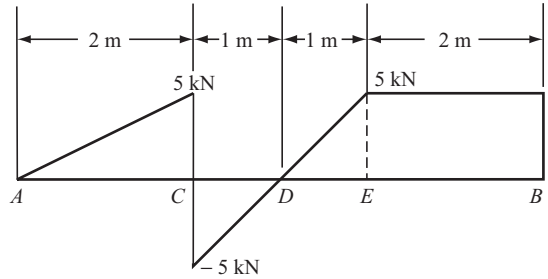


Fig. Prob. S2.7

- 8 The BM diagram for a beam 1 m long is parabolic with a maximum of 20 kN m at the mid-span of a beam as shown in Fig. Prob. S2.8. Sketch the corresponding SF and loading diagrams.

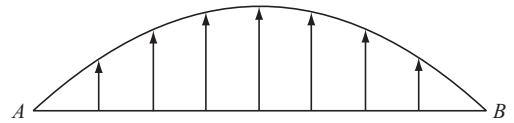


Fig. Prob. S2.8

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

- A thin rigid beam hinged at one end and roller-supported at its mid-points is said to be
- (a) a symmetrical simply supported beam
 - (b) an overhanging simply supported beam
 - (c) a cantilever beam
 - (d) a fixed beam

The shear force at a section in a beam is given by

- (a) the external force at that section
- (b) the transverse component of the external force at that section
- (c) the transverse force from the part of the beam on one side of the section to that on the other side of the section
- (d) the addition of the forces at (b) and (c) above

The bending moment at a section in a beam is given by

- (a) the external moment at that section
- (b) the summation of all the moments about that section
- (c) the summation of moments of all the forces about the section
- (d) the net moment exerted by the part of the beam on one side of the section to that on the other side of the section

The point of contraflexure in a loaded beam is one where

- (a) the bending moment is maximum
- (b) the shear force is maximum
- (c) the bending moment changes sign
- (d) the shear force changes sign

In a symmetrically overhung simply supported beam, the maximum bending moment will be the least possible when

- (a) the supports are near the ends
- (b) the supports coincide to become a single support at the centre
- (c) the distance between the supports becomes one-third of the length of the beam.
- (d) the numerical values of the bending moment at either supports and at the centre of the beam are equal

The maximum bending moment in a simply supported beam length L loaded by a concentrated load W at the mid point is given by

- (a) WL
- (b) $WL/2$
- (c) $WL/4$
- (d) $WL/8$

Answers to the Multiple-Choice Questions

- 1 (b) 2 (c) 3 (d) 4 (c) 5 (d) 6 (c)

Applications with Friction



S3

S3.1 FRICTION AND IMPENDING MOTION

When two surfaces in contact with each other exert force normal to each other and one surface slides or tends to slide with respect to the other. Forces of friction come into play when two surfaces in contact with each other exert force normal to each other and one surface slides or tends to slide with respect to the other. The force of friction is also called *frictional resistance* or simply *friction*. The mechanism of friction can be explained by the interlocking of the roughnesses of the surfaces or by the development of adhesive forces as the molecules of the surfaces come close together or by some other hypothesis; these explanations are of little consequence if the net effect is expressed in terms of a macroscopic parameter, *coefficient of friction*, defined as follows:

Consider a body of mass m resting on a surface as shown in Fig. S3.1. If a normal force F_n acts normal to the lower surface, the body continues to be at rest because an equal and opposite force R acts upon it by the lower surface. The normal force F_n may be due to the weight of the body or by some other action. If now a small tangential force F_t is also acted upon the body, the body may still continue to be at rest. The applied tangential force F_t is balanced by the friction force f due to the lower surface. Until a certain limiting value of F_t

$$f = F_t$$

the body stays at rest. Clearly, the body would slide if the magnitude of F_t is increased beyond this limiting value. This state of the body is called *state of impending motion*; a critical border line condition between the static and dynamic conditions of the body.

The state of impending motion can be modelled in terms of the following laws:

1. *The maximum force of friction is independent of the magnitude of area in contact between the surfaces.*

These laws are stated without proofs at this stage. Premature statements are given merely to introduce the pressure intensity and force concepts in the context of force fields.

2. The maximum force of friction is proportional to the normal force on the area of contact.
3. The maximum force of friction is less and practically constant at low velocities of sliding than that at the state of impending motion.

The first law allows us to ignore the extent of the area in contact. The area does not enter the picture, perhaps because the extent of mechanical interlocking or cohesive forces adjusts with respect to the normal forces along which acts the total force of friction.

The second law provides a proportionality

$$f_{\max} \propto N$$

or

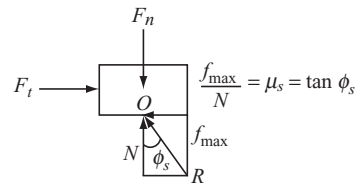
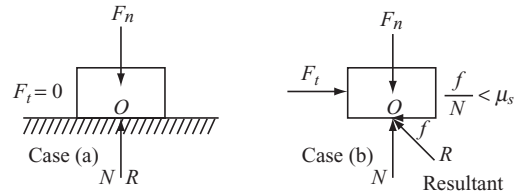
$$f_{\max} = \mu_s N \tag{S3.1}$$

where f_{\max} is the limiting force of friction, N the normal reaction force and μ_s the constant of proportionality.

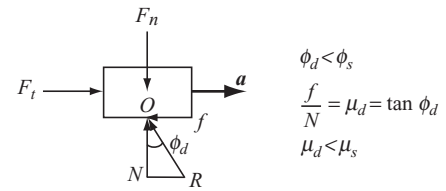
Coefficient of friction between two surfaces is the constant of proportionality μ_s between the limiting f_{\max} and the normal reaction R .

$$\mu_s = \frac{f_{\max}}{N}$$

If two surfaces have the coefficient of friction $\mu_s = 0$, the surfaces are said to be *smooth*. Surfaces with a non-zero coefficient of friction are called *rough surfaces*. Obviously, a smooth surface cannot provide a force of friction and hence the reaction of a smooth surface must always be normal to it. If, however, a body with a smooth surface rests on another smooth surface and a tangential force is applied to it, the body must slide and accelerate under the action of the applied forces.



Case (c) Limiting case of impending motion



Case (d) Motion with friction

Table S3.1 Coefficients of static friction μ_s

| Pair of Surfaces | Range of μ_s |
|---------------------------|------------------|
| Wood and wood | 0.2 – 0.6 |
| Wood and leather | 0.2 – 0.5 |
| Rope and wood | 0.6 – 0.7 |
| Steel and cast iron | 0.4 – 0.5 |
| Steel and leather | 0.4 – 0.6 |
| Mild steel and mild steel | 0.5 – 0.6 |

The coefficient of dynamic friction is about 25% less than that for static friction.

The friction force

$$f_{\max} = \mu_s N$$

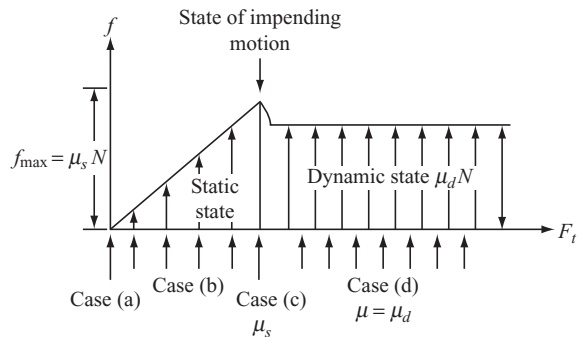


Fig. S3.1 Concept and variation of frictional force

given by the coefficient of friction is the *maximum frictional force* which can be developed between the two surfaces for a given normal reaction force N . In actual practice,

- (i) if a body is at rest on a surface, the friction force at an instant may be less than the limiting value, it being only equal to the applied tangential component of the force such as that shown in case (b) in Fig. S3.1 where

$$f < \mu_s N$$

- (ii) if the body is in motion, the friction force is given by the coefficient of dynamic friction or the coefficient of kinetic friction. This coefficient is indeed less than the coefficient of static friction; the difference being a function of the velocity and the nature of surfaces in contact. The coefficient of dynamic friction is written as μ_d to differentiate it from the coefficient of static friction which is often simply denoted as μ . In the dynamic state, as shown in case (d), Fig. S3.1.

$$F = \mu_d N \tag{S3.2}$$

where

$$\mu_d \approx 75\% \mu_s$$

- (iii) if the body is in an unsteady state, e.g., in intermittent or reversed sliding motion, the coefficient of friction may be quite different from the steady state value.

It is usual to define the term *angle of friction*. Angle of friction is the angle between the line of action of the total reaction by one body on the other and the normal to the common tangent to the surfaces in contact in the state of impending motion.

Angle of friction
$$\phi_s = \tan^{-1} \left(\frac{f_{\max}}{N} \right) = \tan^{-1} \mu_s$$

or

$$\mu_s = \tan \phi_s \tag{S3.3}$$

The angle of friction ϕ_s is, therefore, the maximum angle between the normal reaction R and the resultant reaction at the instant of impending motion.

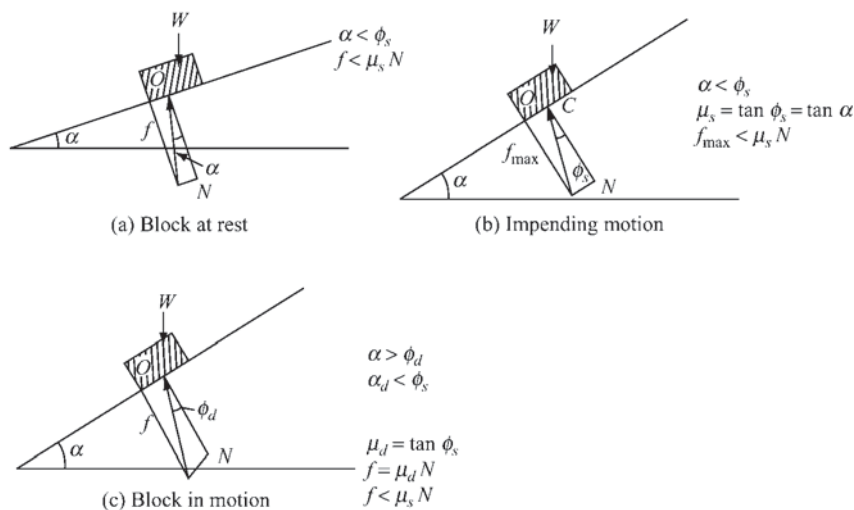


Fig. S3.2 Frictional force on a body on an incline

It can also be shown that the angle to which an inclined plane may be raised before the object resting on it slides under the action of its weight and the reaction of the plane, also called as the *angle of repose*, equals the angle of friction. This is shown in Fig. S3.2. Also shown in the figure are the cases of α less than ϕ_s and greater than ϕ_s .

Fluid Resistance

Surface contact between two material bodies which results in friction forces is permitted only when the frictional forces are usefully employed. This is, however, not the case in most circumstances when the relative velocity of sliding is required and the resistive forces must be minimised. In such case the interspaces must be filled with suitable fluid lubricants. Drag forces resulting from the fluid resistance appear on the moving components. In general, a drag force appears when a solid body moves in a mass of fluid or when a fluid flows around a solid body.

Two different modes of drag formation resulting from the relative velocity between a solid and a fluid must be recognised. First, the *laminar flow drag*, when the relative velocity is low and the fluid flow is characterised by thin sliding laminas sliding over each other; second, the *turbulent flow drag*, when the relative velocity is higher and the fluid flow is characterised by random turbulence and eddy formation.

While the mechanics of drag formation in laminar and turbulent flow is not discussed at this stage, it is worthwhile stating that the variation of drag with the basic parameters is different in the two cases as shown in Table S3.2.

Table S3.2 Variation of drag

| | Laminar Flow | Turbulent Flow |
|------------|---------------------|------------------------------|
| Drag Force | \propto speed | \propto speed ² |
| | \propto viscosity | \propto density |
| | \propto size | \propto size ² |

Example S3.1

A wooden block of mass 1 kg rests on a rough incline at an angle θ as shown in Fig. Ex. S3.1. If the coefficient of friction between the contact surfaces is 0.5, determine the force required to be applied to the string passing through a frictionless pulley to initiate motion of the block down the plane. At what angle θ would the force required be zero? What would happen if the angle is increased beyond that value?

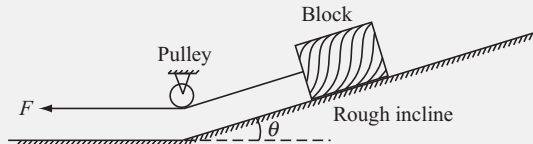


Fig. Ex. S3.1

Solution Consider the free-body diagram of the block shown in Fig. Ex. S3.1 (Solution). It is subjected to a weight force mg acting downward, a string force F down the plane and a frictional force f up the plane because the impending motion is down the plane and the normal reaction R perpendicular to the plane. At the state of impending motion, the frictional force must be such that

$$F = \mu R \tag{i}$$

where R is the normal reaction by the plane on the block.

For equilibrium of the block,

$$\Sigma F_x = 0; \quad F - F - mg \sin \theta = 0 \quad \text{(ii)}$$

$$\Sigma F_y = 0; \quad N - mg \cos \theta = 0 \quad \text{(iii)}$$

From Eq. (iii), $N = mg \cos \theta$
 and from Eq. (i), $f = \mu mg \cos \theta$

which when substituted in Eq. (ii) provides

$$F = \mu mg \cos \theta - mg \sin \theta$$

or $F = (\mu \cos \theta - \sin \theta) mg \quad \text{(iv)}$

Substituting

$$\begin{aligned} \mu &= 0.5 \quad \text{and} \quad m = 1 \text{ kg} \\ F &= (0.5 \cos \theta - \sin \theta) 1 \times 9.81 \\ &= (4.905 \cos \theta - 9.81 \sin \theta) \text{ N} \end{aligned}$$

The force required would be zero when

$$\mu \cos \theta - \sin \theta = 0$$

or $\tan \theta = \mu$

For the present case,

$$\theta = \tan^{-1} 0.5 = 26.56$$

If the angle of incline is increased beyond 26.56°, the force required to initiate the motion is negative which means that an upward force is required to hold the block in equilibrium. If the angle of incline is decreased below 26.56°, the force required to initiate the motion increases.

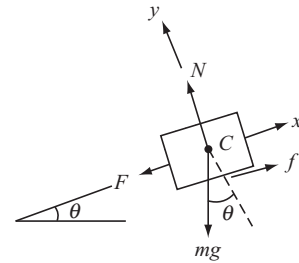


Fig. Ex. S3.1 (Solution)

Example S3.2

Two boxes of weights W_1 and W_2 are stacked on the floor as shown in Fig. Ex. S3.2. The coefficient of friction between the boxes is μ_1 and between the box and the floor is μ_2 . A horizontal force F is applied to the top of the boxes and gradually increased. What is the maximum force before the equilibrium is destroyed? How will the equilibrium be destroyed?

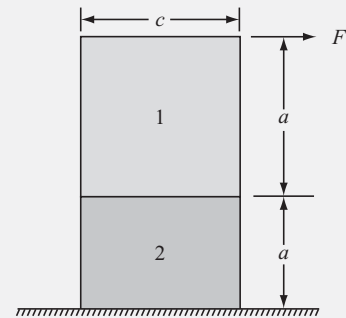


Fig. Ex. S3.2

Solution The free-body diagram of each box is drawn in Fig. Ex. S3.2 (Solution). The upper box 1 is subjected to the applied force F , weight W_1 , normal reaction R_1 from box 2 and frictional force f_1 acting by virtue of contact with box 2. In turn, box 2 is acted upon by a frictional force equal and opposite to f_1 as well as a normal reaction equal and opposite to R_1 at its upper surface. It is, in addition, subjected to a normal reaction R_2 and a friction force f_2 at its base surface due to contact with the floor and its own weight W_2 .

Let us examine the different possibilities of upsetting the equilibrium.

1. Box 1 may slide to the right, box 2 remaining intact.
2. Boxes 1 and 2 may remain stuck and slide together over the ground.
3. Box 1 may tip over its right edge A , box 2 remaining intact.
4. Boxes 1 and 2 may remain stuck and tip together over the bottom right edge B .

Since the force F is increased from a zero value, the mode of upsetting the equilibrium depends upon which of the four possibilities materialises first. We, therefore, evaluate the minimum forces required to materialise each possibility and the least of them would be the decisive force.

For the first possibility, consider the equilibrium of box 1 at the state of impending motion.

$$\Sigma F_x = 0; \quad F_1 = f_1 = \mu_1 N_1$$

$$\Sigma F_y = 0; \quad N_1 = W_1$$

whence $F_1 = \mu_1 W_1$

Similarly for the second possibility,

$$N = W_1 + W_2$$

$$F_2 = f = \mu_2 N = \mu_2 (W_1 + W_2) \quad \text{(ii)}$$

For the tipping of the upper box alone,

$$F_3 \times a = W_1 \times c/2$$

whence $F_3 = \frac{W_1 c}{2a} \quad \text{(iii)}$

For the tipping of the two blocks together,

$$F_4 \times (a + b) = (W_1 + W_2) \times c/2$$

whence $F_4 = \frac{(W_1 + W_2)c}{2(a + b)} \quad \text{(iv)}$

An observation of the results (i) to (iv) reveals that the magnitudes of F_1, F_2, F_3 and F_4 depend upon the values of weights, coefficients of friction and the dimensions of the blocks. In particular, if

$$\mu_1 = \mu_2 = 0.3$$

$$W_1 = W_2 = 100 \text{ N}$$

$$a = b = c = d, \quad \text{say}$$

$$F_1 = 0.3 \times 100 = 30 \text{ N}$$

$$F_2 = 0.3 \times (100 + 100) = 60 \text{ N}$$

$$F_3 = \frac{100 \times d}{2 \times d} = 50 \text{ N}$$

$$F_4 = \frac{(100 + 100) \times d}{2(d + d)} = 50 \text{ N}$$

Since F_1 is the smallest, the boxes will no longer be in equilibrium if the applied force F exceeds 30 N. The equilibrium will be destroyed by the slippage of the upper box, the lower remaining intact.

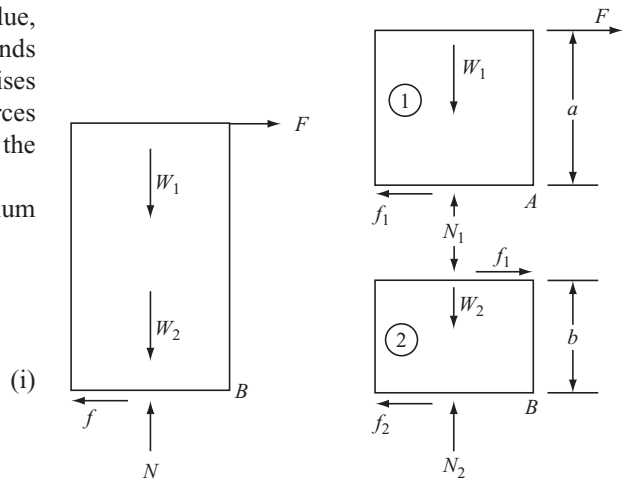


Fig. Ex. S3.2 (Solution)

EXPERIMENT E6 COEFFICIENT OF STATIC FRICTION

OBJECTIVE TO DETERMINE THE COEFFICIENT OF STATIC FRICTION BETWEEN TWO GIVEN MATERIAL SURFACES WITH THE HELP OF AN INCLINED PLANE.

Apparatus An adjustable inclined plane with a frictionless pulley, a block, inextensible string and standard weights.

Background Information The mating surfaces of the block and the incline are faced with sheets of materials between which the coefficient of friction is desired. At a fixed angle of inclination θ , the suspended mass is increased until the block is at the verge of upward slippage, i.e., in the state of impending motion. Referring to the free-body diagram of the block at such a state, as shown for equilibrium,

$$T = Mg = f + mg \sin \theta$$

$$R = mg \cos \theta$$

whence $\mu = f/R = \frac{Mg - mg \sin \theta}{mg \cos \theta} = \frac{M - m \sin \theta}{m \cos \theta}$

Observations and Calculations The mass of the block as well as the mass of the suspended weight together with the chosen angle of inclination should be recorded:

| | | Material 1 | | Material 2 | | |
|--------|----------|------------|-----|---------------|---------------|-------|
| S. No. | θ | m | M | $\sin \theta$ | $\cos \theta$ | μ |
| 1 | | | | | | |
| 2 | | | | | | |

The observations and calculations are repeated with different selections of the two masses or two angles of inclination or both.

Result The average value of the coefficient of static friction μ may be obtained and the range of variation of μ may be noted.

Points for Discussion

Compare the value of μ between the two surfaces with the value given in a standard handbook. Account for the difference, if any.

Would the value of μ be the same if the materials on the incline and block are interchanged?

In the first method, supposing a weight W is placed over the block, would the angle of repose remain the same?

What is the corresponding value of the coefficient of dynamic friction μ_d between the same pair of surfaces? Explain, giving examples, as to where μ is used and where μ_d is to be used?

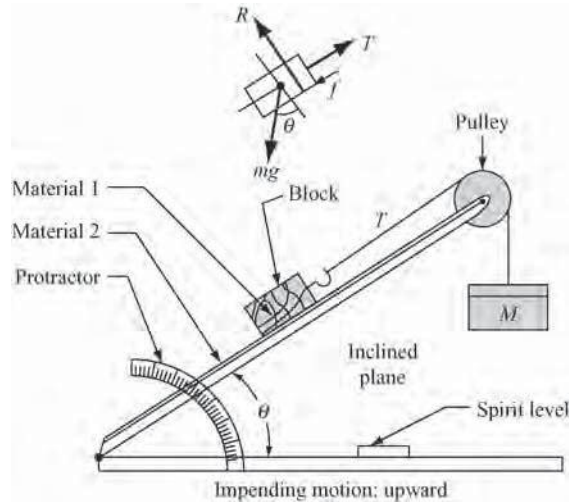


Fig. E6.1 Determination of coefficient of static friction μ

Supposing a block is placed on an incline and there is no pulley and string, etc. Can't we find the coefficient of static friction by creating a condition of downward impending motion? By increasing the inclination θ ; by adding more weight on the block? Can we? How would it compare with our method?

The assumptions made in the analysis of observations are frictionlessness of the pulley, inextensibility of the string, correctness of masses and angles measured. How far are the assumptions justified and what are the possible sources of error?

If the rectangular block were faced with the same material on all faces, would the observations and calculations alter if the block were placed one face or the other resting on the incline?

Example S3.3

A ladder of length 5 m and weight 120 N is placed on a flat floor against a vertical wall as shown in Fig. Ex. S3.3. If the coefficients of friction are 0.3 and 0.2 and the ladder is considered homogeneous, determine the smallest angle θ the ladder can be placed at the floor for equilibrium.

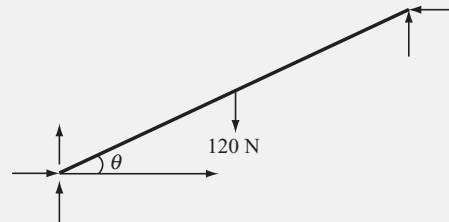


Fig. Ex. S3.3

Solution At the position of the smallest angle θ , the ladder would be in the state of impending skid at A and at B.

$$\mu_A = 0.3 \quad \text{and} \quad \mu_B = 0.2$$

For equilibrium,

$$\Sigma F_x = 0 \quad - \quad \Sigma F_y = 0; \quad \Sigma M = 0$$

$$\Sigma F_x = 0 \quad - \quad 120 + F_B = 0; \quad F_B = 120 -$$

$$\Sigma M = 0 \quad \times 5 \sin \theta + F_B \times 5 \cos \theta - 120 \times 2.5 \cos \theta = 0$$

From these equations,

$$\mu_A = 0.3 \quad \text{and} \quad 120 - F_B = 0.2 F_B$$

whence,

$$F_B = 113.2 \text{ N} \quad \text{and} \quad F_A = 34 \text{ N}$$

and

$$F_A = 34 \text{ N} \quad \text{and} \quad F_B = 6.8 \text{ N}$$

Finally,

$$170 \sin \theta + (34 - 300) \cos \theta = 0$$

$$\tan \theta = 1.565, \quad \theta = 57.4^\circ$$

Example S3.4

A man wishes to climb a 5 m long ladder placed at 60° on a horizontal surface ($\mu = 0.3$) against a vertical wall ($\mu = 0.2$). How far can he climb without the ladder slipping? The man and the ladder weigh 800 N and 150 N respectively.

Solution The free-body diagram of the ladder shows that it is subjected to 6 forces of which 4 are unknown and we wish to find the maximum value of x , i.e., at the state of impending slip at A and B.

Then,

$$\mu_A = 0.3 \quad \text{and} \quad \mu_B = 0.2$$

For equilibrium,

$$\Sigma F_x = 0; \quad - \quad \Sigma F_y = 0; \quad \Sigma M = 0$$

$$\begin{aligned} \Sigma F_y = 0; \quad N_A + f_B - 150 - 800 &= 0; \quad f_B = 950 - N_A \\ \Sigma M_A = 0; \quad 5 N_B \sin 60 + 5 f_B \cos 60 - 800 d \cos 60 - 150 \times \\ & 2.5 \cos 60 = 0; \quad 4.33 N_B + 2.5 f_B - 400 d = 187.5 \end{aligned}$$

From these equations,

$$N_B = 0.3 N_A, \quad 950 - N_A = 0.2 N_B$$

whence $N_A = 896.3$ and $N_B = 268.9$

and $f_A = 268.9$ and $f_B = 53.7$ in N units.

Finally, $-400 d = 187.5 - 1164.3 - 134.3 = -1111.1$
and $d = 2.77$ m

The ladder will skid as the man reaches a distance of 2.77 m up the ladder.

Let us verify this result by considering the summation of moments about point B.

$$\begin{aligned} \Sigma M_B = 800(5 - 2.77) \cos 60 + 150 \times 2.5 \cos 60 + 268.9 \times 5 \sin 60 - 896.3 \times 5 \cos 60 \\ = 889 + 187.5 + 1164.4 - 2240.7 \approx 0.0 \end{aligned}$$

which shows that the distance of 2.77 m up the ladder is the right answer for the equations. Any mistake in the setting up of the equations is not checked by the verification. That can be ensured by checking the results vis-a-vis the data in the problem. From the fact that $\mu = 0.3$ at the floor, $f_A = 0.3 N_A$. The value of f_A as 268.9 N is compatible with that of N_A as 896.3 N. Similarly, the value of f_B as 53.7 N is compatible with that of N_B as 268.9 N for a μ of 0.2 at the wall. The result is, therefore, correct.

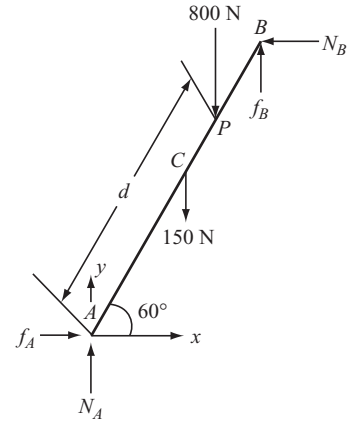


Fig. Ex. S3.4 (Solution)

Example S3.5

A homogeneous ladder is placed on a flat horizontal surface to rest against a vertical wall. Assuming that the coefficient of friction at each surface is μ , determine the minimum possible inclination of the ladder with the horizontal. Can the angle be less than 45° ?

Solution From the free-body diagram of the ladder, the position of minimum inclination corresponds to that of impending skidding.

$$f_A = \mu N_A \quad \text{and} \quad f_B = \mu N_B$$

For equilibrium,

$$\Sigma F_x = 0 \quad f_A - N_B = 0; \quad \mu N_A = N_B$$

$$\Sigma F_y = 0 \quad N_A + f_B - W = 0; \quad N_A + \mu N_B = W$$

$$\Sigma F_A = 0 \quad N_B \times L \sin \theta + f_B \times L \cos \theta - W \times L/2 \cos \theta = 0$$

From these equations,

$$N_A + \mu^2 N_A = W; \quad N_A = \frac{W}{1 + \mu^2}$$

$$N_B = \mu N_A = \frac{\mu W}{1 + \mu^2}$$

and
$$\frac{\mu W}{1 + \mu^2} \sin \theta = \left(\frac{W}{2} - \frac{\mu^2 W}{1 + \mu^2} \right) \cos \theta = \frac{1 - 2\mu^2}{1 + \mu^2} W \cos \theta$$

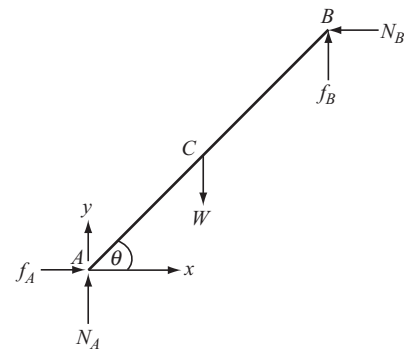


Fig. Ex. S3.5 (Solution)

whence
$$\tan \theta = \frac{1 - 2\mu^2}{\mu}$$

For smooth surfaces, $\mu = 0$, $\tan \theta \rightarrow \infty$, $\theta \rightarrow 90$

For θ to be 45° , $\tan \theta = 1$, $1 - 2\mu^2 = \mu$, or $\mu = 0.5$

Hence it is possible for the inclination to be less than 45° if the coefficient of friction is greater than 0.5.

For example a value of $\mu = 0.6$ implies that $\tan \theta = (1 - 0.72)/0.6 = 0.467$, $\theta = 25^\circ$. Interestingly, $\theta = 0$ corresponds with $1 - 2\mu^2 = 0$; $\mu = 0.707$.

Example S3.6 A small block of mass 100 kg is placed on a 30° incline with a coefficient of friction of 0.25. Determine the horizontal force to be applied on it in order to keep it in equilibrium at rest.

Solution The block would stay in equilibrium at rest for a range of horizontal forces between F_1 and F_2 where F_1 is the minimum force required to prevent it from sliding down and F_2 is the maximum force applied without sliding it up. For the case of impending sliding down the incline,

$$F_1 \cos 30^\circ - 981 \sin 30^\circ + f_1 = 0$$

where $f_1 = 0.25 N_1 = 0.25 \times (F_1 \sin 30^\circ + 981 \cos 30^\circ)$.

$$0.866 F_1 - 490.5 + 0.125 F_1 + 212.4 = 0$$

whence $F_1 = 280.6 \text{ N}$

For the case of impending sliding up the incline,

$$F_2 \cos 30^\circ - 981 \sin 30^\circ - f_2 = 0$$

where $f_2 = 0.25 N_2 = 0.25 \times (F_2 \sin 30^\circ + 981 \cos 30^\circ)$

$$0.866 F_2 - 490.5 - 0.125 F_2 - 212.4 = 0$$

whence $F_2 = 948.6 \text{ N}$

The block would, therefore, be in equilibrium if the horizontal force applied is between 280.6 N and 948.6 N as shown.

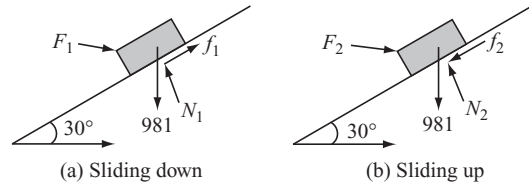


Fig. Ex. S3.6 (Solution)

Example S3.7 A horizontal force F acts on a block at a height h above the surface of the table where it lies. Explain the condition when the body (a) slides (b) overturns.

Solution When a horizontal force F acts at some distance from the base, the point of application of R and f , i.e., point O is not vertically below C . The point O locates itself such that the moment due to couple F and f is balanced by the moment due to couple F_n and N . As long as O lies within the base, the body has a tendency to slide.

Now, if the point O moves to a corner because the frictional force f keeps increasing due to a high value of μ_s , the coefficient of static friction then, the body has a tendency to overturn or tip over that corner. In that case,

$$fh \geq F_n a$$

or
$$\mu_s F_n h \geq F_n a$$

or
$$h \geq a/\mu$$

The tendency to overturn is more if the base width is small, coefficient of friction is large or the height, h is large.

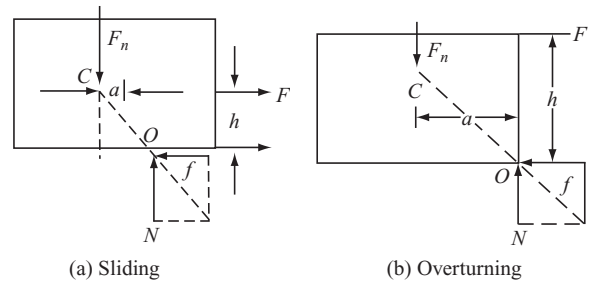


Fig. Ex. S3.7 (Solution)

Example S3.8

A 'levelling plank' is often used in a field to level off the earth by sliding the plank over it. One such plank requires a horizontal force of 600 N to slide it on a horizontal field at a constant velocity when the frictional resistance is 1.5 N per kg mass of the plank. It is employed to level a field inclined at 5° to the horizontal. The span of the plank is 2 m and the force is applied along the direction of movement. It is used for levelling, both going up and coming down the incline end to end, alternately. Estimate the force required each way and compare the average force required with that required for a horizontal field.

Solution The horizontal force of 600 N on a levelling plank must equal the frictional resistance for levelling at a constant velocity.

Mass of the plank = $600/1.5 = 400$ kg

Weight of the plank = $400 \times 9.81 = 3920$ N

The normal reaction on the horizontal surface is 3920 N. The coefficient of friction μ is, therefore, $600/3920 = 0.153$.

For going up the incline, from the free-body diagram of the plank shown in Fig. Ex. S3.8(a) (Solution),

$$N = 3920 \cos 5^\circ = 3900 \text{ N}$$

$$f = \mu R = 0.153 \times 3900 = 597 \text{ N}$$

$$F = 3920 \sin 5^\circ + 597 = 939 \text{ N}$$

For going down the incline,

$R = 3900$ N and $f = 597$ N as shown in the free-body diagram (Fig. Ex. S3.8(b) Solution).

$$F = 597 - 3920 \sin 5^\circ = 255 \text{ N}$$

The average force required for the inclined field is

$$(939 + 255)/2 = 597 \text{ N}$$

The force required for a horizontal lawn is 600 N which is slightly more than 597. In fact, it can be observed that the difference is due to the decrease of the normal component of reaction for the inclined surface.

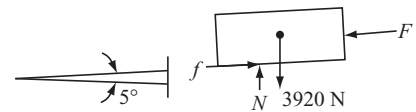


Fig. Ex. S3.8(a) (Solution)

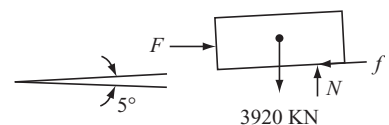


Fig. Ex. S3.8(b) (Solution)

Example S3.9

A block of mass 150 kg is to be raised by means of inserting a 10 wedge weighing 50 kg under it and by applying a horizontal force at it as shown in Fig. Ex. S3.9. Assuming the coefficient of friction between all surfaces of contact as 0.3, determine what minimum horizontal force should be applied to raise the block. What would happen if the horizontal force is removed?

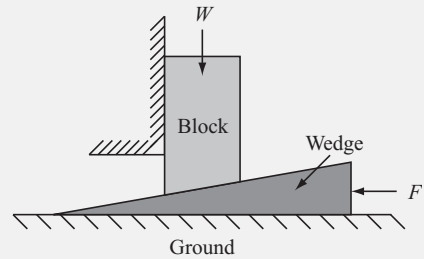


Fig. Ex. S3.9

Solution It is necessary to visualise the forces and to draw the free-body diagrams of the block and of the wedge for the impending motion of the block upwards.

For equilibrium of the block (Fig. Ex. S3.9 (Solution)),

$$W = R_2 \cos \alpha - f_1 - f_2 \sin \alpha$$

$$N_1 = f_2 \cos \alpha + R_2 \sin \alpha$$

where $f_1 = \mu_1 R_1$, $f_2 = \mu_2 R_2$, and W is the weight of the block.

For equilibrium of the wedge,

$$F = f_2 \cos \alpha + f_3 + N_2 \sin \alpha$$

$$N_3 = R_2 \cos \alpha + w - f_2 \sin \alpha$$

where $f_2 = \mu_2 N_2$ $f_3 = \mu_3 N_3$

Putting $W = 150 \text{ kg} = 1470 \text{ N}$

$W = 50 \text{ kg} = 490 \text{ N}$

$\mu_1 = \mu_2 = \mu_3 = 0.3$, and solving.

$$N_1 = 869.8 \text{ N} \quad f_1 = 261.0 \text{ N}$$

$$N_2 = 1854.6 \text{ N} \quad f_2 = 556.4 \text{ N}$$

$$N_3 = 2219.8 \text{ N} \quad f_3 = 665.9 \text{ N}$$

$$F = 1535.94 \text{ N}$$

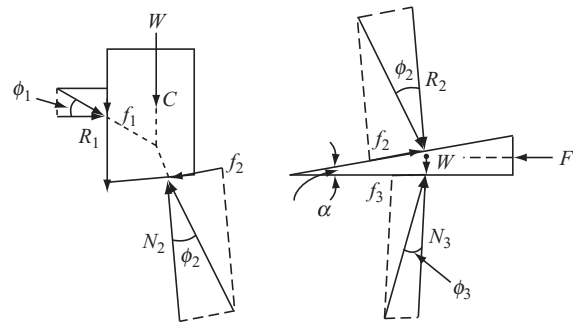


Fig. Ex. S3.9 (Solution)

S3.2 ROLLING RESISTANCE

Rolling resistance occurs as a result of the small deformation of the surface upon which a rolling object rolls. The surface of the rolling object may also be deformed in the process. A cylindrical or spherical object is shown rolling in Fig. S3.3. The force F required parallel to the surface must be adequate to lift the object out of the depression in the surface. The process of deformation of the surfaces is continuous during the rolling movement and hence a constant force F must be applied to overcome the resistance offered by the deformation.

With reference to Fig. S3.3, the sum of the moments about the point O must be zero for equilibrium.

$$W \cdot r \sin \phi - F \cdot r \cos \phi = 0$$

or

$$F = W \tan \phi$$

Since the angle ϕ subtended by the length deformation must be very small,

$$\tan \phi \approx \sin \phi \approx \frac{a}{r}$$

Table S3.3 Coefficient of rolling resistance

| Pair of Surfaces | Range of a (cm) |
|-------------------------|-------------------|
| Steel on steel | 0.02–0.04 |
| Steel on wood | 0.15–0.25 |
| Tyre on road | 0.05–0.15 |
| Hardened steel surfaces | 0.0005–0.0015 |

Hence

$$F = a \frac{W}{r}$$

The distance a reckoned as the ‘forward length of deformation’ is defined as the *coefficient of rolling resistance*. The coefficient of the rolling resistance, unlike the coefficient of friction, has the units of length.

It is interesting to observe that the coefficient of rolling resistance is defined as the forward length of deformation rather than a non-dimensional quantity. The rolling resistance is, therefore, a function of (a) the weight or load W normal to the surface, (b) the radius of the cylinder and (c) the coefficient of rolling resistance.

It follows that less force is required to roll a wheel of bigger radius than a wheel of smaller radius for the same load and the same pair of materials.

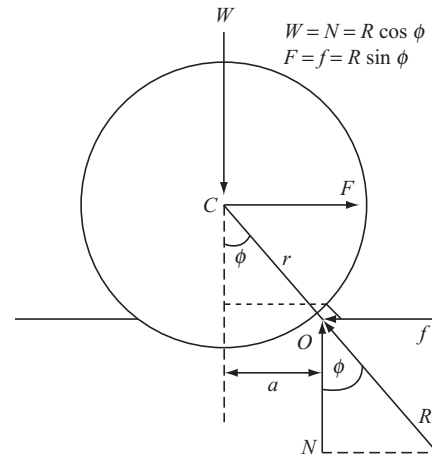


Fig. S3.3 Mechanism of rolling resistance

S3.3 SLIDING AND ROLLING OF CYLINDERS

If a circular cylinder is placed in contact with a surface and subjected to a force, it may either slip or roll about the point of contact. The same is likely to happen when it is placed on an incline. Whether it will actually slip or roll depends upon the friction characteristics, force and angle of incline. In many problems, it is necessary to ascertain whether a cylinder will tend to slip or roll, particularly when it is in contact with two surfaces. Consider, for example a cylinder and a flat-based block on an incline and the angle of incline is gradually raised until the equilibrium is disturbed as shown in Fig. S3.4.

In the limiting condition, the block must slide down the incline at its base whereas the cylinder may slip at B and roll at A or it may slip at A and remain stuck at B . It may be seen that a cylinder cannot roll about two points on it because the meaning of rolling about a point is that the velocity at that point is zero and at all other points non-zero. However, slipping of a cylinder about two points on it cannot be ruled out in general and it is decided by the constraints. In the present case, slipping at A and B would mean rotation about O , i.e., the cylinder rotates about its axis which is obviously not possible without an external moment. Hence, the cylinder may slip at A or B and roll

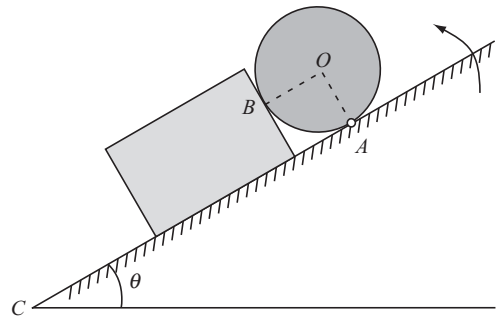


Fig. S3.4 Cylinder and block on an incline

about the other point. There is no simpler way to decide as to which is the actual mode except by considering each mode for the state of impending motion and analysing the situation.

At each contact point, a cylinder experiences a normal reaction directed towards the centre of the cylinder. It also experiences the frictional force at each contact point. At a point where it may slip, the frictional force is related to the normal reaction,

$$f = \mu N$$

but at the point where it may roll, the frictional force is left as an unknown. It should be between zero and the maximum possible value μR at that point. For rolling, therefore,

$$0 < f < \mu N \tag{S3.2}$$

where μ is the coefficient of static friction between the contacting surfaces.

Example S3.10

A circular cylinder of radius 0.5 m and mass 200 kg is placed in contact with a rectangular block of mass 150 kg on an incline at 30° as shown in Fig. Ex. S3.10. If the coefficient of static friction is 0.6, determine the minimum force F to be applied up the plane at the block to prevent the bodies from sliding down.

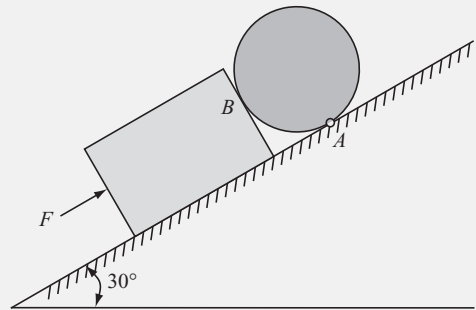


Fig. Ex. S3.10

Solution We should first consider the equilibrium of the cylinder at the state of impending downwards motion. The free-body diagram is constructed by including the normal and frictional reactions by the contact surfaces as well as the weight of the cylinder itself as shown in Fig. Ex. S3.10 (Solution). There are two possible modes of the cylinder coming down the incline. It may roll at A and slide at B or it may slide at A . Let us consider the first possibility, i.e., sliding at B ,

$$f_B = \mu N_B$$

and $f_A = \text{any thing between zero and } \mu R_A$

For equilibrium of the cylinder

$$\Sigma F_x = 0; \quad N_B + f_A - mg \sin 30 = 0$$

$$\Sigma F_y = 0; \quad f_B + N_A - mg \cos 30 = 0$$

$$\Sigma M_O = 0; \quad f_A \times 0.5 - f_B \times 0.5 = 0$$

From these relations,

$$N_B = \frac{mg}{2(1 + \mu)} = \frac{200 \times 9.81}{2(1 + 0.6)} = 613 \text{ N}$$

$$f_B = 0.6 \times 613 = 368 \text{ N}; \quad f_A = 368 \text{ N}$$

$$N_A = 1331 \text{ N}$$

A check on the value of f_A shows that it lies between 0 and 0.6×1331 , i.e., 799 N. The possibility that the cylinder rolls at A and slips at B is, therefore, feasible.

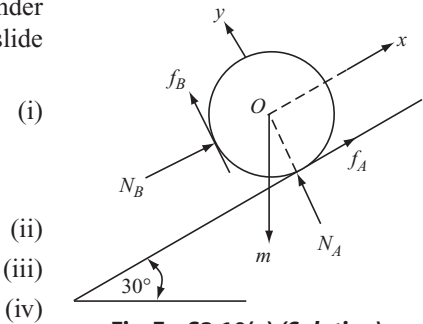


Fig. Ex. S3.10(a) (Solution)

Now, consider the equilibrium of the block at the state of impending downward motion,

$$\Sigma F_x = 0; \quad -W \sin 30^\circ + F = 0$$

$$\Sigma F_y = 0; \quad -W \cos 30^\circ + N = 0$$

Since $W = 613 \text{ N}$, $F = 306.5 \text{ N}$
 $F = \mu N = 0.6 N$

these relations provide

$$N = 680 \text{ N} \quad \text{and} \quad F = 408 \text{ N}$$

A force of 408 N is, therefore, required to be applied up the plane at the block to check the state of impending downward motion; this is the minimum force required to keep the assembly of the block and cylinder in equilibrium on the incline.

The problem could have been formulated in a different way. The block and cylinder could have been shown on the 30° incline in the absence of any applied force and one could be asked to determine whether the bodies are in a state of equilibrium or not. It can now be suggested to attempt the problem in exactly the same way as done here and determine the force required. If F comes out to be positive, as it did come in the given case, the bodies would not be in equilibrium without holding them up by any up-the-incline force. If F comes out to be negative, it means that the bodies were in equilibrium at rest.

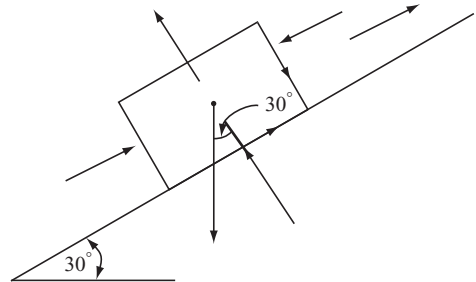


Fig. Ex. S3.10(b) (Solution)

Example S3.11 A circular cylinder of radius 0.5 m and mass 200 kg is placed in contact with a rectangular block of mass 150 kg on an incline at 30° as shown in Fig. Ex. S3.11. If the coefficient of static friction is 0.6, determine the minimum force to be applied up the plane at the block to initiate an upward motion of the bodies.

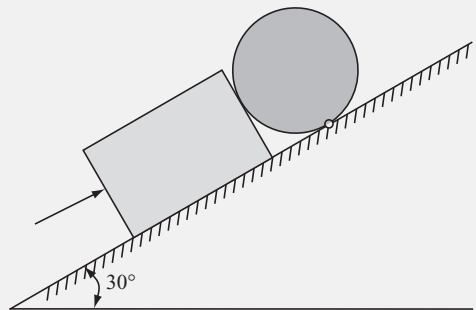


Fig. Ex. S3.11

Solution The cylinder, while tending to move up, may either tend to roll at A and slide at B or slide at A and roll at B . Taking the former possibility first, consider the free-body diagram drawn in Fig. Ex. S3.11(a) (Solution), for equilibrium.

$$\Sigma F_x = 0; \quad -W \sin 30^\circ + F = 0$$

$$\Sigma F_y = 0; \quad -W \cos 30^\circ + N = 0$$

$$\Sigma M_O = 0; \quad W \cos 30^\circ \times 0.5 - N \times 0.5 = 0$$

and $f = \mu N = 0.6 N$

From these relations,

$$F = 245.2 \text{ N}; \quad N = 317.0 \text{ N}$$

$$= 0.6 \times 2452 = 1471 \text{ N}; \quad = 1471 \text{ N}$$

A check on the value of F shows that it lies between 0 and 0.6×3170 , i.e., 1902 N.

It may, therefore, be taken that the cylinder would roll at and slip at $F = 1471 \text{ N}$.

Next, consider the equilibrium of the block with reference to the free-body diagram shown in Fig. Ex. S3.11(b) (Solution).

$$\Sigma F_x = 0; \quad -F + W \sin 30 = 0$$

$$\Sigma F_y = 0; \quad +N - W \cos 30 = 0$$

and $F = \mu N = 0.6$

Since $W = 2452 \text{ N}$ and $F = 1471 \text{ N}$, these relations provide,
 $F = 3070 \text{ N}$

A force of 3070 N is, therefore, required to be applied up the incline at the block in order to initiate an upward motion of the bodies.

It was observed in the previous problem that a force of 380 N is required to hold the bodies in equilibrium against coming down the plane. It has now been seen that a 3070 N force is required to move the bodies up the plane. Obviously, if a force in between 380 N and 3070 N is applied, the bodies would stay in equilibrium.

The block and cylinder would have been shown on the 30° incline and a force of some magnitude shown applied at the block and one could be asked to ascertain whether the bodies are moving down, in equilibrium, or moving up. If the applied force was less than 380 N, the bodies would be moving down; if more than 3070, the bodies would be moving up and if in between these two values, the body must stay at rest in equilibrium.

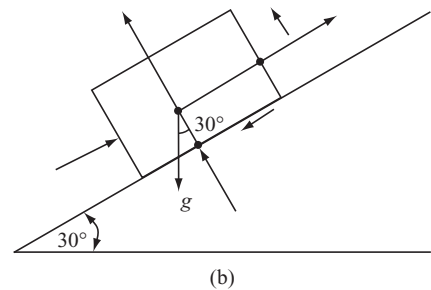
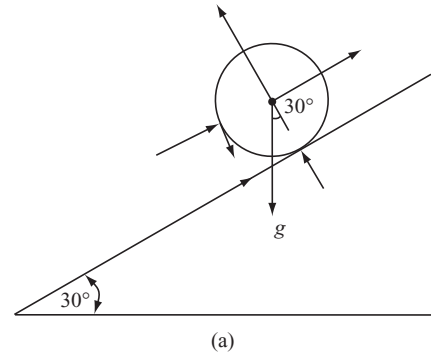


Fig. Ex. S3.11 (Solution)

Example S3.12 A cylinder of radius r and weight W is wedged between a vertical wall and a light-hinged bar as shown in Fig. Ex. S3.12. The coefficient of friction between the cylinder and wall is 0.2 and that between the bar and cylinder is 0.4. What is the force P which, when applied at the end of the bar at right angles to it, is just sufficient to cause the cylinder to move upward?

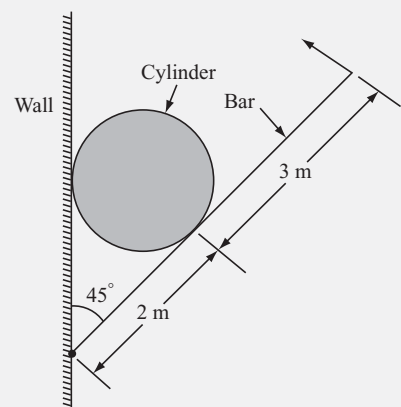


Fig. Ex. S3.12

Solution The cylinder can slip upward at either point; not necessarily at both points simultaneously. In other words, the cylinder may slip at the bar point B and roll about A or it may slip at the wall point A and roll about B . We should, therefore, examine both possibilities.

When there is impending motion at B , the free-body diagram drawn at Fig. Ex. S3.7(a) (Solution) shows the forces acting on the cylinder.

By considering the equilibrium,

$$\begin{aligned} \Sigma F_x = 0; \quad N_A - f_B \cos 45^\circ - 2.5F \cos 45^\circ &= 0 \\ \Sigma F_y = 0; \quad -f_A + 2.5F \sin 45^\circ - f_B \sin 45^\circ - W &= 0 \\ \Sigma M_O = 0; \quad f_B \times \text{radius} = f_A \times \text{radius} \end{aligned}$$

and $f_B = 0.4 \times 2.5F = F$

From these relations,

$$f_A = F = 16.5 W \quad \text{and} \quad N_A = 40.8 W$$

It may be seen that the ratio of the friction to the normal reaction at A is

$$\frac{f_A}{N_A} = \frac{16.5}{40.8} = 0.4$$

but the maximum possible ratio is 0.2. Since the slippage at B leads to this impasse, it is impossible for slippage to occur at B before it occurs at A .

The free-body diagram for the second possibility, i.e., slippage at A is shown in Fig. Ex. S3.12(b) (Solution).

By equilibrium consideration,

$$\begin{aligned} \Sigma F_x = 0; \quad N_A - 2.5F \cos 45^\circ - f_B \cos 45^\circ &= 0 \\ \Sigma F_y = 0; \quad -0.2N_A \times W - f_B \sin 45^\circ + 2.5F \sin 45^\circ &= 0 \\ \Sigma M_O = 0; \quad f_B \times \text{radius} = 0.2N_A \times \text{radius} \end{aligned}$$

From these relations,

$$f_B = 0.39 W, \quad N_A = 1.94 W$$

and $F = 0.94 W$

We may again like to check the ratio of friction to the normal reaction at B .

$$\frac{f_B}{N_B} = \frac{0.39}{2.5 \times 0.94} = 0.17$$

which is considerably less than the limiting value of 0.4.

It may, therefore, be concluded that the force required at the end of the rod is $0.94 W$ and that the cylinder will slip at A , the point of contact with the wall.

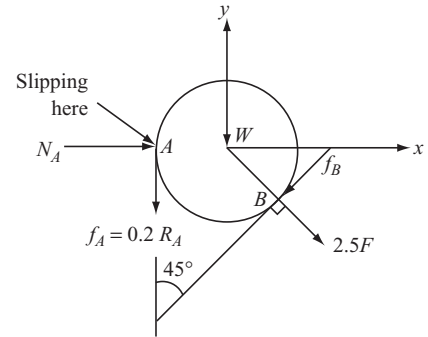


Fig. Ex. S3.12(a) (Solution)

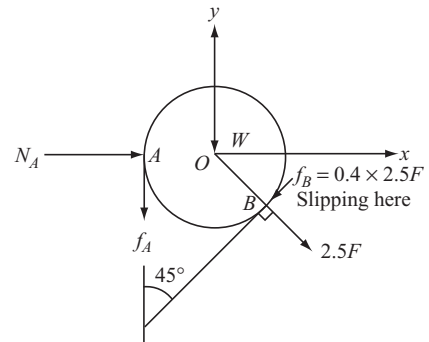


Fig. Ex. S3.12(b) (Solution)

S3.4 BAND-BRAKE AND BELT FRICTION

A flexible member, such as a band, belt or rope passing over or wrapped around a cylinder can be used as band-brake for power absorption or as a belt rope drive for power transmission.

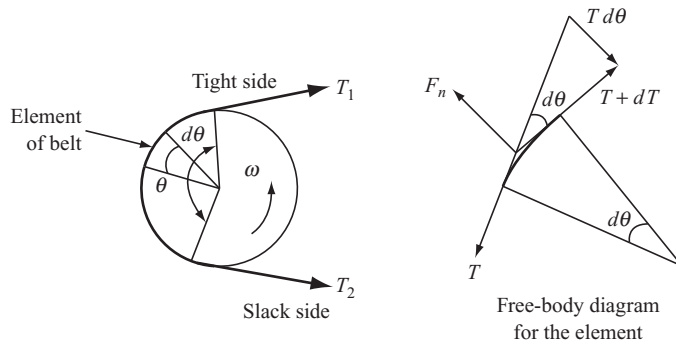


Fig. S3.5 Analysis of belt drive

Consider a flexible member passing over a cylinder as shown in Fig. S3.5. The total angle subtended at the centre by the flexible member is θ . Consider the equilibrium of a small segment over an angle $d\theta$. The tensions on either side are T and $T + dT$ which may be considered collinear over the small segment. The difference in the tensions is due to the force of friction which would be a maximum at the state of impending motion between the flexible member and the cylinder.

$$dT = f = \mu F_n$$

where μ is the static coefficient of friction between the two surfaces. The normal force F_n must be given by

$$F_n = T d\theta$$

Thus

$$dT = \mu T d\theta$$

or

$$\frac{dT}{T} = \mu d\theta$$

Integrating T from T_2 to T_1 over an angle θ

$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^\theta \mu d\theta = \mu\theta$$

or

$$\log_e \frac{T_1}{T_2} = \mu\theta$$

and

$$\frac{T_1}{T_2} = e^{\mu\theta} \tag{S3.3}$$

It may be noticed that the tension T_1 and T_2 can be widely different depending only upon the coefficient of friction μ and the angle of contact θ . The side of the belt with greater tension, T_1 is called the *tight side* and the other with less tension, T_2 is called the *slack side* of the drive. The difference between T_1 and T_2 is responsible for transmitting the power. the difference can be increased by placing the pulleys closer and by employing a cross-belt drive instead of an open belt drive as far as possible. The *power transmitted* is given by

$$\text{Power} = (T_1 - T_2) v$$

where v is the linear speed of the belt.

If T_2 is desired to be much smaller than T_1 , as is the case of drum pulley blocks and capstans, the rope can be wound round the drum a few turns. For n turns round the drum the angle of contact θ becomes 2π times n .

Example S3.13 A rope is placed round a fixed circular post over an angle θ . Determine the ratio of tensions T_1 and T_2 for impending slip in the anticlockwise direction.

Solution Let p be the pressure per unit length of the rope on the post. For an elementary length $r\delta\theta$ of the rope, the normal force is $pr\delta\theta$ and the frictional force is $\mu pr\delta\theta$. It is subjected to tensions T and $T + \delta T$ on either side. For equilibrium as shown in Fig. Ex. S3.13(a) (Solution).

$$T + \delta T - T = \mu p r \delta\theta$$

or $\delta T = \mu p r \delta\theta$ (i)

and $T\delta\theta = p r \delta\theta$

or $T = \mu r$ (ii)

From these two equations

$$\frac{\delta T}{T} = \mu \delta\theta$$

Integrating over the length of contact

$$\log \frac{T_1}{T_2} = \mu\theta$$

or $\frac{T_1}{T_2} = e^{\mu\theta}$

where θ is the angle of lap over the post.

If the post was non-circular, even then the T_1/T_2 ratio would be the same as $e^{\mu\theta}$ where θ is the angle of lap determined w.r.t. a centre located by drawing normals to T_1 and T_2 .

This is because every little element can be approximated a being that of a circle, even though the centre of the circle keeps changing by the additional elements.

Supposing it was a belt going over a shaft instead of a rope going over a post. No difference. The T_1/T_2 ratio for the impending slip of the belt over the shaft would be the same provided the belt was massless. Therefore, for a belt drive,

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

It may, however, be noted that a belt is usually required to transmit power from a driver pulley to a driven pulley which creates a tight side with tension T_2 and a slack side with tension T_1 as shown in Fig. Ex. S3.13(b) (Solution). It may be observed that the angle of lap on the bigger pulley is greater than 180 degrees and that on the smaller pulley is less than 180 degrees. It is, therefore, more likely to slip on the smaller pulley.

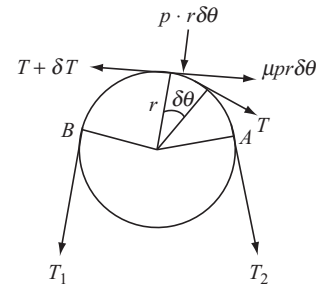


Fig. Ex. S3.13(a) (Solution)

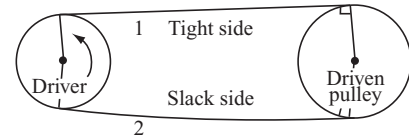


Fig. Ex. S3.13(b) (Solution)

Example S3.14 A horizontal drum of a belt drive carries the belt over a semicircle around it. It is rotated anticlockwise to transmit a torque of 300 Nm. If the coefficient of friction between the belt and the rope is 0.3, calculate the tensions in the limbs 1 and 2 of the belt and the reaction on the bearings. The drum has a mass of 20 kg and the belt is assumed to be massless.

Solution Clearly, T_1 should be more than T_2 .

According to this condition,

$$\frac{T_1}{T_2} = e^{0.3\pi} = 2.57$$

or $T_1 = 2.57 T_2$

From the knowledge of the torque transmitted,

$$(T_1 - T_2) \times 0.5 = 300$$

$$T_1 - T_2 = 600 \text{ N}$$

from (i) and (ii)

$$T_1 = 982 \text{ N}; \quad T_2 = 382 \text{ N}$$

The downward force exerted by the belt on the drum is

$$T_1 + T_2 = 982 + 382 = 1364 \text{ N}$$

Since the weight of the drum is

$$W = 20 \times 9.81 = 196 \text{ N}$$

the vertical reaction on the bearing should be the sum, i.e.,

$$1364 + 196 = 1560 \text{ N}$$

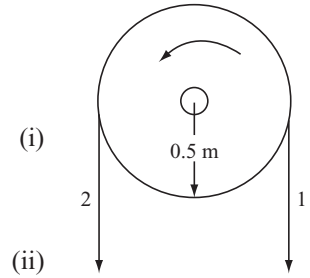


Fig. Ex. S3.14 (Solution)

Example S3.15 Two pulleys, one 450 mm in diameter and the other 200 mm in diameter, are on parallel shafts 1.95 m apart and are connected by a crossed belt as shown in Fig. Ex. S3.15. Find the length of the belt required and the angle of contact between the belt and each pulley.

What power can be transmitted by the belt when the larger pulley rotates at 200 rev/min, if the maximum permissible tension in the belt is 1 kN and the coefficient of friction between the belt and the pulley is 0.25?

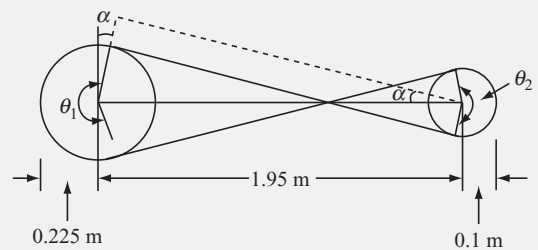


Fig. Ex. S3.15

Solution From the geometry of the problem,

$$\sin \alpha = \frac{0.225 + 0.1}{1.95} = 0.167; \quad \alpha = 9.6^\circ$$

$$\theta_1 = 180 + 2 \times 9.6 = 199.2 = 199.2 \times \frac{\pi}{180} = 3.477 \text{ rad}$$

$$\theta_2 = \theta_1 = 3.477 \text{ rad}$$

Length of the belt is given by

$$L = 0.225(3.477) + 0.1(3.477) + 2 \times 1.95 \cos 9.6 = 4.98 \text{ m}$$

According to the condition of impending slippage of the belt over the pulleys,

$$\frac{T_1}{T_2} = e^{\mu\theta} = e^{0.25 \times 3.477} = 2.385$$

or $T_1 = 2.385 T_2$

If the greater of the tensions is allowed to the maximum permissible, i.e.,

$$T_1 = 1000 \text{ N}$$

then $T_2 = 419 \text{ N}$

The power transmitted must be given by

$$\text{Power} = (T_1 - T_2) \times \text{speed of the belt}$$

$$= (1000 - 419) \times \frac{\pi \times 0.45 \times 200}{60} = 2738 \text{ W} = 2.738 \text{ kW}$$

Example S3.16

A mass of 500 kg is to be maintained in position by pulling a rope taken over a half barrel and wrapped twice around a capstan as shown in Fig. Ex. S3.16. If the coefficient of static friction is 0.2 for all contact surfaces, calculate the minimum force F required to maintain the load.

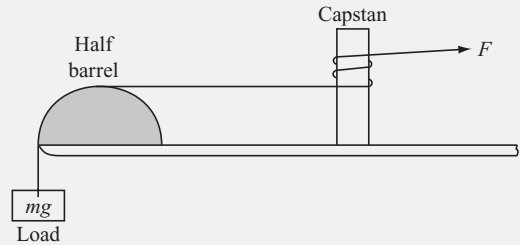


Fig. Ex. S3.16

Solution Consider the equilibrium of the segment of the rope taken over the half barrel. According to the condition of impending slippage of the rope downwards,

$$\frac{T_1}{T_2} = e^{0.2 \times \pi / 2} = 1.37$$

or $T_1 = 1.37 T_2$

$$T_2 = \frac{T_1}{1.37} = \frac{500 \times 9.81}{1.37} = 3583 \text{ N}$$

whence

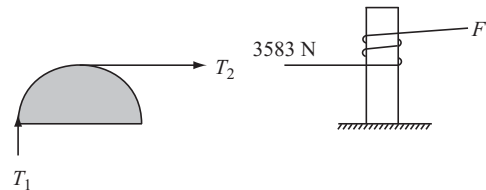


Fig. Ex. S3.16 (Solution)

Now consider the equilibrium of the rope wrapped around the capstan. For the state of impending motion of the rope towards left, i.e., in lowering the load,

$$\frac{3583}{F} = e^{0.2 \times 4\pi} = 12.35$$

whence $F = 290 \text{ N}$

It may be seen that the load of 4905 N can be maintained in position against falling down, by the application of a force which is merely 20% of its magnitude through frictional devices.

Let us also work out the force F which would be required to initiate an upward motion of the load. The impending motion of the rope would then be upwards and towards right and by equilibrium considerations,

$$F = 500 \times 9.81 \times 1.37 \times 12.35 = 82,990 \text{ N} = 82.99 \text{ kN}$$

which is over 16 times the load itself. It follows that the rope passed over the half barrel and wound round the capstan is a good device if the load is to be maintained in position but very inefficient if the load is to be lifted.

Example S3.17 Two blocks *A* and *B* are to be held in position by means of an inextensible rope passing over a fixed drum as shown in Fig. Ex. S3.17. The coefficient of friction between the blocks, between the block and the inclined surface, and between the rope and the drum is 0.2. The mass of *B* is 500 kg. Determine the minimum weight of *A* so that *B* is prevented from moving downwards. The drum cannot rotate.

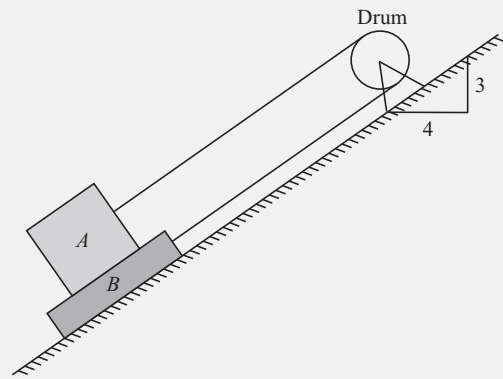


Fig. Ex. S3.17

Solution A tendency of *B* to slip down the incline implies that the tendency of the upper block *A* must be to slip upward. The friction force is, therefore, shown accordingly. Friction on the non-rotatable drum implies that the tensions T_A and T_B on either side of the inextensible string are unequal and from the knowledge of impending motion

$$T_B > T_A$$

The inclination of the slope is specified by

$$\tan \theta = \frac{3}{4} \quad \text{or} \quad \sin \theta = 0.6 \quad \text{and} \quad \cos \theta = 0.8$$

With reference to the free-body diagram for the block *A*, as shown in Fig. Ex. 3.17 (Solution) the conditions of equilibrium are

$$T_A - f_A - m_A g \sin \theta = 0$$

$$R_A - m_A g \cos \theta = 0$$

and the condition of impending motion is

$$f_A = \mu R_A$$

Substituting for the coefficient of friction and θ

$$T_A - 0.2R_A - 0.6 m_A g = 0 \quad \text{(i)}$$

$$R_A - 0.8 m_A g = 0 \quad \text{(ii)}$$

For block *B*,

$$T_B + f_A + f_B - m_B g \sin \theta = 0$$

$$R_B - R_A - m_B g \cos \theta = 0$$

and

$$f_B = \mu R_B$$

whence

$$T_B + 0.2 R_A + 0.2 R_B - (500 \times 9.81) \times 0.6 = 0$$

$$R_B - R_A - (500 \times 9.81) \times 0.8 = 0$$

or

$$T_B + 0.2R_A + 0.2R_B = 2943 \quad \text{(iii)}$$

$$R_B - R_A = 3924 \quad \text{(iv)}$$

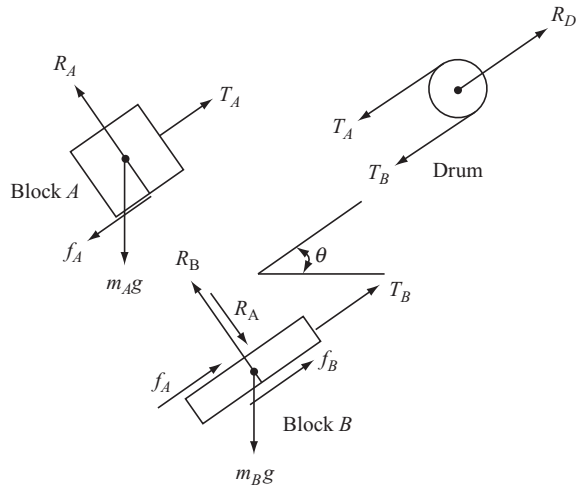


Fig. Ex. S3.17 (Solution)

For the drum,

$$R_D - T_A - T_B = 0$$

Since

$$T_B > T_A$$

$$T_B/T_A = e^{\mu\phi} = e^{0.2\pi} = 1.874 \quad (v)$$

We have now set up five equations, (i) to (v), for the five unknowns T_A , T_B , R_A , R_B and m_A of which only the last one is desired. Even so, the simultaneous set of equations must be solved.

Obtaining

$$T_B = 1.874 T_A \quad \text{from Eq. (v),}$$

$$R_A = 0.8 m_A g \quad \text{from Eq. (ii),}$$

$$R_B = R_A + 3924 = 0.8 m_A g + 3924$$

from Eq. (iv) and substituting in Eq. (i)

$$T_A - 0.16 m_A g - 0.6 m_A g = 0$$

or

$$T_A = 0.76 m_A g$$

and now substituting in Eq. (iii),

$$1.874(0.76 m_A g) + 0.16 m_A g + 0.16 m_A g + 785 = 2943$$

whence

$$m_A g = 1237$$

and

$$m_A = 126 \text{ kg}$$

The minimum mass of block A should, therefore, be 126 kg; for m_A less than 126, the lower block would slide down the incline and the upper block up the incline.

S3.5 LIFTING BY A SCREW JACK

A screw jack consists of a square-threaded central rod called a screw fitted into the internally-threaded collar of a jack. The load W is placed on the screw and the effort F is applied horizontally at the end of a lever of arm L as shown in Fig. S3.6. The lifting action of the screw jack takes place through the normal and frictional forces developed at the threaded surface of contact within the collar. Considering a typical element of the threaded surface in contact, the normal force is dN and the friction force is df opposing the input moment

$$M = FL$$

constituted by the effort F applied at an arm L . The total reaction force at the contact surface must balance the vertical force W and horizontal component H given by

$$H = M/r = FL/r$$

At the state of impending motion, or during the lifting action of the screw, the resultant reaction force makes an angle ϕ with the normal where $\phi = \tan^{-1} \mu$ and μ is the static coefficient or kinetic coefficient of friction as the case may be.

If α is the pitch angle of the screw, then for equilibrium,

$$R \cos (\alpha + \phi) = W$$

$$R \sin (\alpha + \phi) = M/r = FL/r$$

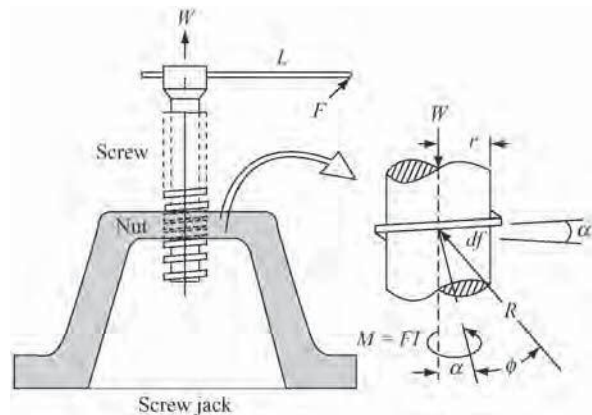


Fig. S3.6 Analysis of a screw jack

and from these equations,

$$F = W \tan (\alpha + \phi) r/L \quad (\text{S3.4})$$

If the load W was being lowered instead of being raised, the friction force would be directed up the incline and the resultant reaction R would be inclined to the line of action of W by an angle $(\alpha - \phi)$. The force F required to lower the load would be given by

$$F = W \tan (\alpha - \phi) \times r/L \quad (\text{S3.5})$$

which is indeed much smaller than that required to raise the load.

The efficiency of a screw jack is defined as the ratio of the work output to the work input over the same period of time. Work input for one revolution of the effort

$$= 2 \pi L F = W \cdot 2 \pi r \tan (\alpha + \phi)$$

Work output in the same duration

$$\begin{aligned} &= W \times \text{pitch of the screw} \\ &= W 2 \pi r \tan \alpha \end{aligned}$$

Efficiency

$$\eta = \frac{W \cdot 2 \pi r \tan \alpha}{W \cdot 2 \pi r \tan (\alpha + \phi)}$$

or

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)} \quad (\text{S3.6})$$

The efficiency can alternatively be interpreted as the ratio of the force required to lift a load in the absence of friction to the actual force required to lift the load.

$$F \text{ (without friction)} = W \tan (\alpha + 0) \cdot r/L$$

$$F \text{ (actual)} = W \tan (\alpha + \phi) \cdot r/L$$

$$\eta = \frac{F \text{ (without friction)}}{F \text{ (actual)}} = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

An expression for the maximum efficiency of a screw jack can be obtained by setting

$$d\eta/d\alpha = 0$$

$$\text{or} \quad \frac{1}{\cos^2 \alpha \tan (\alpha + \phi)} - \frac{\tan \alpha}{\sin^2 (\alpha + \phi)} = 0$$

$$\text{or} \quad \sin 2 (\alpha + \phi) = \sin 2 \alpha$$

$$\text{whence} \quad \alpha = \pi/4 - \phi/2$$

The maximum efficiency is given by

$$\eta_{\max} = \frac{\tan (\pi/4 - \phi/2)}{\tan (\pi/4 + \phi/2)} = \frac{1 - \sin \phi}{1 + \sin \phi} \quad (\text{S3.7})$$

Assuming that the coefficient of friction is 0.15, $\phi = \tan^{-1} 0.15 = 8.53$ degrees; $\alpha = \pi/4 - 8.53/2 = 40.73$ degrees $\eta_{\max} = 74.2\%$.

Let us also examine the condition for a screw jack to be self-locking. A screw jack is called self-locking if, in the absence of the applied moment FL , the screw jack does not unwind to lower the load. Since

$$F \times L = W \tan (\alpha - \phi) \times r$$

Equating $F \times L$ to zero or making it negative implies

$$\tan (\alpha - \phi) \leq 0$$

or

$$\tan \alpha - \tan \phi \leq 0 \quad (\text{S3.8})$$

or
$$\alpha \leq \phi$$

$$\tan \alpha \leq \mu$$

The screw jack will be self-locking if $\tan \alpha$ equals or is less than μ .
From the expression for its efficiency

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

it follows that, if $\alpha = \phi$

$$\begin{aligned} \eta &= \frac{\tan \alpha}{\tan 2\alpha} = \frac{\tan \alpha (1 - \tan^2 \alpha)}{2 \tan \alpha} \\ &= 0.5 - 0.5 \tan^2 \alpha \end{aligned} \tag{S3.9}$$

which must be less than 50%.

Example S3.18 A screw jack requires a force F applied at a radius a on a handle to lift a load W on top of it. Determine the M/F for raising the load for a helix angle α for the screw and coefficient of friction μ . Would the M/F be different for lowering the load?

Solution For an elemental area ds of the surface of the screw, the normal reaction is pds where p is the normal pressure on it. The vertical force is (Ref. Fig. Ex. S3.18 (a) (Solution)).

$$pds \cos \alpha - \mu pds \sin \alpha$$

For the screw jack

$$W = \int (p \cos \alpha - \mu p \sin \alpha) ds = (p \cos \alpha - \mu p \sin \alpha) A$$

and

$$M = \int r (p \sin \alpha + \mu p \cos \alpha) ds = F \cdot a = r (p \sin \alpha + \mu p \cos \alpha) A$$

$$\frac{M}{F} = \frac{r (\tan \alpha + \mu)}{1 - \mu \tan \alpha}$$

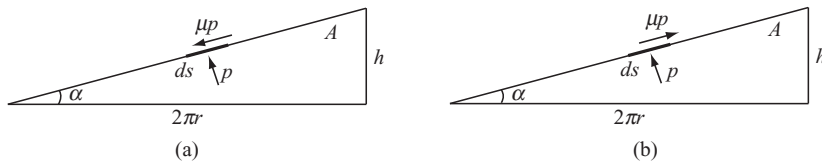


Fig. Ex. S3.18 (Solution)

Now, for lowering the load, the screw is turned in the reverse direction; the frictional force acts in the opposite direction (see Fig. Ex. S3.18(b) (Solution)).

We get $W = (p \cos \alpha + \mu p \sin \alpha) A$ (Fig. Ex. S3.18(b) (Solution))

$$M = r (p \sin \alpha - \mu p \cos \alpha)$$

and hence
$$\frac{M}{F} = \frac{r (\mu - \tan \alpha)}{1 + \mu \tan \alpha}$$

It is, of course, less than the M/F value for raising the load.

A screw jack is said to be self-locking if it does not let the load come down even if the applied moment or force at the lever arm is zero, i.e., it does need a moment or a force at the lever arm to lower the load,

$$\begin{aligned} & \mu > 0 \\ \text{or} & r(\mu - \tan \alpha) > 0 \\ \text{or} & \tan \alpha < \mu \\ \text{or} & \tan \alpha < \tan \phi \\ \text{or} & \alpha < \phi \end{aligned}$$

i.e., when the helix angle is less than the friction angle.

Example S3.19

A screw thread of a screw jack has a mean diameter of 10 cm and a pitch of 1.25 cm. The coefficient of friction between the screw and its nut-housing is 0.25. Determine the force F that must be applied at the end of a 50 cm lever arm to raise a mass of 5000 kg. Is the device self-locking? Also determine its efficiency.

Solution From the definition of the pitch of a screw,

$$\tan \alpha = \frac{p}{2\pi r} = \frac{1.25}{2\pi \times 5} = 0.04$$

$$\alpha = 2.28$$

Also,

$$\mu = 0.25 = \tan \phi; \phi = 14.04$$

The force required at the end of 50 cm long lever is given by

$$\begin{aligned} F &= W \tan (\alpha + \phi) \times r/L \\ &= 5000 \times 9.81 \tan (2.28 + 14.04) \times 5/50 \\ &= 1436 \text{ N} \end{aligned}$$

For the screw jack

$$\alpha < \phi$$

Hence, the screw jack must be self-locking.

The efficiency of the jack is given by

$$\begin{aligned} \eta &= \frac{\tan \alpha}{\tan (\alpha + \phi)} = \frac{0.04}{0.293} = 0.137 \\ &= 13.7\% \end{aligned}$$

The force required at the end of the lever to lower the load may be determined as follows:

$$\begin{aligned} F &= W \tan (\alpha - \phi) \times r/L \\ &= 5000 \times 9.81 \tan (2.28 - 14.04) \times 5/50 \\ &= -1021 \text{ N} \end{aligned}$$

The force required to lower the load is thus 1021 N in a direction opposite to that required to raise it.

S3.6 DISC AND BEARING FRICTION

It is often necessary to estimate the torque required to overcome the frictional resistance offered by a surface to the rotation of the other surface. It may also be of interest to estimate the power lost in friction at a given speed of rotation.

Consider, for example, the end of a rotating cylindrical shaft resting on a flat surface. Such a pair of surfaces constitute a thrust bearing. Let the axial force transmitted by the shaft on the flat pad be P . The force may be distributed uniformly or non-uniformly over the area of contact. The problem is dealt with by considering an elementary area of contact subtending angle $d\theta$ at the centre and of width dr at a radius r as shown in Fig. S3.7. The area of the strip is

$$dA = r dr d\theta$$

The normal force and hence the normal reaction at the strip is

$$dF_n = pr dr d\theta$$

where p is the intensity of normal force over the small area. The elementary frictional force over the area, acting tangent to the radius, must be given by

$$\begin{aligned} df &= \mu_d dF_n \\ &= \mu_d p r dr d\theta \end{aligned}$$

where μ_d is the coefficient of dynamic friction between the contacting surfaces.

The elementary moment exerted by the elementary frictional forces acting at a radius r from the centre and at an angle θ from the x -axis about the axis of the shaft is

$$\begin{aligned} dM &= df r \\ &= \mu_d p r^2 dr d\theta \end{aligned}$$

and the moment exerted by the entire frictional effect at the contacting surfaces is

$$\begin{aligned} M &= \int_0^R dM = \int_0^R \int_0^{2\pi} \mu_d p r^2 dr d\theta = \int_0^R \mu_d \times p 2\pi r^2 dr \\ &= 2\pi \mu_d \int_0^R p r^2 dr \end{aligned} \quad (\text{S3.10})$$

Integration of this expression may be carried out for an assumed or given distribution of p with r . Let us consider two cases:

Case (a)

If
$$p = \text{constant} \frac{P}{\pi R^2}$$

then
$$M = 2\pi \mu_d p \int_0^R r^2 dr$$

or
$$M = 2\pi \mu_d p R^3/3 = 2/3 P R \mu_d \quad (\text{S3.11})$$

Case (b)

If the pressure intensity is assumed to be inversely proportional to the radius,

$$\begin{aligned} p &= \frac{a}{r} \\ P &= \int_0^R 2\pi r p dr = 2\pi a R \end{aligned}$$

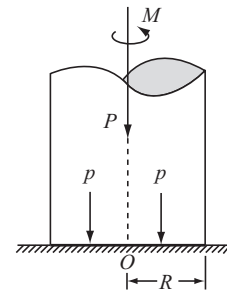
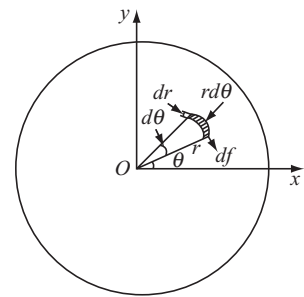


Fig. S3.7 Analysis of disc friction



whence

$$a = \frac{P}{2\pi R}$$

and

$$p = \frac{P}{2\pi Rr}$$

With this expression, the moment is determined as

$$\begin{aligned} M &= 2\pi\mu_d \int_0^R \frac{P}{2\pi Rr} r^2 dr \\ &= 1/2 PR\mu_d \end{aligned} \tag{S3.12}$$

Example S3.20

A shaft of radius R with an axial load F acting on it rests on a flat thrust bearing of radii R_1 and R_2 as shown in Fig. Ex. S3.20. Determine the torque, M for impending rotation of the shaft.

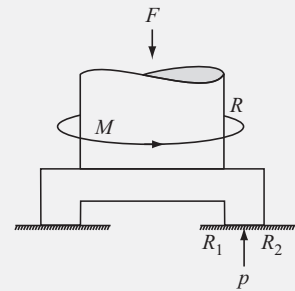


Fig. Ex. S3.20

Solution Consider an elementary area

$$dA = r \delta\theta \times \delta r$$

at a radial distance r , subtending an angle $d\theta$ at the centre as shown in Fig. Ex. S3.20 (Solution).

For a pressure p , the moment is given by

$$M = \int_{R_1}^{R_2} \int_0^{2\pi} r \cdot \mu \cdot p \cdot r \delta\theta \delta r$$

and

$$F = \int_{R_1}^{R_2} \int_0^{2\pi} p \cdot r \delta\theta \delta r$$

If

$$p = \text{constant}$$

$$M = \frac{2}{3} \mu \pi (R_2^3 - R_1^3) p$$

$$F = \pi (R_2^2 - R_1^2) p$$

and

$$M/F = \frac{2}{3} \mu \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

If p varies inversely as r , $p = \frac{C}{r}$

then

$$M/F = \mu \frac{R_1 + R_2}{2}$$

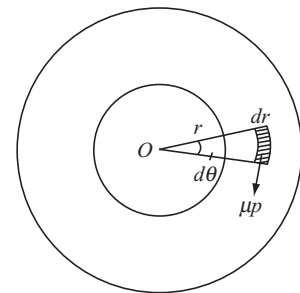


Fig. Ex. S3.20 (Solution)

Example S3.21 A shaft of radius R with axial load F is provided with a conical thrust bearing of radii R_1 and R_2 and cone angle α . Determine the torque needed to rotate it.

Solution Consider an elementary area $dA = r \cdot \delta\theta \cdot \delta r / \sin \alpha$ at a radial distance r , subtending a small angle $\delta\theta$ at the centre.

For a constant pressure p the frictional moment is given by

$$M = \int_{R_1}^{R_2} \int_0^{2\pi} r \cdot \mu \cdot p \cdot r \delta\theta \cdot \delta r / \sin \alpha = \mu p \frac{R_2^3 - R_1^3}{3} \sin \alpha$$

and the thrust is

$$F = \int_{R_1}^{R_2} \int_0^{2\pi} p \cdot r \delta\theta \cdot \delta r / \sin \alpha \cdot \sin \alpha = p \frac{R_2^2 - R_1^2}{2}$$

$$M/F = \frac{2}{3} \mu / \sin \alpha \left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \right)$$

If pressure p varies inversely as the radius,

$$p = \frac{C}{r}$$

then,
$$M/F = \frac{\mu}{\sin \alpha} \frac{R_1 + R_2}{2}$$

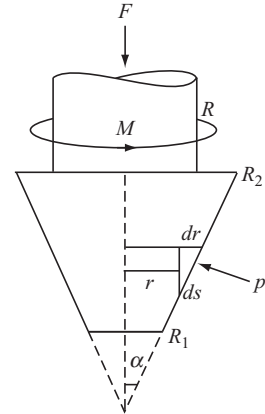


Fig. Ex. S3.21 (Solution)

Example S3.22 The conical end of a shaft of diameter 5 cm rests in a conical bearing of cone angle 60° as shown in Fig. Ex. S3.22. If the coefficient of dynamic friction is 0.3, calculate the frictional torque and power required to rotate the shaft at 1000 revolutions per minute, if the axial load on the shaft is 5 kN.

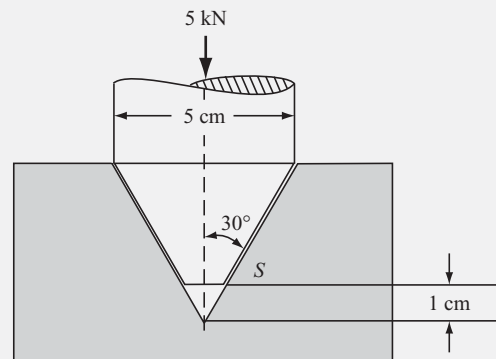


Fig. Ex. S3.22

Solution Consider a strip of the conical surface in contact. For an inclined length ds , the area of the strip is

$$dA = 2 \pi (s \sin 30^\circ) ds$$

where s is the inclined distance on the surface measured from the apex.

The axial force due to pressure p on the strip is

$$dP = dA p \sin 30^\circ = \frac{\pi}{2} p s ds$$

and the total axial force is

$$p = \int dp$$

The limits of integration are from

$$S_1 = \frac{1}{\cos 30^\circ} \text{ or } 1.155 \text{ cm to } S_2 = 5 \text{ cm}$$

$$P = \int_{0.01155}^{0.05} \frac{\pi}{2} ps \, ds = \frac{\pi}{2} p s^2/2 \Big|_{0.01155}^{0.05}$$

Since

$$P = 5 \text{ kN}$$

$$5 = 0.00186 p$$

whence

$$p = 2690 \text{ kN/m}^2$$

The frictional force df on the strip must be given by

$$df = \mu_d p \, dA = 0.3 \times 2690 \cdot 2\pi s \sin 30 \, ds \text{ kN}$$

and the frictional moment dN is such that

$$dM = df s \sin 30 = 1267 s^2 \, ds \text{ kN/m}$$

The total frictional moment is given by

$$M = \int dM = 1267 \Big| s^3/3 \Big|_{0.01155}^{0.05} = 52 \text{ N/m}$$

The speed of 1000 revolutions per minute corresponds to

$$\omega = \frac{2\pi \times 1000}{60} = 104.7 \text{ rad/s}$$

The frictional power is given by

$$\text{Power} = M\omega = 52 \times 104.7 = 5445 \text{ W} = 5.445 \text{ kW}$$

Example S3.23

A simple disc brake consists of four shoes, each subtending an angle α with the inner radius R_1 and outer radius R_2 . The shoes are pressed against a coaxial rotating disc with a force $4P$. Determine the torque exerted on the disc if the coefficient of static frictions is μ .

Solution The area of contact of a single shoe is (Ref. Fig. Ex. S3.23 (Solution))

$$A = \frac{\alpha}{2} (R_2^2 - R_1^2)$$

and the pressure intensity is given by

$$p = \frac{4P}{4A} = \frac{P}{\alpha(R_2^2 - R_1^2)}$$

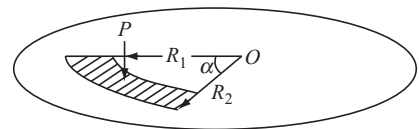


Fig. Ex. S3.23 (Solution)

The elementary frictional moment on an element of area $r d\theta \, dr$ is

$$dM = r \, df = r\mu \, dF_n = r\mu p r \, d\theta \times dr = \frac{2P\mu}{\alpha(R_2^2 - R_1^2)} \times r^2 \, d\theta \, dr$$

and the total moment for one shoe is given by

$$M = \int dM = \frac{2P\mu}{\alpha(R_2^2 - R_1^2)} \int_{R_1}^{R_2} \int_0^\alpha r^2 \, dr \, d\theta$$

$$= \frac{2\mu P (R_2^3 - R_1^3)}{3(R_2^2 - R_1^2)}$$

For the shoe brake with four shoes, the moment is

$$M' = \frac{8\mu P (R_2^3 - R_1^3)}{3(R_2^2 - R_1^2)}$$

It may be noted that the resulting torque is independent of the angle α subtended by each shoe at the centre of the disc brake.

S3.7 INPUT/OUTPUT OF SIMPLE MACHINES

A machine is a device which enables us to employ the input to advantage for achieving a desired output. A machine may consist of a single element or an assemblage of elements. Machines are classified as electrical, mechanical, electro-mechanical and others depending upon the nature of the input and output. We confine ourselves to the consideration of a simple mechanical machine where the input may be a small force or a moment at a convenient point and the output may be a larger load being lifted or moved against resistance.

It is usual to employ the terms mechanical advantage, velocity ratio and mechanical efficiency to describe the features of a lifting machine. The *mechanical advantage* is the ratio of the load lifted to the effort applied:

$$\text{Mechanical advantage} = \frac{\text{Load lifted}}{\text{Effort applied}} = \frac{W}{P} \quad (\text{S3.13})$$

The *velocity ratio* refers to the ratio of the velocities of the points of application of the effort and load. Assuming the lifting process to take place steadily, i.e., at the constant rate, the velocity ratio over an interval of time may also be defined as

$$\text{Velocity ratio} = \frac{\text{Distance moved by the point of application of effort}}{\text{Distance moved by the load}} = \frac{y}{x} \quad (\text{S3.14})$$

where the distances are measured along the directions of the respective forces as shown in Fig. S3.8.

The *efficiency* of a machine is defined as the ratio of the work output to the work input, i.e.,

$$\begin{aligned} \text{Efficiency, } \eta &= \frac{\text{Work output}}{\text{Work input}} = \frac{Wx}{Py} = \frac{W}{P} \bigg/ \frac{y}{x} \\ &= \frac{\text{Mechanical advantage}}{\text{Velocity ratio}} \end{aligned} \quad (\text{S3.15})$$

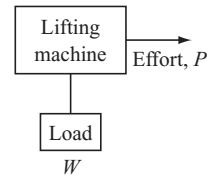


Fig. S3.8 Input and output of a lifting machine

A machine is said to be *ideal* if the efficiency is 100% which may be so in the absence of dissipation actions, such as friction. For an ideal machine, therefore,

$$\text{Mechanical advantage} = \text{Velocity ratio}$$

In actual practice, some energy must be lost in dissipative action and the efficiency is consequently less than 100%. The mechanical advantage must, therefore, be less than the velocity ratio. It follows that

$$\frac{W}{P} \leq \frac{y}{x} \quad (\text{S3.16})$$

depending upon whether a machine is ideal or not. Let us consider the reversed operation of a machine. The reversed operation refers to the movement of the machine components under the application of the load only when the effort is removed. It is possible when the work done by the load overcomes the frictional work over the same time interval,

$$Wx \geq Py - Wx$$

or

$$2Wx \geq Py$$

or

$$\frac{W}{P} \frac{y}{x} \geq 0.5$$

or

$$\eta \geq 0.5 \tag{S3.17}$$

A machine may, therefore, operate in the reverse direction on the removal of the effort if its efficiency exceeds or equals 50%. It is often necessary to stop the reversed operation of a machine. This may be done by reducing the efficiency to less than 50%. The machine is then said to be *self-locking*. The condition of self-locking is contradictory to the condition of an ideal operation.

The relationship between the load lifted and effort required, sometimes called *law of the machine* is such that

$$P = aW + b \tag{S3.18}$$

which shows that the effort bears a linear relationship with the load as shown in Fig. S3.9. The relationship shows that a minimum effort equal to the intercept b is required to lift a load, however small, because the frictional resistance is to be overcome. The effective frictional resistance at the point of application of the effort is defined as the difference between the ideal effort for a frictionless machine and the actual effort for a real machine. The effective frictional resistance F_c increases with the load as also shown in Fig. S3.9. From the expression for mechanical efficiency,

$$\eta = \frac{W}{P} \frac{y}{x}$$

and the law of the machine

$$P = aW + b$$

the efficiency may be written as

$$\eta = \frac{W}{aW + b} \frac{y}{x} = \frac{x/y}{a + b/W} \tag{S3.19}$$

This expression for η shows that the efficiency of a machine must be zero at zero load and that the efficiency increases as the load increases.

As
$$W \rightarrow \infty, \eta \rightarrow \frac{x}{ay}$$

The maximum efficiency is thus given by

$$\eta_{\max} = \frac{1}{a \times \text{velocity ratio}} \tag{S3.20}$$

as shown in Fig. S3.10, and the corresponding mechanical advantage is given by

$$\text{Mechanical advantage} = \eta \text{ velocity ratio} = \frac{1}{a} \tag{S3.21}$$

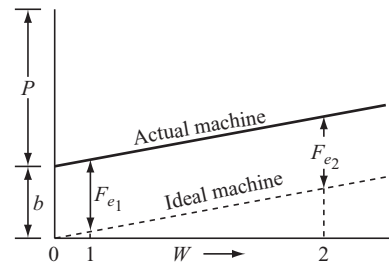


Fig. S3.9 Law of the machine

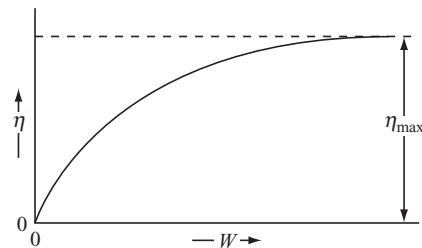


Fig. S3.10 Variation of efficiency with load

Lifting machines may employ one or more of elements such as levers, pulleys, gears, inclined planes and screw-threads.

Example S3.24 A drum of mass 6 kg holding water of mass 40 kg is to be raised from a well by the application of a 120 N human force. Would you recommend the use of a single pulley of diameter 10 cm, a simple wheel and axle of diameters 40 cm and 10 cm or a differential wheel and axle of diameters 40 cm, 10 cm and 5 cm? (Ref. Fig. Ex. 3.24)

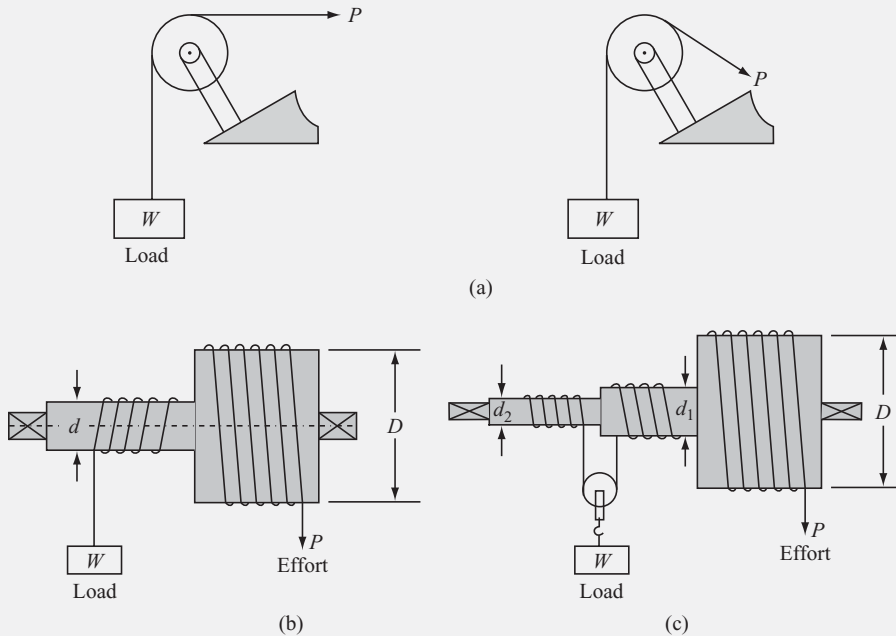


Fig. Ex. S3.24

Solution The requirement of the lifting machine to lift a load

$$W = (6 + 40) \times 9.81 = 451.3 \text{ N}$$

by an effort

$$P = 120 \text{ N}$$

The mechanical advantage is

$$\frac{W}{P} = \frac{451.3}{120} = 3.76$$

A single pulley has a velocity ratio of 1.0. If it operates without friction, its efficiency may be 100% and the mechanical advantage is then 1.0. If the mechanical efficiency is short of 100% due to friction, etc., the mechanical advantage would be less than 1.0. Thus a single pulley may have a mechanical advantage of 1.0 under ideal conditions. It is usefully employed if it is desired to change the direction of the effort applied with respect to the direction of the load being lifted as shown in Fig. Ex. S3.24(a). It is unsuitable for the present problem where the mechanical advantage desired is far beyond 1.0.

A simple wheel and axle of diameters D and d as shown in Fig. Ex. S3.24(b) operates such that the ratio of the distance moved by the effort to the distance moved by the load, i.e.,

$$\text{Velocity ratio} = \frac{\pi D}{\pi d} = \frac{D}{d}$$

which equals $40/10 = 4.0$ in the given case. The efficiency of the machine would be

$$\eta = 3.76/4.0 = 0.94 = 94\%$$

A differential wheel and axle of diameters D , d_1 and d_2 as shown in Fig. Ex. S3.24(c) operates such that

$$\text{Velocity ratio} = \frac{\pi D}{\pi d_1 - \pi d_2} = \frac{D}{d_1 - d_2}$$

which, in the present case, would be $40/(10 - 5) = 8.0$. The efficiency of the machine would then be

$$\eta = 3.76/8.0 = 0.47 = 47\%$$

In order to decide the preference in favour of the simple or differential wheel and axle, it may be noted that both are adequate as far as the lifting action is concerned. A simple wheel and axle is more efficient but the device is capable of operating in the reversed direction if the effort is removed. On the other hand, the differential wheel and axle has less than 50% efficiency, a fact which ensures self-locking when the effort is removed.

For the simple wheel and axle, the effort lost in overcoming the frictional resistance equals the actual effort minus the ideal effort

$$120 - \frac{451.3}{4.0} = 7.2 \text{ N}$$

whereas the effort lost in overcoming friction in the differential wheel and axle is

$$120 - \frac{451.3}{8.0} = 63.6 \text{ N}$$

One would prefer the simple wheel and axle arrangement unless the need for a self-locking arrangement can justify the additional effort lost in friction.

Example S3.25

Determine the effort required at the end of an arm 40 cm long to lift a load of 5 kN by means of a simple screw jack with screw threads of pitch 1 cm if the efficiency at this load is 45%. Also calculate the effort needed if the jack is converted into a differential screw jack with internal threads of pitch 7 mm and the efficiency of operation is 30% (Ref. Fig. Ex. 3.25).

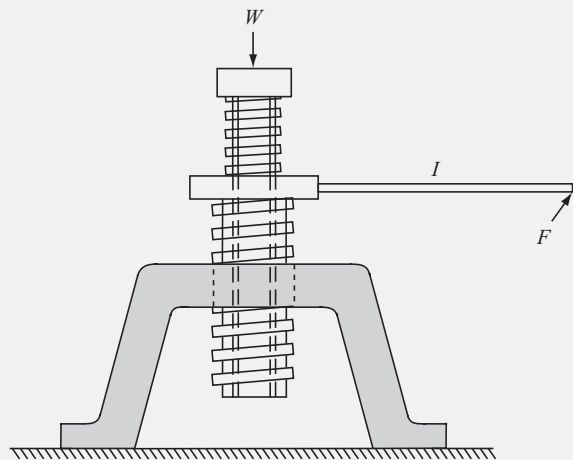


Fig. Ex. S3.25

Solution The velocity ratio of the simple screw jack is given by the ratio of the distance moved by the point of application of the effort, i.e., $2\pi \times 0.40 = 0.8\pi$ m and the distance moved by the load, i.e., 0.01 m if the arm is turned by one revolution

$$\text{Velocity ratio} = \frac{0.8\pi}{0.01} = 251.3$$

The mechanical advantage must be

$$W/P = 251.3 \times 0.45 = 113.1$$

whence

$$P = W/113.1 = 5000/113.1 = 44.2 \text{ N}$$

If the jack is converted into a differential screw jack, the distance moved by the point of application of the effort over one revolution remains the same, i.e., 0.8π but the corresponding distance through which the load moves becomes

$$0.01 - 0.007 = 0.003 \text{ m}$$

and the velocity ratio becomes

$$0.8\pi/0.003 = 837.8$$

The mechanical advantage must now be

$$W/P = 837.8 \times 0.30 = 251.3$$

whence

$$P = 5000/251.3 = 19.9 \text{ N}$$

Example S3.26

A carpenter's hand drill consists of a spindle AB which has a drill at A and a bearing at B as shown in Fig. Ex. 3.26. A rope is wrapped round the spindle 4 turns and the ends of the rope are tied to a handle CD . During operation, the handle is applied an effort P with one hand while the block at B is kept pressed with a force W with the other hand. Calculate the maximum torque which can be produced at the drill if the spindle has a diameter of 5 cm and the coefficient of friction between the rope and the spindle surface is 0.15. The tension in the slack side of the rope may be assumed to be 5 N.

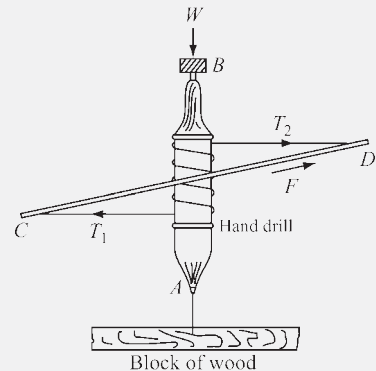


Fig. Ex. S3.26

Solution The torque at the drill would be maximum when the torque applied at the spindle is maximum, the two being equal. In the limiting case,

$$\frac{T_2}{T_1} = e^{\mu\theta} \tag{i}$$

and the torque is given by

$$M = (T_2 - T_1) r \tag{ii}$$

with the usual rotation.

From the data,

$$T_1 = 5 \text{ N} \quad \mu = 0.15$$

$$r = 5/2 = 2.5 \text{ cm} = 0.025 \text{ m}$$

$$\theta = 4 \text{ turns} = 4 \times 2\pi = 8 \text{ rad}$$

From (i),

$$T_2 = 5 \times e^{0.15 \times 8\pi} = 5e^{1.2\pi} = 217 \text{ N}$$

From (ii),

$$M = (217 - 5) \times 0.025 = 5.3 \text{ Nm}$$

Example S3.27

A uniform solid cube of side 8 cm and weight 100 N is resting with an edge on the ground and its base on the edge of a 3 cm step, as shown. An increasing horizontal force F is applied at the edge. In what ways is the cube likely to lose equilibrium? At what minimum value of F would it happen?

Assume the coefficient of friction between the cube and the ground and between the cube and the step as 0.3.

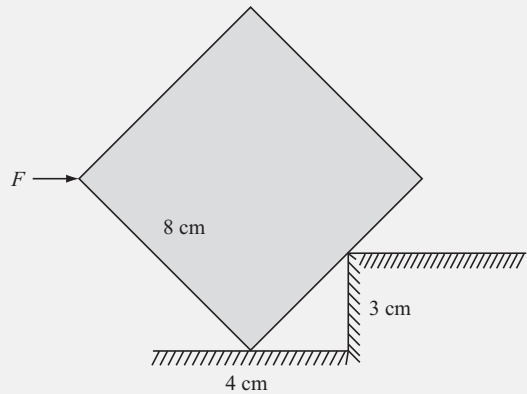


Fig. Ex. S3.27

Solution The cube can lose equilibrium in two possible ways: (a) by sliding at both points of contact or (b) by overturning about the edge of the step. Which one is likely to happen first?

Well, that depends upon the configuration, coefficient of friction, etc., so there is no easier way than to calculate the value of force F in each possibility.

Let us first consider case (a), sliding at both points of contact and draw the fbd of the block in Fig. Sol. Ex. 3.27. It shows all the external forces on the block.

At the verge of equilibrium being lost in sliding,

$$F - 0.3 R_A - 0.3 R_P \cos \theta - R_P \sin \theta = 0$$

and $R_A - 100 + R_P \cos \theta - 0.3 R_P \sin \theta = 0$

Taking moments about A ,

$$F \times 0.08 \cos \theta + 100 \cos \theta \times 0.04 - 100 \sin \theta \times 0.04 - R_P \cdot 0.05 = 0$$

or

$$8 F \cos \theta + 400 \cos \theta - 400 \sin \theta - 5 R_P = 0$$

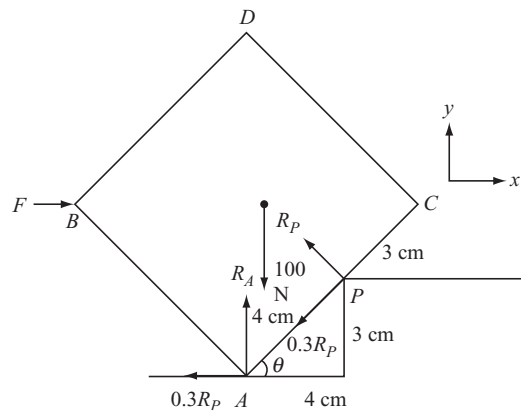


Fig. Ex. S3.27 (Solution)

With $\sin \theta = 3/5$ and $\cos \theta = 4/5$, the equations reduce to

$$\begin{aligned} 5F - 1.5 R_A - 4.2 R_P &= 0 \\ R_A - 100 + 0.62 R_P &= 0 \\ 6.4 F + 80 - 5 R_P &= 0 \end{aligned}$$

From these equations, $F \approx 300$ N

Now, let us consider the case (b), toppling over the step edge P . One way use the same fbd as above except for the fact that, at the instant of toppling over point P , the contact at A is lost, i.e., the reactions, at A become zero. With that in mind, taking moments about P ,

$$F \times 0.08 \cos \theta - F \times 0.05 \sin \theta - 100 \times 0.01 \cos \theta - 100 \times 0.04 \sin \theta = 0$$

or
$$8 F \cos \theta - 5 F \sin \theta - 100 \cos \theta - 400 \sin \theta = 0$$

Using
$$\sin \theta = 3/5 \text{ and } \cos \theta = 4/5$$

$$F = \frac{16}{17} \times 100 = 94 \text{ N}$$

We notice that this value of 94 N for losing the equilibrium in toppling is for less than the value of 300 N for sliding. The cube will, therefore, topple over the step at $F = 94$ N.

Example S3.28 A wooden block is being split by a 20° wedge with a force of 60 N applied horizontally as shown. Taking the coefficient of friction between the wood and the wedge as 0.4, estimate the vertical force tending to split the wood apart.

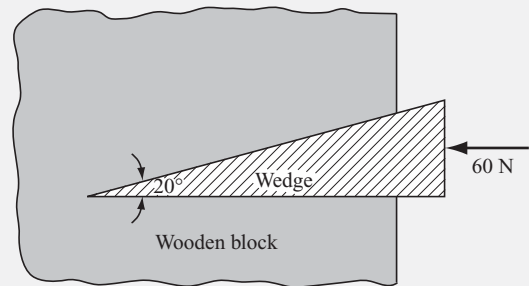


Fig. Ex. S3.28

Solution Draw the free-body diagram, as shown with P and Q forces acting at ϕ with the respective normals n and n to the wedge surfaces above and below it.

From $\mu = 0.4 = \tan \phi$; $\phi = 21.8^\circ$

By equilibrium,

$$P \sin (\phi + 20) + Q \sin \phi - 60 = 0 \quad (i)$$

and
$$-P \cos (\phi + 20) + Q \cos \phi = 0 \quad (ii)$$

Substituting $\phi = 21.8^\circ$, and solving for P and Q , $P = 62.5$ N, $Q = 50$ N

The vertical component of P is $-62.5 \cos 41.8^\circ = -46.5$ N

The vertical component of Q is $50 \cos 21.8^\circ = 46.5$ N.

Of course, the two are equal and opposite for equilibrium!

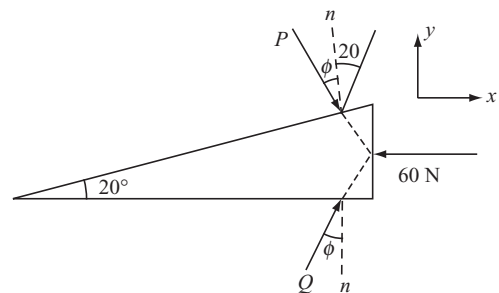


Fig. Ex. S3.28 (Solution)

EXPERIMENT E7 PERFORMANCE OF SIMPLE LIFTING MACHINES

OBJECTIVE TO DETERMINE THE EFFORT REQUIRED TO LIFT A LOAD AND EFFICIENCY OF LIFTING BY SOME SIMPLE MACHINES.

Apparatus Simple lifting machines, such as a screw jack, wheel and differential axle, worm and worm wheel, *winch crab*, as shown in Figs. E7.1 to E7.4; metre rod and standard weights.

Background Information A lifting machine is employed to lift a larger load W at a point by employing a smaller force P at some other point. During the process, the distance y moved by the effort may be much more than the distance x , moved by the load:

$$\begin{aligned} \text{Work input} &= Py \\ \text{Work output} &= Wx \end{aligned}$$

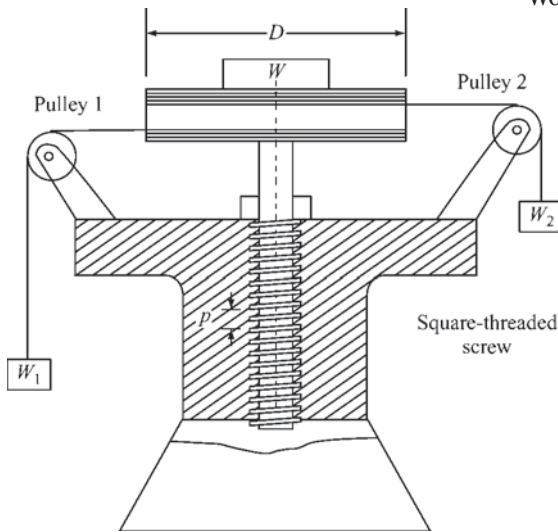


Fig. E7.1 Screw jack

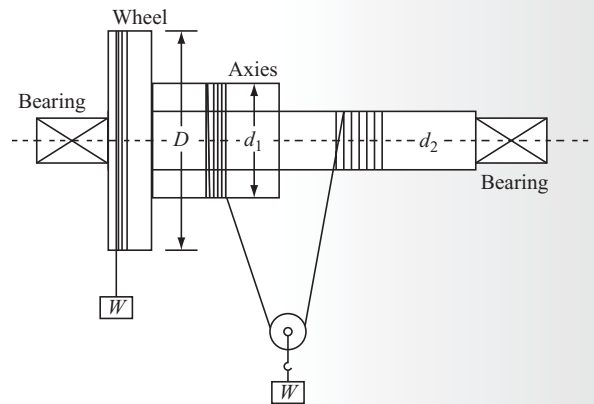


Fig. E7.2 Wheel and differential axle

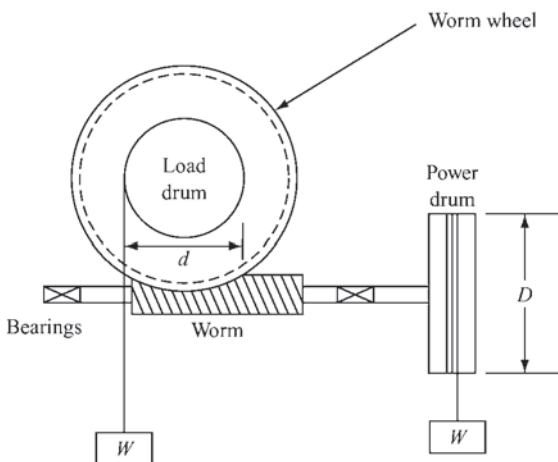


Fig. E7.3 Worm and worm wheel

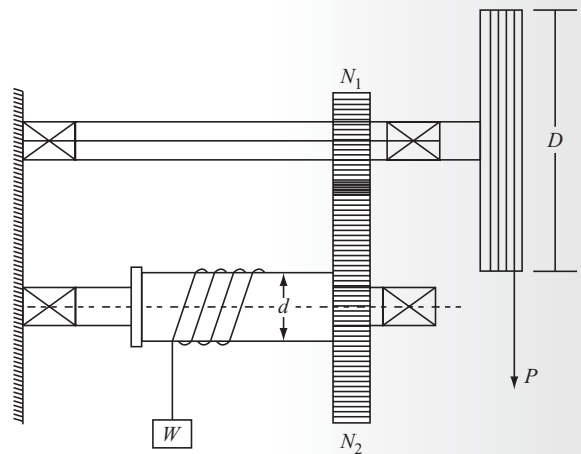


Fig. E7.4 Winch crab

$$\text{Efficiency of the machine, } \eta = \frac{Wx}{Py} = \frac{W/P}{y/x}$$

$$\eta = \frac{\text{Mechanical advantage}}{\text{Velocity ratio}}$$

where the term *mechanical advantage* stands for the ratio of the load to the effort applied to lift the load at a constant velocity and the term *velocity ratio* stands for the ratio of the distance moved by the effort to the distance moved by the load. Conventionally, the load is lifted vertically up and the point of application of effort is moved vertically down. The larger the load, the more the effort required. The plot of P vs. W is usually linear, whereas that between h and W is non-linear as shown in Fig. E7.5.

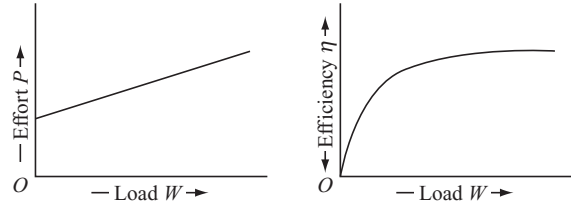


Fig. E7.5 Variation of effort and efficiency with load

While the effort applied to lift a given load at a constant velocity is recorded by measurement, the velocity ratio is determined from a consideration of the distances moved by the points of application of the load and effort.

For a screw jack, the distance moved by the effort is πD for the load to be lifted by a distance p where D is the diameter of the drum and p is the pitch of the screw

$$VR = \frac{\pi D}{p}$$

For a wheel and differential axle, the distance moved by the effort is πD for the load to be lifted by a distance $(\pi d_1 - \pi d_2)$ where D is the diameter of the wheel and d_1 and d_2 are the diameters of the axles on which the string is wound in the opposite directions:

$$VR = \frac{\pi D}{\pi d_1 - \pi d_2} = \frac{D}{d_1 - d_2}$$

For a worm and worm wheel, the distance moved by the effort is πD for the load to be lifted by a distance πd divided by the number of teeth N on the worm wheel where D and d are the diameters of the power drum and the load drum respectively

$$VR = \frac{\pi D}{\pi d/N} = \frac{ND}{d}$$

For a winch crab or lift, the distance moved by the effort is πD for the load to be lifted by a distance πd divided by the ratio N_2/N_1 of the teeth of the wheel and pinion where D and d are the diameters of the power drum and load drum respectively

$$VR = \frac{\pi D}{\pi d/(N_2/N_1)} = \frac{N_2 D}{N_1 d}$$

Observations and Calculations It is necessary to note the parameters which enter into the determination of the velocity ratio for a given machine. The velocity ratio VR is thus obtained. The effort required to lift a load at a constant speed is determined practically and the experiment is repeated for various loads in convenient steps.

| S. No. | W | P | W/P | VR |
|--------|-----|-----|-------|------|
| 1 | | | | |
| 2 | | | | |
| . | | | | |
| . | | | | |
| . | | | | |

Result Plots of P vs. W and η vs. W for the lifting machine may be made from the measured and calculated values. One may like to express the P vs. W plot in the form of a linear relationship.

$$P = aW + b$$

often called the *law of the machine*. The values of a and b are obtained from the plot.

Points for Discussion

A machine is said to be self-locking if the load stays in position even though the effort is removed. What is the condition, in terms of efficiency, for a machine to be self-locking? Which of the machines tested by you are self-locking?

How should the effort vary with the load for an ideal, i.e., a frictionless machine? How does the plot alter due to friction?

Assuming that the law of the machine is linear, what should be the maximum mechanical advantage and maximum efficiency of the machine?

Of the various lifting machines known to you, how would you decide which one to choose for a particular situation? For example, which lifting machine is best suited, in your opinion, for the following jobs:

- (a) lifting a drum of water from a well?
- (b) lifting a heavy consignment from a ship?
- (c) lifting the body of a truck for the purpose of changing a wheel?

Concept Review Questions

Comment on the nature of friction between two surfaces and the concept of impending motion.

A block of base dimensions $a \times b$ and height h is subjected to a horizontal force F at its mid-height. Draw the free-body diagram of the block

- (a) for small F
- (b) for large F

both being before the state of impending action. Examine the difference in the point of application of the resultant of the normal reaction and the frictional force at the base of the block.

Under what conditions can a cylinder roll down and under what conditions can it slide down an inclined plane?

In the analysis of a belt-drive, the relationship between the tensions in the tight side and the slack side incorporates the coefficient of static friction μ . Explain why it is not the coefficient of dynamic friction instead.

Whenever a rope or a wire under tension is to be held, it is wrapped round a tree-trunk or a pole by giving it a number of turns. Why?

Explain why the lifting action of a screw jack is likened to pushing up an incline.

From the expression for the efficiency of a screw jack,

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

where α is the equivalent inclination and ϕ is the friction angle, obtain the expression for the maximum efficiency and also the efficiency for it to be self-locking.

- Define and relate the mechanical advantage velocity ratio and efficiency of a machine. What is meant by the law of the machine?

Tutorial Problems

Figure Prob. S3.1 shows the location of the centre of gravity in a model of a car which can be driven either by its rear or front wheels. If the inertia of the wheels, rolling resistance and aerodynamic drag are ignored, what is the ratio of the maximum accelerations in the two cases when the coefficient of friction between the tyres of the car and ground is 0.7?

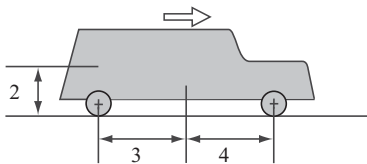


Fig. Prob. S3.1

In Fig. Prob. S3.2, the coefficients of friction between the weight and the wedge is μ_1 and between the wedge and the lower block μ_2 . Ignoring the weight of the wedge and assuming no friction, determine the applied force required to raise by forcing the wedge to the right.

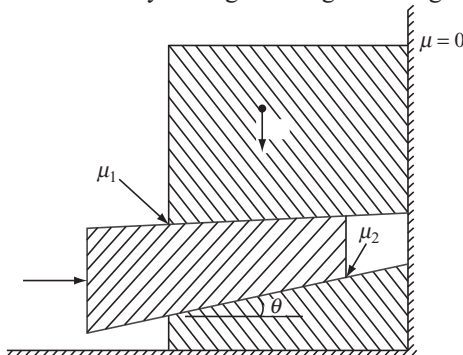


Fig. Prob. S3.2

$$\left[\text{Ans } \left(\mu_1 + \frac{\mu_2 + \tan \theta}{1 - \mu_2 \tan \theta} \right) \right]$$

A homogeneous ladder 6 m long and weighing 400 N rests against a smooth wall. The angle between it and the floor is 70° . The coefficient of friction between the floor and the ladder is 0.25. How far up the ladder can an 80 kg man walk before the ladder slips? (s 4.41 m)

A horizontal force, is applied to a block which rests on an inclined plane, as shown in Fig. Prob. S3.4. Find the force required to initiate motion up the plane.

$$\left(s = \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} \right)$$

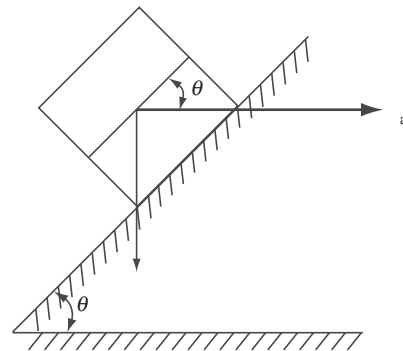


Fig. Prob. S3.4

The block of weight is on a surface (the coefficient of friction is μ) which is inclined at angle θ with the horizontal as shown in Fig. Prob. S3.5. This surface is part of a triangular

block of weight W_1 . A horizontal force P causes the system to have an acceleration a to the right. What value of P will cause the top block to move relative to the surface? Assume no friction of the bottom surface. What is the acceleration?

$$\left(\begin{array}{l} s \ a = \frac{g(\mu - \tan \theta)}{(1 + \mu \tan \theta)}, \\ P = \frac{(W + W_1)(\mu - \tan \theta)}{1 + \mu \tan \theta} \end{array} \right)$$

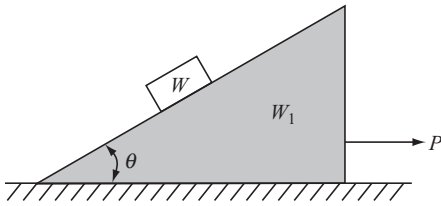


Fig. Prob. S3.5

A uniform rod of mass m and length L is lying on a rough horizontal table. A horizontal force P is applied to the rod perpendicular to its axis at a distance kL ($k > \frac{1}{2}$) from one end so that it just moves. Show that the rod rotates about a point distance hL from the same end and that $P = \mu mg(1 - 2h)$, where $h = k - \left(k^2 - k + \frac{1}{2}\right)^{1/2}$.

A bar rests on two pegs and makes an angle β with the horizontal. The coefficients of friction are μ_1 at one peg, which is at a distance a from G , the centre of gravity of the bar, and μ at the other peg at distance b from G . Show that for an equilibrium condition to exist.

$$\tan \beta \leq \frac{\mu_1 b + \mu_2 a}{(a + b)}$$

- 8 At what height above the surface of a billiard table should a ball of radius r be struck by a horizontal force F in order to have no sliding at the point of contact? (as shown in Fig. Prob. S3.8).

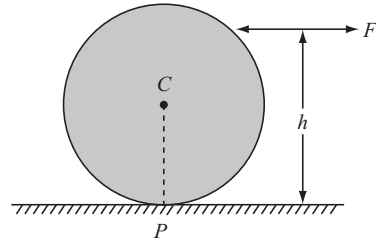


Fig. Prob. S3.8

A cylinder of diameter d weighing W rests at a corner of two surfaces as shown in Fig. Prob. S3.9. Prove that the maximum force P that can be applied as shown without causing the cylinder to rotate is $3/8 W$. Take the coefficient of friction for each pair of contacting surfaces as 0.5.

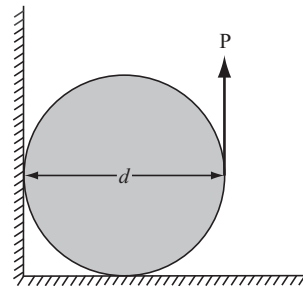


Fig. Prob. S3.9

Find the minimum weight W of the triangular block such that it remains in equilibrium under the action of the force 1 kN applied to it as shown in Fig. Prob. S3.10. Take $\mu = 0.25$.
(s 2.12 kN)

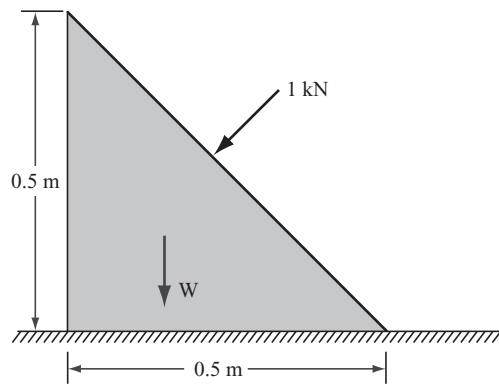


Fig. Prob. S3.10

A cylinder of diameter 0.3 m and mass 25 kg rests on a rough surface as shown in Fig. Prob. S3.11, with $\mu_s = 0.4$ and $\mu_d = 0.35$. Determine the force P to be applied shown to roll the cylinder without slip over the step.

(s $P = 70.5 \text{ N}$)

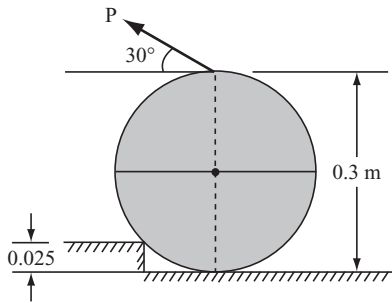


Fig. Prob. S3.11

A cylinder weighing 2670 N, 1.2 m in diameter is acted upon by a force of 445 N, as shown in Fig. Prob. 3.12 with the help of a cord wrapped around it. Determine the coefficient of friction required to prevent slipping. What would happen if the coefficient of friction is reduced?

(s $\mu = 0.21$; it would roll with slip at the point of contact)

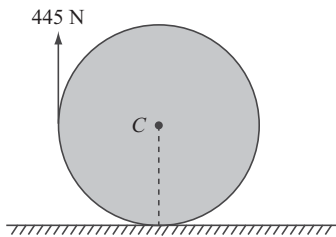


Fig. Prob. S3.12

A rope is wrapped three and a half times around a cylinder, as shown in Fig. Prob. S3.13. Determine the force T_1 exerted on the free end of the rope, that is required to support a 1 kN weight. The coefficient

of friction between the rope and cylinder is 0.25.

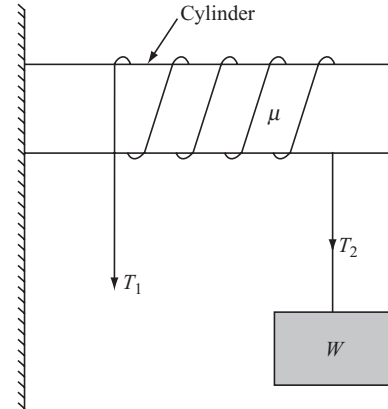


Fig. Prob. S3.13

A belt passes over a number of fixed cylinders, as shown in Fig. Prob 3.14. Find the tension T for a given tension T_0 in terms of the coefficients of frictions and angle.

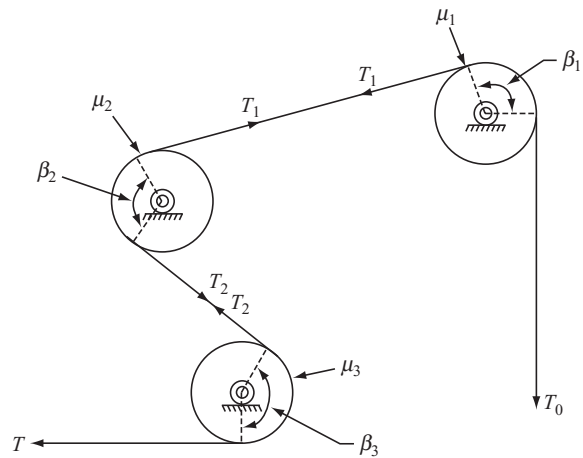


Fig. Prob. 3.14

(s $T = T_0 e^{-(\mu_1 \beta_1 + \mu_2 \beta_2 + \dots)}$)

A square-threaded screw jack has a pitch of 1 cm and a mean diameter of 7.5 cm. The mean diameter of the bearing surface between the cap and the screw is 9 cm.

The coefficient of friction between all surfaces is 0.10. What force is required at the end of a lever 90 cm long to raise 40 kN?

(s 440 N)

Four turns of a rope around a horizontal post are just able to hold a 450 kg mass with a pull of 45 N. Determine the coefficient of friction between the rope and the post.

(s $\mu = 0.18$)

Two equal pulleys, each of diameter 75 cm, are connected by a belt. The tension in the tight side of the belt is 200 N. If the coefficient of friction is 0.25, determine the tension in the slack side of the belt at the instant of impending slip.

(s 91 N)

- 8 A screw jack has a pitch of 6 mm and the mean radius of the threads is 60 mm. The mean diameter of the bearing surface under the cap is 80 mm. What is the turning moment necessary to lift a 680 kg box? Take $\mu = 0.06$.

(s 46.4 N/m)

An axial force P presses the disc brake of radius R onto a flexible elastic surface so that the contact pressure decreases parabolically from p_0 at the centre to zero at the periphery of the disc. Show that the pressure distribution can be expressed as

$$p = \frac{2P}{\pi R^2} (1 - r^2/R^2)$$

Also show that the torque required to cause the impending motion of the disc brake is given by $M = 8/15 \mu_s PR$.

Find the lifting force P required to raise the load 100 N supported as shown in Fig. Prob. S3.20. Take $\mu = 0.3$. What would happen if the force P is less than this value? What is

the minimum force P required to just hold the load in position?

(s 658.6 N; No lifting; 15.2 N)

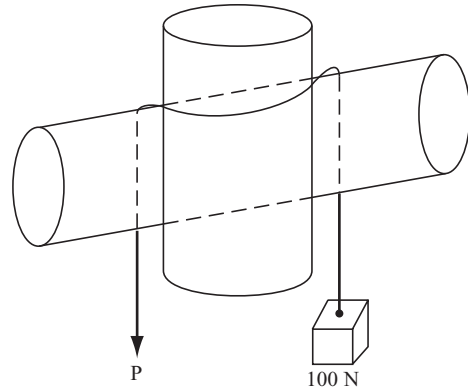


Fig. Prob. S3.20

A drum brake consists of a drum with brake lining over an angle 4α as shown in Fig. Prob. S3. 21. The brake shoes are pressed on it with a force P to produce a braking couple C .

Show that

$$2h \frac{P}{C} = \frac{h(\alpha + \sin \alpha \cos \alpha)}{2\mu R \sin \alpha} \pm 1$$

where the sign depends upon the direction of motion of the drum.

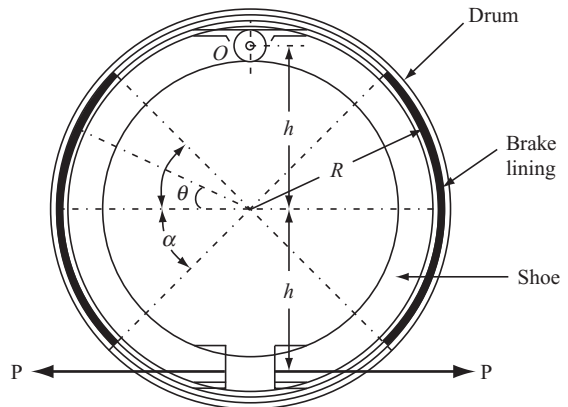


Fig. Prob. S3.21

Look up Hints to Tutorial Problems at the end!**Multiple-Choice Questions**

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

The force of friction between two bodies in contact is

- (a) a function of the relative velocity between them.
- (b) dependent on the areas of contact.
- (c) always normal to the surface of contact.
- (d) never shown in the free-body diagram of the system of these two bodies.

The frictional force is independent of

- (a) the area of contact
- (b) the coefficient of friction
- (c) the normal reaction
- (d) the angle of friction

The coefficient of friction depends upon

- (a) the normal reaction
- (b) the surface roughness
- (c) the tangential force applied
- (d) the speed of movement

Once a body just begins to slide, it continues to slide because

- (a) the body has inertia
- (b) inertia force acts on the body
- (c) the body accelerates
- (d) the frictional force becomes less.

The frictional force on a body acted upon by a force on a rough horizontal surface is

- (a) always equal and opposite to the horizontal component of the force.
- (b) equal and opposite to the applied force.

- (c) equal and opposite to the horizontal component of the applied force if the body is at rest or moving with a constant velocity.
- (d) independent of the vertical component of the force.

The coefficient of friction between two surfaces is the constant of proportionality between the applied tangential force and the normal reaction

- (a) at the instant of application of the force.
- (b) at any instant when the body is at rest.
- (c) at the instant of impending motion.
- (d) at an instant after the motion takes place.

The ratio between the tensions in the tight side and slack side of a flat belt drive increases

- (a) in direct proportion to the angle of lap.
- (b) exponentially as the angle of lap increases.
- (c) in direct proportion to the coefficient of friction.
- (d) proportional to the width of the belt.

8 The condition for a screw jack to be self-locking is that

- (a) its efficiency should be the maximum possible.
- (b) its efficiency should be the minimum possible.
- (c) its efficiency should be more than 50%.
- (d) it should not unwind to lower the load if left to itself.

The maximum efficiency of a machine

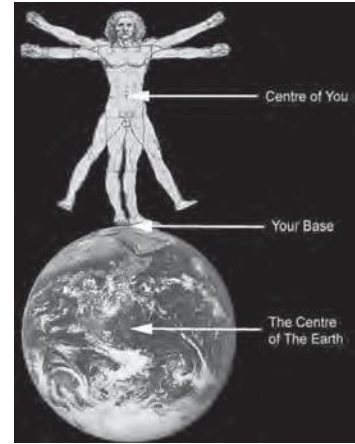
- (a) should be 100% under ideal conditions.
- (b) is directly proportional to the velocity ratio.
- (c) is given by mechanical advantage divided by velocity ratio.
- (d) should occur when the load is 50% of maximum permissible load.

Answers to Multiple-choice Questions

- 1 (d) 2 (a) 3 (b) 4 (d) 5 (c) 6 (c) 7 (b)
8 (d) 9 (c)

Centroid, Centre of Mass and Gravity

4



4.1 INTRODUCTION

It is often necessary to define a point such that the entire length of a curve, area of a surface or volume, mass or gravitational force for a body should be representable to act at that point for some purposes. The point should serve as a convenient origin for the coordinate axes moving with the body. It can be regarded as a convenient base point for the application of the principle of moment of momentum. It can also be taken as the fundamental base for the moment of inertia computations. In view of so many good reasons it is natural to define a central point for a given physical entity, more so, if a single point can satisfy all the above criteria.

Central points or centroids of simple geometrical shapes are well known, as shown in Fig. 4.1. The centre C of a circle, a circular plate or a sphere is also the centroid!

The centre of a straight line segment must also be its centroid. Hence, the point of intersection of the diagonals of a figure must be the centroid of the figure. The centroid of a rectangle or a rectangular plate must, therefore be at C .

Mathematically, a *central point* is that point about which the summation of the first moments of the elements of the body results in zero. Alternatively, if an origin is chosen arbitrarily, then the central point is that where the entire physical quantity may be assumed to be concentrated for the purpose of calculating moment about the origin. This definition, as will be seen later, makes the central point very meaningful since it is unique and invariant with the choice of the origin and the orientation of the set of axes with respect to the body.

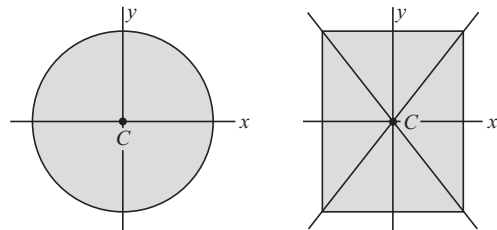


Fig. 4.1 Centroids of simple figures

The terminology of the central point for different physical entities is as follows:

| | | |
|-------------------|-----|-------------------------------|
| | P | |
| Centroid | | Length of a curve |
| Centroid | | Area of a surface |
| Centroid | | Volume of a body |
| Centre of Mass | | Mass of a body |
| Centre of Gravity | | Gravitational force on a body |

4.2 CONCEPT OF FIRST MOMENT

The first moment of an 'element' about an origin is defined as the product of its position vector with the element itself. The element may belong to any physical quantity, such as continuous length, area, volume, mass or distributed gravitational force. The element is correspondingly termed as

| | | |
|----------------|-------------|-------|
| Length element | for length | (i) |
| Area element | for area | (ii) |
| Volume element | for volume | (iii) |
| Mass element | for mass | (iv) |
| Force element | for gravity | (v) |

It may be seen that the element can be a scalar quantity as in (i) to (iv) and a vector quantity as in (v). The product definition of the first moment implies a magnitude multiplication for the scalar elements and a cross product for the vector elements.

Consequently, if the element is located by a position vector \mathbf{r} with respect to an origin O as shown in Fig. 4.2, the first moments are expressed as follows:

$$\begin{aligned}
 & \int \mathbf{r} \, dl && \text{for the length element} && (i) \\
 & \int \mathbf{r} \, dA && \text{for the area element} && (ii) \\
 & \int \mathbf{r} \, dV && \text{for the volume element} && (iii) \\
 & \int \mathbf{r} \, dm && \text{for the mass element} && (iv) \\
 & \int \mathbf{r} \times (-g) && \text{for the gravitational} && \\
 & && \text{force element} && (v)
 \end{aligned} \tag{4.1}$$

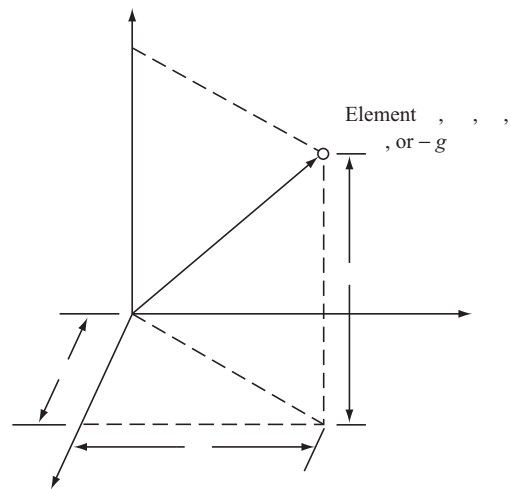


Fig. 4.2 Location of an element

In general, the position vector \mathbf{r} has three rectangular components:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The first moments for the length, area, volume, mass and force elements can be expressed as follows:

$$\begin{aligned}
 & \int x \, dl && \text{for length} && (i) \\
 & \int x \, dA && \text{for area} && (ii) \\
 & \int x \, dV && \text{for volume} && (iii) \\
 & \int x \, dm && \text{for mass} && (iv) \\
 & \int x \, (-g) && \text{for gravity} && (v)
 \end{aligned} \tag{4.2}$$

The first moment of any element can be positive or negative, depending upon the choice of the origin and the contribution of the element. For example, the component of the first moment of an area element, i.e.,

will acquire the sign of x if the area actually exists and it will acquire a sign opposite to that of x if the area in question is a void from the total area counted as positive. The sign and magnitude of x will, of course, depend upon the choice of the origin.

4.3 DEFINITIONS OF CENTROID, CENTRE OF MASS AND GRAVITY

The *central point* is defined as a point where the entire physical quantity can be assumed to be concentrated to give the same first moment as that obtained by considering the elements of the body. The central points for a length, an area and a volume are called the centroids whereas the central points for the distribution of mass and gravitational force are termed as the centre of mass and centre of gravity respectively.

The central point is denoted by C and its position vector by

$$\mathbf{r}_c = x_c \mathbf{i} + y_c \mathbf{j} + z_c \mathbf{k}$$

Thus, for a line in space whose total length is l the first moment is written as $\mathbf{r}_c l$. Equating it to the summation of the first moments of the length elements over the entire length.

$$\mathbf{r}_c l = \int \mathbf{r} dl^*$$

whence

$$\mathbf{r}_c = \left(\int \mathbf{r} dl \right) / l \tag{4.3}$$

Similarly, denoting the total area of a surface by A , its centroid is given by

$$\mathbf{r}_c = \left(\int \mathbf{r} dA \right) / A \tag{4.4}$$

Likewise, the centroid of a volume V is given by

$$\mathbf{r}_c = \left(\int \mathbf{r} dV \right) / V \tag{4.5}$$

and the centre of mass of a body of mass m is located by

$$\begin{aligned} \mathbf{r}_c &= \left(\int \mathbf{r} dm \right) / m \\ &= \left(\int \rho \mathbf{r} dv \right) / m \end{aligned} \tag{4.6}$$

where ρ is the mass density.

The centre of gravity of a body acted upon by a parallel and uniform gravitational force $-mg \mathbf{k}$ is given by

$$\begin{aligned} \mathbf{r}_c \times (-mg \mathbf{k}) &= \int \mathbf{r} \times (-g dm \mathbf{k}) \\ \mathbf{r}_c \times m \mathbf{k} &= \int (\mathbf{r} \times dm \mathbf{k}) \end{aligned} \tag{4.7}$$

The central point, i.e., the centroid, centre of mass or centre of gravity may or *may not lie on the body itself* but it is a point fixed with respect to the body. The choice of the coordinate axes and origin is arbitrary; the centroid is a unique point for the given physical quantity which may be a length, an area, a volume, a mass or a distributed force. The central point, therefore, represents the given physical quantity so far as its first moment about any origin is concerned.

4.4 CENTROID OF A LENGTH

The centroid of length l of a curve in space is given by Eq. (4.3), i.e.,

$$\mathbf{r}_c = \left(\int \mathbf{r} dl \right) / l$$

In general, the limits for such definite integrals are not written for the sake of convenience.

which implies that the coordinates of the centroid are

$$\begin{aligned} &= \left(\int \dots \right) / \dots \\ &= \left(\int \dots \right) / \dots \\ &= \left(\int \dots \right) / \dots \end{aligned} \tag{4.8}$$

and

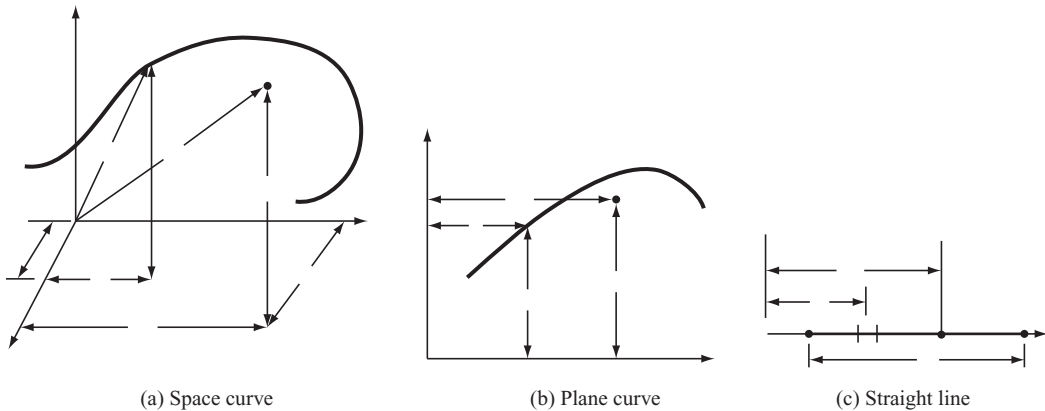


Fig. 4.3 Centroid of a length

For a plane curve, the x and y coordinate axes being chosen in the plane of the curve,

$$= \int (\dots) / \dots \tag{4.9}$$

and

$$= \int (\dots) / \dots$$

For a straight line, the x -axis being chosen along the length,

$$= \left(\int \dots \right) / \dots \tag{4.10}$$

It may be noted that the centroid of a curve may or may not lie on the curve for space and plane curves but it must lie at the mid-point of straight line. Further, the centroid for a curve is independent of the choice of coordinates and the origin. In other words, the centroid is a point for a curve and is not concerned with the choice of the origin and the orientation of the axes.

Example 4.1

Locate the centroid of an arc of a circle of radius r and subtended angle 2α . Hence determine the coordinates of the centroid of

- (i) a quarter circular arc
- (ii) a semicircular arc
- (iii) a complete circular arc.

Solution Consider an arc of a circle with reference to the coordinate axes at the centre of the circle drawn symmetrically with respect to the arc. Symmetry of the x axis ensures that the centroid is located on the x axis or

$$y_c = 0$$

In order to determine x_c , consider a length element

$$dl = R d\theta$$

at an angle θ from the x -axis as shown in Fig. Ex. 4.1. (Solution)

By definition,

$$\begin{aligned} x_c &= \left(\int x dl \right) / l = \left(\int x dl \right) / \int dl \\ &= \left(\int_{-\alpha}^{\alpha} x R d\theta \right) / \left(\int_{-\alpha}^{\alpha} R d\theta \right) \end{aligned}$$

Substituting $x = R \cos \theta$,

$$\begin{aligned} x_c &= \left(\int_{-\alpha}^{\alpha} R^2 \cos \theta d\theta \right) / \left(\int_{-\alpha}^{\alpha} R d\theta \right) \\ &= \frac{R^2 (\sin \alpha - \sin(-\alpha))}{2R\alpha} \\ &= \frac{R \sin \alpha}{\alpha} \end{aligned}$$

(i) For a quarter circular arc,

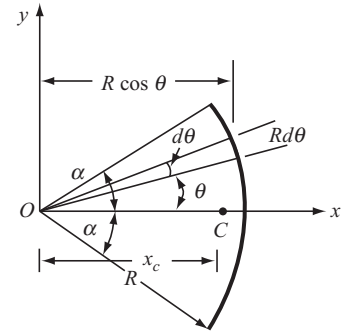
$$2\alpha = \pi/2; \quad \alpha = \pi/4$$

Consequently,

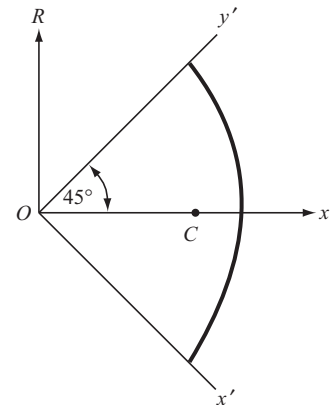
$$x_c = \frac{R \sin \pi/4}{\pi/4} = \frac{2\sqrt{2}}{\pi} R$$

and

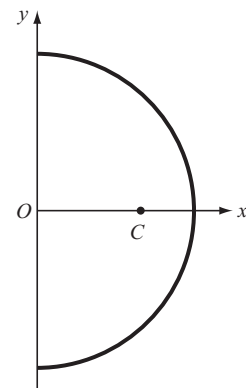
$$y_c = 0$$



(a) Arc of a circle



(b) Quarter-circular arc



(c) Semi-circular arc

Table 4.1 Centroid of Length of an Arc

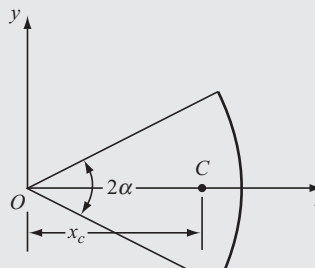
| Description | Shape | Length | x_c |
|---|---|---------------|--------------------------------|
| (Arc of a circle) |  | $2R\alpha$ | $\frac{R \sin \alpha}{\alpha}$ |
| (Quarter circular arc) $2\alpha = \pi/2$ | | $\frac{R}{2}$ | $\frac{2\sqrt{2}R}{\pi}$ |
| (Semicircular arc) $2\alpha = \pi$ | | πR | $\frac{2R}{\pi}$ |
| (Circular arc) $2\alpha = 2\pi$ | | $2\pi R$ | 0 |

Fig. Ex. 4.1 (Solution)

(ii) For a semicircular arc

$$2\alpha = \pi; \quad \alpha = \pi/2$$

$$x_c = \frac{R \sin \pi/2}{\pi/2} = \frac{2R}{\pi}$$

$$y_c = 0$$

(iii) For a complete circular arc,

$$2\alpha = 2\pi; \quad \alpha = \pi$$

$$x_c = \frac{R \sin \pi}{\pi} = 0$$

$$y_c = 0$$

The centre of the circle must be the centroid of the complete circular arc by virtue of its symmetry about the diametral axes.

4.5 CENTROIDS OF A COMPOSITE LENGTH

When the length of a curve can be decomposed into simpler shapes, such as straight lines, arcs of circles, etc., the centroid of the length can be determined by employing the knowledge for these simpler shapes. The principle is:

The first moment of the total length must equal the algebraic sum of the first moments of the lengths of its parts which indeed follows from the definition of the centroid.

If a composite length consists of component lengths l_1, l_2, l_3, \dots with centroids at $(x_{c1}, y_{c1}), (x_{c2}, y_{c2}), \dots$, respectively, then the centroid of the composite area is located by

$$x_c = \frac{x_{c1}l_1 + x_{c2}l_2 + \dots}{l_1 + l_2 + l_3 + \dots}$$

$$= \frac{\sum x_c l}{\sum l} \tag{4.11}$$

and

$$y_c = \frac{y_{c1}l_1 + y_{c2}l_2 + \dots}{l_1 + l_2 + l_3 + \dots} = \frac{\sum y_c l}{\sum l} \tag{4.12}$$

Care must be taken to regard the component lengths as positive or negative depending upon their contribution to the composite length. The coordinates of their individual centroids, referred from the same origin, can be positive or negative.

Example 4.2

A thin wire is bent into a semicircle and its diameter of 2 m.

- Find the centroid of the composite length.
- If there was an additional straight piece of wire attached at O , towards left, what should be its length so that the centroid is at O itself?

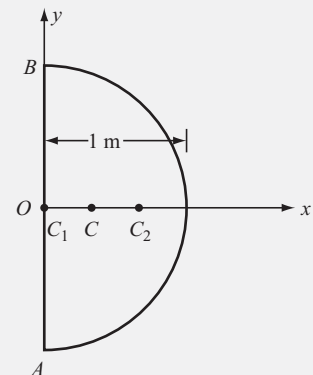


Fig. Ex. 4.2

Solution Referring to the Fig. Ex. 4.2, centroid of part AB is at $C_1(0, 0)$ and the centroid of the semicircle is at $C_2(2R/\pi, 0)$

Therefore, for the composite length,

$$L = 2 + \pi \times 2/2 = 5.14 \text{ m}$$

$$x_c = \frac{0 \times 2 + 2R/\pi \times \pi \times 2/2}{5.14}$$

$$= \frac{2R}{5.14} = 0.389R = 0.389 \times 2/2 = 0.389 \text{ m.}$$

and $y_c = 0$ because both C_1 and C_2 are on the x -axis.

Now, if there was additional straight length of wire x to the left of O assuming Fig. Ex. 4.2 (Solution), the location of its centroid would have been $x/2$ to the left. In that case, for the total composite length.

$$L = x + 2 + \pi \times 2/2 = (5.14 + x) \text{ m}$$

the location of the centroid would have been

$$x_c = \frac{-x \cdot x/2 + 2 \times 0 + \pi \times 2 \times 1/\pi}{(5.14 + x)} = \frac{2 - x^2/2}{5.14 + x}$$

If it is to be at the origin, $x_c = 0$;

$$2 - x^2/2 = 0; \quad x^2 = 4; \quad x = 2 \text{ m.}$$

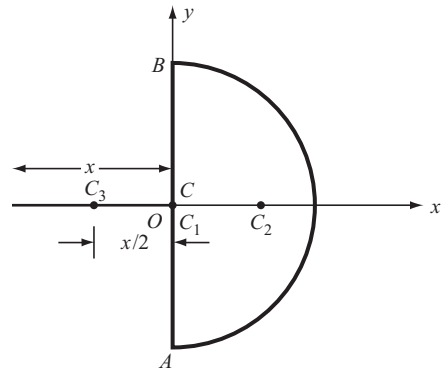


Fig. Ex. 4.2 (Solution)

4.6 CENTROID OF AN AREA

From the definition of the centroid of an area (Eq. (4.4))

$$\mathbf{r}_c = \left(\int dA \right) / A$$

the x , y and z coordinates for a set of axes, with an arbitrary O as shown in Fig. 4.4, are given by

$$\begin{aligned} x_c &= \left(\int x dA \right) / A \\ y_c &= \left(\int y dA \right) / A \\ z_c &= \left(\int z dA \right) / A \end{aligned} \tag{4.13}$$

For a plane area, the x and y coordinate axes being chosen in the plane of the area as shown in Fig. 4.4(b),

$$\begin{aligned} x_c &= \left(\int x dA \right) / A \\ y_c &= \left(\int y dA \right) / A \end{aligned} \tag{4.14}$$

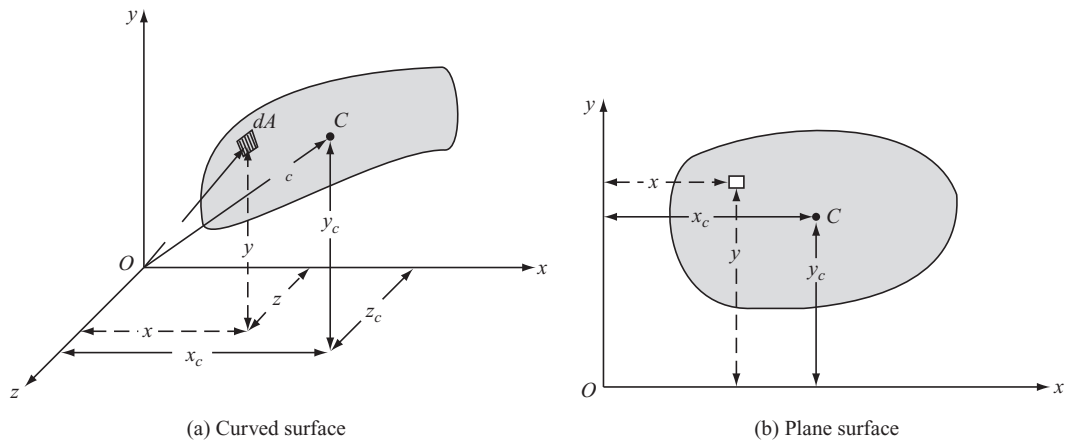


Fig. 4.4 Centroid of area

It may be added to clarify that the centroid of an area may or may not lie on the area in question and that it is a unique point for a given area regardless of the choice of the origin and the orientation of the axes about which we take the first moments.

If an area has an axis of symmetry, the centroid must lie on that axis. If an area has two axes of symmetry, then the point of intersection of the axes must be the centroid. For example, a circle and a rectangle shown in Fig. 4.1 are symmetrical about the x and y axes; point of intersection of x and y is, therefore, the centroid.

Example 4.3

Locate the centroid of a right-angled triangle with base b and height h .

Solution Let us consider an area element dA with respect to the x and y axes drawn from the right-angled vertex O as shown in Fig. Ex. 4.3(a) (Solution).

$$x_c = \left(\int x dA \right) / A \quad \text{and} \quad y_c = \left(\int y dA \right) / A$$

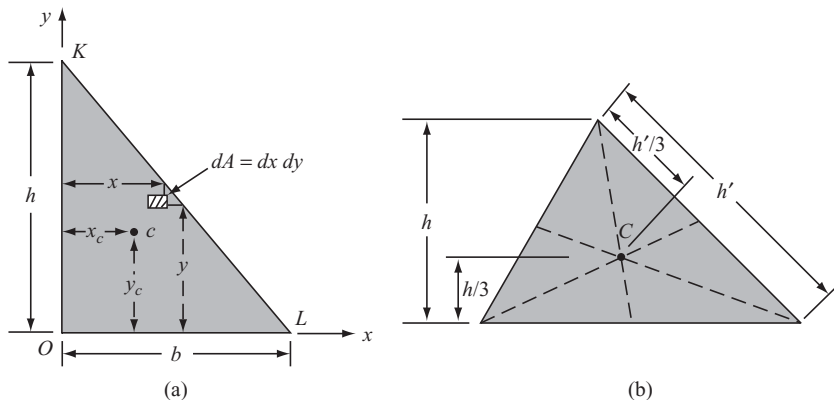


Fig. Ex. 4.3 (Solution)

introducing $dA = dx dy = dy dx$
and recognising the equation of the straight line KL as

or

$$y = -\frac{h}{b}x + h$$

$$x = -\frac{b}{h}y + b$$

and that

$$A = \frac{1}{2}bh$$

$$\begin{aligned} x_c &= \int_0^h \int_0^{\left(-\frac{b}{h}y+b\right)} x \, dx \, dy / A \\ &= \left[\int_0^h \left(-\frac{b}{h}y+b\right)^2 / 2 \, dy \right] / A \\ &= \left[\frac{b^2}{h^2} \frac{y^3}{3} + b^2 y - \frac{b^2}{h} y^2 \right]_0^h / bh \\ &= \frac{b}{3} \end{aligned}$$

Similarly,

$$\begin{aligned} y_c &= \int_0^b \int_0^{\left(-\frac{h}{b}x+h\right)} y \, dy \, dx / A \\ &= \left[\int_0^b \left(-\frac{h}{b}x+h\right)^2 / 2 \, dx \right] / A \\ &= \left[\frac{h^2}{b^2} \frac{x^3}{3} + h^2 x - \frac{h^2}{b} x^2 \right]_0^b / A \\ &= \frac{h}{3} \end{aligned}$$

The centroid of a right-angled triangle is, therefore, located at one-third the distance along the base and one-third the distance along the height. It may be noted that, in general, for any triangle, the centroid is located at one-third the height of the triangle from the base. *Obviously, any side can be taken as the base and the rule of one-third height applies to any orientation.* The centroid of a triangle can, therefore, be located easily by this consideration.

It is also interesting to note that the centroid of a triangle is coincident with the point of intersection of its medians, as shown in Fig. Ex. 4.3(b) (Solution).

Example 4.4

Locate the centroid of the area of a circular sector. Hence, obtain the coordinates of the centroid of a quarter circle and a semi-circle with reference to a set of axes at the centre of the circle.

Solution Consider the sector of a circle as shown in Fig. Ex. 4.4 (Solution) with reference to the coordinate axes at the centre of the circle drawn symmetrically with respect to the sector. Symmetry about the x -axis ensures that the centroid lies on the x -axis or

$$y_c = 0$$

In order to determine x_c , consider an area element

$$dA = \frac{1}{2} R \cdot R d\theta = \frac{1}{2} R^2 d\theta$$

at an angle θ from the x -axis. The centroid of the elementary area must be at $2/3 R$ from the centre such that

$$x = \frac{2}{3} R \cos \theta$$

By the definition of centroid,

$$\begin{aligned} x_c &= \left(\int x dA \right) / \int dA \\ &= \left[\int_{-\alpha}^{\alpha} \frac{2}{3} R \cos \theta \cdot \frac{1}{2} R^2 d\theta \right] / \left[\int_{-\alpha}^{\alpha} \frac{1}{2} R^2 d\theta \right] \\ &= \frac{2}{3} R \frac{|\sin \theta|_{-\alpha}^{\alpha}}{|\theta|_{-\alpha}^{\alpha}} \\ &= \frac{2}{3} R \frac{\sin \alpha}{\alpha} \end{aligned}$$

For a quarter-circle,

$$2\alpha = \frac{\pi}{2}; \quad \alpha = \frac{\pi}{4}$$

Hence, the limits of integration become

$$-\pi/4 \text{ to } +\pi/4$$

The x -coordinate of the centroid is

$$x_c = \frac{2}{3} R \frac{\sin \pi/4}{\pi/4} = \sqrt{2} \frac{4R}{3\pi}$$

Generally, the centroid of a quarter circle is referred to the x' and y' axes:

$$x'_c = y'_c = x_c \cos \frac{\pi}{4} = \frac{4R}{3\pi} = 0.424 R$$

For a semicircle,

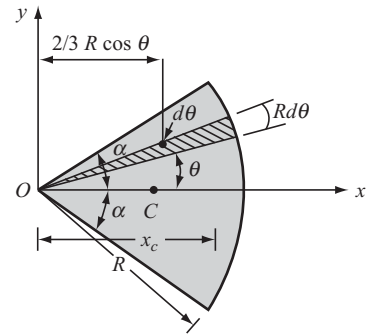
$$2\alpha = \pi; \quad \alpha = \pi/2$$

and the limits of integration become

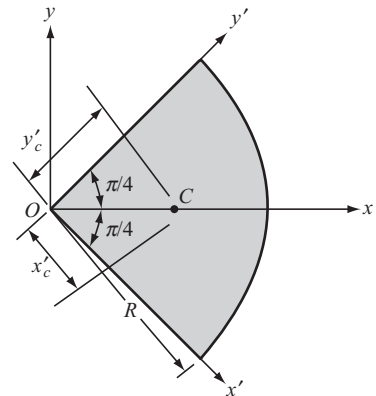
$$-\pi/2 \text{ to } +\pi/2$$

The x coordinate of the centroid is

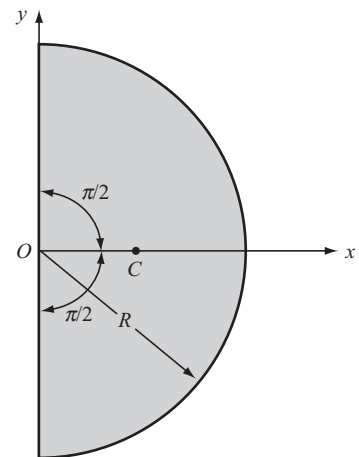
$$x_c = \frac{2}{3} R \frac{\sin \pi/2}{\pi/2} = \frac{4R}{3\pi} = 0.424 R$$



(a) Sector of a circle



(b) Quarter-circle



(c) Semi-circle

Fig. Ex. 4.4 (Solution)

Example 4.5

Two non-viscous, incompressible and immiscible liquids of densities ρ and 1.5ρ are poured into the two limbs of a circular tube of radius R and small cross-section kept fixed in a vertical plane as shown in Fig. Ex. 4.5. Each liquid occupies one-fourth the circumference of the tube.

Find the angle θ that the radius vector to the interface makes with the vertical in equilibrium position.

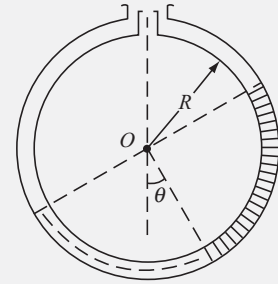


Fig. Ex. 4.5

Solution Centroids C_1 and C_2 of the liquids in the circular tube are located by

$$OC_1 = \frac{2\sqrt{2}}{\pi} R = OC_2$$

Masses of the two liquids are $\rho a \pi R/2$ and $1.5\rho a \pi R/2$ respectively.

The centroid of the composite column BD lies on the vertical line OA . Taking moments, for equilibrium,

$$\begin{aligned} \rho a \frac{\pi R}{2} \cdot g \times \frac{2\sqrt{2}}{\pi} R \sin(45 + \theta) \\ = 1.5 \rho a \frac{\pi R}{2} \cdot g \times \frac{2\sqrt{2}}{\pi} R \sin(45 - \theta) \end{aligned}$$

whence, $\sin(45 + \theta) = 1.5 \sin(45 - \theta)$

$$\sin 45 \cos \theta + \cos 45 \sin \theta = 1.5 \sin 45 \cos \theta - \cos 45 \sin \theta$$

$$\frac{1}{2} \sin 45 \cos \theta - \frac{3}{2} \cos 45 \sin \theta = 0$$

$$\cos \theta - 3 \sin \theta = 0; \quad \tan \theta = \frac{1}{3}; \quad \theta = 18.43^\circ$$

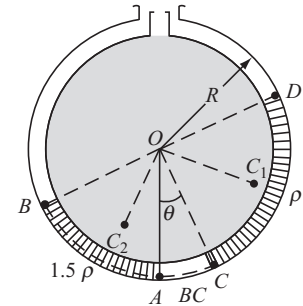


Fig. Ex. 4.5 (Solution)

Example 4.6

Determine the centroid of the area bounded by the x -axis, the line $x = a$ and the parabola $y^2 = kx$ as shown in Fig. Ex. 4.6.

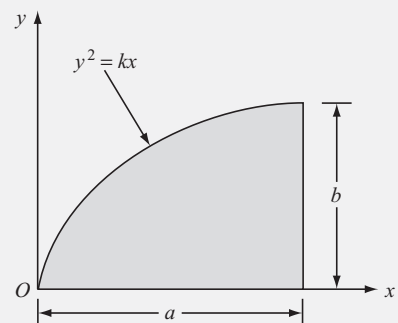


Fig. Ex. 4.6

Solution We may choose to consider a differential area element

$$dA = dx dy$$

located by x, y coordinates, as in the case of a triangle or we may prefer to deal with strip elements, as in the case a circular sector (Fig. Ex. 4.6 (a) and (b)) (Solution). In this case, let us do it both ways and see the equivalence of the procedures.

Method I

Choosing a differential area element

$$dA = dx dy$$

at a location (x, y) , the centroid is given by

$$x_c = \left(\int x dA \right) / A, \quad y_c = \left(\int y dA \right) / A$$

Noting that
$$A = \int dA = \int_0^b \int_{y^2/k}^a dx dy$$

$$= \int_0^b (a - y^2/k) dy$$

$$= \left[ay - \frac{y^3}{3k} \right]_0^b = ab - \frac{b^3}{3k}$$

$$k = \frac{y^2}{x} = \frac{b^2}{a}$$

and

therefore,

$$A = ab - \frac{b^3 a}{3b^2} = \frac{2ab}{3}$$

$$x_c = \left(\int_0^b \int_{y^2/k}^a x dx dy \right) / A$$

$$= \left(\int_0^b (a^2 - y^4/k^2) / 2 dy \right) / A$$

$$= \frac{1}{2} \left(a^2 b - \frac{b^5 a^2}{5b^2} \right) / A$$

$$= \frac{2}{5} a^2 b \frac{3}{2ab} = \frac{3}{5} a$$

$$y_c = \int_0^a \int_{y=0}^{\sqrt{kx}} y dy dx / A$$

$$= \left(\int_0^a \frac{kx}{2} dx \right) / A = \left[\frac{kx^2}{4} \right]_0^a / A$$

$$= \frac{b^2}{a} \cdot \frac{a^2}{4} \frac{3}{2ab} = \frac{3}{8} b$$

The centroid is, hence, at $\left(\frac{3}{5} a, \frac{3}{8} b \right)$.

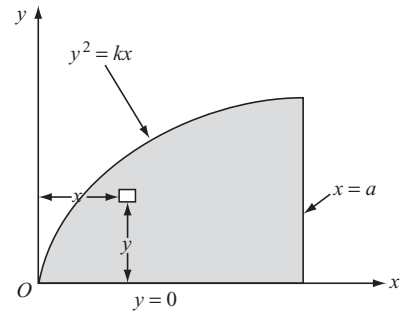


Fig. Ex. 4.6(a) (Solution)

Method II

Choose an elementary strip of area

$$dA = y \, dx$$

at a distance x from the y -axis, the centroid of the strip being at $y/2$ from the base, the centroidal ordinate for the given area is

$$\begin{aligned} y_c &= \left(\int y/2 \cdot y \, dx \right) / A \\ &= \left(\int_0^a \frac{kx}{2} \, dx \right) / A \\ &= \frac{ka^2}{4} / A \\ &= \frac{b^2}{a} \cdot \frac{a^2}{4} \cdot \frac{3}{2ab} = \frac{3}{8} b \end{aligned}$$

Similarly, by taking a strip of area

$$dA = (a - x) \, dy$$

with its centroid at

$$x + \frac{a - x}{2} = \frac{x + a}{2}$$

from the y -axis, the abscissa is given by

$$\begin{aligned} x_c &= \left(\int_0^b \frac{x + a}{2} \cdot (a - x) \, dy \right) / A \\ &= \left(\int_0^b \frac{1}{2} \left(a^2 - \frac{y^4}{k^2} \right) \, dy \right) / A \\ &= \frac{1}{2} \left[a^2 y - \frac{y^5 \cdot a^2}{5b^4} \right]_0^b / A \\ &= \frac{2}{5} a^2 b \cdot \frac{3}{2ab} = \frac{3}{5} a. \end{aligned}$$

The centroid is again observed to be at the point $\left(\frac{3}{5} a, \frac{3}{8} b \right)$

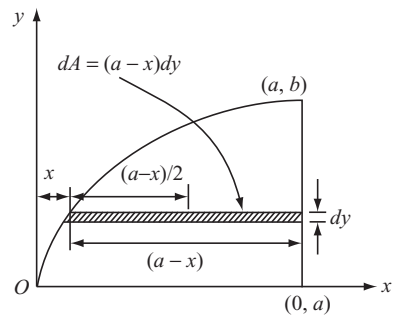
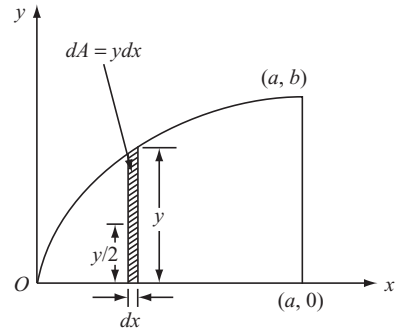


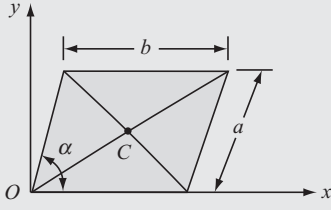

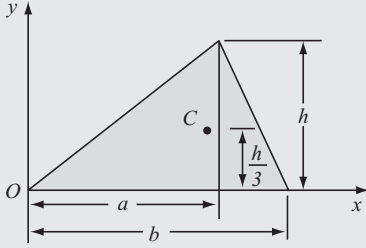
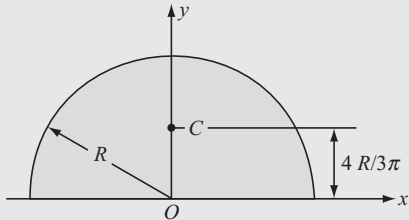
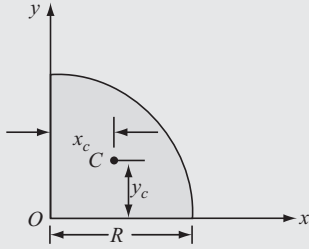
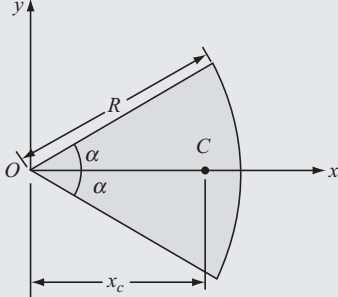
Fig. Ex. 4.6(b) (Solution)

Table 4.2 Centroids of Geometrical Shapes

| Description | Shape | Area | x_c | y_c |
|---------------------------|-------|-------|-------|-------|
| Rectangle | | bh | $b/2$ | $h/2$ |
| Square ($h = b = a$) | | a^2 | $a/2$ | $a/2$ |

(Continued)

Table 4.2 (Continued)

| Description | Shape | Area | x_c | y_c |
|-----------------------------------|---|---------------------|------------------------------------|----------------------------|
| Parallelogram |  | $ab \sin \alpha$ | $\frac{b + a \cos \alpha}{2}$ | $\frac{a \sin \alpha}{2}$ |
| Rectangle ($\alpha = \pi/2$) |  | ab | $b/2$ | $a/2$ |
| Triangle |  | $\frac{1}{2}bh$ | $1/3 (a + b)$ | $h/3$ |
| Semi-circle |  | $\frac{\pi R^2}{2}$ | 0 | $\frac{4R}{3\pi} = 0.424R$ |
| Quarter circle |  | $\frac{\pi R^2}{4}$ | $\frac{4R}{3\pi} = 0.424R$ | $\frac{4R}{3\pi} = 0.424R$ |
| Sector of a circle |  | $R^2 \alpha$ | $\frac{2 R \sin \alpha}{3 \alpha}$ | 0 |

(Continued)

Table 4.2 (Continued)

| Description | Shape | Area | x_c | y_c |
|---|-------|--------------------|--------------------|--------------------------------------|
| Quarter ellipse | | $\frac{\pi ab}{4}$ | $\frac{4a}{3\pi}$ | $\frac{4b}{3\pi}$ |
| Quarter parabola | | $\frac{2ab}{3}$ | $\frac{3}{5}a$ | $\frac{3}{8}b$ |
| General spandrel Parabolic spandrel ($n = 2$) | | $\frac{ab}{n+1}$ | $\frac{n+1}{n+2}a$ | $\frac{n+1}{2n+1} \cdot \frac{b}{2}$ |
| | | $\frac{ab}{3}$ | $\frac{3a}{4}$ | $\frac{3b}{4}$ |

Example 4.7

Locate the centroid of the given composite area with a hole in it as shown in Fig. Ex. 4.7.

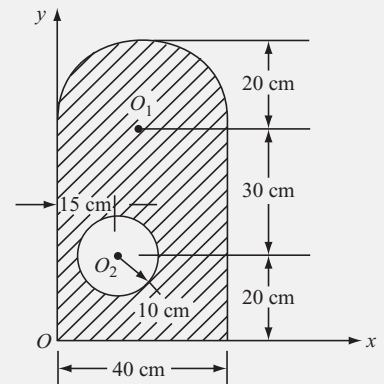


Fig. Ex. 4.7

Solution The given area can be considered to comprise a rectangle 40 cm × 50 cm plus a semicircle of 20 cm radius minus a circle of 10 cm radius. With respect to the x - y axes with the origin at O , we proceed by preparing a table:

| Component | Area (cm ²) | x_c (cm) | y_c (cm) | $x_c A$ (cm ³) | $y_c A$ (cm ³) |
|--------------------------------------|-------------------------------------|------------|------------------------------------|----------------------------|----------------------------|
| Rectangle | 2000 | 20 | 25 | 40 000 | 50 000 |
| Semicircle with centre O_1 | $\frac{\pi \times 20^2}{2} = 628.3$ | 20 | $50 + 0.424 \times 20$ $= 58.5$ | 12 566 | 36 756 |
| Circle with centre O_2 (void area) | $-\pi \times 10^2 = -314.2$ | 15 | 20 | -4712 | -6284 |
| Total | ΣA | — | — | 8 | 8 |
| | | | | $\Sigma x_c A$ | $\Sigma y_c A$ |

Employing the relations for the centroidal point,

$$x_c = \frac{\Sigma x_c A}{\Sigma A} \quad \text{and} \quad y_c = \frac{\Sigma y_c A}{\Sigma A}$$

we obtain,

$$x_c = \frac{47\,854}{2314.1} = 20.68 \text{ cm}$$

$$y_c = \frac{80\,472}{2314.1} = 34.77 \text{ cm}$$

4.7 THEOREMS OF PAPPUS-GULDINUS

There are two very important theorems initially due to the Greek geometer Pappus and later restated by the Swiss mathematician Guldinus which deal with the surfaces and volumes of revolution.

Theorem 1

The area of a surface of revolution is equal to the length of the generating curve times the distance travelled by the centroid of the generating curve while the surface is generated.

Proof of Theorem 1

A variety of generating curves can be employed to generate the surfaces of revolution as shown in Fig. 4.5.

Consider a generating curve, say the general spandrel to generate a surface of revolution as shown in Fig. 4.5(b). An elementary length dl of the curve of length l generates a surface area

$$dA = 2\pi y \times dl$$

The total surface area generated by the given curve is

$$A = \int dA = \int_0^l 2\pi y \, dl$$

$$\begin{aligned}
 &= 2\pi \int_0^l y \, dl \\
 &= 2\pi y_c \cdot l \qquad (4.15)
 \end{aligned}$$

Since $y_c l = \int y \, dl$

by the definition of the centroid of a curve. Hence, the area of the surface generated is given by the product of $2\pi y_c$ and the length of the surface l ; as if the entire length of the generating curve were concentrated at the radius y_c . In other words, the area of the surface generated equals the area of a cylindrical surface of radius y_c and length l .

Theorem 2

The volume of a body of revolution is equal to the generating area times the distance travelled by the centroid of the area while the body is generated.

Proof of Theorem 2 and its scope

Consider a surface area A bounded by a curve, $y=0$ and $x=a$ lines as shown in Fig. 4.6. An area element dA of the surface, when revolved about the x -axis generates a volume

$$dV = 2\pi y \times dA$$

The entire volume generated by the total area is

$$\begin{aligned}
 V &= \int dV = \int 2\pi y \, dA \\
 &= 2\pi \int y \, dA = 2\pi y_c A \qquad (4.16)
 \end{aligned}$$

Since $y_c A = \int y \, dA$

by the definition of the centroid of an area. Hence the volume of the body generated is given by the product of $2\pi y_c$ and the magnitude of area A of the surface as if the entire area of the generating surface was concentrated at its centroid at radius y_c . In other words, the volume of the body generated equals the volume of a right circular cylinder of radius y_c and cross-sectional area A .

The theorems of Pappus–Guldinus provide simple means of relating the areas and volumes of

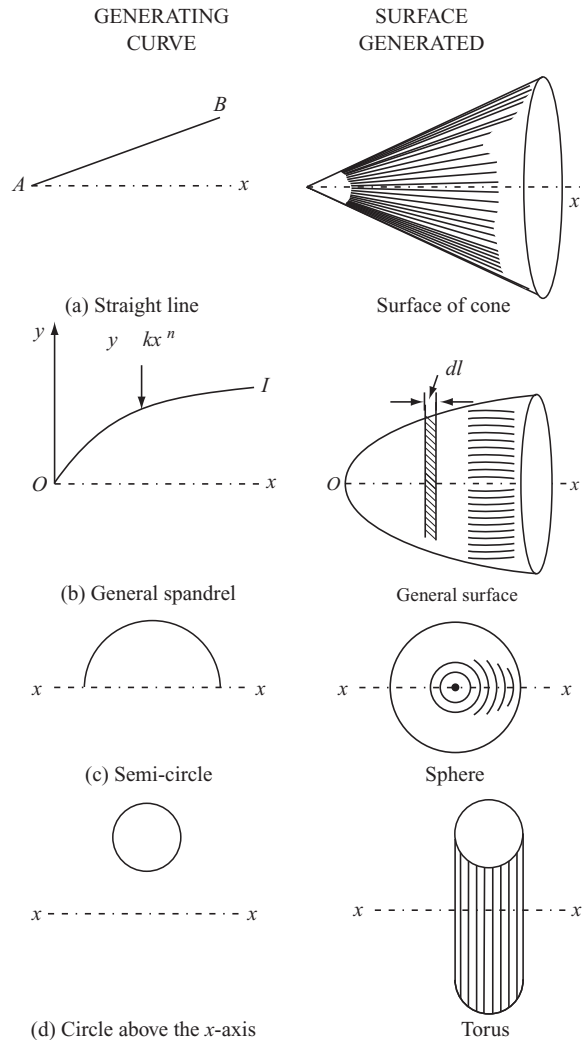


Fig. 4.5 Surfaces of revolution

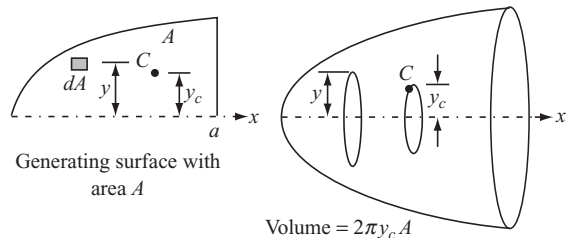


Fig. 4.6 Volume generated by a surface

the surfaces and bodies of revolution to the lengths and areas of the generating curves and surfaces. *These relationships can be used with advantage both ways, i.e., to determine the areas and volumes of the surfaces and bodies of revolution from the given curves and surfaces to generate these or to locate the centroids of the curves and surfaces from the knowledge of the areas and volumes of the surfaces and bodies generated.*

Example 4.8

A semicircle is rotated about its diameter to generate a sphere. Calculate the volume of a sphere of radius R .

Solution The theorems of Pappus-Guldinus can be used to great advantage in this case. The centroid of the semicircle is $4R/3\pi$ above the diametral axis. The area of the semicircle is $\pi R^2/2$. According to the Pappus-Guldinus, the volume of the body of revolution generated should be same as that which would be obtained if the entire area $\pi R^2/2$ were concentrated at a radius $4R/3\pi$. The volume of the sphere is, therefore,

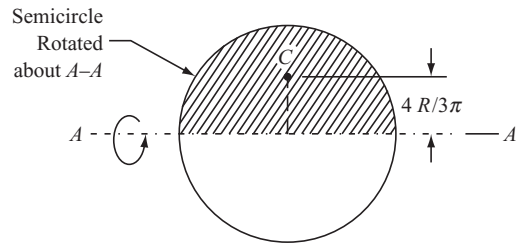


Fig. Ex. 4.8 (Solution)

$$2\pi \frac{4R}{3\pi} \times \frac{\pi R^2}{2} = \frac{4}{3} \pi R^3$$

Example 4.9

Determine the centroid of a quadrant of a circle using the theorems of Pappus and Guldinus.

Solution It is an interesting application of the theorem to locate the centroid of an area if the volume generated by revolving the area over an axis is known. In this case, the volume generated by revolving the quadrant of a circle about either the x -axis or the y -axis is that of a hemisphere, i.e.,

$$\frac{2}{3} \pi R^3$$

Since the area of the quadrant of a circle is

$$\frac{\pi R^2}{4}$$

The distance travelled by the centroid of the quadrant of a circle in generating the hemispherical volume must be

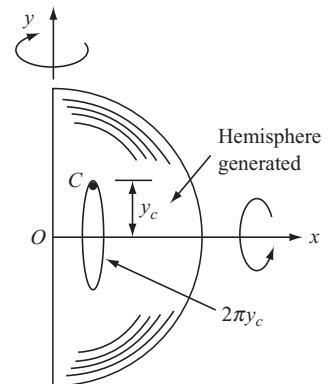


Fig. Ex. 4.9 (Solution)

$$\frac{2}{3} \pi R^3 \bigg/ \frac{\pi R^2}{4} = \frac{8R}{3}$$

Equating it to $2\pi y_c$ for rotation about the x -axis

$$y_c = \frac{4R}{3\pi}$$

Similarly, the x -coordinate of the centroid may be determined by rotating the quadrant about the y axis

$$x_c = \frac{4R}{3\pi}$$

4.8 CENTROID OF A VOLUME

The centroid of a volume, by definition, is given by Eq. (4.5), i.e.,

$$c = (\int dV)/V$$

whence the coordinates of the centroid are obtained as

$$x_c = (\int x dV)/V$$

$$y_c = (\int y dV)/V \tag{4.17}$$

$$z_c = (\int z dV)/V$$

as are shown for the volume of a *matka* in Fig. 4.7.

The *centroid of a volume* is a point about which the first moment of the volume equals the summation of the first moment of the distributed volume elements. It is a unique point for the volume and its location is the body is independent of the choice of the origin or the orientation of the axes.

When a volume V possesses a plane of symmetry, the centroid of the volume must lie in that plane. When a volume possesses two planes of symmetry, the centroid must lie on the line of intersection of the two planes. When a volume has three or more planes of symmetry, the centroid must be located at the point of intersection of these planes. The centroid of the volume of a sphere, a cube, an ellipsoid or a rectangular parallelepiped can be located readily by considering their triple or multiple symmetry.

The *centroid of a volume of revolution* must lie on the axis of symmetry but the distance of the centroid from the apex must be determined by integration. It is cautioned that the centroid of a volume of revolution may not coincide with the centroid of its cross-section containing the axis. A summary of the centroids of some volumes of revolution is given in Table 4.3.

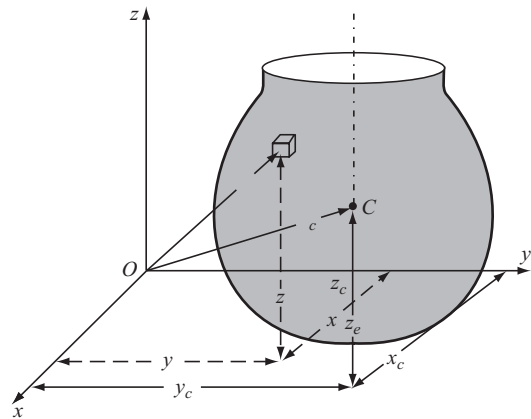
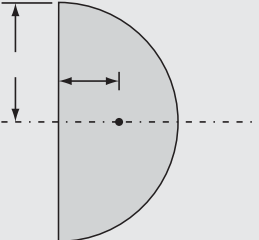
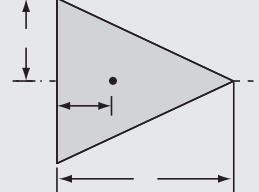
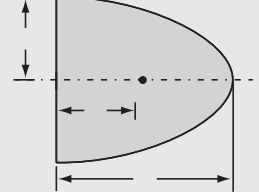
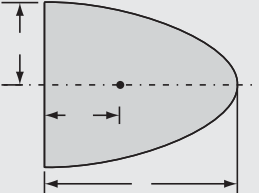


Fig. 4.7 Centroids of volumes of revolution

Table 4.3 Centroids of Volumes of Revolution

| | | | |
|------------------------------|---|---------------|--------------------|
| Hemisphere |  | $\frac{3}{8}$ | $\frac{2}{3}\pi^3$ |
| Right circular cone |  | $\frac{1}{4}$ | $\frac{1}{3}\pi^2$ |
| Semi-ellipsoid of revolution |  | $\frac{3}{8}$ | $\frac{2}{3}\pi^2$ |
| Paraboloid of revolution |  | $\frac{1}{3}$ | $\frac{1}{2}\pi^2$ |

Example 4.10 Locate the centroid of the volume of a right circular cone of base radius r and height h .

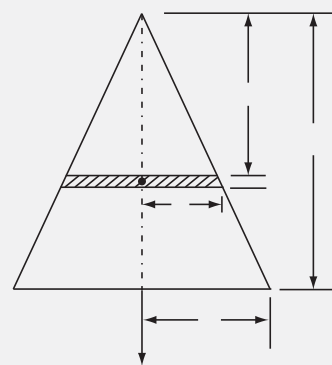


Fig. Ex. 4.10

Solution A right circular cone possesses an axis of symmetry; the centroid must be located on this axis. Let the origin be at O , the apex and x -axis along the axis of symmetry as shown in Fig. Ex. 4.10.

Consider an elementary disc of radius r and width dx at a distance x from the origin. The volume of the elementary disc is

$$dv = \pi r^2 dx$$

and its centroid is at a distance x from O .

For the entire volume of the cone, by definition,

$$\begin{aligned} x_c &= \left(\int x dV \right) / V \\ &= \left(\int_0^h x \pi r^2 dx \right) / V \end{aligned}$$

Substituting $r = \frac{R}{h}x$ and $V = \frac{1}{3}\pi R^2 h$

$$\begin{aligned} x_c &= \left(\int_0^h \frac{\pi R^2}{h^2} x^3 dx \right) / \left(\frac{1}{3} \pi R^2 h \right) \\ &= \frac{\pi R^2 h^2}{4} \times \frac{3}{\pi R^2 h} = \frac{3}{4}h \end{aligned}$$

The centroid of a right circular cone is, therefore, located at quarter height from its base.

Example 4.11

A right circular cone of base radius R and height h is attached to a hemisphere of radius R as shown in Fig. Ex. 4.11. Determine the ratio h/R for which the centroid of the composite volume is located in the plane between the cone and the hemisphere.

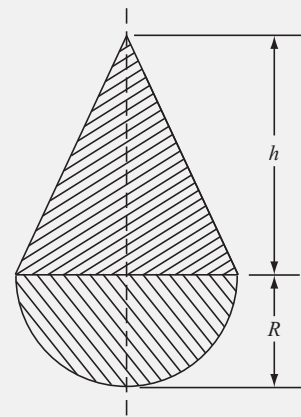


Fig. Ex. 4.11

Solution With reference to the origin O at the apex of the cone, as shown in Fig. Ex. 4.11 (Solution) the centroid of the cone alone lies at

$$x_{c1} = \frac{3}{4}h$$

and the centroid of the semicircle lies at

$$x_{c2} = h + \frac{3}{8}R$$

Recollecting that the volume of the cone is

$$V_1 = \frac{1}{3} R^2 h$$

and that the volume of the hemisphere is

$$V_2 = \frac{2}{3} \pi R^3$$

the centroid of the composite volume must be located such that

$$\begin{aligned} x_c(V_1 + V_2) &= x_{c1}V_1 + x_{c2}V_2 \\ &= \frac{3}{4} h \frac{1}{3} \pi R^2 h + \left(h + \frac{3}{8} R \right) \frac{2}{3} \pi R^3 \end{aligned}$$

In order that the centroid lies in the plane between the cone and the hemisphere,

$$x_c = h$$

Then,

$$h \left(\frac{1}{3} \pi R^2 h + \frac{2}{3} \pi R^3 \right) = \frac{1}{4} \pi R^2 h^2 + \frac{2}{3} \pi R^3 h + \frac{1}{4} \pi R^4$$

or
$$\frac{h^2}{3} + \frac{2}{3} Rh = \frac{h^2}{4} + \frac{2}{3} Rh + \frac{R^2}{4}$$

or
$$h^2 = 3R^2$$

and finally
$$h/R = \sqrt{3}$$

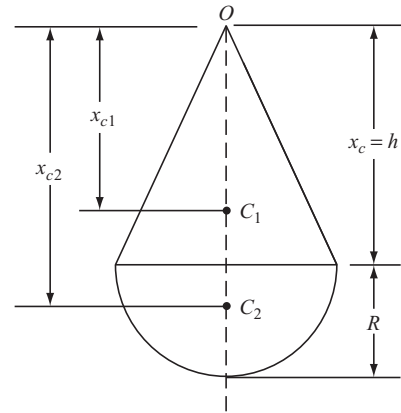


Fig. Ex. 4.11 (Solution)

4.9 CENTRE OF MASS

The *centre of mass* for a body of mass m is a point where the entire mass m can be assumed to be concentrated to give the same first moment as that obtained by considering the element of mass continuously distributed over the body:

$$c = (\mathbf{r} dm)/m$$

In terms of an arbitrarily chosen set of x - y - z axes, as in Fig. 4.8(a),

$$\begin{aligned} x_c &= (\int x dm)/m \\ y_c &= (\int y dm)/m \\ z_c &= (\int z dm)/m \end{aligned} \tag{4.18}$$

The centre of mass is an important point in the study of dynamics. If the external forces acting on a body pass through its centre of mass then the body will behave, for all practical purposes, as if it were, a point mass concentrated at the centre of mass.

For a body with constant mass density ρ , also known as homogeneous body,

$$dm = \rho dv$$

and

$$m = \rho v$$

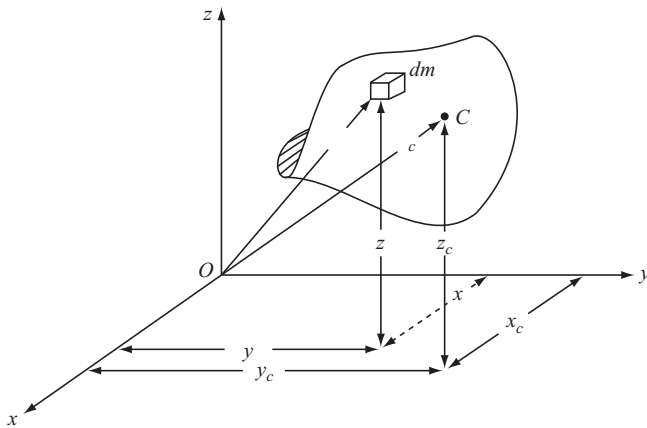


Fig. 4.8(a) Centre of mass

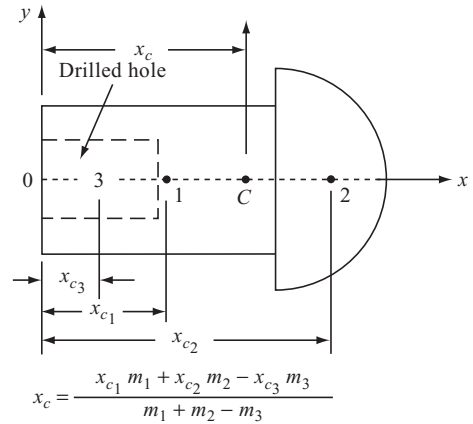


Fig. 4.8(b) Centre of mass of a composite body

hence

$$x_c = (\int x \rho \, dv) / \rho v = (\int x \, dv) / v$$

Similarly,

$$y_c = (\int y \, dv) / v \quad \text{and} \quad z_c = (\int z \, dv) / v \quad (4.19)$$

which means that the centre of mass is coincident with the centroid of volume. It is indeed the variation in density which makes the centre of mass different from the centroid of volume for a body.

If the distribution of the mass of a body is symmetrical about an axis, the centre of mass must lie on that axis. If there are more than one axes of symmetry, the point of intersection of such axes corresponds to the centre of mass. For a body of revolution of uniform mass density, the centroid must lie on the axis of symmetry.

The centre of mass for a composite mass consisting of masses m_1, m_2, m_3, \dots with the mass centres at $(x_{c1}, y_{c1}, z_{c1}), (x_{c2}, y_{c2}, z_{c2}), \dots$, respectively is located by

$$\begin{aligned} x_c &= \frac{x_{c1} m_1 + x_{c2} m_2 + \dots}{m_1 + m_2 + m_3 + \dots} \\ &= \frac{\sum x_c m}{\sum m} \end{aligned} \quad (4.20)$$

$$\begin{aligned} y_c &= \frac{y_{c1} m_1 + y_{c2} m_2 + \dots}{m_1 + m_2 + m_3 + \dots} \\ &= \frac{\sum y_c m}{\sum m} \end{aligned} \quad (4.21)$$

$$\begin{aligned} z_c &= \frac{z_{c1} m_1 + z_{c2} m_2 + \dots}{m_1 + m_2 + m_3 + \dots} \\ &= \frac{\sum z_c m}{\sum m} \end{aligned} \quad (4.22)$$

Care must be taken, however, to ensure the proper sign of coordinate distances and masses of the component bodies. For example, the centre of mass of a composite body shown in Fig. 4.8(b) is determined by selecting the origin at O from where all x -distances are positive but the void mass 3 is considered negative because the mass 1 refers to the undrilled solid body.

Example 4.12 The density at any point of a slender rod varies with the first power of the distance of the point from one end of the rod. Locate the mass centre.

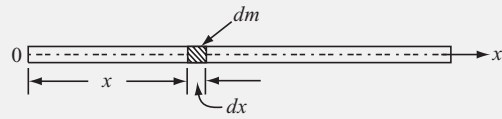


Fig. Ex. 4.12

Solution For the slender rod of length l shown in Fig. Ex. 4.12, the density at a distance x from the left end, say, is

$$\rho = kx$$

where ρ , the density is taken as the mass per unit length of the rod.

The mass dm of an element located at a distance x from the origin is, therefore,

$$dm = \rho dx = kx dx$$

From the definition of the centre of mass,

$$\begin{aligned} x_c &= \left(\int x dm \right) / m = \left(\int x dm \right) / \int dm \\ &= \left(\int_0^l kx^2 dx \right) / \left(\int_0^l kx dx \right) \\ &= \frac{kx^3}{3} \cdot \frac{2}{kx^2} \Big|_0^l = \frac{2}{3}l \end{aligned}$$

The centre of mass for this rod is located at two-third of the length from the end chosen as a reference. This is obviously different from the centroid of volume of the slender rod which is at its mid-point.

Example 4.13 A thin wire of homogeneous material is bent to form an isosceles triangle as shown in Fig. Ex. 4.13. Determine the base angle α for which the centre of mass of the wire coincides with the centroid of the area enclosed by the wire.

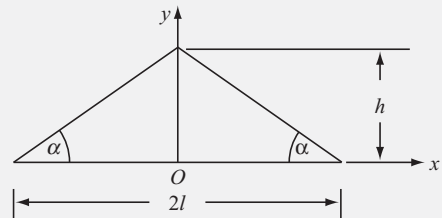


Fig. Ex. 4.13

Solution The centroid of the area of a triangle is at one-third the height of the triangle from the base. Further, by symmetry about the y -axis, the centroid must lie on it. Therefore,

$$x_c = 0 \quad \text{and} \quad y_c = \frac{h}{3}$$

The centre of mass of the wires of the triangle must also lie on the y -axis and the y -coordinate can be obtained by considering it as a composite mass:

$$Y_c = \frac{\rho h \sqrt{l^2 + h^2}}{2\rho(l + \sqrt{l^2 + h^2})}$$

| Component | Length | Mass | y_c | $y_c \times \text{Mass}$ |
|------------------------|--------------------|-------------------------------|---------------|------------------------------------|
| Base | $2l$ | $2\rho l$ | 0 | 0 |
| Each side | $\sqrt{l^2 + h^2}$ | $\rho\sqrt{l^2 + h^2}$ | $\frac{h}{2}$ | $\frac{\rho h}{2}\sqrt{l^2 + h^2}$ |
| Total for the triangle | — | $2\rho(l + \sqrt{l^2 + h^2})$ | Y_c | $\rho h\sqrt{l^2 + h^2}$ |

Equating it to $\frac{h}{3}$ and cancelling ρ ,

$$h\sqrt{l^2 + h^2} = \frac{2h}{3}l + \frac{2h}{3}\sqrt{l^2 + h^2}$$

or
$$h\sqrt{l^2 + h^2} = 2hl$$

or
$$l^2 + h^2 = 4l^2$$

and
$$h = \sqrt{3}l$$

whence
$$\alpha = \tan^{-1} \frac{h}{l} = \tan^{-1}(\sqrt{3}) = 60^\circ$$

Example 4.14

A hollow cylindrical component 15 cm outer diameter and 5 cm inner diameter is 30 cm long as shown in Fig. Ex. 4.14. Its mass density varies from 4000 kg/m³ at the left end to 10 000 kg/m³ at the right end. Locate the centre of mass of the component.

- (a) by assuming a linear variation of density over the length of the component.
- (b) by assuming that it consists of three components A, B and C of mass densities 4000 kg/m³, 7000 kg/m³ and 10 000 kg/m³.

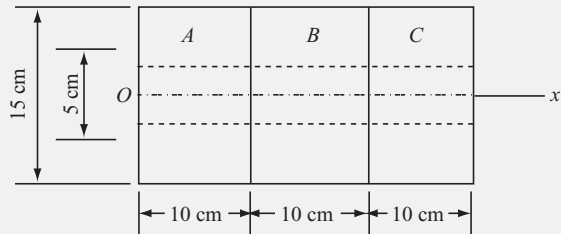


Fig. Ex. 4.14

Solution By virtue of axisymmetry of the component about x -axis as shown in Fig. Ex. 4.14, the mass centre must be located on the x -axis.

The area of the cross-section is

$$\alpha = \frac{\pi(0.15^2 - 0.05^2)}{4}$$

- (a) By definition of the centre of mass,

$$\begin{aligned} x_c &= \left(\int x \, dm \right) / \int dm \\ &= \left(\int x\rho \, dV \right) / \int \rho \, dV \end{aligned}$$

$$= \left(\int x \rho \, dx \right) / \int \rho \, dx$$

For linear variation of density,

$$\begin{aligned} f &= 4000 + (10\,000 - 4000) \times \frac{x}{0.3} \\ &= 4000 + 20\,000x \end{aligned}$$

where x is in m .

With reference to the origin at the left end,

$$\begin{aligned} x_c &= \frac{\int_0^{0.3} (4000 + 20\,000x) x \, dx}{\int_0^{0.3} (4000 + 20\,000x) \, dx} \\ &= \frac{\left[4000x^2/2 + 20\,000x^3/3 \right]_0^{0.3}}{\left[4000x + 20\,000x^2/2 \right]_0^{0.3}} \\ &= \frac{360}{2100} = 0.1715 \, m = 17.15 \, \text{cm from the left end.} \end{aligned}$$

(b) For the components of the composite mass,

| Component | Mass | x_c | $x_c \times \text{Mass}$ |
|-----------|-------------------------------|-------|--------------------------|
| A | $4000 \times a \times 0.1$ | 0.05 | $20a$ |
| B | $7000 \times a \times 0.1$ | 0.15 | $105a$ |
| C | $10\,000 \times a \times 0.1$ | 0.25 | $250a$ |
| Totals | a | | a |

The centre of mass of the composite area is, therefore, located at

$$\begin{aligned} x_c &= \frac{375a}{2100a} = 0.1786 \, m \\ &= 17.86 \, \text{cm from the left end.} \end{aligned}$$

It can be observed that the process of taking the averaged density over the segments yields the same mass, i.e., $2100a$ kg as by assuming a linear variation of the density as expected but the location of the mass centre is different because the first moments of the variable-density mass involve second-order terms.

Example 4.15

A concentric hole of diameter 10 cm is drilled half way through a 20 cm diameter, 30 cm long solid cylinder of brass. The hole is then filled completely with gold and finished flush to make it a complete cylinder again as shown in Fig. Ex. 4.15. Locate the centre of mass of the finished cylinder.

Assume that the density of brass is $8500 \, \text{kg/m}^3$ and density of gold is $19\,500 \, \text{kg/m}^3$.

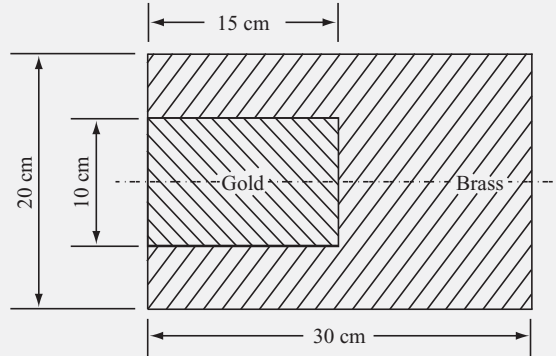


Fig. Ex. 4.15

Solution The composite mass may be visualised as being composed of:
 a solid brass cylinder 20 cm diameter and 30 cm long
 – a solid brass cylinder 10 cm diameter and 15 cm long
 + a solid gold cylinder 10 cm diameter and 15 cm long
 This can also be interpreted as

1. a solid brass cylinder 20 cm diameter and 30 cm long plus
2. a solid cylinder 10 cm diameter and 15 cm long with a density equal to the difference of densities in gold and brass.

Consequently, taking the origin at the left end of the cylinder on the axis, as shown in Fig. Ex. 4.15.

| Component | x_c (cm) | m (kg) | $x_c m$ (kg cm) |
|-----------|------------|---|-----------------|
| 1 | 15 | $\frac{\pi \times 20^2}{4} \times 30 \times \frac{8500}{10^6}$ = 80.1 | 1201.5 |
| 2 | 7.5 | $\frac{\pi \times 10^2}{4} \times 15 \times \frac{(19500 - 8500)}{10^6}$ = 12.96 | 97.2 |
| Total | | | 8 |

The distance of the centre of mass of the finished composite cylinder from the left end of the cylinder on the axis is

$$x_c = \frac{1298.7}{93.06} = 13.96 \text{ cm}$$

4.10 CENTRE OF GRAVITY

The *centre of gravity* for a body of mass m acted upon by a parallel and uniform gravitational force field is a point through which the resultant force due to gravity would act whatever the orientation of the body may be.

For a given orientation of the body, the gravitational force acts vertically downward. The line of action of the resultant force must also be a vertical line and the resultant force must be a summation of the forces acting on the mass elements, i.e.,

$$\begin{aligned} &= \int d\mathbf{F} = -\int g \, dm \, \mathbf{k} \\ &= -g \int dm \, \mathbf{k} = -mg \, \mathbf{k} \end{aligned} \quad (4.23)$$

which is also the total gravitational pull on the body as shown in Fig. 4.9.

The line of action of the resultant gravitational force, i.e., $-mg \, \mathbf{k}$ should be such that the moment by the resultant about any arbitrary origin or about any set of axes must be the same as that exerted by the elemental forces distributed over the mass

$${}_c \times = \int \mathbf{r} \times (-g \, dm \, \mathbf{k})$$

or

$${}_c \times (-mg \, \mathbf{k}) = \int \mathbf{r} \times (-g \, dm \, \mathbf{k})$$

or

$${}_c \times m = \int \mathbf{r} \times dm \, \mathbf{k} \quad (4.24)$$

whence,

$$x_c = \left(\int x \, dm \right) / m = \left(\int x \rho \, dV \right) / m$$

and

$$y_c = \left(\int y \, dm \right) / m = \left(\int y \rho \, dV \right) / m \quad (4.25)$$

If the body is now turned through some angle, say by 90° , about an axis other than the vertical axis, another line of action of the resultant gravitational force can be located and the point of intersection of the two lines of action locates the centre of gravity.

The coordinates of the centre of gravity are the same which locate the centre of mass of a body. It follows that the line of action of the gravitational force on a body of mass m must also pass through its centre of mass.

The centre of gravity can be different from the centre of mass only when the gravitational force field is not parallel and uniform, i.e., if there is a change in the magnitude or the direction of the gravitational force. It can be visualised that the centre of gravity of a large body with the dimensions not negligible in comparison with the radius of the earth or of a body of considerable width where the gravitational force must be taken directed towards the centre of the earth will differ from the centre of mass. For most practical purposes and unless otherwise stated, the centre of gravity and centre of mass are assumed to be identical. It is for this reason that many authors do not distinguish between them.

The centre of gravity of a material body may also become identical with the centroid of volume of the body if the material is homogeneous, i.e., if the density

$$\rho = \text{Constant}$$

Further, for a thin plate of constant thickness and homogeneous material, the centre of gravity may tend to coincide with the centroid of area of the plate. This, then, offers an experimental method of determining the centroid of area of a thin lamina. If the lamina is suspended by a thread from a point at its periphery, the line of suspension of the thread passes through the centroid. If now the lamina is suspended again from another point at its periphery, the new line of suspension of the thread also passes through its centroid. The point of intersection of the two lines locate the centroid as shown in Fig. 4.10. One may, however, suspend the lamina for a third time and ensure the location of the centroid.

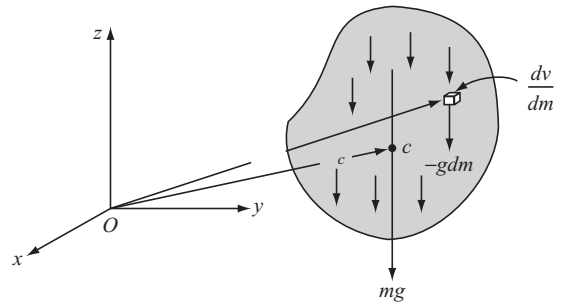


Fig. 4.9 Concept of centre of gravity

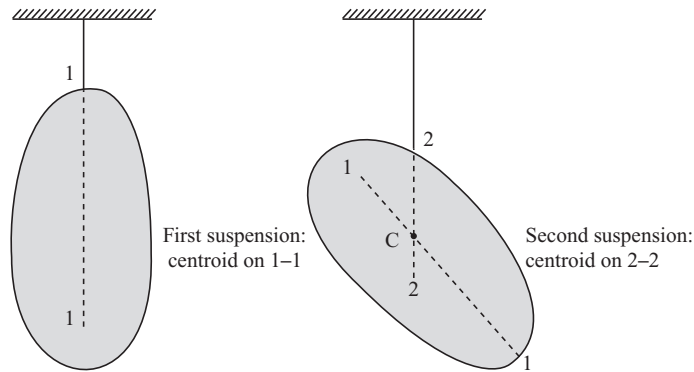


Fig. 4.10 Experimental determination of the centroid for a thin plate

Example 4.16

Locate the centre of gravity of an idealised bullet of 1 cm diameter with a cone in the front and a hemisphere cut from the back as shown in Fig. Ex. 4.16. Assume the material to be homogeneous.

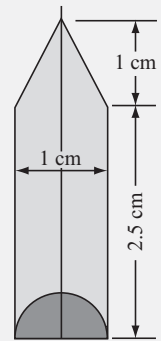


Fig. Ex. 4.16

Solution By virtue of axisymmetry about the x -axis, the centre of gravity which is also the same as the centre of mass or centroid of volume due to homogeneity of the material must lie on the x -axis.

The bullet is idealised to consist of a cone plus a cylinder minus a hemisphere as shown in Fig. Ex. 4.16. The following table is made with reference to O as origin.

The x coordinate of the centroid of the bullet is given by

$$x_c = \frac{3.1258}{1.963} = 1.592 \text{ cm from the base.}$$

| Component | $\frac{\text{Volume}}{(\text{cm}^3)}$ | $\frac{x_c}{(\text{cm})}$ | $\frac{x_c \times \text{Volume}}{(\text{cm}^4)}$ |
|------------|--|--|--|
| Cone | $\frac{\pi}{3}(0.5)^2 \times 1 = 0.262$ | $2.5 + 0.25 = 2.75$ | 0.7205 |
| Cylinder | $\pi \times (0.5)^2 \times 2.5 = 1.963$ | 1.25 | 2.4544 |
| Hemisphere | $-\frac{2}{3}\pi \times (0.5)^3$ $= -0.262$ | $\frac{3 \times 0.5}{8}$ $= 0.1875$ | -0.0491 |
| Total | | | 8 |

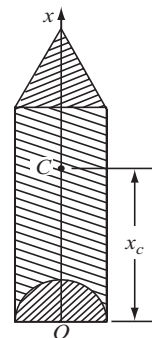


Fig. Ex. 4.16 (Solution)

Example 4.17

An isosceles triangle is to be cut out from one edge of a square piece of thin uniform sheet as shown in Fig. Ex. 4.17 such that the remaining sheet when suspended from the apex P of the cut will remain in equilibrium in any position. Find the area of the triangle cut-out.

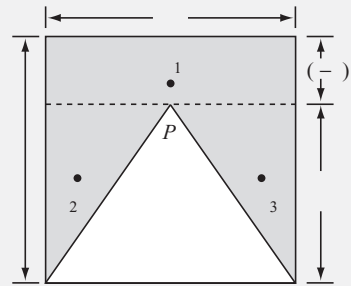


Fig. Ex. 4.17

Solution In order that a body remains in equilibrium in any orientation when suspended from a point, the point must be the centre of gravity. For the piece of thin uniform sheet, the point also qualifies to be the centre of area or the centroid.

The apex point P can, therefore, be located by the definition of the centroid:

$$(6 - 2) \times \frac{(6 - 2)}{2} - 2 \times \frac{1}{2} \times \frac{2}{2} \times \frac{2}{3} = 0$$

or $2^2 - 6 + 3^2 = 0$

whence,
$$\begin{aligned} &= \frac{6 \pm \sqrt{36^2 - 24^2}}{4} \\ &= (3 \pm \sqrt{3}) / 2 \\ &= (3 + \sqrt{3}) / 2 \quad \text{or} \quad (3 - \sqrt{3}) / 2 \end{aligned}$$

Rejecting the first solution which requires to be more than 6,
$$= (3 - \sqrt{3}) / 2 = 0.634$$

The area of the triangle cut out is, therefore,
$$0.634 \times \frac{1}{2} \times 4 = 0.317^2$$

Alternatively, the apex point P can be located by the fact that this is arrived at by removing the cut-out triangle of area 0.317^2 from the given square piece of area 6^2 .

$$6^2 \times \left[\frac{1}{2} - (0.634) \right] \times \frac{2}{3} = 0$$

Using the fact that $0.634 = \frac{3 - \sqrt{3}}{2}$

$$6^2 \times \left[\frac{1}{2} - \left(\frac{3 - \sqrt{3}}{2} \right) \right] \times \frac{4}{3} = 0$$

or
$$\begin{aligned} &8^2 - 12^2 + 3^4 = 0 \\ &= \frac{12^2 \pm \sqrt{144^2 - 96^2}}{16} \\ &= (3 \pm \sqrt{3}) / 4^2 \end{aligned}$$

Again, rejecting the impermissible solution, the answer is
$$= (3 - \sqrt{3}) / 4^2 = 0.317^2$$

Example 4.18

A wooden block of cross-section 10 cm × 10 cm is fixed on top of a semicircular steel cylinder of radius 5 cm as shown in Fig. Ex. 4.18. Determine the maximum height h of the wooden block so that the composite body will be in stable equilibrium at its base. It is given that the density of wood is one-tenth that of steel.

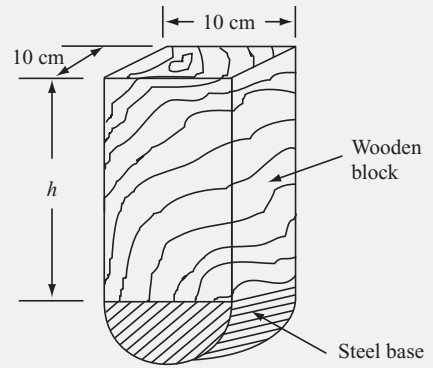


Fig. Ex. 4.18

Solution The composite body should be in stable equilibrium as long as the centre of gravity lies on the semi-cylindrical base. This is so because then a restoring couple will act on the body when it is tipped on either side.

In the parallel and uniform gravitational field of the earth, the centre of mass must be the centre of gravity. We proceed in a tabular form with reference to the mid-point on the base, to determine the centre of mass of the composite body.

| Component | Mass (kg) | y_c (cm) | $y_c \times$ Mass (kg cm) |
|-----------------------|---|--|---|
| Wooden block | $10 \times 10 \times h \times \rho_1$ $= 100h\rho_1$ | $5 + \frac{h}{2}$ | $\left(5 + \frac{h}{2}\right) \times 100 h\rho_1$ |
| Semi-cylindrical base | $\frac{\pi \times 5^2}{2} \times 10\rho_2$ $= 390\rho_2$ | $\frac{4 \times 5}{3 \times \pi}$ $= 2.1$ | $820 \rho_2$ |
| Total | $h\rho + \rho$ | | $h\rho + h\rho + 820 \rho$ |

The centre of mass of the composite body is, therefore, situated at

$$y_c = \frac{500 h\rho_1 + 50 h^2\rho_1 + 820 \rho_2}{100 h\rho_1 + 390 \rho_2}$$

which must be a maximum of 5 cm. Equating it to 5 cm to obtain the maximum height h of the wooden block

$$5 = \frac{500 h\rho_1 + 50 h^2\rho_1 + 820 \rho_2}{100 h\rho_1 + 390 \rho_2}$$

or

$$1950 \rho_2 = 50 h^2\rho_1 + 820 \rho_2$$

whence

$$h = 4.75 \sqrt{\frac{\rho_2}{\rho_1}}$$

Since

$$\rho_2 = 10 \rho_1$$

$$h = 4.75\sqrt{10} = 15 \text{ cm}$$

It may be noted that if the block and base are made of the same material, i.e., $\rho_2 = \rho_1$, then

$$h = 4.75 \text{ cm}$$

but if the block material is lighter than the base material, the length of the block can be made much longer for maintaining stable equilibrium. On the other hand, if the block material is heavier than the base material, the length of the block would be less than 4.75 cm.

Concept Review Questions

State why

- The centroid of a curve, an area or a volume is independent of the choice of the origin or the orientation of the coordinate axes?
- The theorems of Pappus and Guldinus are valid for a complete revolution or a fraction of a revolution of the generating line or area?

Comment on the concept of the first moment of an element about an origin and about a line. Does the first moment have any relationship with the moment of a force?

Compare the location of the centroids of an arc of a circle and a sector of a circle subtending the same angle at the centre of the circle.

Under what conditions do the following coincide?

- Centre of mass and centre of gravity.
- Centroid of volume and centre of mass.
- Centre of gravity and centroid of area.

Would you agree or disagree with the following statements and why?

- The centroid of a body may or may not lie on a material point in the body.
- The centroid of an area symmetrical about two axes must be the point of intersection of these axes.
- The centroid of a parallelogram is located by the point of intersection of its diagonals because the diagonals are the axes of symmetry.
- The vertical line of free suspension of a thin sheet of homogeneous or non-homogeneous material must contain the centre of gravity.

Tutorial Problems

Determine the moment of a semicircular arc about its diameter and hence locate its centroid.

$$\left(\int_0^\pi 2R^2; \frac{2R}{\pi}, 0 \right)$$

Determine the y coordinate of the centroid of the area between the x -axis and the curves $y = \sin x$ between 0 and π . (s. $\pi/8$)

Locate the centroid of the area of a segment of a circle which subtends an angle 2θ at the centre.

$$\left(\int_0^{2R \sin \theta} 0, 2R \sin \theta \cos \theta \right)$$

Locate the centroid of a trapezium with the base b and the parallel sides h_1 and h_2 .

$$\left(\int_0^b x_c = \frac{b(h_1 + 2h_2)}{3(h_1 + h_2)}, y_c = \frac{h_1^2 + h_2^2 + h_1 h_2}{3(h_1 + h_2)} \right)$$

Determine the location of the centroid of the area bounded by the x -axis and the sine curve $y = a \sin \pi x/l$ from $x = 0$ to $x = l$. ($s (l/2, \pi a/8)$)

Find the centroid of the L -section shown in Fig. Prob. 4.6. ($s x_c = 2.6 \text{ cm}, y_c = 3.17 \text{ cm}$)

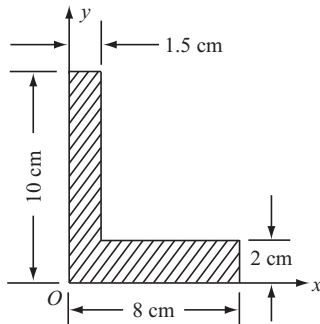


Fig. Prob. 4.6

A flywheel of outside diameter 5 m has a heavy rim of the cross-section shown in Fig. Prob. 4.7. Determine the mass of the rim if the density of the material of the rim is 7000 kg/m^3 .

($s 13\,030 \text{ kg}$)

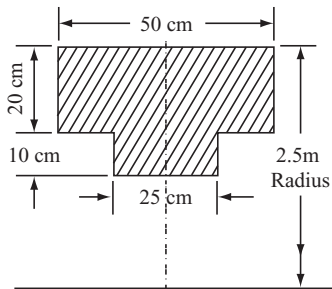


Fig. Prob. 4.7

- 8 From a circular area of diameter $2d$, a smaller circle of diameter d is removed as shown in Fig. Prob. 4.8. Locate the centroid of the remaining area. ($s d/6$ left of O)

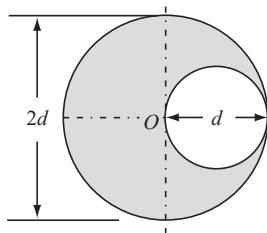


Fig. Prob. 4.8

Find the surface area of the annular torus formed by revolving the circle about the x -axis as shown in Fig. Prob. 4.9. ($s 4 \pi^2 rR$)

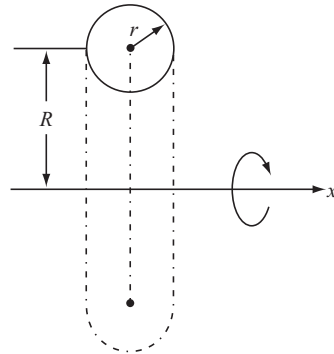


Fig. Prob. 4.9

An area is bounded by the curves $y^2 = 9x$ and $x^2 = 6y$. Sketch the area and find the coordinates of the centroid.

($s x_c = 3.09, y_c = 3.54$)

A concentric hole of 10 cm diameter is drilled to a depth of 15 cm in a perspex cylinder of diameter 20 cm and 40 cm long as shown in Fig. Prob. 4.11. The hole is filled with lead to make it a complete cylinder again. Locate the centre of mass of this cylinder. Take the density of perspex as 1200 kg/m^3 and of lead as $12\,000 \text{ kg/m}^3$.

($s 5.72 \text{ cm below the centre of cylinder}$)

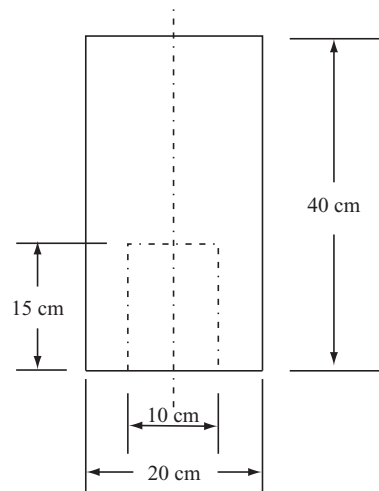


Fig. Prob. 4.11

A square hole is punched out of a thin circular lamina, the diagonal of the square being equal to the radius of the circle as shown in Fig. Prob. 4.12. Find the centre of gravity of the remaining lamina.

(s $0.095 R$ left of O)

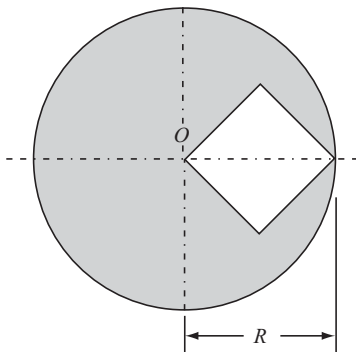


Fig. Prob. 4.12

A frustrum of a solid right circular cone of base diameter 2 m, top diameter 1 m and height 2 m has an axial hole of 0.5 m diameter in it. Locate the centroid of volume of the hollow cone.

(s 0.76 m from the base and on the axis)

Determine the maximum height h of a right circular cylinder mounted on a hemispherical base as shown in Fig. Prob. 4.14 so that the composite body may be in stable equilibrium on its base. Assume that the material for the block and the base is the same. How would the maximum height of the block change if the base was made of steel and the block made of wood, the density of wood being one-tenth that of steel?

(s $R/\sqrt{2}, \sqrt{5} R$)

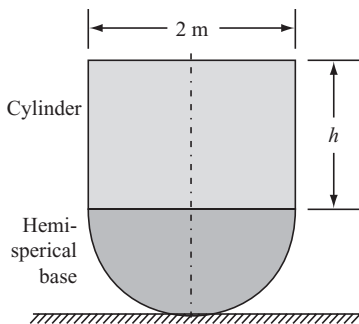


Fig. Prob. 4.14

A thin rectangular plate of length a and width b of homogeneous material is suspended from a corner. Calculate the angle the longer side will make with the vertical in the equilibrium position?

(s $\tan^{-1} \frac{b}{a}$)

Determine the length of a thin homogeneous wire which is bent into a semicircular arc of radius R together with extensions on either end as shown in Fig. Prob. 4.16 such that the centroid is located at the centre O .

(s $(\pi + 2)\sqrt{2} R$)

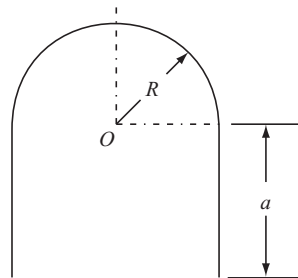


Fig. Prob. 4.16

A thin semicircular bar of weight w is suspended from a hinge at A as shown in Fig. Prob. 4.17. Determine the angle between the diameter and the vertical line. What would the angle be if a weight W is suspended from point B ?

(s $\tan^{-1} \frac{2}{\pi}$ and $\tan^{-1} \frac{2w}{\pi(w + 2W)}$)

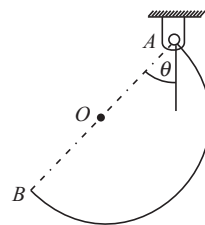


Fig. Prob. 4.17

- 8 Knowing that the surface and volume of a sphere of radius r are $4\pi r^2$ and $4/3\pi r^3$ respectively, deduce the centroid of a semi-circular arc and a semi-circular disc, each of radius r :

(s $2r/\pi, 4r/3\pi$)

A frame consists of a wire bent into a rectangular shape 0.3 m by 0.2 m plus a length of the same wire bent into a semicircle of 0.3 m diameter fixed to a 0.3 m side. Find the distance of the mass-centre of the frame from the 0.3 base.

(s 0.163 m)

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

The length of a line can be considered to be concentrated at the centroid of that line for the purpose of calculating.

- (a) the area of the surface of revolution generated by revolving the line about any axis outside it.
- (b) the average distance of the line from any axis outside it.
- (c) the volume of the body generated by revolving the line about any axis outside it.
- (d) the weightage factor for all purposes.

The centre of volume and centre of mass of a body coincide

- (a) if and only if the body is of uniform density.
- (b) if the body is geometrically symmetrical about the centre of mass.
- (c) if the density variation is symmetrical about the centroid.
- (d) if and only if the body is made of homogeneous material.

The centroid of a body

- (a) must be a point on that body.
- (b) is a point which can be made to lie on or outside the body by changing the coordinate system.

- (c) is a fixed point in space regardless of the orientation of the body.
- (d) is a unique point fixed with respect to the body.

The first moment of area of a semicircular area about its diameter d is given by

- (a) $d^3/12$.
- (b) $d^3/24$.
- (c) $d^3/6$.
- (d) $d^3/36$.

The first moment of a triangular area of base b and height h taken about an axis coincident with the base is given by

- (a) $bh^3/12$.
- (b) $b^2h/6$.
- (c) $bh^2/6$.
- (d) $h/3$.

Given that there is a rectangle and a triangle, each of base b and area A , the first moment of the area of the rectangle about its base

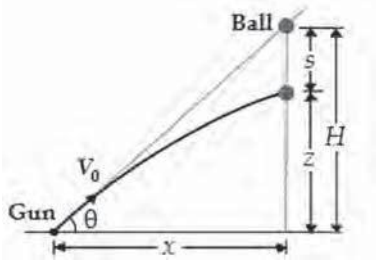
- (a) equals the first moment of the triangular area about its base b .
- (b) is more than the first moment of the triangular area about its base b .
- (c) is less than the first moment of the triangular area about its base b .
- (d) equals twice that of the triangular area about its base b .

Answers to Multiple-Choice Questions

- 1 (a) 2 (c) 3 (d) 4 (a) 5 (c) 6 (c)

Kinematics of a Particle

5



5.1 KINEMATIC CONCEPTS

Kinematics refers to the study of motion i.e., displacement, velocity and acceleration of bodies without reference to mass or force. It deals with ‘displacement, velocity and acceleration’ of a point of interest at a particular time or with the passage of time.

The simplest case of ‘kinematics of a particle’ is a point object moving along a straight line, called rectilinear motion.

A single coordinate, say x can describe the displacement, velocity and acceleration with time t .

$$V = \frac{dx}{dt}$$

$$a = \frac{dV}{dt} = \frac{d^2x}{dt^2}$$

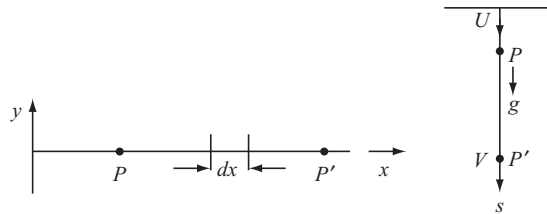


Fig. 5.1(a) Examples of rectilinear motion

Another common example of a rectilinear motion is the free fall of a particle along the s -direction.

$$a = \frac{dV}{dt} = g = 9.81 \text{ m/s}^2$$

By integration,

$$V = U + gt$$

Also,

$$V^2 - U^2 = 2g s.$$

A particle may trace a plane trajectory, e.g., a circle or an ellipse or an arbitrary plane curve as shown in Fig. 5.1(b).

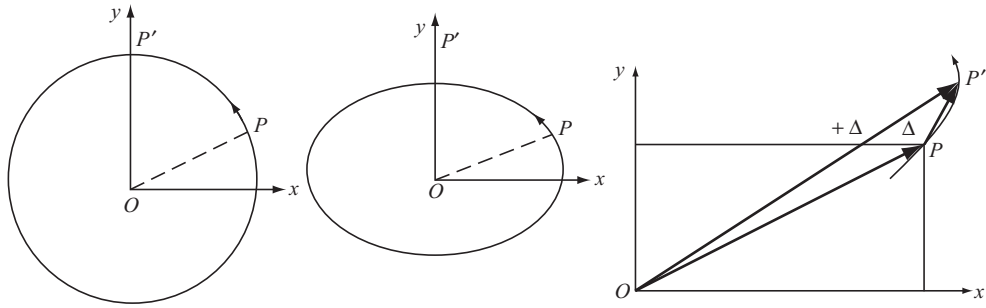


Fig. 5.1(b) Plane motion of a particle

A point may be displaced from its initial position in any direction, i.e., the displacement of a particle can have arbitrary components along the three mutually perpendicular directions. The number of the degrees of freedom of a particle in a given configuration equals the minimum number of coordinates required to describe its configuration.

5.2 MOTION REFERRED TO FIXED RECTANGULAR COORDINATES

Consider a point P moving in space with respect to a rectangular or cartesian frame of reference fixed in space as shown in Fig. 5.1(c). The *position vector* of the point is denoted by a vector \mathbf{r} .

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (5.1)$$

at a certain instant of time t . In words, the point is located by tracing x along the x -axis, y along the y -axis and z along the z -axis.

Over a short interval of time Δt , the point moves over to a new position P' described by the position vector $\mathbf{r} + \Delta \mathbf{r}$,

$$\mathbf{r} + \Delta \mathbf{r} = (x + \Delta x)\mathbf{i} + (y + \Delta y)\mathbf{j} + (z + \Delta z)\mathbf{k}$$

The point is said to have been displaced by $\Delta \mathbf{r}$ such that

$$\Delta \mathbf{r} = (\mathbf{r} + \Delta \mathbf{r}) - \mathbf{r} = (x + \Delta x)\mathbf{i} + (y + \Delta y)\mathbf{j} + (z + \Delta z)\mathbf{k} - (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$$

In words, the displacement $\Delta \mathbf{r}$ of a point is composed of displacements Δx along the x -axis, Δy along the y -axis and Δz along z -axis.

The *velocity* of the point is the rate of change of displacement:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) \\ &= \frac{d}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\ &= V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} \end{aligned} \quad (5.2)$$

where

$$V_x = \frac{dx}{dt} = x\text{-component of the velocity}$$

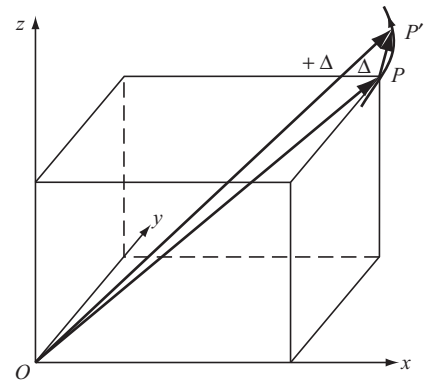


Fig. 5.1 (c) Space motion of a particle

$$V_y = \frac{dy}{dt} = \text{y-component of the velocity}$$

$$V_z = \frac{dz}{dt} = \text{z-component of the velocity}$$

and

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2} = \text{speed, the magnitude of the velocity.}$$

The *acceleration* of the point is the rate of change of velocity:

$$\begin{aligned} \mathbf{a} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \\ &= \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} + \frac{d^2z}{dt^2} \mathbf{k} \\ &= \frac{dV_x}{dt} \mathbf{i} + \frac{dV_y}{dt} \mathbf{j} + \frac{dV_z}{dt} \mathbf{k} \\ &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \end{aligned} \tag{5.3}$$

where

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} = \text{x-component of the acceleration}$$

$$a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2} = \text{y-component of the acceleration}$$

$$a_z = \frac{dV_z}{dt} = \frac{d^2z}{dt^2} = \text{z-component of the acceleration}$$

and $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$, magnitude of the acceleration vector.

It is possible to rewrite Eqs. (5.2) and (5.3) in the integral form

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} dt + \mathbf{r}_0 \\ \mathbf{v} &= \int \mathbf{a} dt + \mathbf{v}_0 \end{aligned}$$

where \mathbf{r}_0 and \mathbf{v}_0 are the vector constants of integration.

In terms of definite integrals, the change in velocity is given by

$$\mathbf{v} - \mathbf{v}_0 = \int_{t_0}^t \mathbf{a} dt \tag{5.4}$$

and the displacement is expressed as

$$\mathbf{r} - \mathbf{r}_0 = \int_{t_0}^t \mathbf{v} dt \tag{5.5}$$

where \mathbf{v}_0 and \mathbf{r}_0 are the velocity and displacement respective at time t_0 .

It may also be noted that a differential element of the arc-length is given by

$$ds = (d\mathbf{r} \cdot d\mathbf{r})^{1/2} = (dx)^2 + (dy)^2 + (dz)^2)^{1/2} \tag{5.6}$$

The distance travelled by a point along its trajectory can be calculated by integrating this expression over the appropriate time interval.

The plane motion of a particle may be studied by referring to two coordinates as shown in Fig. 5.1(b).

5.3 RECTILINEAR MOTION OF A POINT

In order to study the rectilinear motion of a point, we choose the x -axis along the line of motion as shown in Fig. 5.1(b). From the expressions for the position vector, displacement, velocity and acceleration, we write

$$\begin{aligned} &= x ; \Delta = \Delta x \\ &= \frac{dx}{dt} = V_x \\ \mathbf{a} &= \frac{d^2x}{dt^2} = \frac{dV_x}{dt} = a_x \end{aligned}$$

Since all these vectors are directed along the x -axis only, the vector notation and suffixes may be dropped. Hence,

$$V = \frac{dx}{dt}, \quad x = \int V dt + C$$

and

$$a = \frac{d^2x}{dt^2} = \frac{dV}{dt}, \quad V = \int a dt + K$$

If the acceleration a is constant, $V = at + K$ and if the initial velocity of the point at time $t = 0$ is U , then at any time t ,

$$V = U + at \tag{5.7}$$

From the expression for x , the distance moved by the point

$$s = \int (U + at) dt$$

or

$$s = Ut + \frac{1}{2} at^2 \tag{5.8}$$

Eliminating t in Eqs. (5.7) and (5.8)

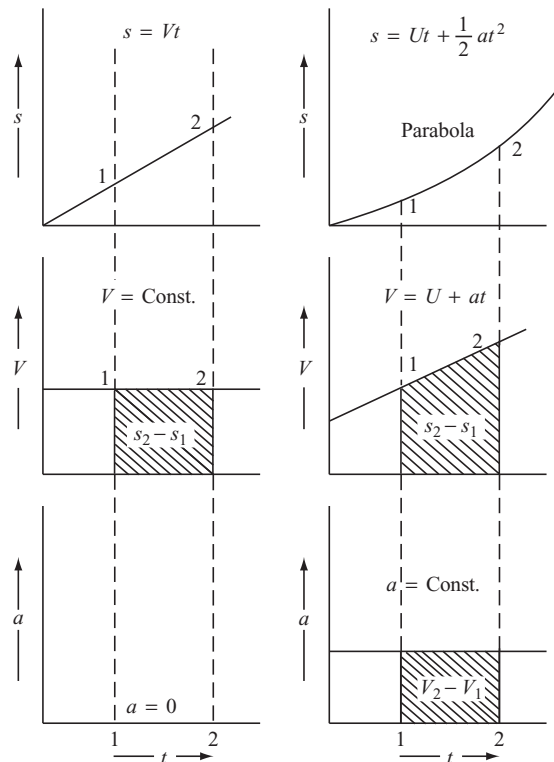
$$V^2 - U^2 = 2as$$

Motion curves or motion diagrams are drawn to show the variation of displacement, velocity and acceleration with time for the rectilinear motion of a particle. An important aspect of the motion curves is that these are mutually related. Drawing of motion curves is useful to obtain a graphical picture of the distance traversed, velocity at any instant, the average velocity and the effect of acceleration particularly when the motion occurs in distinct phases.

The motion curves of some typical motions of a particle are shown in Fig. 5.2. In order to explore the relationship between the curves, consider the motion of the particle over a time interval $(t_2 - t_1)$.

From the relationship,

$$\Delta V = V_2 - V_1 = \int_1^2 a dt$$



(a) Constant velocity

(b) Constant acceleration

Fig. 5.2 Motion curves

it is clear that the area under the acceleration diagram for a time interval must equal the change of the velocity during that interval.

From the relationship,

$$\Delta s = s_2 - s_1 = \int_1^2 V dt$$

it follows that the area under the velocity diagram for a time interval must equal the displacement over the time interval.

Some comments on the shape of the motion curves can now be made:

1. The linear displacement diagram corresponds to uniform velocity, i.e., zero acceleration.
2. Constant acceleration implies linear velocity variation and a parabolic displacement diagram.
3. The point of zero acceleration must correspond to the point of maxima or minima or inflexion on the velocity curve and the point of inflexion on the displacement diagram.
4. The slope of the velocity curve must be the maximum at the point of the maximum acceleration.
5. The reversal of the motion of a particle corresponds to a drop in the displacement curve, reversal of velocity and maximum acceleration.

The *rectilinear motion of a particle due to gravitational acceleration of the earth* is of a special interest. In such a case,

$$a = g = 9.81 \text{ m/s}^2 \text{ directed towards the centre of the earth.}$$

A particle approaching the earth in this manner is said to be in a state of *free fall*.

During a free fall, as shown in Fig. 5.3,

and
$$\begin{aligned} V &= U + gt \\ V^2 - U^2 &= 2gh \end{aligned}$$

In particular, if a particle is dropped from rest, $U = 0$, then

and
$$\begin{aligned} V &= gt \\ V^2 &= 2gh \quad \text{or} \quad V = \sqrt{2gh} \end{aligned}$$

whence
$$t = \sqrt{2h/g}$$

For an upward motion of a particle in the gravitational field,

and
$$\begin{aligned} V &= U - gt \\ V^2 - U^2 &= -2gh \end{aligned}$$

since it is acted upon by a deceleration $-g$.

When it comes to rest, $V = 0$

$$\begin{aligned} U &= gt \\ U^2 &= 2gh \end{aligned}$$

and again,
$$t = \sqrt{2h/g}$$

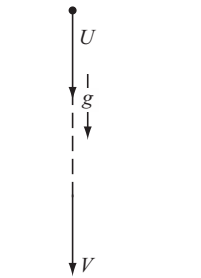


Fig. 5.3 Free fall

Example 5.1

A point moving in a straight line is observed to accelerate as

$$a = 12t - 20$$

It passed through a reference point at $t = 0$ and another point 20 cm away after an interval of 5 s. Calculate the displacement, velocity and acceleration of the point after a further interval of 5 s.

Solution

$$a = 12t - 20$$

Integrating with respect to time t ,

$$V = 6t^2 - 20t + C_1$$

and integrating once again,

$$S = 2t^3 - 10t^2 + C_1t + C_2$$

Using the boundary conditions,

$$\text{at } t = 0; S = 0; C_2 = 0 \quad \text{and} \quad \text{at } t = 5 \text{ s, } S = 20 \text{ cm, } C_1 = 4$$

The expressions for the displacement, velocity and acceleration are, therefore,

$$S = 2t^3 - 10t^2 + 4t \text{ cm}$$

$$V = 6t^2 - 20t + 4 \text{ cm/s}$$

$$a = 12t - 20 \text{ cm/s}^2$$

Substituting

$$t = 5 + 5 = 10 \text{ s}$$

$$S = 1040 \text{ cm} = 10.4 \text{ m}$$

$$V = 404 \text{ cm/s} = 4.04 \text{ m/s}$$

$$a = 100 \text{ cm/s}^2 = 1 \text{ m/s}^2$$

Example 5.2

A particle, while at rest at the position (5, 6, 2) is accelerated at

$$= 6t - 24t^2 + 10 \text{ m/s}^2$$

Determine the acceleration; velocity and displacement of the particle after a lapse of one second.

Solution

At $t = 0$

$$= 5 + 6 + 2$$

and $= 0$

From the acceleration

$$= 6t - 24t^2 + 10$$

(i)

the velocity at any instant must be

$$\begin{aligned} \mathbf{V} &= \int dt + \\ &= 3t^2 - 8t^3 + 10t + \end{aligned}$$

Now $= 0$ from the initial condition,

$$= 0 \text{ at } t = 0$$

Hence,

$$= 3t^2 - 8t^3 + 10t$$

(ii)

Integrating Eq. (ii) with respect to time t again,

$$\begin{aligned} \mathbf{r} &= \int dt + \\ &= t^3 - 2t^4 + 5t^2 + \end{aligned}$$

From the initial condition

$$= 5 + 6 + 2 \text{ at } t = 0$$

$$= 5 + 6 + 2$$

and

$$= (t^3 + 5) + (6 - 2t^4) + (5t^2 + 2) \tag{iii}$$

$$t = 1 \text{ s}$$

the acceleration is obtained from Eq. (i)

$$= (6 - 24 + 10) \text{ m/s}^2$$

the velocity is obtained from Eq. (ii)

$$= (3 - 8 + 10) \text{ m/s}$$

and the position vector is obtained from Eq. (iii),

$$= (6 + 4 + 7) \text{ m}$$

The displacement from the initial position must be

$$= (6 + 4 + 7) - (5 + 6 + 2)$$

$$= (-2 + 5) \text{ m}$$

5.4 MOTION REFERRED TO CYLINDRICAL POLAR COORDINATES

Consider the motion of a point on a circular trajectory in the x - y plane, i.e., the r - θ plane as shown in Fig. 5.4.

The position of a point P at any time t can be specified by the x and y coordinates or the r and θ coordinates such that

$$x = r \cos \theta$$

$$y = r \sin \theta$$

or $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$

Let the angular displacement of the point be $\Delta\theta$ over a differential time interval Δt . The angular velocity is, therefore,

$$\omega = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta\theta}{\Delta t} \right) = \frac{d\theta}{dt}$$

directed about an axis normal to the plane; anticlockwise positive for the right-handed triad.

The linear velocity V of the point is given by

$$V = r\omega = r \frac{d\theta}{dt} \tag{5.9}$$

directed tangentially to the circular path.

Over a differential time interval dt , the velocity of the point changes both in magnitude and in direction as shown in the sketch.

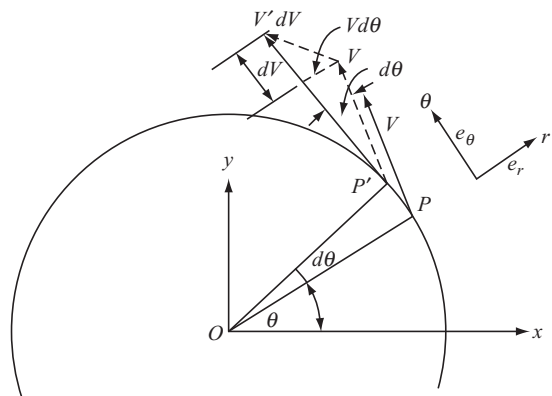


Fig. 5.4 Circular motion of a point

| Change | Direction | Remarks |
|------------|--------------------------------|---|
| dV | Tangentially forward; θ | +ve, if the speed increases |
| $Vd\theta$ | Radially inwards; $-r$ | -ve always, because r is +ve radially out |

The acceleration of the point is the rate of change of velocity with time. The components of acceleration are

Tangential:
$$\frac{dV}{dt} = r \frac{d}{dt} \left(\frac{d\theta}{dt} \right) = r \frac{d^2\theta}{dt^2} = r\alpha \quad (5.10a)$$

Radial:
$$-\frac{Vd\theta}{dt} = -V\omega = -\frac{V^2}{r} = -\omega^2 r \quad (5.10b)$$

The tangential component of acceleration dV/dt is by virtue of a change in speed only whereas the radial component called the centripetal acceleration is by virtue of the circular trajectory of radius r traced at a speed V . Obviously,

1. For a rectilinear motion, the ‘radial’ component is zero because the radius r of the ‘circle’ is infinite.
2. For a circular motion at a constant speed V , the tangential component $dV/dt = 0$ and the point undergoes only the centripetal acceleration $\omega^2 r$ directed radially inwards towards the centre or axis of rotation. For such a case,

$$= \omega^2 r$$

In terms of angular quantities alone,

angular displacement = $d\theta$

angular velocity = $\omega = \frac{d\theta}{dt}$

angular acceleration = $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

and in the integral form,

$$\omega = \int \alpha dt + K'$$

$$\theta = \int \omega dt + C'$$

and

If the angular acceleration α is constant,

$$\omega = \omega_0 + \alpha t \quad (5.11)$$

where the initial angular velocity at time $t = 0$ is ω_0

and

$$\theta = \int (\omega_0 + \alpha t) dt + C'$$

or

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad (5.12)$$

where the angular displacement θ is referred to $\theta = 0$ at $t = 0$.

Eliminating t between the equations

$$\omega^2 - \omega_0^2 = 2\alpha\theta \quad (5.13)$$

The relations for the angular quantities are similar to those for the linear quantities derived earlier. Table 5.1 brings out a systematic comparison.

Let us now consider the general motion of a point in space referred to cylindrical coordinates as shown in Fig. 5.5(a).

At any instant of time t , the position vector of the point P is

$$\mathbf{R} = r_r + z_z \quad (5.14)$$

Table 5.1 Linear vs. Angular Motion

| | Linear motion | Angular motion | Relationship |
|--|---------------------|--|--------------------------------------|
| Displacement | s | θ | $ds = r d\theta$ |
| Velocity | $v = \frac{ds}{dt}$ | $\omega = \frac{d\theta}{dt}$ | $V = r\omega$ |
| Acceleration | $a = \frac{dv}{dt}$ | $\alpha = \frac{d\omega}{dt}$ | $a = r\alpha$ |
| Initial velocity | U | ω_0 | Valid only for constant acceleration |
| Expressions relating the displacement, velocity, acceleration and time | $V = U + at$ | $\omega = \omega_0 + \alpha t$ | |
| | $s = Ut + 1/2 at^2$ | $\theta = \omega_0 t + 1/2 \alpha t^2$ | |
| | $v^2 - U^2 = 2as$ | $\omega - \omega_0 = 2\alpha\theta$ | |
| | $s = \int V dt + C$ | $\theta = \int \omega dt + C'$ | |
| | $V = \int a dt + K$ | $\omega = \int \alpha dt + K'$ | |

and at a later instant $t + \Delta t$, the position vector for the location Q of the point is

$$\mathbf{R} + \Delta \mathbf{R} = (r + \Delta r)(\mathbf{e}_r + \Delta \mathbf{e}_r) + (z + \Delta z)\mathbf{e}_z$$

The displacement of the point is, therefore, given by $\Delta \mathbf{R}$, which by subtraction is,

$$\Delta \mathbf{R} = \Delta r \mathbf{e}_r + r \Delta \mathbf{e}_r + \Delta z \mathbf{e}_z$$

neglecting the term $(\Delta r \Delta \mathbf{e}_r)$ which is smaller in order of magnitude than the other terms.

The velocity of the point is expressed as

$$= \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{R}}{\Delta t} = \frac{d\mathbf{R}}{dt}$$

or

$$= \frac{dr}{dt} \mathbf{e}_r + r \frac{d\mathbf{e}_r}{dt} + \frac{dz}{dt} \mathbf{e}_z \tag{5.15}$$

This expression requires the knowledge of the temporal derivative of the unit vector \mathbf{e}_r . It can be seen from Fig. 5.4(b) that a small change in \mathbf{e}_r takes place in the direction of $(+\mathbf{e}_\theta)$ and a small change in θ takes place

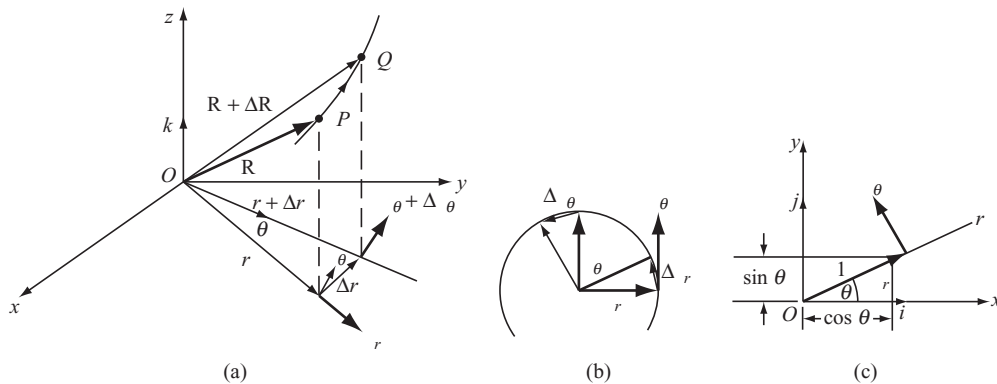


Fig. 5.5 Motion referred to cylindrical coordinates

only along the direction of $(-e_r)$. The magnitude of the change in each case is given by the product of the magnitude of the unit vector, i.e., unity and the small angle $\Delta\theta$ through which the unit vector is turned;

$$1 \quad \Delta\theta = \Delta\theta$$

Consequently,

$$\begin{aligned} \frac{d e_r}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta e_r}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} e_\theta \\ &= \omega e_\theta \end{aligned} \tag{5.16}$$

$$\begin{aligned} \frac{d e_\theta}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta e_\theta}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} (-e_r) \\ &= -\omega e_r \end{aligned} \tag{5.17}$$

The temporal derivatives of the unit vectors e_r and e_θ may alternatively be obtained as follows with reference to Fig. 5.5(c). If, at a particular instant, the radial direction makes an angle θ with the x direction,

$$e_r = \cos \theta e_x + \sin \theta e_y$$

since the magnitude of e_r is unity.

Differentiating with respect to time t

$$\begin{aligned} \frac{d e_r}{dt} &= -\sin \theta \frac{d\theta}{dt} e_y + \cos \theta \frac{d\theta}{dt} e_x \\ &= \frac{d\theta}{dt} (-\sin \theta e_y + \cos \theta e_x) \\ &= \omega e_\theta \end{aligned}$$

Similarly,

$$\begin{aligned} e_\theta &= -\sin \theta e_x + \cos \theta e_y \\ \frac{d e_\theta}{dt} &= -\frac{d\theta}{dt} (\cos \theta e_x + \sin \theta e_y) \\ &= -\omega e_r \end{aligned}$$

Returning to the equation for the velocity of the point,

$$\mathbf{V} = \dot{r} e_r + r\omega e_\theta + \dot{z} e_z$$

or

$$= V_r e_r + V_\theta e_\theta + V_z e_z \tag{5.18}$$

Hence, the acceleration of the point,

$$\begin{aligned} \mathbf{a} &= \frac{d}{dt} \mathbf{V} = \frac{d}{dt} (\dot{r} e_r + r\omega e_\theta + \dot{z} e_z) \\ &= \ddot{r} e_r + \dot{r}\omega e_\theta + \dot{r}\omega e_\theta + r\dot{\omega} e_\theta - r\omega^2 e_r + \ddot{z} e_z \\ \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2) e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) e_\theta + \ddot{z} e_z \end{aligned} \tag{5.19}$$

where dot ($\dot{\quad}$) stands for derivative, and double dot ($\ddot{\quad}$) stands for second derivative with respect to time.

Observation of the results for \mathbf{V} and \mathbf{a} lead us to some interesting and useful conclusions:

For plane motion in the x - y plane

$$z = \text{Const.}, \quad \dot{z} = 0 = \ddot{z}$$

the point can, therefore, have neither velocity nor acceleration in a direction perpendicular to the plane.

For constant speed of rotation ω

$$\omega = \dot{\theta} = \text{Const.}, \quad \alpha = \dot{\omega} = 0$$

the point can, therefore, not have any angular acceleration.

For plane circular motion in the x - y plane

$$z = \text{Const.}, \quad \dot{z} = 0 = \ddot{z}$$

and

$$r = \text{Const.}, \quad \dot{r} = 0 = \ddot{r}$$

the expressions for the velocity and acceleration reduce to

$$\begin{aligned} \mathbf{v} &= r\omega \boldsymbol{\theta} \\ \mathbf{a} &= -r\omega^2 \mathbf{r} + r\alpha \boldsymbol{\theta} \end{aligned}$$

which shows that the velocity of the point is wholly tangential at any instant whereas the acceleration has two components:

$r\omega^2$ radially inwards, towards the centre of the circle, due to the rotational speed of the point about the centre

$r\alpha$ tangentially directed due to the angular acceleration α of the point.

In particular, for a plane circular motion at constant speed, the tangential acceleration vanishes and the point is only subjected to a tangential velocity together with a radially inwards acceleration often termed as the centripetal acceleration.

For helical motion, i.e., particle moving along a helix, as shown in Fig. 5.6, dealing in terms of cylindrical coordinates,

$$\begin{aligned} r &= R, \text{ a constant} \\ z &= kR\phi \end{aligned}$$

where k is the tangent of the helix angle. Note that $k = 0$ corresponds to plane circular motion.

Now, let

$$\dot{\phi} = \omega$$

and

$$\dot{z} = kR\dot{\phi} = kR\omega$$

With these values, velocity $\mathbf{v} = \dot{r} \mathbf{r} + r \dot{\phi} \boldsymbol{\theta} + \dot{z} \mathbf{z}$

$$\mathbf{v} = \dot{r} \mathbf{r} + r \dot{\phi} \boldsymbol{\theta} + \dot{z} \mathbf{z}$$

If the particle moves at a uniform speed along the helix, the angular acceleration is zero and only the radial acceleration remains.

Then, $\mathbf{a} = -R\omega^2 \mathbf{r}$.

The radius of curvature ρ of the helix is given by

$$\rho = R(1 + k^2).$$

For a helix angle of 45° , $\rho = 2R$

For a helix angle of zero, $\rho = R$, i.e., the radius itself since it reduces to plane circular motion.

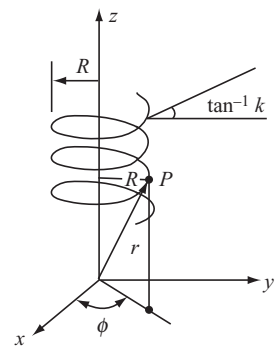


Fig. 5.6

Example 5.3

A Scotch yoke mechanism consists of a crank CA of radius r turning with a constant angular velocity ω rad/s and a reciprocating slotted sliding member S as shown in Fig. Ex. 5.3. Obtain the expression for the displacement, velocity and acceleration of the sliding member.

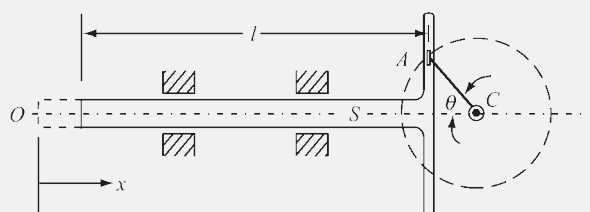


Fig. Ex. 5.3

Solution Let the motion of the slider be referred to its extreme left position O as the origin when the angle of the crank is zero at $t = 0$

$$\begin{aligned} x &= OC - l - CA \cos \theta \\ &= (l + r) - l - r \cos \theta = r(1 - \cos \theta) \end{aligned}$$

Differentiating with respect to time t and recognising that

$$\frac{d\theta}{dt} = \omega \text{ the constant rotational speed}$$

and

$$\theta = \omega t$$

$$V = \frac{dx}{dt} = r\omega \sin \omega t$$

and differentiating again,

$$a = \frac{dV}{dt} = r\omega^2 \cos \omega t.$$

Example 5.4

A 3 m long arm OA rotates in a plane such that $\theta = 0.15 t^2$ where θ is the angle with x -axis in radians and t is in seconds. A slider collar B slides along the arm in such a way that its distance from the hinge O is given by $r = 3 - 0.4 t^2$ where r is in metres. Determine the velocity and acceleration of the collar at an instant the arm has turned through 30° .

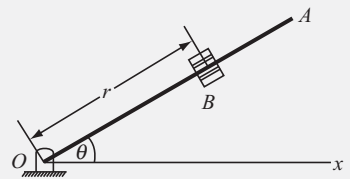


Fig. Ex. 5.4

Solution

For $\theta = 30^\circ = \frac{\pi}{6}$ radians,

$$\frac{\pi}{6} = 0.15 t^2$$

whence

$$t = 1.87 \text{ seconds}$$

For

$$r = 3 - 0.4 t^2 = 3 - 0.4 \times 1.87^2 = 1.6 \text{ m}$$

$$\dot{r} = -0.8t = -0.8 \times 1.87 = -1.5 \text{ m/s}$$

$$\ddot{r} = -0.8 \text{ m/s}^2$$

Correspondingly,

$$\theta = 0.15 t^2 = \frac{\pi}{6} = 0.524 \text{ rad}$$

$$\dot{\theta} = 0.3t = 0.3 \times 1.87 = 0.561 \text{ rad/s}$$

$$\ddot{\theta} = 0.3 \text{ rad/s}^2$$

Velocity of the collar B is such that

$$V_r = \dot{r} = -1.5 \text{ m/s}$$

$$V_\theta = r\dot{\theta} = 1.6 \times 0.561 = 0.9 \text{ m/s}$$

which is 1.75 m/s at an angle α with the arm as shown in Fig. Ex. 5.4 (Solution) (a) such that

$$\alpha = \tan^{-1} \frac{0.9}{1.5} = 31^\circ$$

The acceleration of the collar is such that

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -0.8 - 1.6 \times (0.561)^2 = -1.3 \text{ m/s}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ &= 1.6 \times 0.3 + 2 \times (-1.5) \times 0.561 \\ &= -1.2 \text{ m/s}^2 \end{aligned}$$

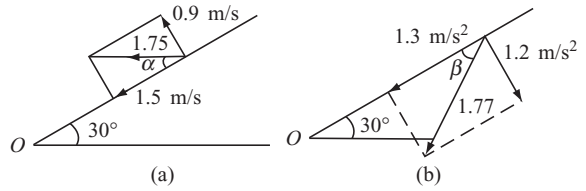


Fig. Ex. 5.4 (Solution)

which is 1.77 m/s^2 at an angle β with the arm as shown in Fig. Ex. 5.4 (Solution) (b) such that

$$\beta = \tan^{-1} \frac{1.2}{1.3} = 42.7.$$

Example 5.5

A wheel of radius 0.5 m is turned to advance up on a right-handed screw of pitch 1 cm as shown in Fig. Ex. 5.5. At an instant when the wheel is turned at a rotational speed of 2 rad/s, determine the velocity and acceleration of the hand held at A. If the wheel was accelerated rotationally at 0.6 rad/s^2 , what would be the velocity and acceleration of the hand?

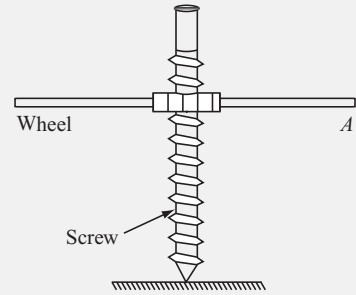


Fig. Ex. 5.5

Solution For the point A on the wheel (Ref. Fig. Ex. 5.5 (Solution))

$$r = 0.5 \text{ m} \quad \text{and} \quad \omega = 2 \text{ rad/s} = \dot{\theta}$$

The velocity is given by

$$\begin{aligned} \mathbf{V} &= \dot{r} \mathbf{e}_r + r\dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{e}_z \quad (i) \\ &= 0 \mathbf{e}_r + 0.5 \times 2 \mathbf{e}_\theta + \frac{0.01 \times 2}{2 \times \pi} \mathbf{e}_z \\ &= \dot{\theta} \mathbf{e}_\theta + 0.0032 \mathbf{e}_z \text{ m/s} \end{aligned}$$

The acceleration is obtained as

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta + \ddot{z} \mathbf{e}_z \\ &= (0 - 0.5 \times 2^2) \mathbf{e}_r + (0.5 \times 0 + 2 \times 0 \times 2) \mathbf{e}_\theta + 0 \mathbf{e}_z \\ &= -2 \mathbf{e}_r \text{ m/s}^2, \text{ i.e., } 2 \text{ m/s}^2 \text{ radially inwards} \end{aligned}$$

For the second case,

$$r = 0.5 \text{ m}, \omega = \dot{\theta} = 2 \text{ rad/s} \text{ and } \ddot{\theta} = 0.6 \text{ rad/s}^2$$

The velocity is given by Eq. (i) as

$$= \dot{\theta} \mathbf{e}_\theta + 0.3 \mathbf{e}_z \text{ m/s}$$

the same as before and the acceleration from Eq. (ii) becomes,

$$\begin{aligned} &= (0 - 0.5 \times 2^2) \mathbf{e}_r + (0.5 - 0.6 + 2 \times 0 \times 2) \mathbf{e}_\theta + 0.0032 \times 0.6 \mathbf{e}_z \\ &= -2 \mathbf{e}_r + 0.3 \mathbf{e}_\theta + 0.0019 \mathbf{e}_z \text{ rad/s}^2 \end{aligned}$$

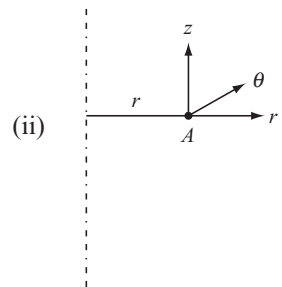


Fig. Ex. 5.5 (Solution)

It may be noted that the speed and acceleration of advance of the wheel along the z -axis are related to the rotational speed and rotational acceleration respectively of the wheel. This fact has been used to evaluate \dot{z} and \ddot{z} terms in the analysis.

5.5 MOTION REFERRED TO PATH COORDINATES

It is sometimes very convenient to describe the kinematics of a point in terms of the path coordinates, i.e., the geometric features of the curve traced by the point as shown in Fig. 5.7(a). It may be appreciated that the path of a point may not be known *a priori* and hence the path coordinates cannot be specified until the point traces a curve in the vicinity of the position of interest. For this reason, the path coordinates are also known as *intrinsic coordinates*.

The orthogonal triad in the path coordinates consists of the following unit vectors:

- t tangential to the path
- n directed towards the instantaneous centre of curvature
- b perpendicular to the plane containing t and n so as to form a right-hand triad.

In terms of the path coordinates, the velocity of a point is given by

$$= \lim_{\Delta t \rightarrow 0} \frac{\Delta R}{\Delta t} = \frac{dR}{dt} = \frac{dR}{ds} \frac{ds}{dt} = V_t \quad (5.20)$$

which means that the velocity must be tangential to the path at any instant as shown in Fig. 5.7(b).

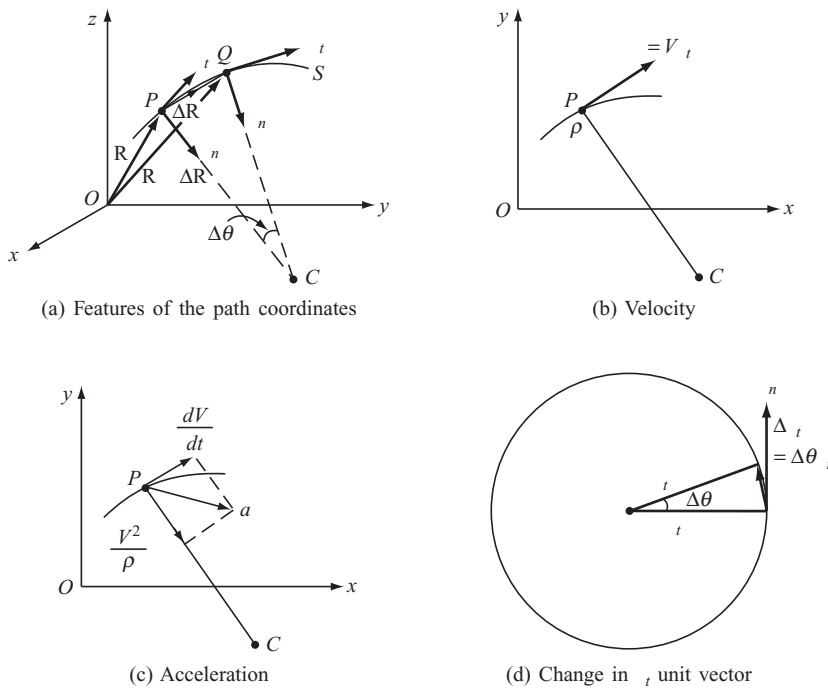


Fig. 5.7 Motion referred to path coordinates

The acceleration of the point is expressed as

$$\begin{aligned}
 \mathbf{a} &= \frac{d}{dt} \left(V \mathbf{e}_t \right) \\
 &= \frac{dV}{dt} \mathbf{e}_t + V \frac{d \mathbf{e}_t}{ds} \frac{ds}{dt} \\
 &= \frac{d^2 s}{dt^2} \mathbf{e}_t + k V^2 \mathbf{e}_n \\
 \mathbf{a} &= \frac{d^2 s}{dt^2} \mathbf{e}_t + \frac{V^2}{\rho} \mathbf{e}_n
 \end{aligned} \tag{5.21}$$

where k , the curvature of the path $= \frac{d\theta}{ds}$ and ρ , the radius of curvature $= \frac{ds}{d\theta} = \frac{1}{k}$. For a plane curve,

$$\rho = \left| \frac{(1 + (dy/dx)^2)^{3/2}}{d^2 y/dx^2} \right| = \left| \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| \tag{5.22}$$

The components of acceleration are shown in Fig. 5.7(c). The fact that

$$\frac{d \mathbf{e}_t}{d\theta} \cdot \frac{d\theta}{ds} = \frac{d \mathbf{e}_t}{ds} = k \mathbf{e}_n = \frac{1}{\rho} \mathbf{e}_n$$

used in the above derivation can be appreciated with reference to a circular diagram (Fig. 5.7(d)). The unit vector \mathbf{e}_t suffers a small change $\Delta \mathbf{e}_t$ along the \mathbf{e}_n direction

$$\frac{d \mathbf{e}_t}{ds} = \lim_{\Delta s \rightarrow 0} \frac{1 \cdot \Delta \theta}{\Delta s} \mathbf{e}_n = k \mathbf{e}_n = \frac{1}{\rho} \mathbf{e}_n \tag{5.23}$$

Let us examine the relations for the simple case of motion of point on a circular trajectory. The curvature of a circle is the inverse of the radius

$$k = \frac{1}{r}$$

The velocity and acceleration of the point are

$$\begin{aligned}
 \mathbf{v} &= V \mathbf{e}_t \\
 \mathbf{a} &= \frac{d}{dt} \left(V \mathbf{e}_t \right) + \frac{V^2}{r} \mathbf{e}_n = a_t \mathbf{e}_t + a_n \mathbf{e}_n
 \end{aligned}$$

and

It may be seen that the velocity is directed along a tangent to the circle at any instant. The acceleration, on the other hand, is made up of a tangential component, dV/dt ; sense forward and normal component, V^2/r ; sense inwards. This is shown in Fig. 5.8. In particular, if the point moves at a constant speed on a circular path, the tangential component of acceleration vanishes and only the radially inward centripetal acceleration remains.

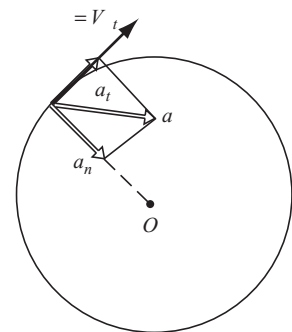


Fig. 5.8 Velocity and acceleration for a circular trajectory

Example 5.6

parabola

$$y^2 = 4x$$

At a certain instant, when passing through a point $P(4, 4)$ its speed is 5 m/s and the rate of increase of its speed is 3 m/s^2 along the path. Express the velocity and acceleration of the particle in terms of rectangular coordinates.

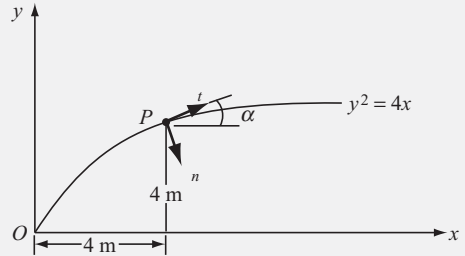


Fig. Ex. 5.6

Solution Since the data relate to the path of the particle, the path coordinates may be used to advantage. The unit vectors are related as follows:

$$t = \cos \alpha \quad + \quad \sin \alpha \tag{i}$$

$$n = \sin \alpha \quad - \quad \cos \alpha \tag{ii}$$

where

$$\tan \alpha = \frac{dy}{dx} \text{ at } P$$

From the equation of the path,

$$y^2 = 4x$$

differentiation with respect to x yields

$$2y \frac{dy}{dx} = 4$$

and

$$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{\sqrt{x}}$$

which, at P , is $\frac{1}{\sqrt{4}}$ or 0.5

and

$$\alpha = \tan^{-1} 0.5 = 26.57^\circ = 0.464 \text{ rad}$$

Equations (i) and (ii) at point P become

$$t = 0.894 \quad + \quad 0.447$$

$$n = 0.447 \quad - \quad 0.894$$

The velocity of the particle is given by

$$\begin{aligned} &= V \quad t = 5(0.894 \quad + \quad 0.447) \\ &= 4.47 \quad + \quad 2.235 \quad \text{m/s} \end{aligned}$$

The tangential component of acceleration is

$$a_t = 3(0.894 \quad + \quad 0.447) = (2.68 \quad + \quad 1.34) \text{ m/s}^2$$

The normal component of acceleration is

$$a_n = \frac{V^2}{\rho}$$

The radius of curvature r is given by

$$\begin{aligned} \rho &= \frac{(1 + (dy/dx)^2)^{3/2}}{d^2y/dx^2} \\ &= \frac{(1 + 0.5^2)^{3/2}}{0.0625} = 22.36 \text{ m} \end{aligned}$$

because
$$\frac{d^2y}{dx^2} = (-1/2x^{-3/2}) = 0.0625$$

The normal component of acceleration is

$$\begin{aligned} a_n &= \frac{V^2}{\rho} \\ &= \frac{5^2}{22.36} \times (0.447 \quad -0.894) \\ &= 0.5 \quad - \end{aligned}$$

The acceleration is, therefore, given by

$$\begin{aligned} a &= a_n + a_t \\ &= 2.68 \quad +1.34 \quad +0.5 \quad - \\ &= (3.18 \quad +0.34) \text{ m/s}^2 \end{aligned}$$

Example 5.7

A particle moves in the xy plane with a velocity of 30 m/s directed at an angle of $\tan^{-1} 4/3$ as shown in Fig. Ex. 5.7. It accelerates as $a_x = -1.8 \text{ m/s}^2$ and $a_y = -9 \text{ m/s}^2$. Compute the radius of curvature of the path and the rate of change of speed along the path.

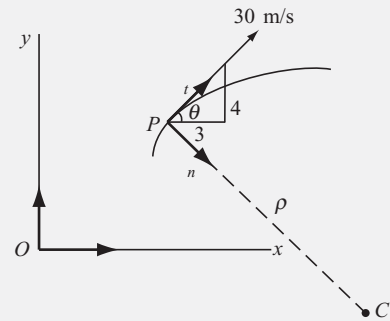


Fig. Ex. 5.7

Solution The unit vector along the velocity vector is

$$\mathbf{t} = \cos \theta \quad + \sin \theta = 0.6 \quad +0.8$$

The unit vector along the inward normal is

$$\mathbf{n} = 0.8 \quad -0.6$$

In terms of path coordinates, the acceleration is expressed as

$$a = f \quad \mathbf{t} + V^2/\rho \quad \mathbf{n}$$

where f is the rate of change of speed along the path and ρ is the radius of curvature.

$$\text{or} \quad -1.8 \quad -9 = f (0.6 \quad 0.8) + \frac{900}{\rho} (0.8 \quad -0.6)$$

which results in two equations:

$$0.6f + 720/\rho = -1.8$$

and
$$0.8f - 540/\rho = -9$$

whence
$$f = -8.3 \text{ m/s} \text{ and } \rho = 227 \text{ m.}$$

5.6 PLANE MOTION OF A POINT: GRAVITATIONAL FIELD

A point is said to be in plane motion if the point continues to move in one and the same plane, i.e., its trajectory is in a plane. Let us choose the xz plane to coincide with the plane of motion as shown in Fig. 5.1(b). From the expressions for the position vector, displacement, velocity and acceleration,

$$\begin{aligned} \Delta &= x + z \\ \Delta &= \Delta x + \Delta z \\ &= \frac{dx}{dt} + \frac{dz}{dt} = V_x + V_z \\ &= \frac{d^2x}{dt^2} + \frac{d^2z}{dt^2} = \frac{dV_x}{dt} + \frac{dV_z}{dt} = a_x + a_z \end{aligned}$$

Of special interest here is the case of a particle projected at an angle α to the horizontal in the gravitational field of the earth close to its surface as shown in Fig. 5.9. Then,

$$\begin{aligned} a_x &= 0 \quad \text{and} \quad a_z = -g \\ &= -g \quad ; \text{ constant acceleration.} \end{aligned}$$

Starting with the equation

$$= \int dt +$$

and substituting

$$\begin{aligned} &= -gt \\ &= -gt + \\ V_x + V_z &= -gt + V_{x0} + V_{z0} \\ &= V_{x0} + (V_{z0} - gt) \end{aligned}$$

where V_{x0} and V_{z0} are the initial components of velocity at $t = 0$.

It follows that,

$$V_x = V_{x0}$$

i.e., the horizontal component of velocity remains the same during motion

and

$$V_z = V_{z0} - gt$$

i.e., the vertical component of velocity undergoes a linear change.

Starting now with the equation

$$= \int dt +$$

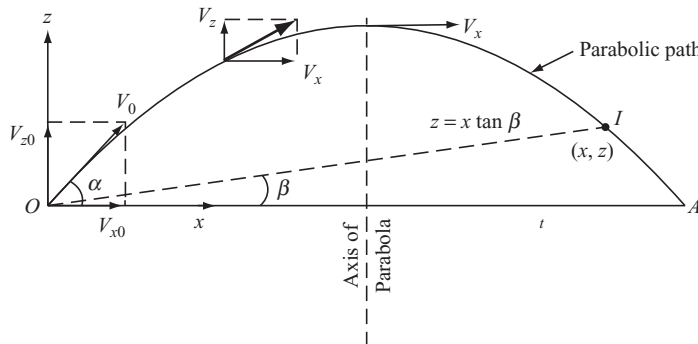


Fig. 5.9 Parabolic trajectory of a particle

and substituting

$$x + z = V_{x0} t + \left(V_{z0} t - \frac{1}{2} g t^2 \right)$$

where x and z are measured from the initial point O as the origin at $t = 0$

It follows that,

$$x = V_{x0} t \tag{5.24}$$

$$z = V_{z0} t - \frac{1}{2} g t^2 \tag{5.25}$$

Eliminating t and using the relations

$$\frac{V_{z0}}{V_{x0}} = \tan \alpha \quad \text{and} \quad V_{x0} = V_0 \cos \alpha$$

$$z = x \tan \alpha - \frac{g x^2}{2 V_0^2 \cos^2 \alpha} \tag{5.26}$$

This is the equation of the trajectory. Since it represents a parabola, the path of a particle projected in some oblique direction must be a plane parabolic trajectory.

The assumptions in this analysis are:

1. *Air-resistance is negligible*
2. *The gravitational acceleration g is constant*
3. *The point stands for the particle or centre of mass of a body.*

These assumptions restrict the analysis to short-range and low-altitude motion of small objects in the atmosphere.

Let us now inspect the equation to provide some further information.

(a) *Maximum attainable height z_{\max} occurs when*

$$V_z = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = 0$$

From the expression for z ,

$$\frac{\partial z}{\partial x} = \tan \alpha - \frac{g x}{V_0^2 \cos^2 \alpha} = 0$$

whence

$$\alpha = \frac{1}{2} \sin^{-1} \left(\frac{2 g x}{V_0^2} \right)$$

$$x = V_0^2 \frac{\sin 2\alpha}{2g}$$

and the ordinate of the vertex is given by

$$z_{\max} = \frac{V_0^2 \sin^2 \alpha}{2g} \tag{5.27a}$$

(b) *Range of the particle on a horizontal plane equals twice the x coordinate for z_{\max} .*

$$\text{Range } OA = \frac{V_0^2 \sin 2\alpha}{g} \tag{5.27b}$$

This is a maximum for $\alpha = 45^\circ$, Maximum range = $\frac{V_0^2}{g}$

- (c) Range of the particle on an inclined plane OI is determined by solving for (x, z) , the point of interaction of the parabola and the inclined plane $z = x \tan \beta$.
- (d) Variation of z_{\max} with α is observed from the relationship

$$z_{\max} = \frac{V_0^2 \sin^2 \alpha}{2g}$$

and z_{\max} is the maximum possible when $\alpha = 90^\circ$, i.e., when the particle is projected vertically upwards.

- (e) For a particle projected horizontally,

$$\alpha = 0$$

$$z = -\frac{gx^2}{2V_0^2} \quad (5.28)$$

- (f) For a particle projected vertically upwards,

$$\alpha = 90^\circ, \quad \tan \alpha \rightarrow \infty \quad \text{and} \quad x = 0$$

the problem reduces to that of a rectilinear motion along the gravitational acceleration due to the earth.

- (g) The time taken to reach a particular point on the trajectory can be estimated from the relations,

$$x = V_{x0} t$$

$$z = V_{z0} t - \frac{1}{2} g t^2$$

If the initial velocity V_0 (in terms of V_{x0} and V_{z0}) is known, the time taken can be estimated by knowing either the x or z coordinate of the point on the trajectory.

The time of flight t_{\max} of the particle from the origin O to the end point A , at the initial level can be calculated by observing that

$$z = 0 \quad \text{at} \quad t = t_{\max}$$

$$0 = V_{z0} t_{\max} - \frac{1}{2} g t_{\max}^2$$

whence

$$t_{\max} = \frac{2V_{z0}}{g} = \frac{2V_0 \sin \alpha}{g} \quad (5.29)$$

Alternatively,

$$\begin{aligned} t_{\max} &= \frac{x_{\max}}{V_{x0}} \\ &= \frac{V_0^2 \sin 2\alpha}{gV_{x0}} = \frac{2V_0 \sin \alpha V_0 \cos \alpha}{gV_{x0}} \\ &= \frac{2V_{z0} V_{x0}}{gV_{x0}} = \frac{2V_{z0}}{g} = \frac{2V_0 \sin \alpha}{g} \end{aligned}$$

The time of flight on an inclined plane can be seen to be

$$t_{\max} = \frac{2V_0 \sin(\alpha - \beta)}{g \cos \beta} \quad (5.30)$$

since the initial velocity component normal to the inclined plane is $V_0 \sin(\alpha - \beta)$ instead of $V_0 \sin \alpha = V_{z0}$ and the acceleration due to gravity is $g \cos \beta$ instead of g .

The range of flight on an inclined plane can be calculated by working out the horizontal coordinate x first,

$$\begin{aligned} x &= V_{x0} t_{\max} = V_0 \cos \alpha t_{\max} \\ &= \frac{2V_0^2 \cos \alpha \sin(\alpha - \beta)}{g \cos \beta} \end{aligned}$$

The range s on the incline is given by

$$s = \frac{x}{\cos \beta}$$

or

$$s = \frac{2V_0^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta} \quad (5.31)$$

It may be checked that when $\beta = 0$, it reduces to the expression

$$s = \frac{V_0^2 \sin 2\alpha}{g}$$

as obtained earlier for the horizontal plane.

(h) At any time t , the velocity of the particle can be computed from

$$V_x = V_{x0}$$

$$V_z = V_{z0} - gt$$

and hence,

$$= V_x + V_z = V_{x0} + (V_{z0} - gt) \quad (5.32)$$

which is $\sqrt{(V_{x0}^2 + (V_{z0} - gt)^2)}$ in magnitude

and directed at an angle $\tan^{-1} \left(\frac{V_{z0} - gt}{V_{x0}} \right)$

(i) At a certain height h , the velocity of the particle can be computed from

$$V_x = V_{x0}$$

$$V_z = (V_{z0}^2 - 2gh)^{1/2}$$

and hence,

$$\begin{aligned} &= V_x + V_z \\ &= V_{x0} + (V_{z0}^2 - 2gh)^{1/2} \end{aligned} \quad (5.33)$$

(j) If the angle of projection α is negative, the z -coordinate continues to decrease, V_z continues to increase and the particle tends to drop down closer to the vertical line with the passage of time.

Example 5.8

A gun is fired, aimed at a ball, from a ground position simultaneously as the ball is let go vertically down. Show that the shot will hit the ball regardless of the initial velocity of the shot and the distances.

Solution Let the ball be at a horizontal distance x and be at an angle of elevation θ with respect to the gun as shown in Fig. Ex. 5.8 (a) (Solution). Let the velocity of the shot at the instant of firing be V_0 at an angle of elevation of θ in the line of sight of the ball.

The time taken for the shot to travel a horizontal distance x is

$$t = \frac{x}{V_{x0}} = \frac{x}{V_0 \cos \theta} = \frac{H/\tan \theta}{V_0 \cos \theta} = \frac{H}{V_0 \sin \theta} \quad (i)$$

The vertical coordinate z of the shot in that time is given by

$$z = V_{z0}t - \frac{1}{2}gt^2$$

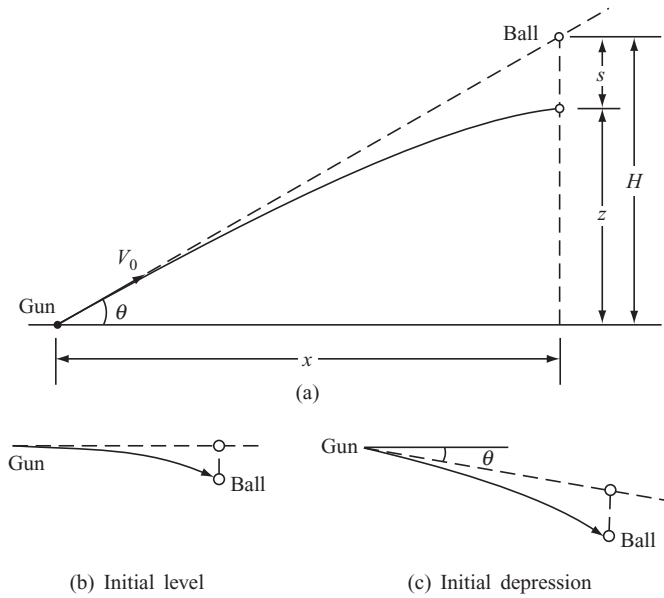


Fig. Ex. 5.8 (Solution)

$$= V_0 \sin \theta \frac{H}{V_0 \sin \theta} - \frac{1}{2} g \left(\frac{H}{V_0 \sin \theta} \right)^2$$

or

$$z = H - \frac{gH^2}{2V_0^2 \sin^2 \theta} \tag{ii}$$

During the same time interval t given by Eq. (i), the ball drops through a distance s where

$$s = \frac{1}{2} gt^2 = \frac{1}{2} g \left(\frac{H}{V_0 \sin \theta} \right)^2 = \frac{gH^2}{2V_0^2 \sin^2 \theta}$$

and the z coordinate of the ball becomes

$$z = H - \frac{gH^2}{2V_0^2 \sin^2 \theta}$$

which is the same as that of the shot. The shot should, therefore, hit the ball irrespective of the initial coordinates of the ball with respect to the gun and initial velocity of the shot.

It may be noted that the drop of the ball before it is shot is

$$s = \frac{gH^2}{2V_0^2 \sin^2 \theta} = \frac{g(x \tan \theta)^2}{2V_0^2 \sin^2 \theta} = \frac{gx^2}{2V_0^2 \cos^2 \theta}$$

which does vary with all the parameters. The drop is less if the initial velocity V_0 of the bullet is more, range angle θ is more or horizontal distance x is less.

It may also be noted that the shot will hit the ball whether the ball is initially at an angle of elevation, level, or an angle of depression with respect to the gun so long as the gun is fired at the initial line of sight of the ball. This is illustrated in Figs. Ex. 5.8(b) and (c) (Solution).

Example 5.9

The world records for the shot put and discus throw are 20 m and 70 m respectively. Assuming that their respective masses are 7 kg and 2 kg respectively, compare the work done by the champions in making their record throws if each trajectory starts at an elevation of 2 m and has an initial inclination of 45° with the horizontal. Neglect air resistance.

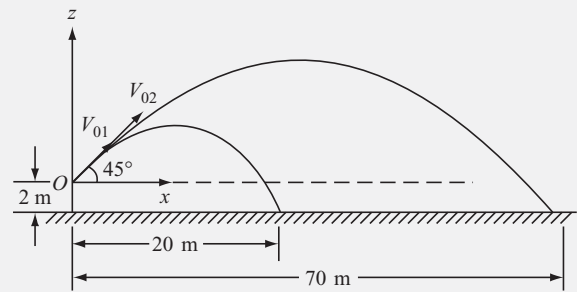


Fig. Ex. 5.9

Solution The equation of the trajectory of an object projected at an angle α with an initial speed V_0 from the point O in the gravitational field of the earth is

$$z = x \tan \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha}$$

For the shot put

$$-2 = 20 \tan 45^\circ - \frac{9.81 \times 20^2}{2V_{01}^2 \cos^2 45^\circ}$$

whence $V_{01} = 13.36$ m/s

The kinetic energy at the instant of its projection is

$$\frac{1}{2} m_1 V_{01}^2 = \frac{1}{2} \times 7 \times 13.36^2 = 625 \text{ J}$$

For the discus throw

$$-2 = 70 \tan 45^\circ - \frac{9.81 \times 70^2}{2V_{02}^2 \cos^2 45^\circ}$$

whence $V_{02} = 26.2$ m/s

and the kinetic energy at the instant of projection is

$$\frac{1}{2} m_2 V_{02}^2 = \frac{1}{2} \times 2 \times 26.2^2 = 687 \text{ J}$$

Neglecting that the discus has a tendency of spinning during flight, the work done by a champion must equal the kinetic energy imparted to the object at the instant of projection, i.e., at the same datum. The champion throwing the discus, therefore, does more work than the champion putting the shot.

Example 5.10

A large balloon is rising up with a velocity of 9.81 m/s at an altitude of 39.2 m from the ground. At that instant, a stone of mass 5 kg is dropped from it. After how many seconds will the stone reach the ground?

Solution The acceleration due to gravity $g = 9.81$ m/s². Considering all quantities positive downwards,

$$U = -9.81 \text{ m/s}, g = +9.81 \text{ m/s}^2$$

and employing,

$$S = Ut + \frac{1}{2} gt^2$$

$$39.2 = -9.81t + \frac{1}{2}9.81t^2$$

whence

$$t^2 - 2t - 8 = 0$$

$$t = 4 \quad \text{or} \quad t = -2$$

Rejecting the negative value of time, $t = 4$ s.

Example 5.11

A car A is travelling on a straight level road with a uniform speed of 60 km/hr. It is followed by another car B moving at a speed of 70 km/hr. When the distance between them is 2.5 km, the car B is decelerated at 20 km/hr². Will the car B catch up with A ? If not, why not? If yes, at what distance and time?

Solution Let us suppose that car B catches up the car A in t hours. (If it doesn't, t will turn out to be negative or imaginary!)

In that time, A travels $60t$ km. The distance travelled by B will be given by

$$S = ut + \frac{1}{2}at^2$$

$$= 70t - \frac{1}{2} \times 20t^2 = 70t - 10t^2$$

Since the car A is already leading by 2.5 km, the condition for B to catch up with A is

$$70t - 10t^2 = 60t + 2.5$$

whence, $10t^2 - 10t + 2.5 = 0$, $t = 0.5$ hr.

In that time, A travels $60 \times 0.5 = 30$ km and B travels $70 \times 0.5 - 10 \times 0.5^2$ i.e., 32.5 km.

Example 5.12

A stone is dropped gently from the top of a tower. During its last one second of motion it falls through 64% of the height. Find the height of the tower.

Solution Assuming that the total time of fall is t seconds,

$$h = \frac{1}{2}gt^2 \tag{i}$$

In $(t - 1)$ seconds it falls through a distance given by $\frac{1}{2}g(t - 1)^2$ which is only 36% of its height,

$$0.36h = \frac{1}{2}g(t - 1)^2 \tag{ii}$$

Dividing (i) by (ii),

$$\frac{1}{0.36} = \frac{t^2}{t^2 + 1 - 2t}$$

whence

$$0.64t^2 - 2t + 1 = 0 \quad \text{and} \quad t = 2.5 \text{ seconds}$$

From (i),

$$h = \frac{1}{2} \times 9.81 \times 2.5^2 = 30.65 \text{ m.}$$

Example 5.13

Two motor cars start from A simultaneously and pass through B after 2 hours on the same road. The first travelled half the distance at a speed of 30 km/hr and the other half at a speed of 60 km/hr. The other car covered the entire distance with a constant acceleration. At what instants of time were the speeds of both the vehicles the same?

Solution Let the distance AB be $2x$.

From the data for the first car,

$$\frac{x}{30} + \frac{x}{60} = 2 \text{ hrs}; \quad x = 40, \quad 2x = 80 \text{ km.}$$

For the second car,

$$80 = 0 + \frac{1}{2}at^2 = \frac{1}{2}az^2$$

$$a = 40 \text{ km/hr}^2.$$

$$V = 0 + 40 \times 2 = 80 \text{ km/hr at } B$$

At any instant of time, for car B

$$V = 40 t$$

It becomes 30 km/hr at $t = 3/4$ hours and it becomes 60 km/hr at $t = 3/2$ hours after departure from A .

Let us check the state of the first car at these timings. At $t = 3/4$ hours, it was $30 \times 3/4$, i.e., 22.5 km from A running at 30 km/hr. At $t = 3/2$, it was indeed running at 60 km/hr, having crossed the 40 km mark. Hence, the two had the same speed at these two timings. It can be checked that there was no overtaking!

Example 5.14

An elevator ascends with an upward acceleration of 1.2 m/s^2 . At the instant when the upward speed is 2.4 m/s , a loose bolt drops from the ceiling of the elevator located 2.75 m from its floor. Calculate:

- the time of flight of the bolt from ceiling to floor of the elevator
- the displacement and the distance covered by the bolt during the free fall relative to the elevator shaft.

Solution

(a) The bolt, initially travelling up with a velocity of 2.4 m/s drops under gravity; its downwards displacement is

$$S_1 = -2.4t + \frac{1}{2} \times 9.81t^2$$

The floor of the elevator is displaced up in the same time,

$$S_2 = 2.4t + \frac{1}{2} \times 1.2t^2$$

The sum of the two must be 2.75 m .

$$\text{Hence,} \quad (4.9 + 0.6)t^2 = 2.75; \quad t = 0.707 \text{ s.}$$

(b) Displacement of the bolt is given by

$$S_1 = -2.4 \times 0.707 + \frac{1}{2} \times 9.81 \times 0.707^2 = 0.75 \text{ m}$$

The distance travelled by the bolt is the sum of distances it goes up first and then comes down. It goes up until its velocity becomes zero,

$$v^2 - u^2 = -2as$$

$$0 - 2.4^2 = -2 \times 9.81 \times s ; \quad s = 0.293 \text{ m}$$

It comes down such that the final downward displacement is 0.75 m, i.e., it traverses 0.293 m down and then 0.75 m down, making a total distance of $0.293 + 0.293 + 0.75 = 1.34 \text{ m}$.

Example 5.15

A bullet is projected so as to graze the tops of two walls each of height 20 m located at distances of 30 and 170 m in the same line from the point of projection as shown in Fig. Ex. 5.15. Find the angle and the speed of projection of the bullet.

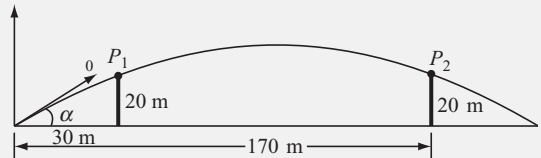


Fig. Ex. 5.15

Solution

From the equation,

$$y = x \tan \alpha - \frac{g x^2}{2 u_0^2 \cos^2 \alpha}$$

for the two points P_1 and P_2 ,

$$20 = 30 \tan \alpha - \frac{9.81 \times 30^2}{2 u_0^2 \cos^2 \alpha}$$

$$= 30 \tan \alpha - \frac{4415}{u_0^2 \cos^2 \alpha} \tag{i}$$

and

$$20 = 170 \tan \alpha - \frac{9.81 \times 170^2}{2 u_0^2 \cos^2 \alpha} = 170 \tan \alpha - \frac{141755}{u_0^2 \cos^2 \alpha} \tag{ii}$$

we can determine both α and u_0 .

Calling ' $u_0^2 \cos^2 \alpha$ ' as a parameter

$$642 = 963 \tan \alpha - \frac{141755}{u_0^2 \cos^2 \alpha}$$

$$20 = 170 \tan \alpha - \frac{141755}{u_0^2 \cos^2 \alpha}$$

whence

$$\tan \alpha = 0.784 ; \alpha = 38.1$$

Substituting the value of α in (i),

$$u_0 = 45 \text{ m/s}$$

Example 5.16

A shell bursts on contact with the ground and pieces fly off in all directions with speeds up to 30 m/s. A person is standing 40 m away. What is the time duration over which he can be hit by a piece?

Solution With the maximum initial velocity of 30 m/s, the horizontal range of 40 m requires that

$$40 = \frac{30^2 \times \sin 2\alpha}{9.81}; \quad \sin 2\alpha = 0.435$$

whence

$$2\alpha = 25.8 \text{ or } 154.2; \quad \alpha = 12.9 \text{ or } 77.1$$

The first piece that can hit the person has an angle of projection of 12.9° and initial velocity 30 m/s takes time

$$t_1 = \frac{2 \times 30 \times \sin 12.9}{9.81} = 1.36 \text{ seconds.}$$

The last piece that can hit the person has an angle of projection of 77.1° and initial velocity 30 m/s which takes time $t_2 = \frac{2 \times 30 \times \sin 77.1}{9.81} = 5.96$ seconds.

How about the maximum time taken by a piece at less than 30 m/s? Well, then the higher value of α would be less than 77.1° and hence t_2 would be less.

The duration over which the person can be hit by a piece is a period of 4.6 seconds, beginning 1.36 seconds and ending 5.96 seconds after the bursting of the shell.

Example 5.17

The maximum horizontal range of a gun is R_{\max} . Determine the firing angle which should be used to hit a target located at a distance $R_{\max}/2$ on the same level.

Solution The range of a bullet fired with a velocity V_0 at an angle α is given by

$$R = \frac{V_0^2 \sin 2\alpha}{g} \quad (i)$$

This would be maximum for $\alpha = 45^\circ$

$$R_{\max} = V_0^2/g$$

The target is located at

$$R = R_{\max}/2 = V_0^2/2g \quad (ii)$$

Equating it to (i),

$$\begin{aligned} \frac{V_0^2 \sin 2\alpha}{g} &= \frac{V_0^2}{2g} \\ \sin 2\alpha &= \frac{1}{2} \end{aligned}$$

whence

$$2\alpha = 30^\circ \quad \text{and} \quad \alpha = 15^\circ$$

The gun should therefore, be fired at an inclination of 15° to hit the target.

One may check the result by considering the equation of trajectory,

$$z = x \tan \alpha - \frac{g x^2}{2V_0^2 \cos^2 \alpha}$$

For the target,

$$z = 0, \quad x = V_0^2/2g$$

$$0 = \frac{V_0^2}{2g} \tan \alpha - \frac{V_0^2}{8g \cos^2 \alpha}$$

whence,
$$\tan \alpha = \frac{1}{4 \cos^2 \alpha}; \sin 2\alpha = 1/2$$

which is the same as determined earlier.

5.7 MOTION REFERRED TO MOVING FRAMES OF REFERENCE

In order to arrive at the description of the motion of a point with reference to an inertial frame for the application of Newton's laws it may be necessary to first ascertain the motion in relation to a moving frame and then refer it to an inertial frame for the sake of convenience. The moving frame may, in general, translate and rotate as well as accelerate linearly or angularly. It is the purpose of the following treatment to arrive at a systematic procedure for referring the space motion of a point with respect to a frame of reference if its motion is known in relation to another frame moving with respect to the former.

This task is achieved through a series of simple steps for the sake of clarity and understanding.

(a) Relative Motion of Two Points

Consider a pair of points, P_1 and P_2 moving with the velocities v_1 and v_2 and the accelerations a_1 and a_2 with respect to a fixed frame of reference. The velocity of P_1 with respect to P_2 is given by

$$v_{12} = v_1 - v_2 \quad (5.34)$$

and the acceleration of P_1 with respect to P_2 is given by

$$a_{12} = a_1 - a_2 \quad (5.35)$$

This is demonstrated in Fig. 5.10. It can also be appreciated that

$$v_{21} = v_2 - v_1 = -v_{12}$$

and

$$a_{21} = a_2 - a_1 = -a_{12}$$

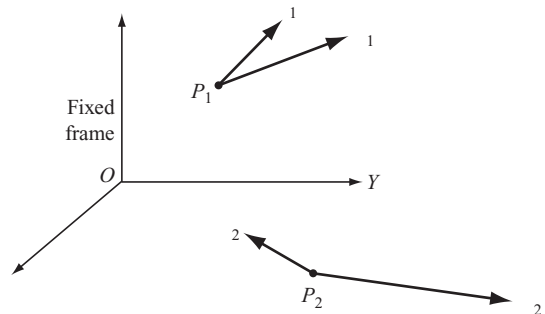


Fig. 5.10 Relative velocity and relative acceleration

Example 5.18

Two roads cross at 60° angle at an intersection at O. At an instant of time, a scooter 1 at 100 m east of the intersection moving at a velocity of 50 km/hr decelerates at 5 km/hr/s as it approaches the intersection O. At the same instant, a car 2 passes the intersection at a velocity of 20 km/hr and accelerates at 10 km/hr/s as shown in Fig. Ex. 5.18. Determine (a) the velocity of the scooter with respect to the car and (b) the acceleration of the car with respect to the scooter at the instant of observation as well as after a lapse of 3 s and, in general after t seconds.

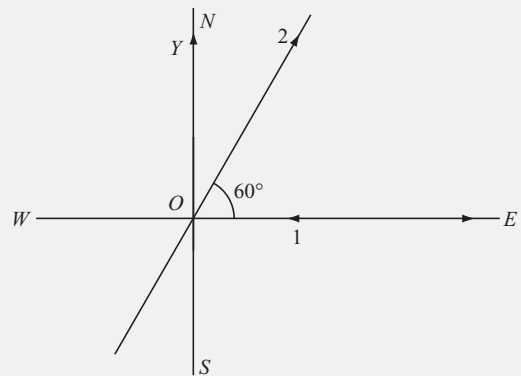


Fig. Ex. 5.18

Solution Choosing the x and Y axes as shown,
 At the instant of observation,

$$\begin{aligned} v_1 &= -50 \text{ km/hr} \\ a_1 &= +5 \text{ km/hr/s} \\ v_2 &= 20(\sin 60^\circ + \cos 60^\circ) \\ &= (17.32 + 10) \text{ km/hr} \\ v_2 &= 10(\sin 60^\circ + \cos 60^\circ) \\ &= (8.66 + 5) \text{ km/hr/s} \end{aligned}$$

(a) $v_{12} = v_1 - v_2 = -50 - 17.32 - 10$
 $= (-67.32 - 10) \text{ km/hr}$

(b) $a_{21} = a_2 - a_1 = 8.66 + 5 - 5$
 $= (3.66 + 5) \text{ km/hr/s}$

At an instant after a lapse of 3 s

$$\begin{aligned} v_1 &= -50 + 3 \times 5 = -35 \text{ km/hr} \\ v_2 &= 17.32 + 10 + 3(8.66 + 5) \\ &= (43.3 + 25) \text{ km/hr} \end{aligned}$$

(a) $v_{12} = v_1 - v_2 = -35 - 43.3 - 25$
 $= (-78.3 - 25) \text{ km/hr}$

(b) the relative acceleration remains the same;

$$a_{21} = (3.66 + 5) \text{ km/hr/s}$$

In general, after a lapse of t s

$$\begin{aligned} v_1 &= -50 + 5t = (-50 + 5t) \text{ km/hr} \\ v_2 &= 17.32 + 10 + (8.66 + 5)t \\ &= (17.32 + 8.66t) + (10 + 5t) \text{ km/hr} \end{aligned}$$

The relative velocity of the scooter with respect to the car is

$$\begin{aligned} v_{12} &= (-50 + 5t - 17.32 - 8.66t) - (10 + 5t) \text{ km/hr} \\ &= (-67.32 - 3.66t) - (10 + 5t) \text{ km/hr} \end{aligned}$$

The relative acceleration remains the same as at the earlier instants because the acceleration of each vehicle is unaltered with time.

$$a_{21} = (3.66 + 5) \text{ km/hr/s}$$

Example 5.19 A rod follower AB is subjected to a vertical up-and-down movement while resting on the circular contour of radius 30 cm of a cam. The cam moves to the right with a velocity of 5 cm/s and an acceleration of 10 cm/s². Find the velocity and acceleration of the point B on the rod at the instant of interest as shown in Fig. Ex. 5.19.

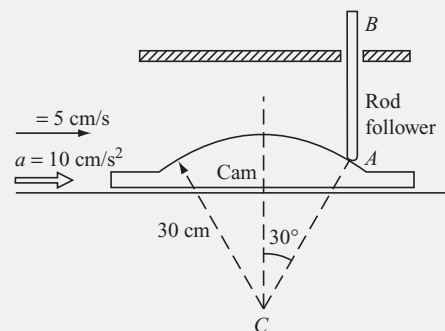


Fig. Ex. 5.19

Solution It is an interesting and a fairly simple problem which can be solved by a variety of methods.

Let us demonstrate the power of two different methods by solving this problem.

Method 1

The point B moves with the same velocity and acceleration as A . The motion of A may in turn be related to the motion of A' , a coincident point on the cam

$$\begin{aligned} \mathbf{v}_A &= \mathbf{v}_{A'} + \mathbf{v}_{AA'} \\ \mathbf{v}_A &= 0.05 \mathbf{j} + \mathbf{v}_{AA'}(-0.866 \mathbf{i} + 0.5 \mathbf{j}) \end{aligned} \quad (i)$$

since the velocity of A must be along the y -axis, that of A' is given along the x -axis and the velocity of A with respect to A' is assumed up the tangent intuitively.

From Eq. (i),

$$\begin{aligned} 0.05 - 0.866 V_{AA'} &= 0 \\ V_A - 0.5 V_{AA'} &= 0 \end{aligned}$$

whence

$$\begin{aligned} V_{AA'} &= 0.0577 \text{ m/s} \\ V_A &= 0.0289 \text{ m/s} = V_B \end{aligned}$$

Similarly, for the acceleration,

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_{A'} + \mathbf{a}_{AA'} + \frac{V_{AA'}^2}{0.3}(-0.5 \mathbf{i} - 0.866 \mathbf{j}) \\ &= 0.1 \mathbf{i} + \mathbf{a}_{AA'}(-0.866 \mathbf{i} + 0.5 \mathbf{j}) + \frac{0.0577^2}{0.3}(-0.5 \mathbf{i} - 0.866 \mathbf{j}) \end{aligned} \quad (ii)$$

From this, the x -component of acceleration is equated to zero.

$$0.1 - 0.866 a_{AA'} - 0.0055 = 0$$

Then,

$$a_A = 0.5 a_{AA'} + 0.0096$$

whence

$$\begin{aligned} a_{AA'} &= 0.109 \text{ m/s}^2 \\ a_A &= -0.045 \text{ m/s}^2 = a_B \end{aligned}$$

Hence, the point B moves up with a velocity of 0.0289 m/s and decelerates at 0.045 m/s² while moving upward.

Method 2

One may like to visualise the cam to be at rest and the point A of the rod follower to slide up the circular path with the x -component of velocity -5 cm/s and x -component of acceleration -10 cm/s², the equation of the path, then being

$$x^2 + y^2 = (0.3)^2$$

Differentiating it with respect to time t ,

$$x\dot{x} + y\dot{y} = 0 \quad (i)$$

and differentiating it again with respect to time t ,

$$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0 \quad (ii)$$

The position of A is given by

$$x = 0.3 \sin 30^\circ = 0.15 \text{ m}$$

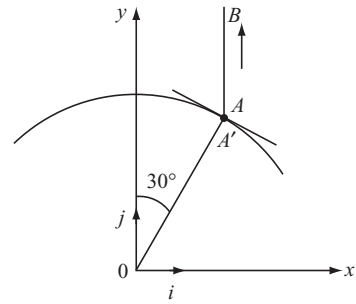


Fig. Ex. 5.19 (Solution)

$$y = 0.3 \cos 30 = 0.26 \text{ m}$$

From Eq. (i)

$$\dot{y} = -\frac{0.15 \times (-0.05)}{0.26} = 0.0288 \text{ m/s}$$

which must be the velocity of *A* and of *B*.

From Eq. (ii)

$$\ddot{y} = -\frac{0.15 \times (-0.1) + (-0.05)^2 + 0.0288^2}{0.26} = 0.045 \text{ m/s}^2$$

which must be the acceleration of *A* and *B*.

(b) Translation of a Moving Frame

If a moving frame of reference *x-y-z* translates with respect to a fixed frame *X-Y-Z*, as shown in Fig. 5.11, then any pair of parallel lines in the two frames remain parallel to each other with the passage of time. It follows that the unit vectors *i*, *j*, and *k* in the moving frame remain parallel to themselves and their magnitudes remaining unity, *the unit vectors remain invariant*.

If the velocity of the moving frame is \dot{O} and the velocity of a point with respect to the moving frame is P_m , the velocity of the point with respect to the fixed frame is given by

$$P_f = P_m + \dot{O} \tag{5.36}$$

Similarly, the acceleration of a point with respect to the fixed frame is

$$P_{f''} = P_{m''} + \ddot{O} \tag{5.37}$$

where $P_{m''}$ is the acceleration of the point with respect to the moving frame and \ddot{O} is the linear acceleration of the moving frame with respect to the fixed frame.

It should be understood that a translating frame may translate either in space or in a plane or along a straight line. The velocity \dot{O} and acceleration \ddot{O} of the moving origin may, in general, be in different directions at any instant of time. Only for the case of rectilinear translation, i.e., for motion along a straight line would \dot{O} and \ddot{O} be collinear.

(c) Rotation of a Moving Frame

Consider a moving frame of reference *x-y-z* and a fixed or inertial reference frame *X-Y-Z*. Infinitesimal rotations $d\theta_x$, $d\theta_y$ and $d\theta_z$ of the moving frame are specified about the *x*, *y* and *z* axes respectively and are represented by vector components along the respective axes with *i*, *j*, *k* unit vectors as shown in Fig. 5.12.

Angular velocity components ω_x , ω_y and ω_z refer to the rates of rotation about *x*, *y* and *z* axes respectively.

$$\begin{aligned} \omega &= \omega_x i + \omega_y j + \omega_z k \\ &= \frac{d\theta_x}{dt} i + \frac{d\theta_y}{dt} j + \frac{d\theta_z}{dt} k \end{aligned}$$

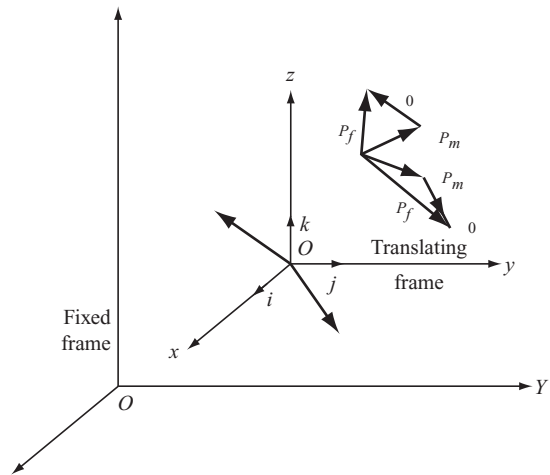


Fig. 5.11 Translation of a moving frame

(d) Derivatives of Moving Unit Vectors

Consider the change in the unit vector \mathbf{i} observed from the fixed frame $(d)_f$, due to the differential rotation $(d\theta_x, d\theta_y, d\theta_z)$ to the moving frame as shown in Fig. 5.12:

- $d\theta_x$ produces no change in \mathbf{i}
- $d\theta_y$ produces a change $= -d\theta_y \mathbf{j}$ in \mathbf{i}
- $d\theta_z$ produces a change $= d\theta_z \mathbf{k}$ in \mathbf{i}
- Total change $(d)_f \mathbf{i} = d\theta_z \mathbf{k} - d\theta_y \mathbf{j}$

and the rate of change of \mathbf{i} with respect to t is given by

$$\left(\frac{d}{dt}\right)_f \mathbf{i} = \frac{d\theta_z}{dt} \mathbf{k} - \frac{d\theta_y}{dt} \mathbf{j} = \omega_z \mathbf{k} - \omega_y \mathbf{j}$$

which is the same as $\omega \times \mathbf{i}$

Hence
$$\left(\frac{d}{dt}\right)_f \mathbf{i} = \omega \times \mathbf{i}$$

Similarly, expressions for the rates of change of \mathbf{j} and \mathbf{k} can be obtained:

$$\left(\frac{d}{dt}\right)_f \mathbf{j} = \omega \times \mathbf{j}$$

$$\left(\frac{d}{dt}\right)_f \mathbf{j} = \omega \times \mathbf{j}$$

$$\left(\frac{d}{dt}\right)_f \mathbf{j} = \omega \times \mathbf{j}$$

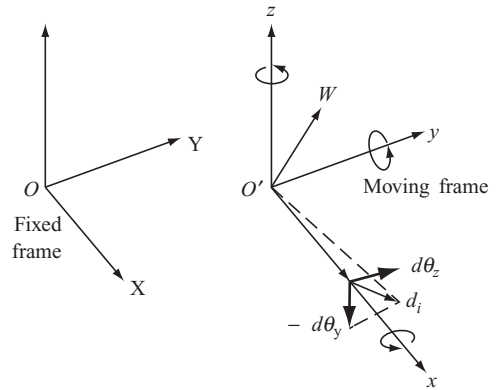


Fig. 5.12 Rotation of a moving frame

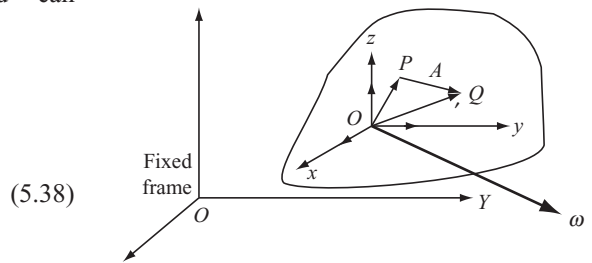


Fig. 5.13 Reference to fixed and moving frames

(e) Derivative of a Constant Vector in a Moving Frame

Consider a fixed position vector \mathbf{r} of a point P fixed with respect to a moving frame x - y - z which is rotating at an angular velocity ω as shown in Fig. 5.13.

The axis of rotation must be along the ω vector. Writing the position vector of P and differentiating it with respect to t as observed from the moving frame,

$$\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

$$\left(\frac{d}{dt}\right)_f \mathbf{r} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} + x \left(\frac{d}{dt}\right)_f \mathbf{i} + y \left(\frac{d}{dt}\right)_f \mathbf{j} + z \left(\frac{d}{dt}\right)_f \mathbf{k}$$

Since the position vector is fixed with respect to the moving system,

$$\left(\frac{d}{dt}\right)_m \mathbf{r} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} = 0$$

$$\left(\frac{d}{dt}\right)_f \mathbf{r} = x(\omega \times \mathbf{i}) + y(\omega \times \mathbf{j}) + z(\omega \times \mathbf{k})$$

$$= \omega \times (x \mathbf{i} + y \mathbf{j} + z \mathbf{k})$$

$$= \omega \times \mathbf{r}$$

Hence,

$$\left(\frac{d}{dt}\right)_f = \omega \times$$

Let there be a vector \mathbf{r} fixed with respect to the x - y - z frame rotating at ω . If the position vectors of its end are \mathbf{r} and \mathbf{r}' , then $\mathbf{r}' = \mathbf{r} + \delta\mathbf{r}$.

$$\begin{aligned} \left(\frac{d}{dt}\right)_f \mathbf{r} &= \omega \times \mathbf{r} \\ \left(\frac{d}{dt}\right)_f \mathbf{r}' &= \omega \times \mathbf{r}' \\ \left(\frac{d}{dt}\right)_f (\mathbf{r}' - \mathbf{r}) &= \omega \times (\mathbf{r}' - \mathbf{r}) \\ \left(\frac{d}{dt}\right)_f \delta\mathbf{r} &= \omega \times \delta\mathbf{r} \end{aligned} \tag{5.39}$$

(f) Derivatives of a Position Vector for Different References

Let the position vector of a moving point P be referred to the moving frame x - y - z and \mathbf{R} referred to the fixed frame X - Y - Z .

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Differentiating \mathbf{R} with respect to t observed from the moving frame and fixed frame respectively,

$$\begin{aligned} \left(\frac{d}{dt}\right)_m \mathbf{R} &= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \\ \left(\frac{d}{dt}\right)_f \mathbf{R} &= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} + x \frac{d\mathbf{i}}{dt} + y \frac{d\mathbf{j}}{dt} + z \frac{d\mathbf{k}}{dt} \\ &= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} + x(\omega \times \mathbf{i}) + y(\omega \times \mathbf{j}) + z(\omega \times \mathbf{k}) \\ &= \left(\frac{d}{dt}\right)_m \mathbf{R} + \omega \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \left(\frac{d}{dt}\right)_m \mathbf{R} + \omega \times \mathbf{R} \end{aligned} \tag{5.40}$$

(g) Velocity of a Point

The velocity of a point relative to a reference frame is the time derivative, as seen from the reference of the position vector with respect to that reference. Referring to Fig. 5.14.

$$\mathbf{v}_f = \left(\frac{d\mathbf{R}}{dt}\right)_f$$

$$\mathbf{v}_m = \left(\frac{d}{dt}\right)_m \mathbf{R}$$

Differentiating \mathbf{R} with respect to time for the f reference

$$\left(\frac{d\mathbf{R}}{dt}\right)_f = \left(\frac{d\mathbf{R}_0}{dt}\right)_f + \left(\frac{d}{dt}\right)_f \mathbf{R}$$

or

$$\mathbf{v}_f = \mathbf{v}_0 + \left[\left(\frac{d}{dt}\right)_m \mathbf{R} + \omega \times \mathbf{R} \right]$$

$$P_f = P_m + a_0 + \omega \times R \tag{5.41}$$

This is interpreted by stating that the velocity P_f of a particle as observed from a fixed reference frame must be the vector sum of the velocity P_m as observed from a moving reference frame, the velocity a_0 of the moving origin and the term $\omega \times R$ due to rotation of the moving frame at an angular velocity ω with respect to the fixed frame.

(h) Acceleration of a Point

The acceleration of a point relative to a reference frame is the time derivative, as seen from the reference frame of the velocity relative to that reference. Referring to Fig. 5.14 again,

$$\begin{aligned}
 P_f &= \left(\frac{d}{dt} P_f \right)_f \\
 P_m &= \left(\frac{d}{dt} P_m \right)_m = \left(\frac{d^2}{dt^2} \right)_m \\
 P_f &= \left(\frac{d}{dt} P_f \right)_f \\
 &= a_0 + \left(\frac{d}{dt} P_m \right)_f + \left(\frac{d}{dt} (\omega \times R) \right)_f \\
 \left(\frac{d}{dt} P_m \right)_f &= \left(\frac{d}{dt} P_m \right)_m + \omega \times P_m \\
 &= P_m + \omega \times P_m \\
 \left(\frac{d}{dt} (\omega \times R) \right)_f &= \omega \times \left(\frac{d}{dt} R \right)_f + \left(\frac{d\omega}{dt} \right)_f \times R \\
 &= \omega \times \left[\left(\frac{d}{dt} \right)_m + \omega \times \right] R + \alpha \times R \\
 &= \omega \times P_m + \omega \times (\omega \times R) + \alpha \times R
 \end{aligned}$$

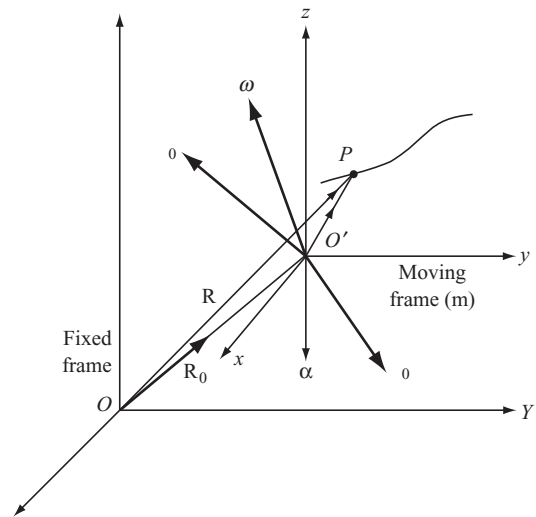


Fig. 5.14 Motion observed from fixed and moving frames

Finally,

$$P_f = P_m + a_0 + \alpha \times R + 2\omega \times P_m + \omega \times (\omega \times R) \tag{5.42}$$

The interpretation of the terms is as follows:

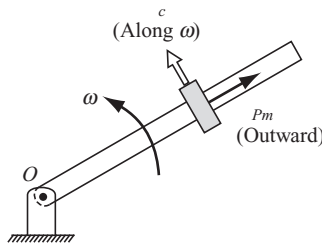
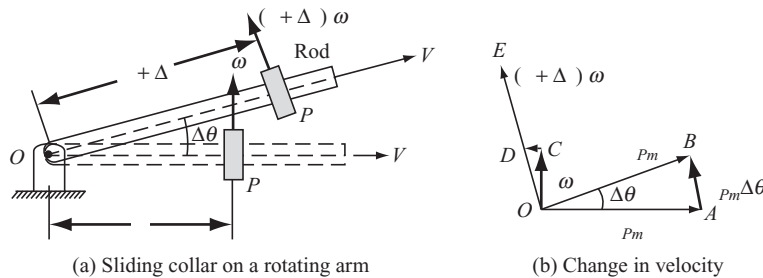
- P_f acceleration of the point P with respect to the fixed frame of reference.
- P_m acceleration of the point P with respect to the moving frame of reference.
- a_0 translational acceleration of the origin of the moving frame with respect to the fixed origin.
- $\alpha \times R$ acceleration due to the moving frame accelerating with an angular acceleration α ; the tangential component as seen from the fixed origin. It vanishes when α and R are parallel, collinear or when either is zero.
- $2\omega \times P_m$ Coriolis component of acceleration due to ω , the rotation of the moving frame and P_m , the relative velocity of the point P in the moving frame. It is zero when ω is parallel to or collinear with P_m or either of them vanishes.

$\omega \times (\omega \times r)$ normal component of acceleration as seen from the fixed frame; also called centripetal acceleration, which for plane circular motion of the point becomes $\omega^2 r$. It vanishes where ω and r are collinear or when either ω or r is zero.

(i) A Note on Coriolis Acceleration

It is interesting to demonstrate the origin of Coriolis acceleration and the concept of its direction physically by a simple example.

Consider a slidable collar P made to slide at a constant velocity $v_{P/m}$ with respect to a rod rotating at a constant angular velocity ω about O as shown in Fig. 5.15(a). Placing a frame of reference on the rod, we notice that the relative acceleration of P with respect to the rotating rod, angular acceleration of the rotating frame and the absolute acceleration of the origin on the rotating frame are specified as zero. For this simple case of plane motion only, the centripetal component $\omega \times (\omega \times r)$ and the Coriolis component $2\omega \times v_{P/m}$ are non-zero. The former, $\omega^2 r$ in magnitude, is directed radially towards O whereas the latter, $2\omega \times v_{P/m}$ is directed at right angles to $v_{P/m}$ and ω .



(c) Sign convention 'OUT-ALONG'

Fig.5.15 Coriolis acceleration

In order to appreciate the origin of the Coriolis acceleration, consider the change in system configuration over a small time interval Δt as shown in Fig. 5.15(b). Initially, the velocity of P was made up of two components:

- Radial: $v_{P/m}$ shown by OA
- Tangential: ωr shown by OC

Finally, after a lapse of time Δt , the components of the velocity of P are:

- Radial: $v_{P/m}$ shown by OB
- Tangential: $(\omega + \Delta\omega) r$ shown by OE

The initial velocity vector P can be obtained by adding OA and OC and the final velocity vector by adding OB and OE , but these are purposely not shown to avoid unnecessary details. It is preferred to show the change of the velocity of P in three stages:

1. Due to the rotation of the relative sliding velocity

$$AB = v_{P/m} \Delta\theta$$

- Due to the rotation of the rod

$$CD = \omega \Delta\theta$$

- Due to the sliding effect on the rotational velocity

$$DE = (\omega + \Delta\omega) \omega - \omega = \Delta\omega$$

It is the sum of AB and DE , divided by Δt , in the limiting case which is called Coriolis acceleration:

$$\begin{aligned} \text{Coriolis acceleration } a_c &= \lim_{\Delta t \rightarrow 0} \left(\frac{P_m \cdot \Delta\theta + \Delta\omega \cdot \omega}{\Delta t} \right) \\ &= \omega \frac{d\theta}{dt} + \frac{d\omega}{dt} \omega = 2\omega \frac{d\theta}{dt} + \omega \frac{d\omega}{dt} \end{aligned}$$

or

$$a_c = 2\omega v_{pm}$$

directed perpendicular to the rod as indicated by the limiting case of the velocity diagram for $\Delta t \rightarrow 0$ and $\Delta\omega$ getting smaller.

Similarly, the rate of change of CD corresponds to the centripetal acceleration:

$$\text{Centripetal acceleration } a_r = \lim_{\Delta t \rightarrow 0} \left[\frac{\omega \cdot \Delta\theta}{\Delta t} \right]$$

or

$$a_r = \omega \frac{d\theta}{dt} = \omega^2 r$$

directed towards the centre of rotation of the rod as also observed from the limiting case of the velocity diagram.

It is sometimes difficult to visualise the direction of the Coriolis acceleration. Actually, there is no difficulty about the direction of any component of acceleration when proceeding vectorially but a rule or sign convention (shown in Fig. 5.15(c)) may be stated for the graphical procedure applicable to the plane motion of rigid bodies:

If the slider collar moves radially with respect to the centre of rotation of the arm on which it slides, the Coriolis acceleration of the collar is in the direction of rotation. This is called the Out-Along convention.

Example 5.20

A platform as shown in Fig. Ex. 5.20 rotates about its axis with an angular speed of 2 rad/s counterclockwise and decelerates with an angular deceleration of 3 rad/s². A rod OA rotates about the hinge O with an angular velocity 4 rad/s and accelerates at a rate 5 rad/s² with respect to the platform. A collar B slides outward on the rod OA with a velocity of 2 m/s and an acceleration of 3 m/s² with respect to the rod. Compute the absolute velocity and acceleration of the collar if it is at 0.5 m from O .

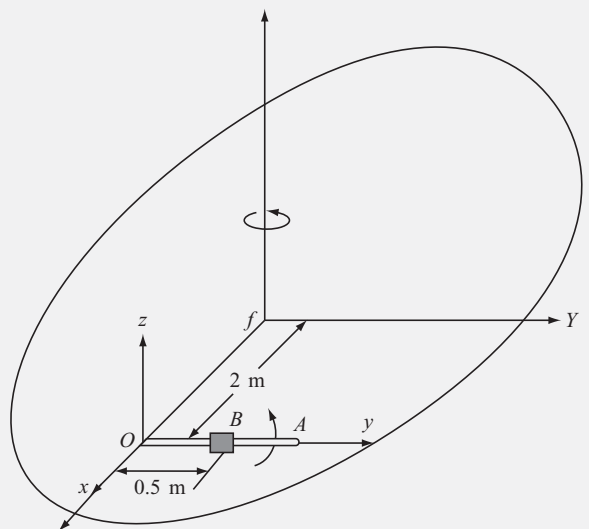


Fig. Ex. 5.20

Solution Let the fixed frame Y be fixed with the ground with its origin coincident with the centre of the platform and let the moving frame xyz be attached to the rod OA with its origin at O .

Next, it is necessary to identify the motion of the collar with respect to the moving frame and the motion of the moving frame with respect to the fixed frame.

The collar moves with respect to the moving frame identified with the rod such that

$$= 0.5 \text{ m/s}, \quad v_{Pm} = 2 \text{ m/s} \quad \text{and} \quad v_{Pm} = 3 \text{ m/s}$$

The moving frame moves with respect to the fixed frame such that

$$\omega = 2 \mathbf{i} + 4 \mathbf{j} = 6 \mathbf{k}$$

$$\alpha = \dot{\omega} = 5 \mathbf{k} - 3 \mathbf{k} = 2 \mathbf{k}$$

$$R_0 = 2 \mathbf{i}$$

$$V_0 = \dot{R}_0 = 2 \mathbf{k} \times 2 \mathbf{i} = 4 \mathbf{j}$$

$$a_0 = \ddot{R}_0 = -3 \mathbf{j} \times 2 \mathbf{i} + 2 \mathbf{k} \times 4 \mathbf{j} = -6 \mathbf{i} - 8 \mathbf{j}$$

The absolute velocity and acceleration of the collar are determined by recalling the corresponding expressions:

$$v_{Pf} = v_{Pm} + v_0 + \omega \times R_0 = 2 \mathbf{i} + 4 \mathbf{j} + 6 \mathbf{k} \times 0.5 \mathbf{i}$$

$$= (-3 \mathbf{j} + 6 \mathbf{k}) \text{ m/s}$$

$$a_{Pf} = a_{Pm} + a_0 + \alpha \times R_0 + 2 \omega \times v_{Pm} + \omega \times (\omega \times R_0)$$

$$= 3 \mathbf{i} - 6 \mathbf{j} - 8 \mathbf{k} + 2 \mathbf{k} \times 0.5 \mathbf{i} + 2(6 \mathbf{k} \times 2 \mathbf{i}) + 6 \mathbf{k} \times (6 \mathbf{k} \times 0.5 \mathbf{i})$$

$$= (-33 \mathbf{i} - 21 \mathbf{j}) \text{ m/s}^2$$

Alternatively, the moving frame xyz could have been attached to the platform with the origin either at O or at the centre of the platform. If the origin is located at O , the collar moves with respect to the moving frame such that

$$v_{Pm} = 0.5 \mathbf{i}$$

$$v_{Pm} = 2 \mathbf{i} + 4 \mathbf{j} \times 0.5 \mathbf{i} = (-2 \mathbf{j} + 2 \mathbf{k}) \text{ m/s}$$

$$a_{Pm} = (\ddot{r} - r\dot{\theta}^2) \mathbf{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \boldsymbol{\theta} + \ddot{z} \mathbf{z}$$

$$= (3 - 0.5 \times 4^2) \mathbf{i} - (0.5 \times 5 + 2 \times 2 \times 4) \mathbf{j}$$

$$= (-18.5 \mathbf{i} - 5 \mathbf{j}) \text{ m/s}^2$$

The motion of the moving frame with respect to the fixed frame is such that

$$\omega = 2 \mathbf{i} + 4 \mathbf{j} = 6 \mathbf{k}$$

$$\alpha = \dot{\omega} = -3 \mathbf{k}$$

$$R_0 = 2 \mathbf{i}$$

$$v_0 = \dot{R}_0 = 4 \mathbf{j}$$

$$a_0 = \ddot{R}_0 = -6 \mathbf{j} - 8 \mathbf{i}$$

Again, from the expressions for the absolute velocity and acceleration,

$$v_{Pf} = v_{Pm} + v_0 + \omega \times R_0$$

$$= -2 \mathbf{j} + 2 \mathbf{k} + 4 \mathbf{j} + 2 \mathbf{k} \times 0.5 \mathbf{i}$$

$$= (-3 \mathbf{j} + 6 \mathbf{k}) \text{ m/s}$$

$$a_{Pf} = a_{Pm} + a_0 + \alpha \times R_0 + 2 \omega \times v_{Pm} + \omega \times (\omega \times R_0)$$

$$= -18.5 \mathbf{i} - 5 \mathbf{j} - 6 \mathbf{k} - 8 \mathbf{i} - 3 \mathbf{k} \times 0.5 \mathbf{i}$$

$$+ 2(2 \mathbf{k} \times (-2 \mathbf{j} + 2 \mathbf{k})) + 2 \mathbf{k} \times (2 \mathbf{k} \times 0.5 \mathbf{i})$$

$$= (-33 \mathbf{i} - 21 \mathbf{j}) \text{ m/s}^2$$

Example 5.21

A crane rotates about a vertical axis with a constant angular velocity of 0.4 rad/s while the boom is being raised with a constant angular velocity of 0.5 rad/s relative to the cab as shown in Fig. Ex. 5.21. If the length of the boom is 10 m, determine

- (a) the angular velocity of the boom,
- (b) the angular acceleration of the boom,
- (c) the velocity of the tip of the boom and
- (d) the acceleration of the tip of the boom.

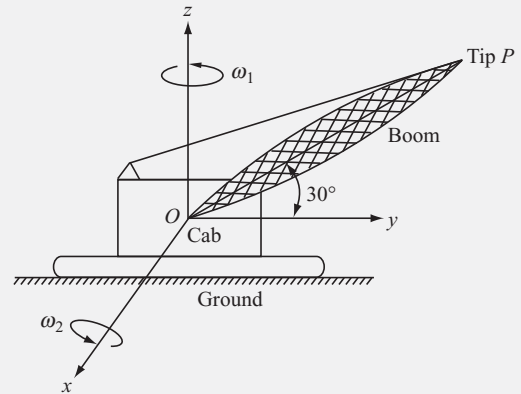


Fig. Ex. 5.21

Solution The problem may be tackled in a variety of ways by considering the moving frame attached to any moving component at any desired point. We shall consider some of the different possibilities with a view to gaining experience of analysing such motions.

Method 1

Moving frame $x-y-z$ attached to the rotating cabin with its origin at O and the fixed frame $X-Y-Z$ fixed in space but with its origin also at O .

The angular velocity ω of the moving frame is then 0.4 rad/s about the z axis, i.e.,

$$\omega = 0.4$$

From the given length $OP = 10$ m, the tip of the boom P is located by the position vector

$$= 10 (\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 8.66 \mathbf{i} + 5 \mathbf{j}$$

The velocity of P with respect to the fixed frame is given by

where

$$\begin{aligned} \mathbf{v}_{Pf} &= \mathbf{v}_{Pm} + \mathbf{v}_0 + \omega \times \mathbf{r}_{Pm} \\ \mathbf{v}_{Pm} &= 0.5 \times (8.66 \mathbf{i} + 5 \mathbf{j}) \\ &= 4.33 \mathbf{i} - 2.5 \mathbf{j} \end{aligned}$$

and

$$\mathbf{v}_0 = 0, \text{ the origin being coincident with } O$$

$$\omega \times \mathbf{r}_{Pm} = 0.4 \times (8.66 \mathbf{i} + 5 \mathbf{j}) = -3.465 \mathbf{j}$$

Hence,

$$\mathbf{v}_{Pf} = (-3.465 \mathbf{j} - 2.5 \mathbf{j} + 4.33 \mathbf{i}) \text{ m/s}$$

The angular velocity of the boom must be the sum of the angular velocity of the moving frame on the cab and the angular velocity of the boom with respect to the cab, i.e.,

$$\omega_{\text{boom}} = \omega_1 + \omega_2 = (0.4 + 0.5) \text{ rad/s}$$

The angular acceleration of the boom can be likewise visualised as that due to the rotation of the boom on a rotating cab.

$$\alpha_{\text{boom}} = \omega_1 \times \omega_2 = 0.4 \times 0.5 = 0.2 \text{ rad/s}^2$$

The acceleration of P with respect to the fixed frame is given by

where

$$\begin{aligned} \mathbf{a}_{Pf} &= \mathbf{a}_{Pm} + \mathbf{a}_0 + \alpha \times \mathbf{r}_{Pm} + 2 \omega \times \mathbf{v}_{Pm} + \omega \times (\omega \times \mathbf{r}_{Pm}) \\ \mathbf{a}_{Pm} &= 0.5 \times (0.5 \times (8.66 \mathbf{i} + 5 \mathbf{j})) = -2.16 \mathbf{i} - 1.25 \mathbf{j} \\ \mathbf{a}_0 &= 0 \end{aligned}$$

$$\begin{aligned} \alpha \times r_{Pm} &= 0 \\ 2 \omega \times r_{Pm} &= 2 \times 0.4 \times (4.33 \mathbf{i} - 2.5 \mathbf{j}) = 2 \\ \omega \times (\omega \times r_{Pm}) &= 0.4 \times (0.4 \times (8.66 \mathbf{i} + 5 \mathbf{j})) = -1.385 \end{aligned}$$

Hence,

$$a_{Pf} = -2.165 \mathbf{i} - 1.25 \mathbf{j} + 2 \mathbf{k} - 1.385 \mathbf{k} = (2 \mathbf{i} - 3.55 \mathbf{j} - 1.25 \mathbf{k}) \text{ rad/s}^2$$

Method 2

Moving frame x - y - z attached to the boom with its origin at O and the fixed frame fixed in space but with its origin also at O .

In this case,

$$\omega = 0.4 \mathbf{i} + 0.5 \mathbf{j}$$

is the angular velocity of the moving frame attached to the boom.

$$r_{Pm} = 8.66 \mathbf{i} + 5 \mathbf{j} \text{ as before}$$

$$a_{Pf} = a_{Pm} + a_o + \omega \times r_{Pm}$$

where $a_{Pm} = 0$, the point P being on the boom itself and

$$a_o = 0, \text{ the origin being coincident with } O.$$

$$\omega \times r_{Pm} = (0.4 \mathbf{i} + 0.5 \mathbf{j}) \times (8.66 \mathbf{i} + 5 \mathbf{j}) = -3.465 \mathbf{i} - 2.5 \mathbf{j} + 4.33 \mathbf{k}$$

Hence,

$$a_{Pf} = (-3.465 \mathbf{i} - 2.5 \mathbf{j} + 4.33 \mathbf{k}) \text{ m/s}$$

The angular acceleration of the boom is obtained by $\omega_1 \times \omega_2$ by imagining the moving frame undergoing a relative rotation with respect to a rotating frame.

$$\alpha_{\text{boom}} = 0.4 \mathbf{i} \times 0.5 \mathbf{j} = 0.2 \mathbf{k} \text{ rad/s}^2$$

This is also the angular acceleration of the moving frame in this case.

$$\alpha = \alpha_{\text{boom}} = 0.2 \mathbf{k} \text{ rad/s}^2$$

The acceleration of P with respect to the fixed frame is

$$a_{Pf} = a_{Pm} + a_o + \alpha \times r_{Pm} + 2\omega \times r_{Pm} + \omega \times (\omega \times r_{Pm})$$

where

$$a_{Pm} = 0$$

Now,

$$a_o = 0$$

$$\alpha \times r_{Pm} = 0.2 \mathbf{k} \times (8.66 \mathbf{i} + 5 \mathbf{j}) = 1 \mathbf{i} - 1.732 \mathbf{j}$$

$$2 \omega \times r_{Pm} = 0$$

$$\begin{aligned} \omega \times (\omega \times r_{Pm}) &= (0.4 \mathbf{i} + 0.5 \mathbf{j}) \times ((0.4 \mathbf{i} + 0.5 \mathbf{j}) \times (8.66 \mathbf{i} + 5 \mathbf{j})) \\ &= 1 \mathbf{i} - 3.55 \mathbf{j} - 1.25 \mathbf{k} \end{aligned}$$

Hence,

$$a_{Pf} = (2 \mathbf{i} - 3.55 \mathbf{j} - 1.25 \mathbf{k}) \text{ m/s}^2$$

Method 3

No moving frame at all - X - Y - Z fixed with reference to frame fixed in space with its origin at O .

The angular velocity of the boom is due to ω_1 of the cab and ω_2 of the boom with respect to the cab, i.e.,

$$\omega_{\text{boom}} = \omega_1 + \omega_2 = (0.4 \mathbf{i} + 0.5 \mathbf{j}) \text{ rad/s}$$

The angular acceleration of the boom is obtained by

$$\alpha_{\text{boom}} = \left(\frac{d\omega_{\text{boom}}}{dt} \right)_f = \left(\frac{d\omega_1}{dt} \right)_f + \left(\frac{d\omega_2}{dt} \right)_f$$

Now,

$$\left(\frac{d\omega_1}{dt} \right)_f = 0$$

because the cab rotates at a constant angular velocity with respect to the fixed frame but the boom has a constant angular velocity with respect to the cab. Observing that ω_2 rotates with the cab at ω_1

$$\alpha_{\text{boom}} = \left(\frac{d \omega_2}{dt} \right)_f = \omega_1 \times \omega_2$$

$$= (0.4 \times 0.5 = 0.2) \text{ rad/s}^2$$

The velocity of P is given by

$$v_P = \omega \times r = (0.4 + 0.5) \times (8.66 + 5)$$

$$= (-3.465 \hat{i} - 2.5 \hat{j} + 4.33 \hat{k}) \text{ m/s}$$

The acceleration of P is likewise computed:

$$a_P = \alpha_{\text{boom}} \times r + \omega_{\text{boom}} \times (\omega_{\text{boom}} \times r)$$

$$= 0.2 \times (8.66 + 5) + ((0.4 + 0.5) \times (0.4 + 0.5) \times (8.66 + 5))$$

$$= (2 \hat{i} - 3.55 \hat{j} - 1.25 \hat{k}) \text{ m/s}^2$$

Method 4

By multiple references. A moving frame $-Y-$ attached to the boom with its origin at O and another moving frame attached to the cab with its origin at O and a fixed frame $-Y-$ also with its origin at O .

The velocity and acceleration are now determined first with respect to the intermediate reference m_1 and then with respect to the fixed frame in the next step.

For the first step in respect of velocity.

$$v_{Pm1} = v_{Pm} + v_{Om} + \omega \times r$$

where

$$v_{Pm} = 0$$

$$v_{Om} = 0$$

$$\omega \times r = 0.5 (8.66 + 5) = 4.33 \hat{i} - 2.5 \hat{j}$$

ω being the angular velocity of the boom with respect to the cab.

Hence,

$$v_{Pm1} = -2.5 \hat{j} + 4.33 \hat{i}$$

and for the second step,

$$v_{Pf} = v_{Pm1} + v_{Om1} + \omega \times r$$

where

$$v_{Pm1} = -2.5 \hat{j} + 4.33 \hat{i}$$

$$v_{Om1} = 0$$

$$\omega \times r = 0.4 (8.66 + 5) = -3.465 \hat{i}$$

ω being the angular velocity of the cab with respect to the ground.

Hence,

$$v_{Pf} = (3.465 \hat{i} - 2.5 \hat{j} + 4.33 \hat{i}) \text{ m/s}$$

Similarly, for the first step in respect of acceleration,

$$a_{Pm1} = a_{Pm} + a_{Om} + \alpha \times r + 2 \omega \times v_{Pm} + \omega \times (\omega \times r)$$

where

$$a_{Pm} = 0$$

$$a_{Om} = 0$$

$$\alpha \times r = 0$$

$$2 \omega \times v_{Pm} = 0, \text{ as } v_{Pm} = 0$$

$$\omega \times (\omega \times r) = 0.5 \times (0.5 \times (8.66 + 5)) = -2.165 \hat{i} - 1.25 \hat{j}$$

Hence,

$$a_{Pm1} = -2.165 \hat{i} - 1.25 \hat{j}$$

and for the second step,

$$\begin{aligned} \mathbf{a}_f &= \mathbf{a}_{pm1} + \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_{pm1} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= -2.165 \mathbf{i} - 1.25 \mathbf{j} + 0 + 0 + 2 \times 0.4 \mathbf{k} \times (-2.5 \mathbf{i} + 4.33 \mathbf{j}) + 0.4 \mathbf{k} \times (0.4 \mathbf{k} \times (8.66 \mathbf{i} + 5 \mathbf{j})) \\ &= (2 \mathbf{i} - 3.55 \mathbf{j} - 1.25 \mathbf{k}) \text{ m/s}^2 \end{aligned}$$

Some comments can be made on the choice of a method. Methods 1 and 2 invoke a single moving frame whereas method 3 does not require any moving frame and method 4 requires more than one moving frame. Method 3 tends to be difficult because it requires thinking of the complete motion in one go. Method 4 offers simplicity of understanding and is indeed the choice if the number of moving components is large. There can be a number of intermediate moving frames for convenience. Methods 1 and 2 combine two steps into one and reduce the length of the procedure at the cost of clarity. However, in many problems, it is necessary to choose one moving frame for expressing the motion of every two rotating members. Methods 1 and 2 are, therefore, representative of a typical choice.

The origin of every moving frame was fixed at O only for convenience because then $\mathbf{a}_0 = 0$ and $\boldsymbol{\omega}_0 = 0$. There is no bar to fix the origin of the moving frame at any point. For example, it can be fixed at P , the tip of boom itself. Then, $\mathbf{a}_0 = 0$ and $\boldsymbol{\omega}_0 \neq 0$ and $\boldsymbol{\alpha}_0 \neq 0$. Expressions for position vector, velocity and acceleration in rectangular and cylindrical coordinates are summarized in Table 5.2 for ready reference.

Example 5.22

- (a) A tennis ball has a velocity of 30 m/s at angle of +30° with the horizontal, just after being struck by a player. Determine the radius of curvature of its trajectory.
- (b) What would be the radius of curvature when it reaches the maximum height?

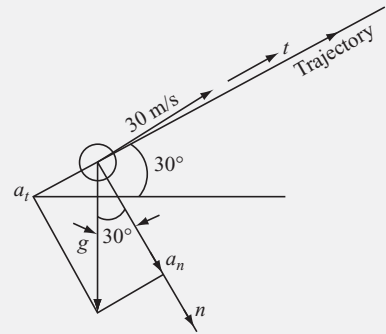


Fig. Ex. 5.22

Solution

P

At this instant, the ball has a velocity of 30 m/s and it is going up the trajectory, while subjected to the gravitational acceleration g , 9.81 m/s^2 downwards. This acceleration can be resolved into a ‘normal’ and a ‘tangential’ component:

$$\begin{aligned} a_n &= g \cos 30 = 8.5 \text{ m/s}^2 \\ a_t &= -g \sin 30 = -4.9 \text{ m/s}^2 \end{aligned}$$

In terms of path coordinates,

$$a_n = \frac{v^2}{\rho} = 8.5 \text{ m/s}^2$$

whence

$$\rho = 30^2 / 8.5 = 105.9 \text{ m}$$

is the radius of curvature.

Also, a_t = rate of change of speed equals -4.9 m/s^2 .

P

At the given instant, the horizontal component of velocity is $30 \cos 30^\circ$, i.e., 26 m/s. In the absence of horizontal forces, this component remains unaltered throughout the trajectory.

When it reaches the maximum height, it has just the horizontal component of velocity, i.e., 26 m/s only. It is still subjected to the gravitational acceleration g , 9.81 m/s^2 downwards. At that instant, the 'normal' is also downwards;

$$a_n = g = 9.81 \text{ m/s}^2$$

$$= \frac{v^2}{\rho} = \frac{26^2}{\rho}$$

whence $\rho = 26^2/9.81 = 68.9 \text{ m}$, which is the radius of curvature at that instant.

Further, since g is wholly equal to a_n , there is no tangential acceleration,

$$a_t = 0$$

which means that the rate of change of speed is zero!

Example 5.23

Determine the horizontal and vertical components of the acceleration of a particle describing a circle of 10 m radius, with a constant speed of 20 m/s, while passing the position P at $+30^\circ$ with the horizontal.

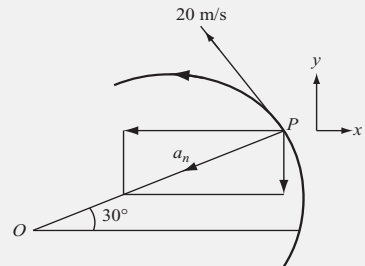


Fig. Ex. 5.23

Solution The particle describing a circular path must be subjected to a normal, i.e., centripetal acceleration

$$a_n = \frac{20^2}{10} = 40 \text{ m/s}^2, \text{ as shown. The horizontal and vertical components are determined as follows, with regard to } x \text{ and } y \text{ taken +ve as shown:}$$

Horizontal component = $-40 \cos 30^\circ = -34.64 \text{ m/s}^2$

$$\text{Vertical component} = -40 \sin 30^\circ = -20 \text{ m/s}^2$$

Example 5.24

A motor-cycle accelerates linearly, starting with an initial acceleration of 1.5 m/s^2 to 6 m/s^2 over a distance of 50 m. Find the velocity acquired and the time taken, assuming the initial velocity to be 5 m/s at $t = 0$.

Solution It is a question of varying acceleration; we must use the general equations

$$a = \frac{dv}{dt} \quad \text{and} \quad v = \frac{ds}{dt}; \quad a = v \frac{dv}{ds}$$

or

$$v = \int a \, dt \quad \text{and} \quad s = \int v \, dt$$

From the data,

$$a = 1.5 + (6 - 1.5)s/50 = 1.5 + 4.5 \frac{s}{50}$$

$$v \frac{dv}{ds} = 1.5 + 4.5 \frac{s}{50}$$

$$\frac{1}{2} \frac{d(v^2)}{ds} = 1.5 + \frac{4.5}{50} s$$

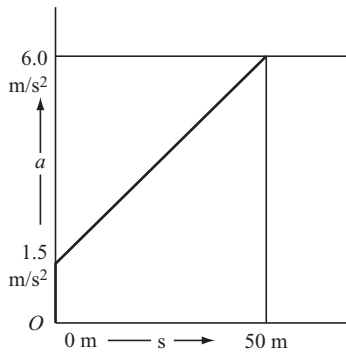
$$|v^2|_5^V = 2 \left[1.5s + \frac{4.5}{50} \frac{s^2}{2} \right]_{s=0}^{50}$$

$$V^2 - 25 = 2[75 + 112.5] = 375$$

$$V^2 = 375 + 25 = 400$$

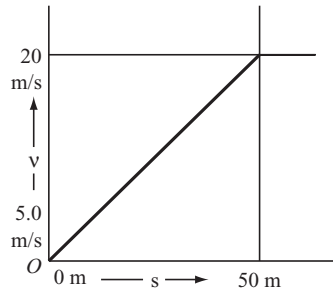
$$V = 20 \text{ m/s}$$

Let us sketch the acceleration-distance and the velocity distance graphs for the sake of interest.



Acceleration-distance graph

Fig. Ex. 5.24(a) (Solution)



Velocity-distance graph

Fig. Ex. 5.24(b) (Solution)

Table 5.2 Expressions for position vector, velocity and acceleration in different coordinate systems

| Entity | Cartesian coordinates | Cylindrical coordinates |
|--------|---|--|
| | $x + y + z$ | $r + z$ |
| | $u + v + w$ | $V_r + V_\theta + V_z$ |
| a_x | $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$ | $\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}$ |
| a_y | $\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$ | $\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}$ |
| a_z | $\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$ | $\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z}$ |

Concept Review Questions

Comment on the truth of the following statements:

(a) The displacement of a particle during an interval of time may not be the same as the distance moved by it.

(b) Directions of velocity and acceleration of a point at any instant may not be the same whereas the directions of displacement and velocity at any instant must be the same.

- (c) The acceleration of a particle undergoing simple harmonic motion must be directed towards the centre of oscillations whereas the velocity may either be towards or away from it.
- (d) A vector may be constant in a moving frame of reference but it may not be constant as observed from a fixed frame of reference.

State the assumptions made in the derivations of the relations

$$V = U + at$$

$$S = Ut + \frac{1}{2}at^2$$

and $V^2 - U^2 = 2as$

and hence state when these relations are not applicable.

Explain the meaning of the terms: centripetal acceleration, Coriolis acceleration and normal acceleration as applied to the motion of a particle.

Show that

$$\frac{d}{dt} r = \omega_{\theta}$$

$$\frac{d}{dt} \theta = -\omega_r$$

$$\frac{d}{dt} n = \omega_t$$

$$\frac{d}{dt} t = -\omega_n$$

Under what circumstances are the cylindrical coordinates preferred to the rectangular coordinates and under what conditions are the path coordinates preferred to both?

Would the unit vectors r , θ and n of a moving frame appear to change with time as observed from a fixed frame, if the moving frame

- (a) translates at a constant velocity?
- (b) translates rectilinearly with a variable velocity?
- (c) rotates at a constant rotational velocity?
- (d) rotates at a variable rotational velocity?

Tutorial Problems

The acceleration of a particle is given by

$$a = t^3 - 3t^2 + 5 \text{ m/s}^2$$

where the time t is in s. If the velocity of the particle at $t = 1$ s is 6.25 m/s and the displacement is 8.80 m, calculate the velocity and the displacement at $t = 2$ s.

($v = 8$ m/s and 16.1 m)

A particle, starting from rest, moves in a straight line and its acceleration is given by

$$a = 50 - 36t^2 \text{ m/s}^2$$

where t is in s. Determine (a) the velocity of the particle when it has travelled 52 m, and (b) the time taken by it before it comes to rest again.

($v = 4$ m/s, 2.04 m)

A particle passes through a point (3, 4, 5) with a velocity of

$$V = 10i + 11j + 12k$$

at a time $t = 1$ s. A constant acceleration

$$= 2i - 3j$$

is impressed upon it for 10 s. Compute the position and velocity of the particle at the final instant.

($r = 203i, -36j, 125k; 30i - 19j + 12k$)

A particle is observed at $t = 0$, and 2 s when it passes through positions (0, 3, 0) and (20, 3, -4) respectively. If the velocity vector has the form

$$= At + B + C$$

determine its position and velocity at $t = 5$ s.

($r = (125, 3, -10), (25i - 2j)$ m/s)

A point moving with simple harmonic motion has an amplitude of 1 m and the period of one complete oscillation is 2 s. Determine its displacement, velocity and acceleration at an instant 0.4 s after passing an extremity.

(s 0.309 m, 2.99 m/s, 3.05 m/s²)

A particle moving with simple harmonic motion performs 10 complete oscillations per minute and its speed, when at a distance of 8 cm from the centre of oscillation is 60% of the maximum speed. Find the speed of the particle when it is 6 cm from the centre of oscillation.

(s $x_0 = 10$ cm, $V = 8.38$ cm/s)

A particle moving with simple harmonic motion, has a time period of 0.6 s. Its speed at its mean position is 1.5 m/s. Determine its speed when it is half way between its mean position and an extremity.

(s 1.3 m/s at $x = 0.0715$ m)

- 8 Determine from first principles, the angle at which a bullet must be fired over a horizontal plane such that the greatest height attained by it equals the range on the plane.

(s $\alpha = 76^\circ$)

Two guns are projected at each other, one upward at an angle of 30° and the other at the same angle of depression, the muzzles being 30 m apart as shown in Fig. Prob. 5.9. If the guns are shot with velocities of 350 m/s upward and 300 m/s downward respectively, find when and where the bullets will meet.

(s 0.0462 s, (14 m, 8.07 m))

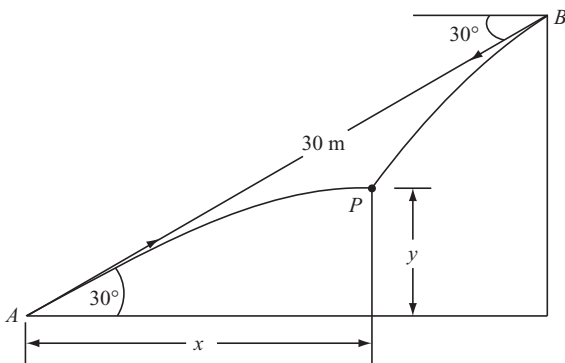


Fig. Prob. 5.9

Show that there are two directions in which a particle may be projected at the same velocity so that it passes through a given target. Establish the minimum velocity-of-projection requirement such that the particle does reach the target.

(s $\theta_1 = 1/2 \sin^{-1}(gx/v^2), \pi/2 - \theta_1$;
 $V_{\min} \geq [gx^2/2(x \tan \theta - y)^{1/2}]$ for θ_1 and θ_2)

The horizontal distance of a target to be hit by a projectile is 10,000 m. The shell leaves the gun with a velocity of 600 m/s as shown in Fig. Prob. 5.11. What must be the angle of elevation α of the gun if a mountain 2000 m high intervening midway between the gun and the target is to be cleared?

(s $\alpha = 82.1^\circ$)

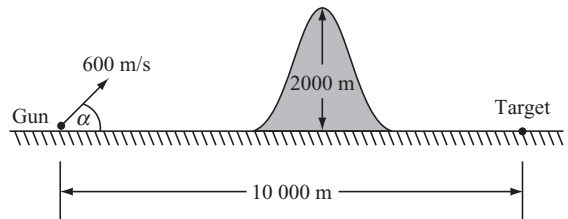


Fig. Prob. 5.11

A projectile is fired from a cliff 120 m above sea level with an initial velocity of 500 m/s directed at an angle of elevation of 30° to the horizontal. Estimate the time of flight and the horizontal range if the target is at the sea level.

(s 51.4 s, 22 276 m)

Two particles are projected simultaneously from two points A and B such that h is the horizontal distance and k the vertical distance between them. They are projected at the same inclination α to the horizon with the same velocity V as shown in Fig. Prob. 5.13. Show that their distance from each other will be minimum after a time.

$$t = \frac{n}{2V \cos \alpha}$$

and that the minimum distance will be k .

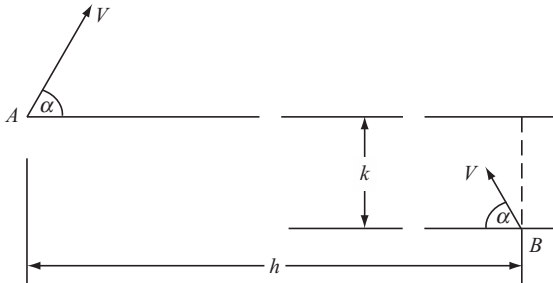


Fig. Prob. 5.13

If a body travels half its total path in the last second of its free fall, starting from rest, find the total time and height of its fall.

(s 3.414 seconds; 57.2 metres)

A ball rolls off the top of a stairway with a horizontal velocity of 1.5 m/s. The steps are 20 cm wide and 20 cm high. Which step will the ball hit first?

(s The path $y = -2.18x^2$ intersects the line $y = -x$ at $x = 0.45$ m; hence, 3rd step)

Determine the minimum speed with which the motorcycle must leave the 30° ramp at A to reach the point B, clearing the pond in between.

(s 37 km/hour)

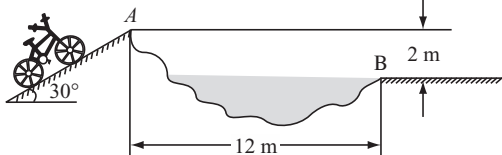


Fig. Prob. 5.16

A helicopter is descending vertically downward with a uniform velocity. At a certain instant, a food packet is dropped from it which takes 5 seconds to reach the ground. As this packet strikes the ground, another food packet is dropped from it, which takes 4 seconds to reach the ground. Find the velocity with which the helicopter is descending and its height, when second packet is dropped. Also find the distance travelled by the helicopter during the interval of dropping the packets.

(s 11.04 m/s downwards at 177.8 m height; 55.2 m downwards)

- 8 A point moves on a curve $xy = 16$ according to the law $x = 4t^2$ where x and y are expressed in meters and t is in seconds. Find the magnitude and direction of the velocity of the point (a) when $t = 1$ second and (b) when $x = 2$ metres.

(s $V_x = 8t, V_y = -8/t^3$ in m/s;
 (a) $t = 1, x = y = 4; V_x = 8, V_y = -8$ m/s
 (b) $x = 2, y = 8, t = 0.707$ s; $V_x = 5.66, V_y = -22.6$ m/s)

A gun fires a bullet with such an initial velocity and such an angle of elevation that the maximum height to which it rises is h . Find the maximum range that can be obtained with the same initial velocity.

(s $h = z_{\max} = \frac{V_0^2}{2g}$ at $\alpha = 90^\circ$;
 $x_{\max} = V_0^2 \sin 2\alpha / g = 2h$ at $\alpha = 45^\circ$)

A body travels a distance s in a duration of t seconds. It starts from rest and ends at rest. In the first part of its journey it moves at a constant acceleration a and in the second part with a constant retardation r . Show that

$t = \sqrt{2s(1/a + 1/r)}$
 (: Set up the three equations;
 $t = t_1 + t_2, s = 1/2(a t_1^2 + r t_2^2)$ and
 $a t_1 = r t_2$ and eliminate t_1 and t_2)

The rotor of a motor has an angular acceleration which is directly proportional to the time t . The motor starts from rest at time $t = 0$. After 3 seconds, the rotor has completed 5 revolutions. Obtain the equation of motion of the rotor and estimate its angular velocity at $t = 2$ seconds

(s $\alpha = d\omega/dt = 20/9 \pi t; 14$ rad/s)

A 250 m long railway train is travelling along a curved track of 1 km radius at a speed of 60 km/hour and decelerates at $0.2g$. Calculate the velocity and acceleration of the engine as seen by the guard at the tail end of the train.

(s 16.67 (-) m/s; $(-1.68 \hat{i} - 2.24 \hat{j})$ m/s²)

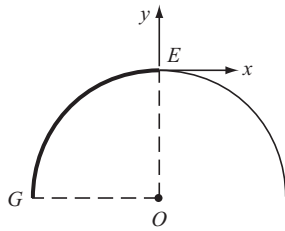


Fig. Prob. 5.22

The path of a particle P is an Archimedean spiral. The motion of the particle is defined by the relations,

$$r = 10t \quad \text{and} \quad \theta = 2t$$

where r is in metres, t is in seconds and θ is in radians. Determine the velocity and acceleration of the particle (a) when $t = 0$ and (b) when $t = 0.25$ seconds.

(s (a) $10i, 40\pi j$; (b) $10j - 5\pi i, -40\pi i - 10\pi^2 j$)

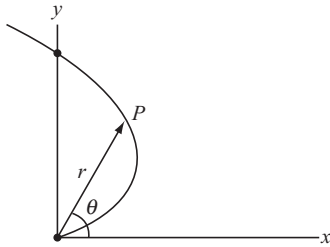


Fig. Prob. 5.23

An aeroplane is flying with a constant velocity v at a constant height h . Show that, if a gun is fired point blank at the aeroplane as it passes directly over the gun with an angle of elevation α , the shell will hit the aeroplane provided

$$2(V \cos \alpha - v) v \tan^2 \alpha = gh$$

where V is the initial velocity of the shell.

A bomber is flying horizontally at a speed of 500 km/h at an altitude of 3 km such that a ship lies in a vertical plane through the line of sight as shown in Fig. Prob 5.25. Determine the angle of the line of sight of the bomber with the ship at the instant a bomb is released so as to hit the ship. Where would the bomber be at the instant the ship is wrecked?

s $\theta = 48.9^\circ$; Over the ship)

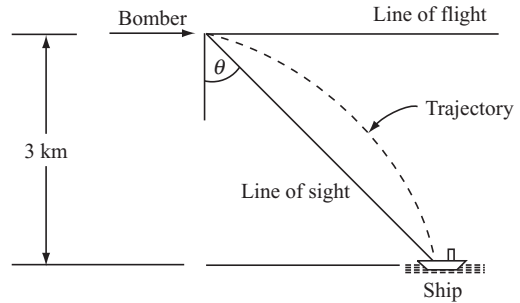


Fig. Prob. 5.25

The motion of a point in the vertical plane is given by

$$r = 3t^2, \quad \theta = 0.5 \sin \pi t/4$$

where r is in cm and θ in radian and t in s. Determine the velocity and acceleration of the point when $t = 3$ s.

(s 19.5 cm/s, 16.4 cm/s²)

A wheel rotates at an angular speed 10 rad/s and the rotational speed increases at 2 rad/s². A collar C moves out on a horizontal spoke such that its speed and acceleration with respect to the spoke are 3 m/s and 2 m/s² respectively as shown in Fig. Prob. 5.27. Compute the absolute velocity and acceleration of the collar if it is at 0.5 m from the centre of rotation.

(s 5.83 m/s; 77.62 m/s²)

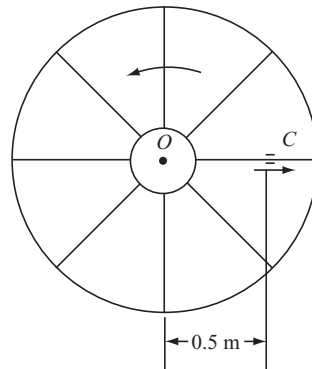


Fig. Prob. 5.27

- 8 A tracking device stationed at the launching point of a missile records the r and θ coordinates of the missile with the passage of time.

It is noticed that if r is in km, θ in degrees and t in s, the following expressions represent the motion closely

$$r = 2t - t^2/20$$

$$\theta^2 = 1300 - t^2$$

for the plane trajectory of the missile. Estimate the position, velocity and acceleration of the missile at $t = 20$ s.

$$(\quad \text{s } 20 \text{ km, } -0.2333 \text{ km/s, } 0.104 \text{ km/s}^2)$$

A particle moves on a frictionless wire bent into a cubic $y = 2x^3$. At a point (1, 2), the speed of the particle is 3 m/s and it decreases at a rate of 2 m/s. Compute its velocity and acceleration in terms of the rectangular coordinates.

$$(\quad \text{s } 0.493, 2.96 \text{ m/s; } -0.79, -1.84 \text{ m/s}^2)$$

A particle P slides down an incline of 30° frictionlessly and then moves up a circular arc of 1 m radius as shown in Fig. Prob. 5.30. Compute the velocity and acceleration of the particle at A just before the start of the arc and at B midway on the arc.

$$(\quad \text{s } 5.18 \text{ m/s; } 8.5, 27.3 \text{ m/s}^2)$$

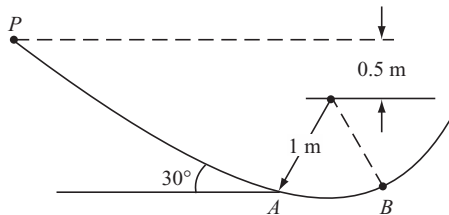


Fig. Prob. 5.30

A load P is being raised by means of an assemblage of two links AB and BP as shown in Fig. Prob. 5.31. At the instant of interest link AB rotates and accelerates at 3 rad/s and 4 rad/s^2 respectively with respect to the ground whereas link BP with an angle of 90° to AB rotates and accelerates at 5 rad/s and 2 rad/s^2 with respect to link AB . If AB is 5 m long and BP is 2 m long with an angle of 60° at the instant of interest, determine the acceleration of the load with respect to the ground reference.

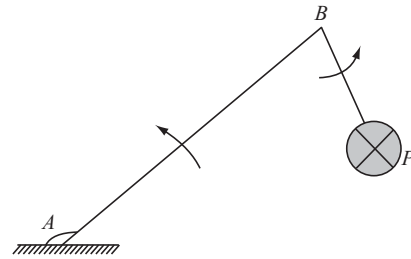


Fig. Prob. 5.31

Two boys A and B stand diametrically opposite on a plane horizontal table of diameter 5 m rotating anticlockwise at 10 radians per second as shown in Fig. Prob. 5.32. If A throws a ball towards B at a speed of 5 m/s, how will it tend to move on the table and why?

$$(\quad \text{s } 50 \text{ m/s tangential, } 5 \text{ m/s radius})$$

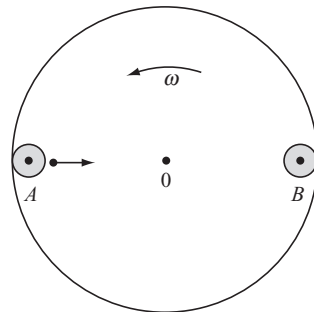


Fig. Prob. 5.32

A rotating spotlight is at a perpendicular distance l from a horizontal floor. The light revolves at constant N revolutions per minute about a horizontal axis perpendicular to the plane representing it in Fig. Prob. 5.33. Derive expressions for the velocity and acceleration of the light spot travelling along the floor. Let θ be the angle between the vertical line l and the light beam at time t .

$$(\quad \text{s } \omega = 0.1051 N \text{ sec}^2 \theta, \alpha = 0.0221 N^2 \text{ sec}^2 \theta \tan \theta)$$

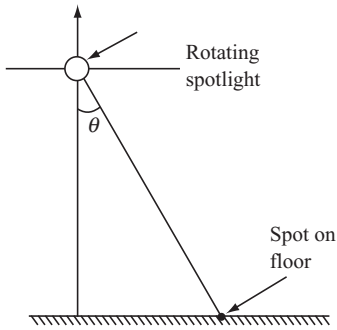


Fig. Prob. 5.33

A flexible chain of length l rests on a smooth table with length a overhanging the edge as shown in Fig. Prob 5.34. The system originally at rest is released. Describe the motion. The chain weighs w N/m.

$$\ddot{s} + \frac{w}{2l} s = \frac{w}{2} \left(\frac{l-a}{l} \right)$$

$$\exp(\sqrt{\frac{w}{2l}} s) + \frac{1}{2} \exp(-\sqrt{\frac{w}{2l}} s)$$

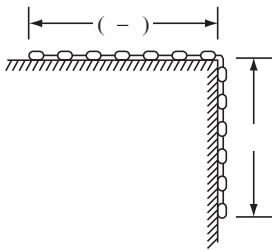


Fig. Prob. 5.34

A straight tube is attached to a vertical shaft at a fixed angle α as shown in Fig. Prob. 5.35. The shaft rotates with a constant angular velocity ω . A particle moves along the tube with a constant velocity v relative to the tube. find the magnitude of the acceleration of the particle when it is at a distance r along the tube from the centre.

$$a = \omega^2 r \sin \alpha \sqrt{1 + \left(\frac{2v}{\omega r} \right)^2}$$

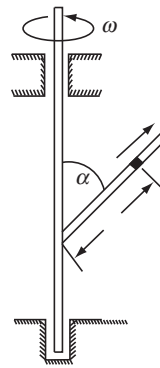


Fig. Prob. 5.35

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

The displacement of a point

- (a) implies the distance moved by the point
- (b) is a vector, from the initial to the final position of the point

- (c) is always less than the distance traversed by the point
- (d) is independent of the distance and the direction of movement of the point

The relationship $v^2 - u^2 = 2as$, with conventional notation, is applicable for

- (a) all possible motions of a point
- (b) constant velocity of a point

- (c) constant acceleration of a point
- (d) variable acceleration of a point

One of the following assumptions is *not* necessary in obtaining the equation for parabolic trajectory of a particle:

- (a) Air resistance is negligible
- (b) The gravitational acceleration g is constant
- (c) The body can be represented by a particle
- (d) The body must not change its mass during the motion

The unit vector 'normal' to a curve

- (a) is directed towards the local centre of curvature
- (b) is directed outward along the join of the centre of curvature and the point

- (c) is the same as the radial unit vector
- (d) must only be perpendicular to the path of the point

The Coriolis acceleration may *not* vanish if the

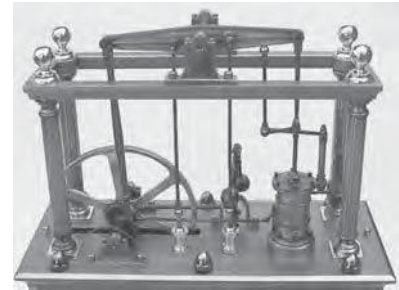
- (a) relative velocity of the moving point becomes zero
- (b) rotational velocity of the moving frame becomes zero
- (c) rotational velocity of the moving frame and the relative velocity become collinear
- (d) angular acceleration of the point becomes zero

Answers to Multiple-Choice Questions

- 1 (b) 2 (c) 3 (d) 4 (a) 5 (d)

Kinematics of a Rigid Body

7



7.1 INTRODUCTION

A body is said to be a **rigid body** when its dimensions and the relative positions of points within it do not change during the course of events. Mathematically, distances between any pairs of points within the body remain constant. For example, two arbitrary points P_1 and P_2 shown in a rigid body in Fig. 7.1(a) remain a constant distance apart no matter what happens to the body. Mathematically, $P_1P_2 = \text{constant}$. In other words, the body is undeformed under the static and dynamic actions. The idealisation of a rigid body allows the distribution of matter, uniformly or non-uniformly over the extent of the body.

Kinematics of a rigid body refers to the relationship of position, velocity and acceleration with time for a rigid body. We first visualise the three modes of linear displacement and three modes of angular displacement and hence six types of motion a rigid body can undergo and classify them as **translation** and **rotation**. We then proceed to study the kinematics in the same order for two reasons: one, to go from simple to not-so-simple and two, because the more general motions can be thought of as superpositions of the simpler motions namely translation and rotation.

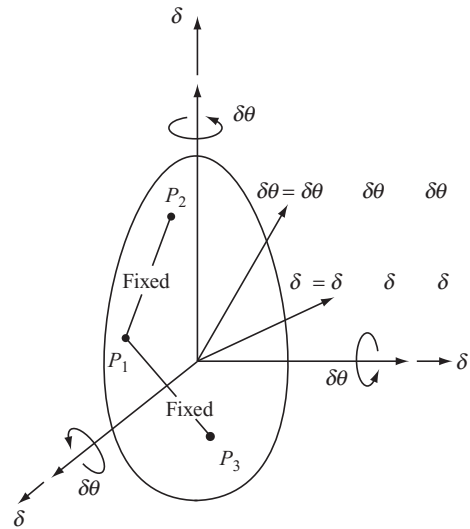


Fig. 7.1(a) Definition of a rigid body and six modes of displacement

A great deal of attention is paid to the relative motion of rigid bodies with reference to the moving and fixed frames of reference. A number of examples on the analysis of motion, i.e., on the determination of velocity and acceleration, both analytically and graphically, are given.

7.2 TYPES OF MOTION

A rigid body may be displaced from its initial position *rectilinearly* along one or more of the three axes or *angularly* about one or more of the three axes, the set of axes being chosen arbitrarily. There are altogether six modes of displacement and hence six coordinates are needed to specify the motion; a rigid body is said to have six degrees of freedom as shown in Figs. 7.1(a) and (b). Correspondingly, a rigid body may have six components each of velocity and of acceleration as shown in Table 7.1.

The degrees of freedom of a rigid body can be thought of in another way. Suppose that a rigid body is to be located in space during its motion. The number of independent coordinates required to specify its location at a time t is equal to the number of degrees of freedom it enjoys. A little reflection will show that if a body is held at one point, it can rotate about that point; if it is held at two points, it can rotate about the axis passing through the two points and if it is held at three non-collinear points, it is held fully constrained. It follows that the position of the body is fully specified if the coordinates of any three non-collinear points on the body are specified such as that shown in Fig. 7.1(c). Each of the three points requires three coordinates to be specified which makes a total of nine coordinates for the entire body. Of these nine coordinates, only six are independent because the distance between the points remains fixed by the definition of the rigid body. In other words, the nine coordinates are

$$x_1, y_1, z_1 \quad x_2, y_2, z_2 \quad \text{and} \quad x_3, y_3, z_3$$

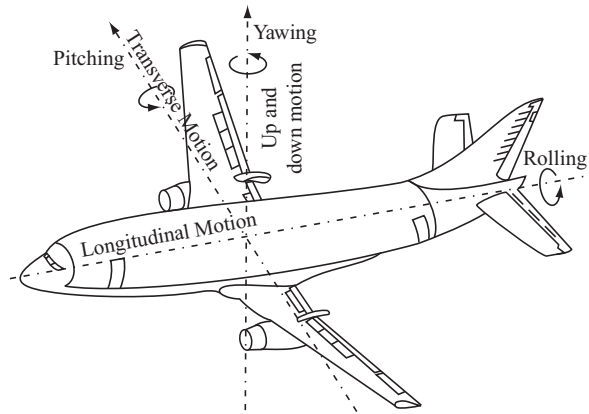


Fig. 7.1(b) Degrees of freedom

Table 7.1 Displacement, velocity and acceleration of a rigid body

| | Cartesian components | | | Numbers |
|----------------------|----------------------|-------------------|-------------------|---------|
| | x | y | z | |
| Linear displacement | δx | δy | δz | 6 |
| Angular displacement | $\delta \theta_x$ | $\delta \theta_y$ | $\delta \theta_z$ | |
| Linear velocity | V_x | V_y | V_z | 6 |
| Angular velocity | ω_x | ω_y | ω_z | |
| Linear acceleration | a_x | a_y | a_z | 6 |
| Angular acceleration | α_x | α_y | α_z | |

and the three constraints are

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = C_1$$

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = C_2$$

$$(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 = C_3$$

The number of independent coordinates is thus reduced to six.

The motion of a rigid body is usually subjected to certain kinematic constraints, thus bringing down the number of degrees of freedom.

A rigid body allowed to rotate about a fixed axis has only one degree of freedom, i.e., it can only undergo angular displacement about that axis. A rigid body allowed to translate in general has three degrees of freedom, i.e., it can acquire displacements along the x , y and z axes. If the translation is restricted in a plane, it has only two degrees of freedom and if it is required to go along a specified curve, it is left with only one degree of freedom.

The concept of the degrees of freedom of a rigid body can be extended to a number of connected rigid bodies. Each rigid body has six degrees of freedom. From the total, the number of constraints must be subtracted to find the degrees of freedom of the system of bodies.

A *mechanism* is an assemblage of rigid linkages which have relative motion between them but the motion of every linkage is uniquely determined if one of them is given a prescribed motion. A mechanism has, therefore only one degree of freedom.

A *structure* is an assemblage of members capable of withstanding loads without any change in their dimensions. No relative motion is allowed between the structural members. A structure can, therefore, be thought of as a rigid body with zero degree of freedom.

The definition of a rigid body has far-reaching implications. There can be no component of velocity of a point relative to another point along the line joining the two points in the rigid body because the distance between them cannot change. The velocity of a point relative to another point in the rigid body must, therefore, be wholly perpendicular to the join of the two points. For example, with reference to Fig. 7.1(c), point 1 can have any arbitrary velocity V_1 but points 2 and 3 cannot have arbitrary velocities. The component of V_1 along 1-2 must be the same as that of V_2 along 1-2. Similarly, the component of V_3 along 1-3 must equal the component of V_1 along 1-3 and of V_3 along 3-2 must equal the component of V_2 along 3-2. In other words,

$$V_{21} \text{ is perpendicular to line 1-2}$$

$$V_{12} \text{ is perpendicular to line 1-2}$$

$$V_{13} \text{ is perpendicular to line 1-3, etc.}$$

It implies that a point on a rigid body can only undergo an angular displacement with respect to another point on the body at any instant. The absence of relative velocity along the join of two points on a rigid body does not imply the absence of relative acceleration along the join of the points. In fact, whenever there is relative velocity between two points in a rigid body, there must be a component of acceleration along the join. In addition, there may be a component of acceleration normal to the join of the points resulting from the rate of change of the relative velocity of the points.

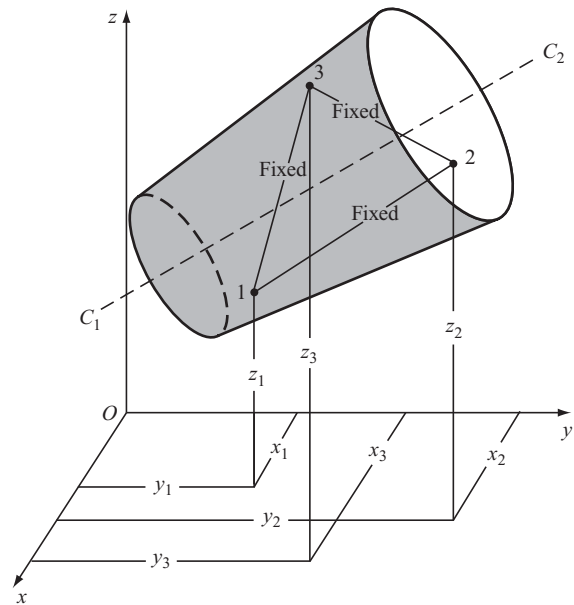


Fig. 7.1(c) Degrees of freedom of a rigid body

Let us look at different types of motion of a rigid body. It is important to understand their characteristics and how they help us simplify our solutions to problems.

Translation A rigid body is said to be in *translation* if the linear displacement of every point in the rigid body is the same. Translatory motion is characterised by the movement of a typical line element PQ parallel to itself. In other words, the translational motion of a rigid body is characterised by each point on the body to have the same velocity and also the same acceleration at an instant. It should be understood that a rigid body can undergo a change in velocity both in magnitude and direction during translation. In *rectilinear translation*, a typical point P translates along a straight line PP_1P' and an element PQ moves to $P_1Q_1, P'Q'$ as shown in Fig. 7.2(a). In *curvilinear translation*, a typical point P may trace a plane or a space curve PP_1P' and a line PQ moves to $P_1Q_1, P'Q'$ as shown in Fig. 7.2(b). The curve traced by each point must be identical on a rigid body in a translation. It follows that all the points of the body have the same linear displacement, same velocity and same acceleration at a given instant.

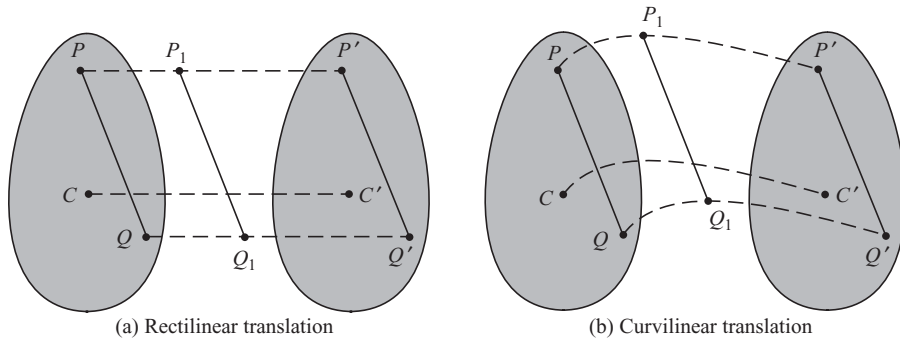


Fig. 7.2 Translation of rigid body

Rotation *Rotational motion* is characterised by the same angular displacement of all the points in the rigid body. It follows that the angular displacement, angular velocity and angular acceleration at a given instant are the same for all the points in the body with reference to the axis of rotation. The axis of rotation, chosen as $z-z$, may lie within the body or outside it as shown in Figs 7.3 (a) and (b). The trajectory of each point on the rigid body in rotation must be a circle with its centre on the axis of rotation. It may be noted that the velocity of a point Q with a position vector \mathbf{r} on a body in rotation at angular velocity $\boldsymbol{\omega}$ is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{7.1}$$

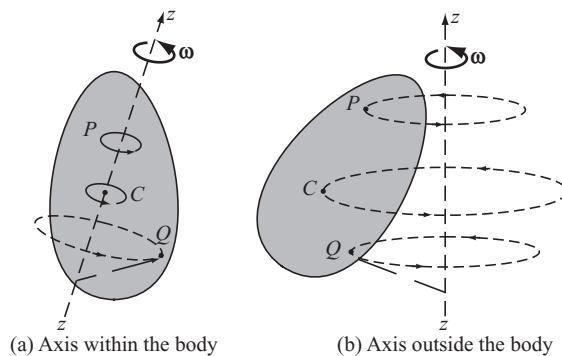


Fig. 7.3 Rotational motions

The acceleration of a point is likewise given by

$$= \alpha \times r + \omega \times (\omega \times r) \quad (7.2)$$

The words *pure translation* and *pure rotation* are often used to stress the absence of rotation and translation respectively.

In general, a rigid body may *translate* and *rotate* simultaneously. Conversely, a general rigid body motion may be thought of as a superposition of a pure rotation and a pure translation, viz., (a) pure rotation about an axis through a chosen point and (b) pure translation along the join of the initial and final positions of the chosen point.

A rigid body is said to be in fixed-axis rotation if there exists a fixed straight line within or outside the body such that the points identified with the body but on that line have zero velocity and zero acceleration. The straight line thus qualified is called *axis of rotation* of the body. Rotation may also be specified to be about a point if there exists only a fixed point identified with the body where both the velocity and acceleration vanish. An example of fixed-axis rotation is a shaft rotating in a fixed journal bearing and an example of fixed-point rotation is a spinning top rotating about the tip in steady or unsteady states.

Plane Motion The motion of a rigid body is said to be *plane motion* if all the points in the body stay in the same parallel planes. A plane motion may be composed of translation and rotation. Examples of plane rotation are given as follows:

1. Curvilinear translation is a plane motion if the curve traced by any point on it is a plane curve. A thin plate hanging by two equal inextensible strings is an example of plane curvilinear translation, e.g., AP and BQ oscillating in its own plane as shown in Fig. 7.4(a).
2. Linear translation of a rigid body must be a plane motion.
3. Rotation of a rigid body about a fixed axis must be a plane motion.
4. Translation and rotation can result in a plane motion if the rotation takes place about an axis perpendicular to a plane of motion as shown in Fig. 7.4(b).
5. Rolling without slipping of a cylindrical object on a flat or curved surface must be a plane motion as shown in Fig. 7.4(c).

Space Motion *Space motion* of a rigid body is a general type of motion, with 6 degrees of freedom, not constrained to be categorised in any of the restricted motions. Examples of such motions are the flying

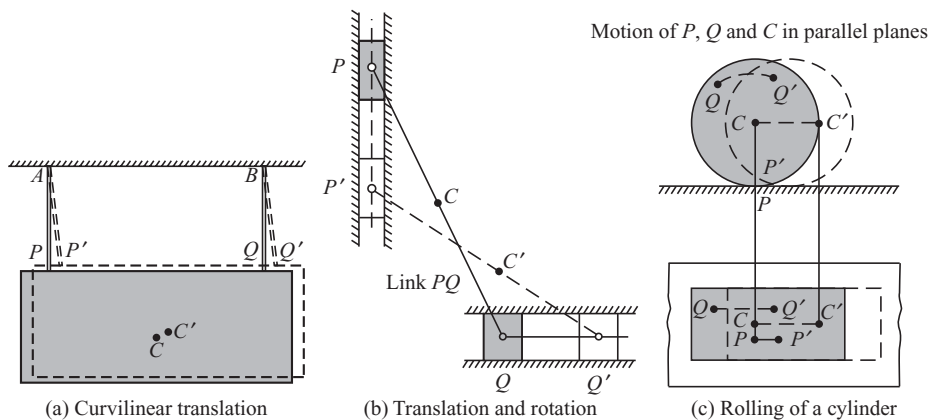


Fig. 7.4 Examples of plane motion

objects, vehicles on land, etc. The rolling of a cone on a flat or curved surface and the final stage of the motion of a spinning top are space motion. The space motion of a body in which one point remains fixed in space is called motion about a fixed point. An example of such a motion is the entire span of spinning motion of a top on a rough floor.

7.3 FIXED-AXIS ROTATION

For a rigid body rotating about a fixed axis z within the body as in Fig. 7.3(a) or outside the body as in Fig. 7.3(b).

$$\text{Angular velocity } \omega = \frac{d\theta}{dt} \quad (7.3)$$

$$\text{Angular acceleration } \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega \frac{d\omega}{d\theta} \quad (7.4)$$

Two cases of fixed-axis rotation are considered as follows.

Rotation with Constant Angular Velocity

$$\omega = \text{Const.}, \quad \alpha = 0$$

since
$$\omega = \frac{d\theta}{dt}$$

whence
$$\theta = \theta_0 + \omega t \quad (7.5)$$

where θ_0 is the initial angular displacement and θ is the angular displacement at any instant.

Rotation with Constant Angular Acceleration

$$\alpha = \frac{d\omega}{dt} = \text{Const.}$$

whence
$$\omega = \omega_0 + \alpha t \quad (7.6)$$

Since
$$\omega = \frac{d\theta}{dt}$$

it follows that
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad (7.7)$$

and also
$$\omega^2 = \omega_0^2 + 2 \alpha (\theta - \theta_0) \quad (7.8)$$

where ω_0 is the initial angular velocity at θ_0 .

7.4 PLANE MOTION OF A RIGID BODY

Let us now focus our attention to the plane motion of a rigid body. By definition, the motion is said to be plane if all the points in the body stay in the same and parallel planes. In other words, for a plane motion, any chosen point in the body should continue to be in the same plane once a plane is preferred by it during the motion. The other points are then obliged to move in planes parallel to this plane because the distance of every other point from this point must remain the same by the definition of a rigid body. The concept of motion of the points to stay in parallel planes for the plane motion of a rigid body enables us to consider *only one of the parallel planes* and analyse the motion of the points lying in that plane. The parallel planes may not appear to be identical in shape and size but there is no difficulty because any plane of the body can be hypothetically extended by a massless extension of the rigid body for the purpose of kinematic studies.

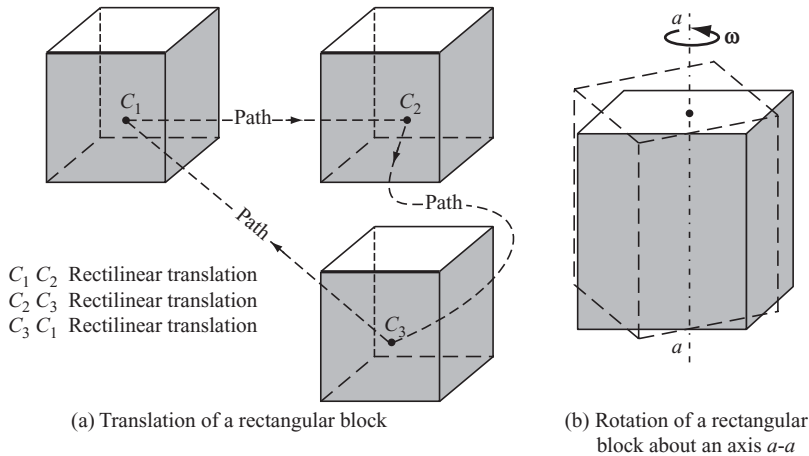


Fig. 7.5 Types of plane motion of a rigid body

Consider, for example, the motion of a rectangular rigid block as shown in Fig. 7.5. It is said to undergo a translation if it is displaced such that an arbitrary line, say the top front horizontal edge, remains parallel to itself. The translation is rectilinear if any point, say the centre of mass C of the block, traces a straight line such as C_1C_2 or C_1C_3 . On the other hand, the translation is said to be curvilinear if the point traces a curve such as C_2C_3 . In contrast to translation, the block is said to have undergone pure rotation about an axis as identified to be on the block as also shown in Fig. 7.5.

The general plane motion of the block would be such that it may both translate and rotate; the rotation being allowed about an axis perpendicular to the parallel planes. Once the body starts moving, the parallel planes remain the same thereafter as shown in Fig. 7.6. These remarks apply equally well to the general plane motion of a rigid body of arbitrary shape and size as shown in Fig. 7.7. *A general plane motion of a rigid body may, therefore, be considered as the sum of a plane translation and a rotation about an axis perpendicular to the plane of motion.*

A rigid element PQ is shown displaced to a position $P'Q'$ in a general plane motion in Fig. 7.8. The displacement may be considered to comprise a translation of an arbitrary point on the element plus a rotation about an axis perpendicular to the plane and passing through that point. For example, the following 8 combinations are equivalent:

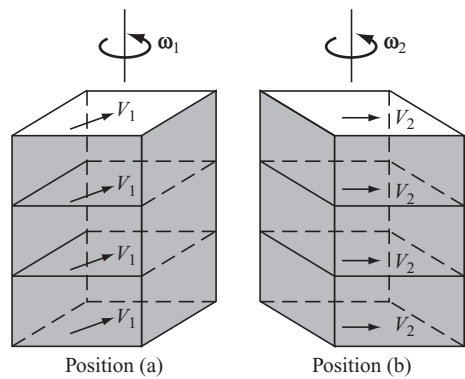


Fig. 7.6 General plane motion of a rectangular block

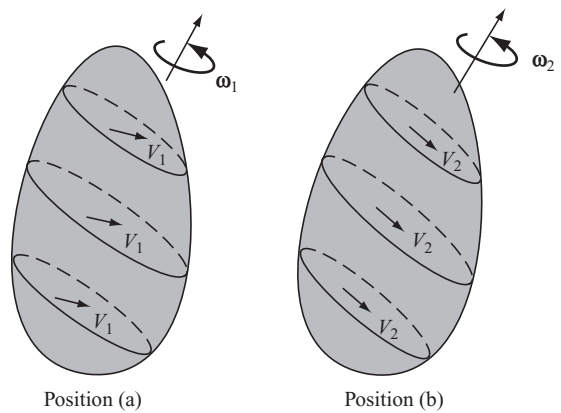


Fig. 7.7 General plane motion of a rigid body

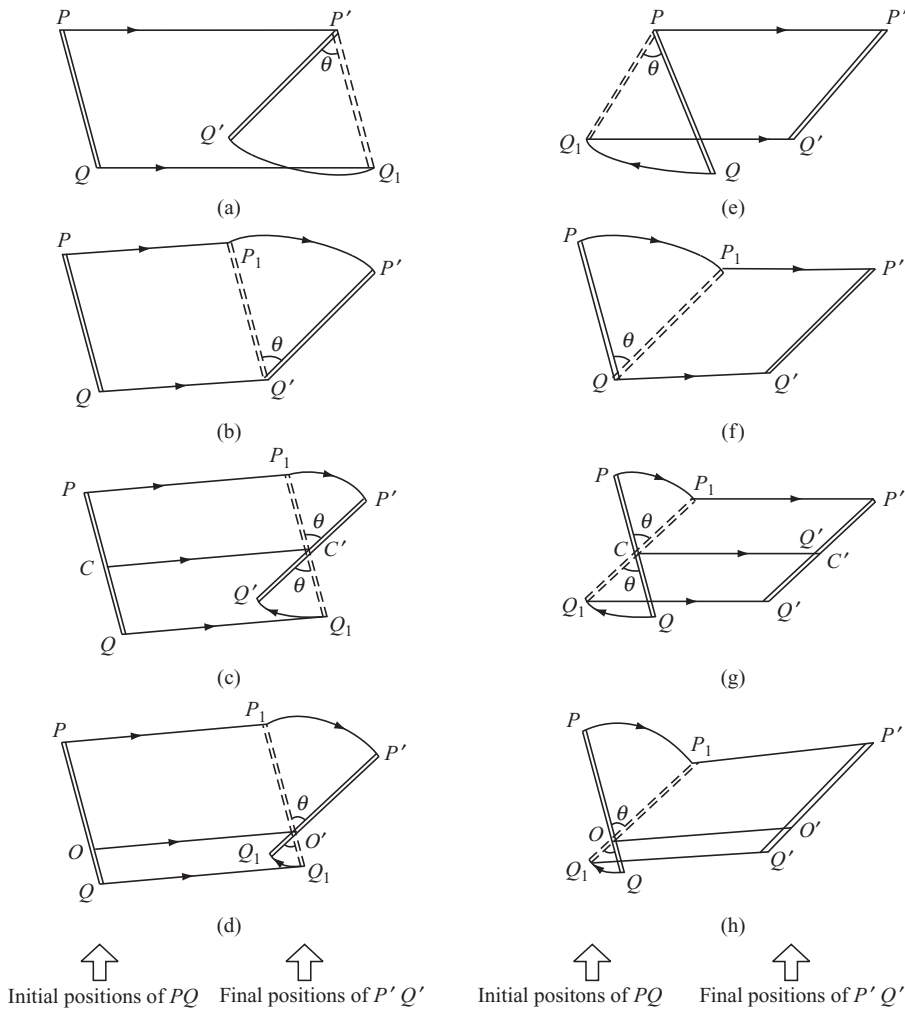


Fig. 7.8 Different equivalent combinations of translation and rotation

The equivalent combinations are described as follows:

Figure 7.8(a) Translation from PQ to $P'Q_1$ and rotation about P'

Figure 7.8(b) Translation from PQ to P_1Q' and rotation about Q'

Figure 7.8(c) Translation from PQ to P_1Q_1 and rotation about C'

Figure 7.8(d) Translation from PQ to P_1Q_1 and rotation about O'

Figure 7.8(e) Rotation from PQ to PQ_1 about P and translation to $P'Q'$

Figure 7.8(f) Rotation from PQ to P_1Q about Q and translation to $P'Q'$

Figure 7.8(g) Rotation from PQ to P_1Q_1 about C and translation to $P'Q'$

Figure 7.8(h) Rotation from PQ to P_1Q_1 about an arbitrary point O and translation to $P'Q'$

Of course, *translation and rotation for a plane motion are commutative*; the order of rotation and translation in the summation is immaterial as can be verified geometrically. For example, Figs. 7.8(e) and (f) show equal amount of translation and rotation but in Fig. 7.8(h) translation is done first and rotation later and in Fig. 7.8(f) it is the other way round. Similarly, Figs. 7.8(a) and (e), (c) and (f); (d) and (h) also commutative.

It is also important to note that, whatever be the mode of combination, the amount of rotation is the same. The angular velocity ω of every point on the element is therefore the same. It is, therefore, in order to use the term *angular velocity of the link or of the rigid body* rather than about any particular point on it.

The fact that a general plane motion can be thought of as a superposition of translation and rotation is a special case of *Chasle's theorem*. The theorem, in general, states that any general motion of a rigid-body can be considered as an appropriate superposition of a translational motion and a rotational motion. In particular, if a rigid body is displaced from position 1 to position 2 in space, then it is possible to visualise the body to have undergone translation from position 1 to an intermediate position in regard to a certain point O on the body and then rotation about the point O . The extent of translation up to the intermediate position depends upon the choice of the reference point on the body whereas the extent of rotation is independent of the choice. Alternatively, a rigid body can be visualised to have rotated about a reference point first and then translated to the final position. In other words, *the order of superposition of the translational and rotational motions is immaterial*.

7.5 RIGID-BODY MOTION OF FLUIDS

It is possible that fluids, particularly liquids, may undergo rigid-body translation and rigid-body rotation.

An example of translation is a liquid mass in a container subjected to a constant linear acceleration a along any direction. For the simplest case of constant horizontal acceleration a , the liquid orients itself with its free surface inclined at θ as shown in Fig. 7.9(a) such that

$$\tan \theta = \frac{a}{g} \quad (7.9)$$

An example of rotation is a forced-vortex flow of air in the core region of a tornado or cyclone. Another case of solid-body rotation is that of a liquid in a container subjected to a constant angular velocity ω about any axis, passing through the liquid or outside it. The surface of the liquid orients itself in the form of a paraboloid of revolution as shown in Fig. 7.9(b) such that

$$z = z_{\min} + \frac{r^2 \omega^2}{2g} \quad (7.10)$$

where z is the depth of the liquid at a radial distance r from the axis of rotation and z_{\min} is the minimum depth. Equations (7.9) and (7.10) have been stated without proofs.

7.6 INSTANTANEOUS CENTRE OF ROTATION

Recalling the statement that *a general plane motion of a rigid body may be considered as the sum of a plane translation and a rotation about an axis perpendicular to the plane of motion*, the velocity of a rigid body is completely specified by stating the translational velocity V_P of a point P and the rotational velocity ω about an axis through the point as shown in Fig. 7.10(a).

The fact that at an instant, the rotational velocity ω is the same for every point in the body and the extent of translation is different for different choices of the points of rotation suggests that a point *can* exist such that the body may be assumed to rotate about an axis through that point *at the instant*. Such a point is called

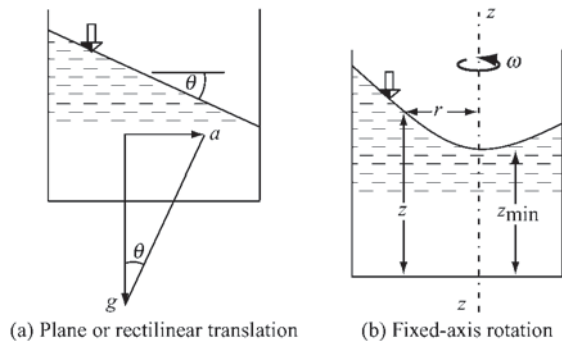


Fig. 7.9 Rigid body motion of fluids

the *instantaneous centre of rotation*. The velocity of the instantaneous centre of rotation is zero at that instant. Location of such an instantaneous point I requires that the perpendicular distance from it to the velocity at a point P in one of the parallel planes should be such that

$$V_P = IP \times \omega \quad \text{or} \quad V_P = r\omega \quad \text{and} \quad r = V_P/\omega \quad (7.11)$$

The velocity of the other points in the body can be determined by considering the body to rotate about an axis passing through I and normal to the parallel planes of motion. For example,

$$V_Q = (IQ) \times \omega \text{ perpendicular to } IQ$$

and $V_R = (IR) \times \omega$ perpendicular to IR

This fact provides another method for locating the instantaneous centre of rotation I if the directions of the velocities at any two points on a rigid body are known. The point of intersection of the perpendiculars to the directions of the velocities must be I . In case the directions of the two points are the same, the magnitudes of the velocities are required to locate I by the join of the extremities of the velocity vectors as at P and R in Figs. 7.10(b) and (c). The instantaneous centre of rotation is a point identified with the body where the velocity is zero; conversely if a point identified with the rigid body is at rest at an instant *within or outside a rigid body*, it must be the instantaneous centre of rotation.

The instantaneous centre of rotation of a body undergoing pure translation must be at infinity since the directions of velocities of all the points in the body are the same in pure translation. The instantaneous centre of rotation of a cylindrical body rolling without slip must be I , the instantaneous point of contact between the body and the surface as shown in Fig. 7.10(d) because this point on the cylinder is instantaneously at rest.

It should also be noted that the acceleration at I may not be zero; only the velocity is zero.

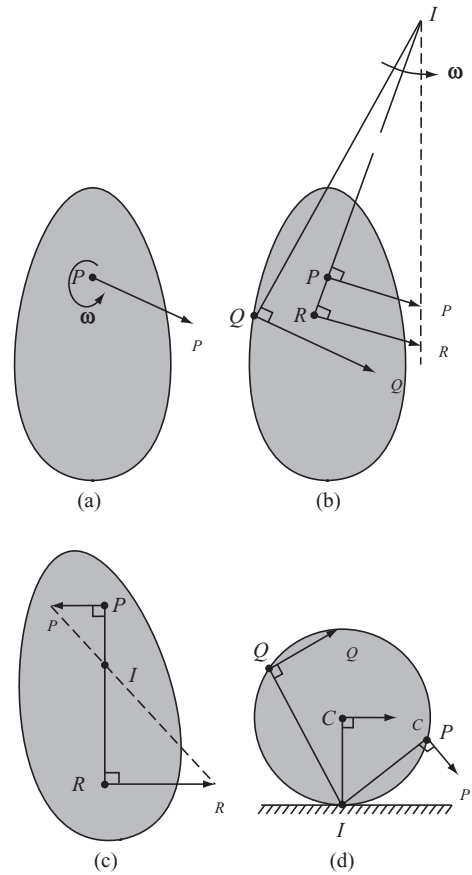


Fig. 7.10 Instantaneous centre of rotation

Example 7.1

A wheel of radius 0.5 m rolls without slipping down an incline as shown in Fig. Ex. 7.1. At the instant of interest, the velocity and acceleration of the centre are 10 m/s and 5 m/s² respectively. Determine the velocity of the instantaneous centre of rotation and the velocities of the point P , Q and R on the periphery.

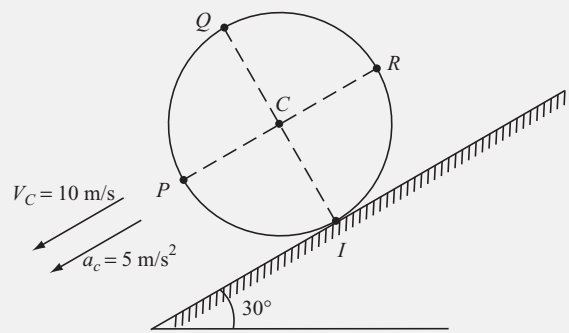


Fig. Ex. 7.1

Solution The rolling of a wheel without slip on a surface requires the point of contact to have zero velocity and hence to be the instantaneous centre of rotation I . By definition, the instantaneous velocity of I is zero. The velocity of any other point on the wheel can be determined from the knowledge of the velocity of one point, say C . The rate of rotation about I is

$$\omega = \frac{V_P}{IC} = \frac{10}{0.5} = 20 \text{ rad/s}$$

Consequently,

$$V_P = 20 IP = 20 \times 0.707 = 14.14 \text{ m/s}$$

perpendicular to the join IP as shown in Fig. Ex. 7.1(a) (Solution).

$$V_Q = 20 IQ = 20 \times 2 \times 0.5 = 20 \text{ m/s}$$

perpendicular to IQ or parallel to V_C as shown in Fig. Ex. 7.1(b) (Solution).

$$V_R = 20 IR = 20 \times 0.707 = 14.14 \text{ m/s}$$

perpendicular to IR as shown in Fig. Ex. 7.1(b) (Solution).

The motion of the wheel can alternatively be visualised as that of rotation about C and translation down the incline. In that case the velocity of a point on the periphery is composed of a component V_θ tangential to the periphery and a translational component of 10 m/s. The tangential component must also be 10 m/s for rolling without slip. Vector addition of the tangential velocity of 10 m/s to the down-slope component of 10 m/s at P , Q , R and I result in the velocities of 14.14 m/s, 20 m/s and 0 respectively.

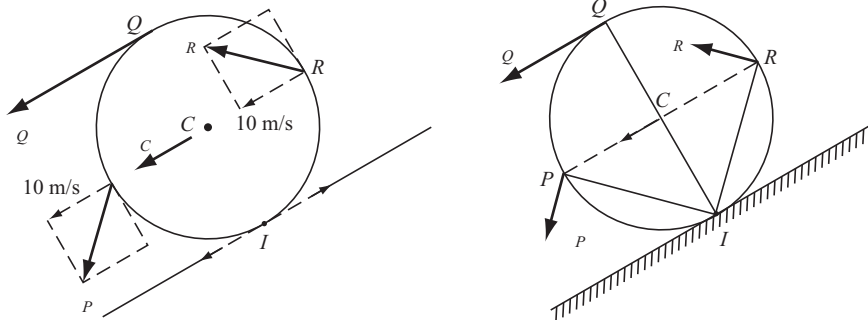


Fig. Ex. 7.1(a), (b) (Solution)

Example 7.2

A straight rigid link AB 40 cm long has, at a given instant, end B moving along a line O at 4 m/s and the other end A moving along YO , OY being a right angle. Find the velocity of the end A and of the mid-point of the link when inclined at 30° with O .

Solution The instantaneous centre of rotation of the link AB can be located by the knowledge of the directions of the velocities of the ends A and B . The point of intersection of the dotted lines drawn normal to the direction of velocities at A and B , as shown in Fig. Ex. 7.2 (Solution) should be the instantaneous centre I . The instantaneous angular speed with which the entire link AB rotates about the instantaneous centre I is given by

$$\begin{aligned} \omega &= \frac{V_A}{IA} = \frac{V_B}{IB} \\ &= \frac{V_C}{0.3464} = \frac{4}{0.20} = 20 \text{ rad/s} \end{aligned}$$

The velocity of the point A must, therefore, be

$$V_A = 20 \times 0.3464 = 6.93 \text{ m/s}$$

Once the angular velocity of a link about an instantaneous centre is known, the velocity of any point on the link can be determined. The velocity of the midpoint C of the link is

$$\begin{aligned} V_C &= \omega \times (IC) \\ &= 20 \times 0.20 = 4 \text{ m/s} \end{aligned}$$

in magnitude and its direction is perpendicular to the join IC .

It may be noted that the lengths IA , IB and IC are determined either by measurement if the whole drawing is made to scale or by trigonometry.

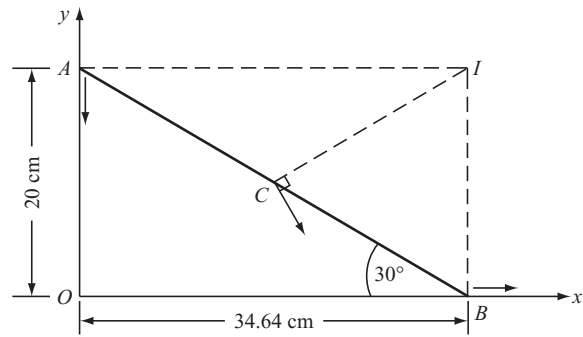


Fig. Ex. 7.2 (Solution)

7.7 RELATIVE VELOCITY AND ACCELERATION FOR POINTS ON A RIGID BODY

From the fact that a general plane motion is made up of a translation of a reference point P and a rotation about P , the absolute velocity of a point Q is given by

$$v_Q = v_P + \omega \times r_{PQ} \quad (7.12)$$

where r_{PQ} is the position vector of Q with respect to P as shown in Fig. 7.11.

Alternatively,

$$\begin{aligned} v_Q &= v_P + \omega \times r_{PQ} \\ v_{QP} &= \omega \times r_{PQ} \end{aligned} \quad (7.13)$$

Hence which implies that the velocity of a point Q with respect to a point P on a rigid link must be perpendicular to the plane containing ω and r_{PQ} , i.e., must be in the plane of the motion and directed perpendicular to the line joining P and Q .

Similarly, the absolute acceleration of a point Q in terms of the acceleration of a reference point P is given by

$$a_Q = a_P + \alpha \times r_{PQ} + \omega \times (\omega \times r_{PQ}) \quad (7.14)$$

The acceleration of Q with respect to a point P may be made up of

Tangential component: $\alpha \times r_{PQ}$

Normal component: $\omega \times (\omega \times r_{PQ})$

In the plane motion of a rigid body, attention is focussed on one of the parallel planes the velocity of a point in the plane must stay in that plane and the rotation of the body must be about an axis normal to that plane. It follows that the velocity of P and Q as well as the relative velocity of Q with respect to P must lie in that plane. Likewise, the accelerations of P and Q and the components of the relative acceleration of Q with respect to P must all lie in the same plane. From the fact that ω and α are collinear and r is perpendicular to either of them, the magnitudes of the vectors are identified easily:

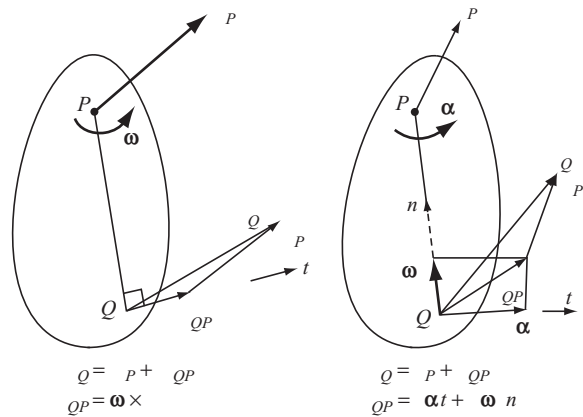


Fig. 7.11 Relative velocity and acceleration

| Vector | Magnitude | Direction |
|--|------------|------------------|
| $Q_P = \omega \times$ | ω | Perpendicular to |
| $Q_P' = \alpha \times$ | α | Perpendicular to |
| $Q_P'' = \omega \times (\omega \times)$ | ω^2 | From Q to P |

There may, in addition, be a Coriolis component of acceleration if there is a velocity of sliding $v_{B/S}$, of an element S sliding over the link PQ and being coincident with Q at that instant. The Coriolis acceleration of S would be

$$a_{cor} = 2\omega \times v_{B/S} \tag{7.15}$$

Some facts regarding the graphical or vectorial representation of the velocities and accelerations for link-motions are summarised as follows:

1. The velocity of a point on a link with respect to another point on the same link must be perpendicular to the line joining the two points.
2. The acceleration of a point on a link with respect to another point on the same link may have a component perpendicular to the line joining the points and a component along the line joining them.
3. The velocity of a point on a member sliding relative to a link must be along the line and the velocity of a point on a link sliding relative to a surface must be along the tangent to that surface at that point.
4. If a member slides along a link rotating about a point of an axis in space, the member is subjected to the Coriolis component of acceleration in addition to the other components. If the member slides outward from the centre of rotation; the Coriolis component is along the direction of rotation and vice versa.
5. The velocity and acceleration diagrams are drawn for a known configuration of the linkage or mechanism and give results which are valid *for that instant only*. The velocity and acceleration diagrams are, therefore, instantaneous diagrams and keep changing from instant to instant.

Example 7.3

A straight bar AB is placed in a semi-cylindrical trough of radius 20 cm and released to slide in it such that the end A slides inside the trough as shown in Fig. Ex. 7.3, while the bar touches and slips at the corner O . At an instant when the bar makes 45° with the diametral axis Ox and the end A is known to slide at 5 m/s, determine the velocity of sliding of the bar at point P .

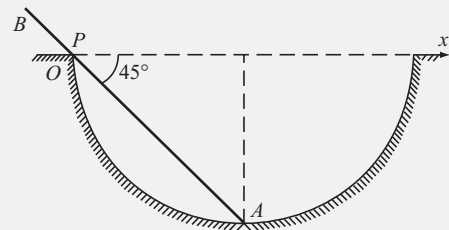


Fig. Ex. 7.3

Solution This example will be solved both graphically and vectorially to demonstrate the methods as well as to appreciate the simplicity offered by the graphical method for the plane motion of rigid bodies.

The bar AB slides at the corner point O of the trough. The point P on the bar is coincident with the point O on the trough such that P has a relative velocity with respect to O as shown in Fig. Ex. 7.3(a) (Solution). Since O is a stationary point, this is also the absolute velocity of P . It is related to the velocity of A as

$$v_P = v_A + v_{PA}$$

Absolute velocity of P = Absolute velocity of A + Velocity of P relative to A .

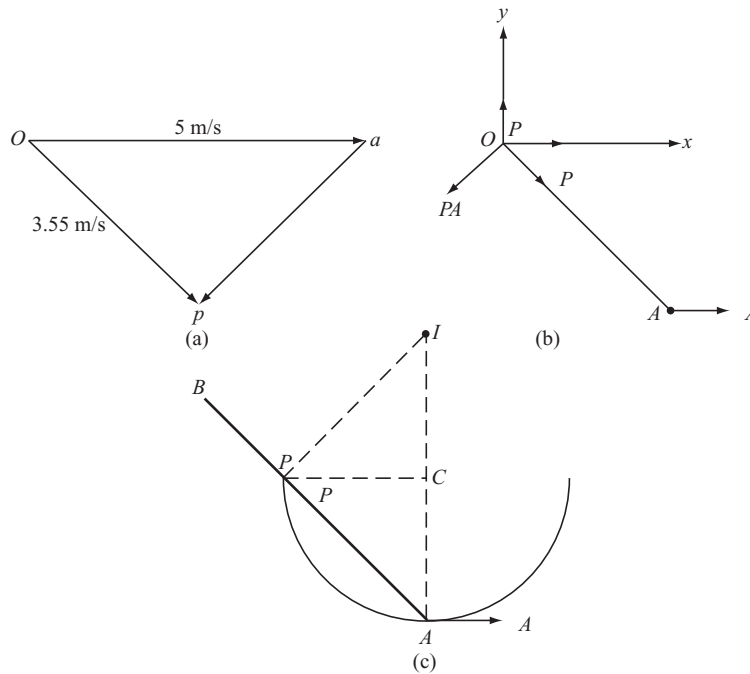


Fig. Ex. 7.3(a), (b), (c) (Solution)

By Graphical Method

Draw a line oa to represent the velocity of A as 5 m/s tangential to the semicircumference at A . Draw Pa perpendicular to PA to represent the relative velocity of P with respect to A . Since P can slide past O only along the bar, the absolute velocity of P must be shown by op drawn parallel to the bar. Now that point p is located, by measurement,

$$v_p = op = 3.55 \text{ m/s}$$

down along the bar.

By Vector Analysis

Let the coordinate axes xy be drawn through the point O as shown in Fig. Ex. 7.3(b) (Solution). In terms of the unit vectors \mathbf{i} and \mathbf{j} ,

$$\mathbf{v}_A = 5\mathbf{j}$$

The velocity of P with respect to A , i.e., \mathbf{v}_{PA} must be perpendicular to the joint of P and A . The unit vector along PA is

$$(\mathbf{i} - \mathbf{j})/\sqrt{2}$$

such that

$$\mathbf{v}_{PA} = -v_{PA}(\mathbf{i} - \mathbf{j})/\sqrt{2}$$

where v_{PA} is the magnitude of \mathbf{v}_{PA} .

One could write the unit vector as $(\mathbf{i} + \mathbf{j})/\sqrt{2}$, assuming it positive and the sign of v_{PA} would take care of it automatically. However, proceeding with respect to the Fig. Ex. 7.3(c) (Solution), the velocity of P should be along the bar and assuming it P to A as shown, the unit vector along it is

$$(\mathbf{i} - \mathbf{j})/\sqrt{2}$$

such that

$$v_P = v_P(\mathbf{i} - \mathbf{j})/\sqrt{2}$$

Substituting these relations in the vector equation,

$$v_P = v_A + v_{PA}$$

$$v_P(-\mathbf{i})/\sqrt{2} = 5 + v_{PA}(-\mathbf{i} - \mathbf{j})/\sqrt{2}$$

or $(v_P + v_{PA} - 5\sqrt{2})\mathbf{i} + (v_{PA} - v_P)\mathbf{j} = 0$

whence $(v_P + v_{PA} - 5\sqrt{2}) = 0$

and $v_{PA} - v_P = 0$ or $v_P = v_{PA}$

From these relations,

$$v_P = \frac{5\sqrt{2}}{2} = 3.54 \text{ m/s}$$

Finally, the velocity vector for P where the bar slides at the corner of the trough is

$$v_P = 3.54(\mathbf{i} - \mathbf{j})/\sqrt{2}$$

$$= 2.54(-\mathbf{i} - \mathbf{j})$$

Comparison of the Two Methods

It is probably quite clear from this simple example that the graphical construction offers great simplicity. This is indeed true for the plane motion of rigid bodies in general. However, it may be remarked that the velocity and acceleration diagrams are only valid for the instant they are drawn. *For every instant of motion, we may require to draw separate diagrams.* Analytically, on the other hand, one can write the equations for the velocity and acceleration in terms of the variables and obtain the results by numerical substitution for different instants. For example, in this case, vector equations for v_P and v_{PA} can be written for any angle θ instead of 45° and at different instants, substitution for θ would give the results whereas the graphical construction will have to be repeated many times.

The same problem can be solved by employing the concept of the instantaneous centre of rotation. The centre of rotation can be located by knowing that the velocity of A is tangential to the trough at A and the velocity of P is along the bar at O . Lines IA and IP drawn normal to the directions of velocity at A and P respectively to locate the instantaneous centre of rotation I as shown in Fig. Ex. 7.3(c) (Solution).

By measurement from the figure drawn to scale or from trigonometry,

$$IA = 2 CA = 40 \text{ cm}$$

$$IP = \sqrt{(20^2 + 20^2)} = 28.28 \text{ cm}$$

Considering the bar to be in pure rotation about I at this instant,

$$\frac{v_P}{IP} = \frac{v_A}{IA}$$

$$v_P = 5 \times 28.28/40 = 3.54 \text{ m/s}$$

It may also be remarked that the ‘instantaneous centre of rotation’ concept can be employed to advantage in some cases.

Example 7.4

A straight rigid link AB 40 cm long has, at a given instant, end B moving along a line O at 0.8 m/s and accelerating at 4 m/s^2 and the other end moving along YO , OY being a right angle as shown in Fig. Ex. 7.4. Find the velocity and the acceleration of the end A and of the mid-point C of the link when inclined at 30° with O .

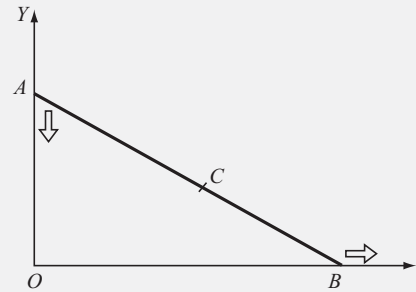


Fig. Ex. 7.4

Solution For the velocity diagram

$$A = B + AB$$

$$C = B + CB$$

Draw a line ob representing the absolute velocity of B along O to some scale. Point a can be located by drawing a line from O parallel to YO and a line from b perpendicular to AB as shown in Fig. Ex. 7.4(a) (Solution). This is because the velocity of A is along YO as well as the velocity of A relative to B on a link AB must be perpendicular to the link AB . The absolute velocity of a is, therefore, represented by oa to the same scale as ob represents the velocity of B . By measurement,

$$A = oa = 1.39 \text{ m/s}$$

The mid-point C on the link must have its velocity with respect to B perpendicular to the link and half in magnitude to the velocity of A with respect to B . The point c is, therefore, located by the fact that

$$cb = ab/2$$

Join o to c to give the absolute velocity of point C as

$$C = 0.8 \text{ m/s}$$

directed at an angle of 60° with O as shown in Fig. Ex. 7.4(a) (Solution).

For the acceleration diagram

$$A = B + AB$$

The acceleration of B is given as 4 m/s^2 along O and drawn to some scale as ob whereas the relative acceleration AB has two components:

(i) Normal or centripetal component given by

$$bd' = \frac{v_{AB}^2}{AB} = \frac{1.6^2}{0.4} = 6.4 \text{ m/s}^2$$

directed along AB towards B where ω_{AB} is the angular velocity of the link AB .

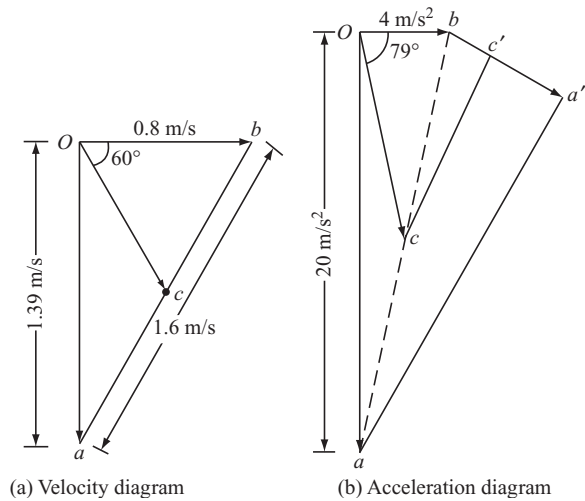


Fig. Ex. 7.4 (Solution)

(ii) Tangential component given by

$$a_t = r \alpha$$

directed perpendicular to OB where α is the angular acceleration of the link OB . The acceleration of the end B is known in direction, i.e., it must be along the line AB . The point a on the acceleration diagram is thus located by the point of intersection of a line parallel to AB drawn from a' and a line perpendicular to OB drawn from O as shown in Fig. Ex. 7.4(b) (Solution). Finally, by measurement,

$$a_t = 20 \text{ m/s}^2$$

The link rotates with an angular velocity given by

$$\omega = \frac{v_B}{OB} = \frac{1.6}{0.4} = 4 \text{ rad/s}$$

and accelerates with an angular acceleration given by

$$\alpha = \frac{a_t}{OB} = \frac{4.8}{0.4} = 12 \text{ rad/s}^2$$

The acceleration of any point on the link can now be determined. For the mid-point C , in particular,

$$a_C = a' + a''$$

where the components of a'' are

Centripetal: $a''_c = 0.2 \times 4^2 = 3.2 \text{ m/s}^2$

Tangential: $a''_t = 0.2 \times 12 = 2.4 \text{ m/s}^2$

The acceleration of mid-point C is, therefore, represented by ac . By measurement

$$ac = 10 \text{ m/s}^2$$

directed at an angle of 79° with OC as shown in the acceleration diagram.

It may be appreciated that the point c can alternatively be located on the line aa' which represents the relative acceleration of C with respect to A . Since C is the mid-point of AB , the point c must be the mid-point of aa' . This is because both the centripetal and tangential components of the relative acceleration of C with respect to A are halved for the point C .

Example 7.5

A reciprocating engine mechanism shown in Fig. Ex. 7.5 has a crank of radius 150 mm rotating at 10 revolutions per second in the clockwise direction. The connecting rod is 700 mm long and its centre of gravity is 200 mm from O . Find the velocity and acceleration of the piston and of the centre of the connecting rod when the crank is 45° past the inner dead centre as shown. Find also the angular velocity and the angular acceleration of the connecting rod.

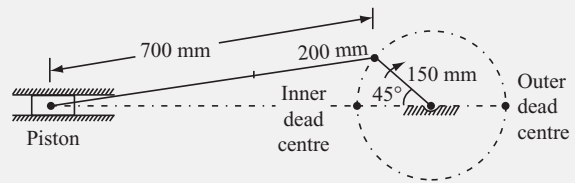


Fig. Ex. 7.5

Solution The solution is attempted both by the graphical method and by vectorial analysis.

The link diagram is first drawn to a suitable scale as shown. From the data,

$$\omega = 2\pi \times 10 = 62.83 \text{ rad/s}$$

$$v_G = \omega \times OG = 62.83 \times 0.15 = 9.42 \text{ m/s}$$

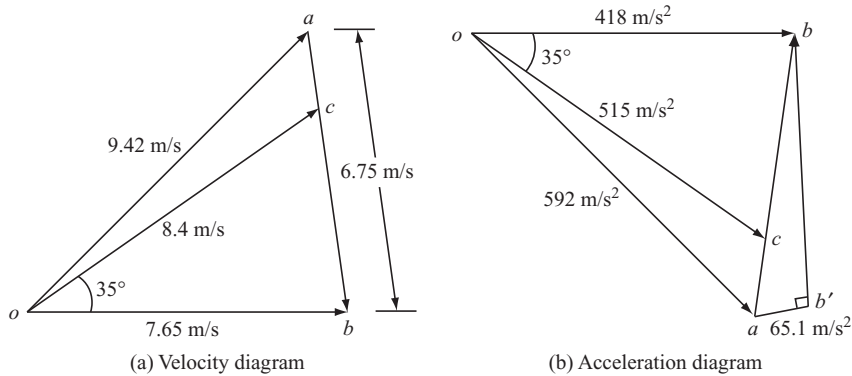


Fig. Ex. 7.5 (Solution)

The velocity diagram is now constructed to a suitable scale. Line oa is drawn perpendicular to OA to represent the absolute velocity of A . The velocity of B must be along BO and the velocity of B with respect to A on the link AB must be perpendicular to AB . Point b is thus located by drawing a line from o parallel to BO and a line from a perpendicular to AB as shown in Fig. Ex. 7.5(a) (Solution). The velocity of the piston at B is, therefore, given by

$$V_B = ob = 7.65 \text{ m/s}$$

directed towards O .

The angular velocity of AB is

$$\omega_{AB} = \frac{V_{AB}}{AB} = \frac{ab}{AB} = \frac{6.75}{0.7} = 9.64 \text{ rad/s}$$

The velocity of the centre of gravity C can be determined by locating the point c on ab such that

$$\frac{ac}{ab} = \frac{AC}{AB}$$

By measurement, the velocity of C is

$$V_C = oc = 8.4 \text{ m/s}$$

at an angle of 34.5° with respect to the line of dead centres as shown in the velocity diagram.

For the acceleration diagram,

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{BA} \\ \mathbf{a}_C &= \mathbf{a}_A + \mathbf{a}_{CA} \end{aligned}$$

The acceleration of A is known as

$$\begin{aligned} a_A &= \omega_{OA}^2 OA = 62.83^2 \times 0.15 \\ &= 592 \text{ m/s}^2 \end{aligned}$$

directed from A towards O , the centre of rotation of the crank. This is represented by the line oa drawn parallel to OA as shown in the acceleration diagram (Fig. Ex. 7.5(b) (Solution)).

The piston B is constrained to slide along BO ; the acceleration of B must also be along BO , i.e., the point b must lie on a line parallel to BO drawn from o . The acceleration of B with respect to A has two components.

Centripetal:
$$\begin{aligned} \omega_{AB}^2 AB &= 9.64^2 \times 0.7 \\ &= 65.1 \text{ m/s}^2 \end{aligned}$$

directed from B to A , shown by line ab' drawn parallel to BA .

Tangential: α_{AB} directed normal to ab shown by $b'b$ which locates the point b .

The acceleration of the piston B is found by measurement as

$$a_B = ob = 418 \text{ m/s}^2$$

directed towards O .

The acceleration of the centre of gravity C is determined by locating the point c such that

$$\frac{ac}{ab} = \frac{AC}{AB}$$

The acceleration of the centre of gravity of the connecting rod is, therefore, $a_C = oc = 515 \text{ m/s}^2$ directed at -35° with the line of dead centre.

By vectorial analysis

From the data and the geometry,

$$\begin{aligned} \omega_{OA} &= -2\pi \times 10 = -62.83 \text{ rad/s} \\ \dot{O}A &= 0.15 \times (-\cos 45^\circ + \sin 45^\circ) \\ &= -0.106 + 0.106 \\ \dot{A} &= \omega_{OA} \times \dot{O}A = -62.83 \times (-0.106 + 0.106) \\ &= 6.66 + 6.66 \text{ m/s} \\ \dot{B} &= \dot{B} \end{aligned}$$

At the instant of interest, the connecting rod is inclined at an angle ϕ with the x -axis such that

$$0.7 \sin \phi = 0.15 \sin 45^\circ$$

$$\sin \phi = 0.1515$$

whence

$$\phi = 8.71^\circ \quad \text{and} \quad \cos \phi = 0.988$$

The unit vector along AB is

$$e_1 = \cos \phi + \sin \phi = 0.988 + 0.1515$$

and the unit vector normal to AB is

$$e_2 = -\sin \phi + \cos \phi = -0.1515 + 0.988$$

The velocity of B must be given by

$$\dot{B} = \dot{A} + \dot{B}A$$

Substituting the values of the terms to the extent these are known,

$$\dot{B} = 6.66 + 6.66 + (-0.1515 + 0.988) \dot{B}A$$

which may be written as two scalar equations,

$$\dot{B} = 6.66 - 0.1515 \dot{B}A$$

$$0 = 6.66 + 0.988 \dot{B}A$$

whence

$$\dot{B}A = -6.741 \text{ m/s}$$

and

$$\dot{B} = 7.68 \text{ m/s}$$

It may be noted that the negative sign of $\dot{B}A$ implies that the velocity of B with respect to A is opposed to the unit vector e_2 whereas the positive sign of \dot{B} means that the velocity of B is indeed directed towards O .

The angular velocity of the connecting rod is given by

$$\omega_{BA} = \frac{V_{BA}}{BA} = -\frac{6.741}{0.7} = -9.63 \text{ rad/s}$$

The velocity of C must be given by

$$\begin{aligned} \mathbf{v}_C &= \mathbf{v}_A + \boldsymbol{\omega}_{BA} \times \mathbf{r}_{CA} \\ &= 6.66 \mathbf{i} + 6.66 \mathbf{j} + \frac{CA}{BA} (-6.741) \times (-0.1515 \mathbf{i} + 0.988 \mathbf{j}) \\ &= 6.95 \mathbf{i} + 4.76 \mathbf{j} \end{aligned}$$

The magnitude of v_C is, therefore,

$$V_C = \sqrt{(6.95^2 + 4.76^2)} = 8.42 \text{ m/s}$$

and it is inclined to the x -axis at an angle

$$\theta = \tan^{-1} \left(\frac{4.76}{6.95} \right) = 34.4^\circ$$

The acceleration of B may be obtained from the relation

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{BA} \times \mathbf{r}_{BA}$$

which, upon substitution of the known facts, becomes

$$\begin{aligned} \mathbf{a}_B &= \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_{OA}) + \boldsymbol{\omega}_{BA} \times (\boldsymbol{\omega}_{BA} \times \mathbf{r}_{BA}) + \boldsymbol{\alpha}_{BA} \times \mathbf{r}_{BA} \\ \text{or } \mathbf{a}_B &= -62.83 \mathbf{i} \times (-62.83 \mathbf{i} \times (-0.106 \mathbf{i} + 0.106 \mathbf{j})) + (-9.63 \mathbf{k}) \times (-9.63 \mathbf{k}) \\ &\quad \times 0.7 (-0.988 \mathbf{i} - 0.1515 \mathbf{j}) + \alpha \mathbf{k} \times 0.7 (-0.988 \mathbf{i} - 0.1515 \mathbf{j}) \end{aligned}$$

$$\text{or } \mathbf{a}_B = 418.4 \mathbf{i} - 418.4 \mathbf{j} + 64.14 \mathbf{i} + 9.84 \mathbf{j} + 0.106\alpha \mathbf{i} - 0.69\alpha \mathbf{j}$$

which may be written as two scalar equations,

$$\begin{aligned} a_B &= 482.54 + 0.106\alpha \\ 0 &= -408.4 - 0.69\alpha \end{aligned}$$

whence $\alpha = -592 \text{ rad/s}^2$

and $a_B = 420 \text{ m/s}^2$

The acceleration of C may be obtained by utilising the fact that

$$\mathbf{a}_C = \mathbf{a}_A + \boldsymbol{\alpha}_{CA} \times \mathbf{r}_{CA}$$

or

$$\begin{aligned} \mathbf{a}_C &= 418.4 \mathbf{i} - 418.4 \mathbf{j} + \frac{CA}{BA} (64.14 \mathbf{i} + 9.84 \mathbf{j} + 0.106\alpha \mathbf{i} - 0.69\alpha \mathbf{j}) \\ &= 418 \mathbf{i} - 300 \mathbf{j} \end{aligned}$$

which shows that the magnitude of the acceleration of C is

$$a_C = \sqrt{418^2 + 300^2} = 514.5 \text{ m/s}^2$$

and it is directed at an angle of

$$\beta = \tan^{-1} \left(\frac{-300}{418} \right) = -35.67^\circ$$

with the line of dead centres.

Example 7.6

A quick-return shaping mechanism consists of a crank CA rotating clockwise, as shown in Fig. Ex. 7.6, at 50 revolutions per minute. At an instant when the crank makes 30° with the x -axis, determine the velocity of the ram F moving in the horizontal direction. Determine also the stroke length of the ram and the velocity of the ram, which carries the cutting tool, during the cutting stroke of the ram when the oscillating link OE and the crank CA are vertical.

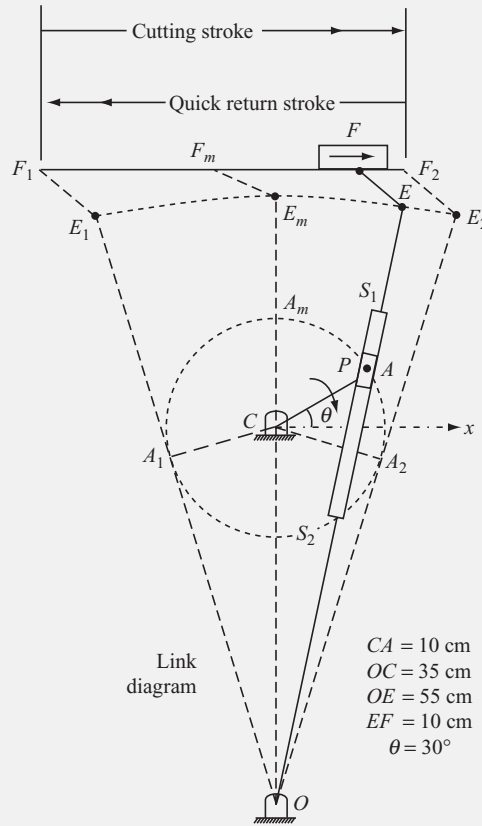


Fig. Ex. 7.6

Solution The link diagram is first drawn to a suitable scale as shown in Fig. Ex. 7.6 (Solution). The angular velocity of the crank CA is

$$\omega = \frac{2\pi \times 50}{60} = 5.24 \text{ rad/s}$$

and the linear velocity of point A is

$$V_A = 5.24 \times 0.1 = 0.524 \text{ m/s}$$

Consider P as a point on link OE coincident with the point A on the crank CA . The points A and C have, therefore, a relative motion of sliding in the slot s_1s_2 shown in the line diagram.

At the instant shown,

$$\begin{aligned} \theta &= 30^\circ \\ OA = OP &= 41 \text{ cm} \end{aligned}$$

For the velocity diagram, draw $ca = 0.524$ m/s perpendicular to CA to represent the velocity of A . Since P can slide with respect to A along the slot, draw pa parallel to the slot as shown in Fig. Ex. 7.6 (Solution). Knowing that P is on the rotating link OPE , the absolute velocity of P must be perpendicular to OPE ; this is shown by drawing op perpendicular to OPE and thus locating P . From the link diagram,

$$\frac{V_E}{V_P} = \frac{OE}{OP}$$

oe is, therefore, $55/41$ times op .

$$oe = 0.355 \times 55/41 = 0.48 \text{ m/s}$$

Since F can only move horizontally, of is drawn a horizontal line. The velocity of F with respect to E on the link EF must be perpendicular to EF ; ef is drawn thus locating the point f . By measurements from the velocity diagram drawn to scale,

$$V_F = of = 0.55 \text{ m/s towards right}$$

The stroke length of the ram is

$$F_1F_2 = E_1E_2$$

which, by measurement in the link diagram is 31.5 cm.

The velocity of the ram in the cutting stroke when the crank is in the vertical position as shown by CA_m is now determined. The velocity diagram at that instant would be as shown in Fig. Ex. 7.6 (Solution). A quick reflection will show that the velocity of P equals the velocity of A and also that the velocity of F equals the velocity of E .

Thus,

$$cp = oa; \quad pa = 0$$

$$V_F = V_E; \quad ef = 0$$

and

$$oe = of = 55/45 \times 0.524 = 0.64 \text{ m/s}$$

The cutting velocity of the shaper machine at that instant is, therefore,

$$V_F = 0.64 \text{ m/s}$$

This is very nearly the maximum velocity of the tool.

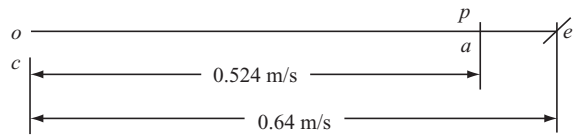
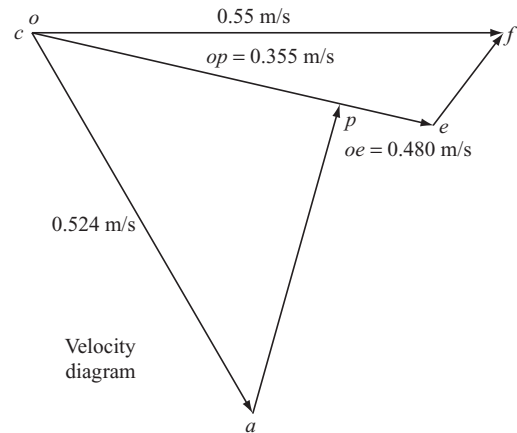


Fig. Ex. 7.6 (Solution)

Example 7.7

Two links O_1P_1 and O_2P_2 rotate in parallel planes, one immediately above the other as shown in Fig. Ex. 7.7. Determine the velocity and the acceleration of the 'point of crossing' P for the given orientation. Assume that $\omega_1 = 5$ rad/s and $\omega_2 = 10$ rad/s.

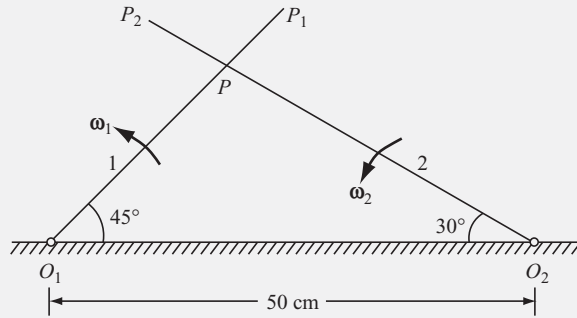


Fig. Ex. 7.7

Solution By drawing the link diagram to scale as shown and by measurement,

$$O_1P = 0.259 \text{ m}$$

$$O_2P = 0.366 \text{ m}$$

The point P is the point of crossing of the two links; it can be considered to be a point on either link and moving with respect to the other. If it is considered moving along the rotating link O_1P_1 it has two component of velocity:

- (i) $\omega_1 O_1P$ perpendicular to O_1P_1
- (ii) a component along O_1P_1 due to the sliding of P .

If it is considered moving along the rotating link O_2P_2 , the velocity components are:

- (i) $\omega_2 O_2P$ perpendicular to O_2P_2
- (ii) a component along O_2P_2 due to the sliding of P .

Consequently, from a point o , a line op'_1 is drawn perpendicular to O_1P_1 as shown in Fig. Ex. 7.7(a) (Solution)

$$op'_1 = \omega_1 O_1P = 5 \times 0.259 = 1.295 \text{ m/s}$$

and from p_1 , a line p'_1p is drawn parallel to the link O_1P_1 . Similarly, op'_2 and p'_2p are drawn such that

$$op'_2 = \omega_2 O_2P = 10 \times 0.366 = 3.66 \text{ m/s}$$

and the point p is located by drawing p'_2p parallel to O_2P_2 .

The absolute velocity of the point of crossing of the links is, therefore, given by

$$V_p = op = 4.38 \text{ m/s}$$

at an angle of 22° with the base line O_1O_2 as shown in the velocity diagram.

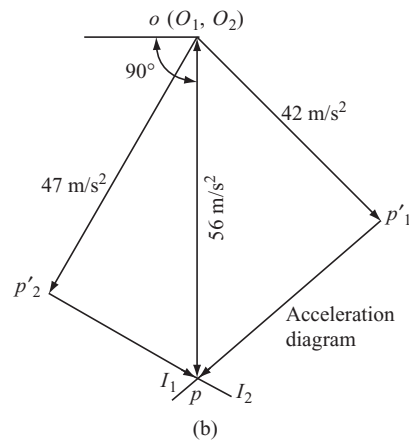
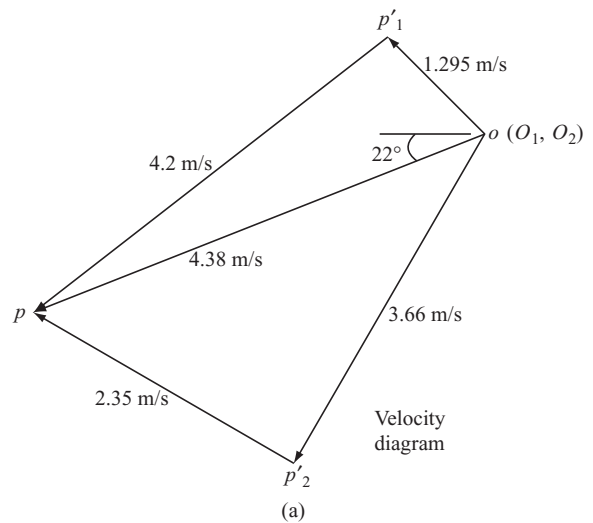


Fig. Ex. 7.7 (Solution)

Also, from the velocity diagram, the velocities of sliding of P are

$$p_{s1} \text{ with respect to link 1} = 4.20 \text{ m/s}$$

$$p_{s2} \text{ with respect to link 2} = 2.35 \text{ m/s}$$

The acceleration of P is also obtained in a similar manner. First, consider P to be moving in relation to the rotating link O_1P_1 ; it has three components of acceleration.

(i) Centripetal component given by

$$5^2 \times 0.259 = 6.48 \text{ m/s}^2$$

directed from P_1 to O_1 .

(ii) Coriolis component given by

$$2\omega_1 p_{s1} = 2 \times 5 \times 4.2 = 42.0 \text{ m/s}^2$$

directed perpendicular to O_1P_1 and

(iii) Sliding component along O_1P_1 .

Draw line op'_1 perpendicular to OP showing the Coriolis component of acceleration as shown in Fig. Ex. 7.7(b) (Solution). Draw a line l_1 perpendicular to OP_1 at p'_1 containing the centripetal and the sliding component of acceleration.

Now, consider the point P moving in relation to the rotating link O_2P_2 ; it has three components of acceleration:

(i) Centripetal component given by

$$10^2 \times 0.366 = 36.60 \text{ m/s}^2$$

directed from P_2 to O_2 ,

(ii) Coriolis component given by

$$2\omega_2 p_{s2} = 2 \times 10 \times 2.35 = 47.0 \text{ m/s}^2$$

directed perpendicular to O_2P_2 and

(iii) Sliding component along O_2P_2 .

The Coriolis component of acceleration is shown by op_2 . Draw a line l_2 perpendicular to OP_2 at P_2 containing the centripetal and sliding component of acceleration.

The point P can now be located at the intersection of the lines l_1 and l_2 and the absolute acceleration of P may be read as 56.0 m/s^2 directed along op .

Example 7.8

A right circular cone of base 1 m diameter and height 2 m rolls without slip on a flat horizontal surface such that its centreline OA rotates at a constant angular velocity of 3 radians per second about the vertical axis O as shown in Fig. Ex. 7.8. Determine the angular velocity and angular acceleration of the cone relative to the ground when it is in the yz plane as shown.

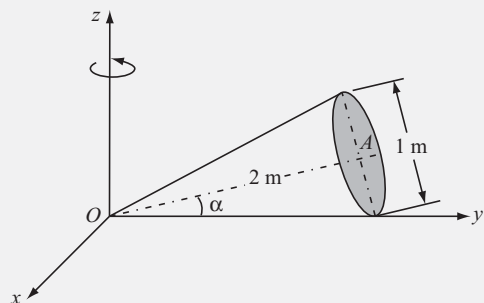


Fig. Ex. 7.8

Solution From the geometry,

$$\tan \alpha = \frac{0.5}{2} = 0.25$$

$$\alpha = 14.05 \text{ ; } \sin \alpha = 0.243, \cos \alpha = 0.97$$

The instantaneous axis of rotation of the cone must be the line of contact of the cone with the flat surface and the angular velocity relative to the ground is given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \tag{i}$$

where $\boldsymbol{\omega}_1$ is the angular velocity of the centreline of the cone about the z axis and $\boldsymbol{\omega}_2$ is the angular velocity of the cone about its centreline. From the data,

$$\boldsymbol{\omega}_1 = 2\pi \times 3 = 18.85 \tag{ii}$$

and

$$\begin{aligned} \boldsymbol{\omega}_2 &= \boldsymbol{\omega}_2 (\cos \alpha + \sin \alpha) \\ &= 0.97 \boldsymbol{\omega}_2 + 0.243 \boldsymbol{\omega}_2 \end{aligned} \tag{iii}$$

Substituting Eqs. (ii) and (iii) in Eq. (i)

$$\boldsymbol{\omega} = 18.85 + 0.97 \boldsymbol{\omega}_2 + 0.243 \boldsymbol{\omega}_2$$

which results into the scalar equations,

$$\begin{aligned} \omega &= 0.97 \omega_2 \\ 0 &= 18.55 + 0.243 \omega_2 \end{aligned}$$

whence

$$\omega_2 = -77.57 \text{ rad/s}$$

and

$$\omega = -75.25 \text{ rad/s}$$

The angular acceleration α of the cone relative to the ground must be given by

$$\begin{aligned} \boldsymbol{\alpha} &= \frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt} (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \\ &= \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = 0 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \\ &= 18.85 \times (-77.57 \times (0.97 + 0.243)) \\ &= 1420 \text{ rad/s}^2 \end{aligned}$$

The motion of the cone relative to the ground is such that while it rotates about OY with an angular velocity of 75.25 rad/s anticlockwise as seen from O , it accelerates about O with an acceleration of 1420 rad/s² clockwise when seen from O .

Example 7.9

A wheel is made to roll without slip, towards right, by means of pulling a string wrapped around a coaxial spool as shown in Fig. Ex. 7.9. Determine the velocity of pulling the string such that the centre of the wheel has a velocity of 10 m/s.

Compute the angular velocity of the wheel and velocity of the point marked B on the wheel.

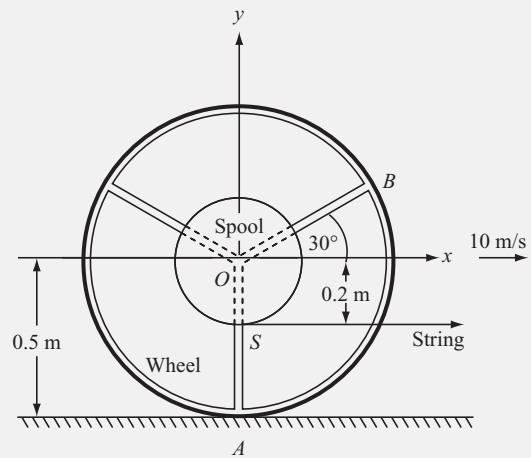


Fig. Ex. 7.9

Solution The condition for a wheel rolling without slip on a surface is that the point of contact A must be the instantaneous centre of rotation, i.e.,

$$v_A = 0$$

From the data,

$$v_O = 10$$

Therefore, the velocity of O relative to A must be

$$v_{OA} = v_O + v_A = 10$$

which should also be equal to $\omega \times OA$

Hence $\omega \times 0.5 = 10$

or $\omega = -20 \text{ rad/s}$

which implies that the wheel rotates at 20 rad/s clockwise as seen from this end.

The velocity of point S or of the string would be given by

$$\begin{aligned} v_S &= v_A + \omega \times SA \\ &= 0 + (-20) \times 0.3 \\ &= 6 \text{ m/s} \end{aligned}$$

The velocity of point B marked on wheel would be given by

$$\begin{aligned} v_B &= v_A + \omega \times BA \\ &= 0 + (-20) \times 0.5 (\cos 30^\circ + \sin 30^\circ) \\ &= (5 - 8.66) \text{ m/s} \end{aligned}$$

Example 7.10

A link OAR rotates anticlockwise at an angular velocity of 1 rad/s. Another link BCS at right angles to it has a collar at B which slides over OAR at 1 m/s and decelerates at 2 m/s^2 with respect to OAR as shown in Fig. Ex. 7.10. A collar D slides over BCS with a velocity of 3 m/s and decelerates at 4 m/s^2 with respect to BCS as shown. Obtain the velocity and acceleration of the collar D with respect to ground reference at the instant of interest.

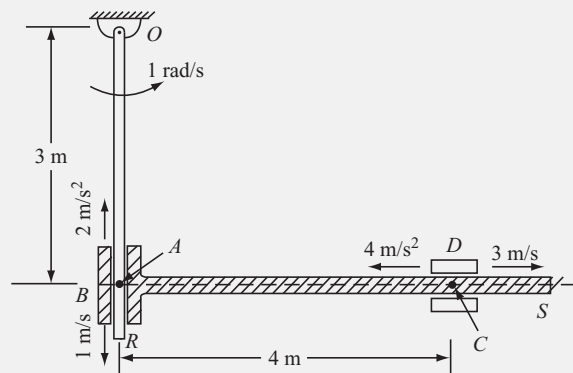


Fig. Ex. 7.10

Solution There are many alternative methods of locating the moving frame of reference. Accordingly, the methods of solution differ.

Method I

The moving frame can be fixed at point C on the link BCS such that

$$\omega = 1 \text{ rad/s}$$

$$\alpha = 0$$

and

$$v_D = 0$$

since the points C and D are coincident.

$$a_{Dm} = 3 \text{ m/s}^2$$

$$\begin{aligned} D_f &= D_m + C + \omega \times D \\ &= 3 + (1 \times (4 - 3) - 1) + 0 \\ &= 6 + 3 \text{ m/s} \end{aligned}$$

Similarly, $D_m = -4 \text{ m/s}$

and

$$\begin{aligned} D_f &= D_m + C + \alpha \times D + 2\omega \times D_m + \omega \times (\omega \times D) \\ &= -4 + (2 + 1 \times (1 \times (4 - 3))) + 2 \\ &\quad \times 1 \times (-1) + 0 + 2 \times 1 \times 3 + 0 \\ &= -6 + 11 \text{ m/s}^2 \end{aligned}$$

Method II

Let the moving frame be fixed at O on the link OAR such that

$$\omega = 1 \text{ rad/s}$$

$$\alpha = 0$$

and $D = 4 - 3$

The velocity of the collar D with respect to the link OAR is

$$D_m = (3 -) \text{ m/s}$$

and the acceleration of the collar with respect to the link OAR is

$$D_m = (-4 + 2) \text{ m/s}^2$$

Velocity of the collar D with respect to ground reference is

$$\begin{aligned} D_f &= D_m + 0 + \omega \times D \\ &= (3 -) + 0 + 1 \times (4 - 3) \\ &= 3 - + 4 + 3 = 6 + 3 \text{ m/s} \end{aligned}$$

Acceleration of the collar D with respect to the ground reference is

$$\begin{aligned} D_f &= D_m + 0 + \alpha \times D + 2\omega \times D_m + \omega \times (\omega \times D) \\ &= (-4 + 2) + 0 + 0 + 2 \times 1 \times (3 -) + \\ &\quad 1 \times (1 \times (4 - 3)) \\ &= -4 + 2 + 6 + 2 - 4 + 3 \\ &= -6 + 11 \text{ m/s}^2 \end{aligned}$$

Method III

Let us now fix the moving frame at point A on the link OAR such that

$$\omega = 1 \text{ rad/s}$$

$$\alpha = 0$$

and $D = 4$

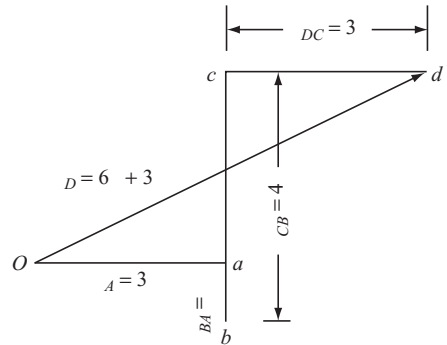
Velocity and acceleration of the collar D with respect to the link OAR are first obtained:

$$D_m = (3 -) \text{ m/s}$$

$$D_m = (-4 + 2) \text{ m/s}^2$$

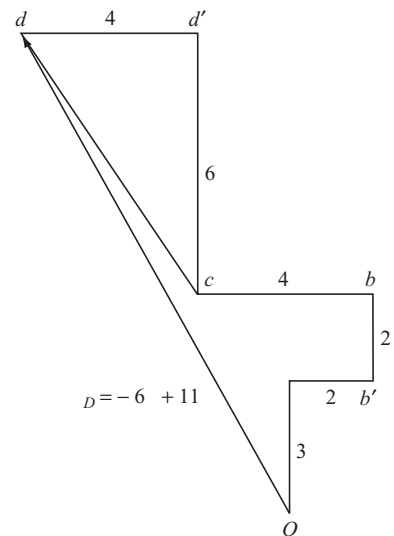
Velocity of the collar D with respect to the ground reference is then expressed as

$$\begin{aligned} D_f &= D_m + A + \omega \times D \\ &= (3 -) + 1 \times (-3) + 1 \times (4) \\ &= 3 - + 3 + 4 = 3 + 6 \text{ m/s} \end{aligned}$$



Velocity diagram

Fig. Ex. 7.10(a) (Solution)



Acceleration diagram

Fig. Ex. 7.10(b) (Solution)

Acceleration of the collar D with respect to the ground reference is given by

$$\begin{aligned} D_f &= D_m + A + \alpha \times D + 2 \omega \times D_m + \omega \times (\omega \times D) \\ &= -4 + 2 + 1 \times (1 \times (-3)) + 0 \\ &\quad + 2 \times 1 \times (3 -) + 1 \times (1 \times 4) \\ &= -4 + 2 + 3 + 6 + 2 - 4 \\ &= -6 + 11 \text{ m/s}^2 \end{aligned}$$

Method IV

The moving frame may alternatively be fixed on the intermediate collar B but at a location coincident with A such that

$$\begin{aligned} \omega &= 1 \text{ rad/s} \\ \alpha &= 0 \end{aligned}$$

and

$$D = 4$$

The velocity and acceleration of the collar D with respect to the collar B on link BCR are

$$\begin{aligned} D_m &= 3 \text{ m/s} \\ D_m &= -4 \text{ m/s} \end{aligned}$$

The velocity of the collar D with respect to the ground reference is expressed as

$$\begin{aligned} D_f &= D_m + B + \omega \times D \\ &= 3 + (1 \times (-3) - 1) + 1 \times (4) \\ &= 3 + 3 - 1 + 4 \\ &= 6 + 3 \text{ m/s} \end{aligned}$$

The acceleration of the collar D with respect to the ground reference is given by

$$\begin{aligned} D_f &= D_m + B + \alpha \times D + 2 \omega \times D_m + \omega \times (\omega \times D) \\ &= -4 + (2 + 1 \times (1 \times (-3))) + 2 \times 1 \times (-) \\ &\quad + 2 \times 1 \times (3) + 1 \times (1 \times 4) \\ &= -4 + 2 + 3 + 2 + 6 - 4 \\ &= -6 + 11 \text{ m/s}^2 \end{aligned}$$

Method V

Let us, in this case, proceed by choosing a moving frame of reference at C on link BCS and an intermediate moving frame at A on link OAR . Let these frames be denoted m_1 and m_2 respectively and

For m_1 with respect to m_2

$$\begin{aligned} \omega_1 &= 0 \\ \alpha_1 &= 0 \end{aligned}$$

For m_3 with respect to ground reference

$$\begin{aligned} \omega_2 &= 1 \text{ rad/s} \\ \alpha_2 &= 0 \end{aligned}$$

The velocity and acceleration of the collar D with respect to the moving frame m_1 are, respectively.

$$\begin{aligned} D_{m1} &= 3 \text{ m/s} \\ D_{m1} &= -4 \text{ m/s}^2 \end{aligned}$$

Since D and C are coincident points,

$$\begin{aligned} D_{m1} &= 0 \\ D_{m2} &= D_{m1} + C_{m2} + \omega_1 \times D_{m1} \\ &= 3 - 1 + 0 = 3 - 1 \end{aligned}$$

and

$$\begin{aligned} D_{m2} &= D_{m1} + C_{m2} + \alpha_2 \times D_{m1} + 2 \omega_1 \times D_{m1} + \omega_1 \times (\omega_1 \times D_{m1}) \\ &= -4 + 2 + 0 + 0 + 0 = -4 + 2 \end{aligned}$$

These are indeed the values of the velocity and acceleration of the collar D with respect to the link OAR as intuitively written in method III. Now the moving reference is fixed at A on link OAR as in the method III and the velocity and acceleration with respect to the ground reference are obtained,

$$\begin{aligned} D_f &= 3 + 6 \text{ m/s} \\ D_f &= -6 + 11 \text{ m/s}^2 \end{aligned}$$

Method VI

There is no end to the choice and location of the moving frames. Moving frames may be fixed at intermediate locations on the links but no advantage can be gained by such choices. Finally, let us see how we can solve the problem by two extreme choices of the moving frames:

- (a) Without fixing the moving frame anywhere.
- (b) By fixing the moving frame at the collar D itself.

In case (a), the moving frame is coincident with the fixed frame.

$$\begin{aligned} \omega &= 0 \\ \alpha &= 0 \end{aligned}$$

The velocity and acceleration in the moving and the fixed frame are such that

$$\begin{aligned} D_m &= D_f \\ D_m &= D_f \end{aligned}$$

The entire thinking of the components and the constitution of D_f and D_f is, therefore, done mentally in one long step. This is clearly inconvenient and unmanageable.

In case (b), the moving frame is at the collar D itself such that

$$\begin{aligned} D_m &= 0 \\ D_m &= 0 \end{aligned}$$

which leave the entire job of finding out D_f and D_f to the application of the formulae in one long step. The method is as inconvenient as that in case (a). Thus, we can state that if a moving frame is desired to be used to advantage then it may not be chosen to be at the object. The moving frame should be located at a convenient point intermediate between the object and the ground reference.

Example 7.11

A cylinder of radius R rolls over a flat surface. Show that the motion is equivalent to 'pure translation plus pure rotation' if there is no slip. State an expression for the kinetic energy of the rolling cylinder.

Solution A body in pure translation has the same velocity for all particles. For a cylinder, therefore, the centre of mass, a point at the top and a point at the bottom (or a point any where on it) has the same velocity v_C , as shown in Fig. Ex.7.11(a) (Solution).

A body in pure rotation at ω , as shown should have its centre of mass at rest (i.e., zero velocity at centre of mass) and linearly increasing speed, with radius, in the direction of rotation. Hence, a cylinder in pure rotation should have speeds at a point at the top and a point at the bottom as shown in Fig. Ex. 7.11(b) (Solution).

If the rotational speed ω is such that $v = R\omega = v_C$ in magnitude then, the combination of pure translation and pure rotation results in rolling motion about the bottom point B at that instant.

Clearly, velocity of a rolling cylinder is zero at the bottom, v_C at the centre of mass and $v_C + R\omega$ i.e., $2v_C$ at the top point. This is what is meant by stating that B is the instantaneous centre of rotation' because B is at rest and the velocity at other points increases linearly as we go away from B .

Velocity at $C = R\omega$

Velocity at $T = 2R\omega$

Velocity at any other point must be perpendicular to the line joining B to that point and be equal to ω times the distance to that point. For example,

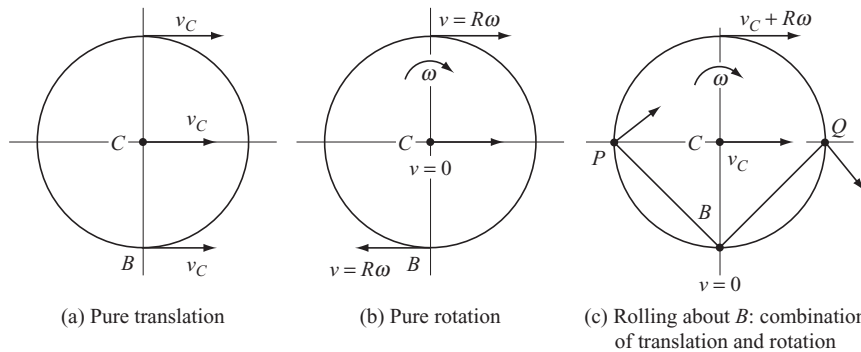


Fig. Ex. 7.11 (Solution)

Velocity at $P = PB \cdot \omega$ perpendicular to PB

Velocity at $Q = QB \cdot \omega$ perpendicular to QB , etc.

Having shown the equivalence,

$$\begin{aligned}
 KE_{\text{rolling}} &= KE_{\text{trans}} + KE_{\text{rotation}} \\
 &= \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2 \\
 &= \frac{1}{2} M (r\omega)^2 + \frac{1}{2} I_C \omega^2 \\
 &= \frac{1}{2} (I_C + Mr^2) \omega^2 \\
 &= \frac{1}{2} I_B \omega^2
 \end{aligned}$$

where I_B is the moment of inertia of the cylinder about the instantaneous centre of rotation.

Concept Review Questions

State with justifications if the following statements are true or false:

- A rigid body may move along a curved path but may not be in a state of rotation.
- A general plane motion must be a combination of a translation and a rotational motion about an axis perpendicular to the plane.
- There must always be an instantaneous centre of rotation and a centre of angular acceleration of a rigid body no matter what the mode of motion of the body may be.

- The plane motion of a rigid body is characterised by the locus of the instantaneous centre of rotation to be a plane curve.

Classify the following motions as translation, rotation, plane motion or space motion:

- A cone rolling on a flat surface.
- A cone sliding on a flat surface.
- A door being shut by turning about the hinges.
- A spherical ball rolling down an incline.

- (e) An aeroplane banked and taking a steady turn.
- (f) A compound pendulum oscillating about the fulcrum.
- (g) A spinning top in an unsteady motion.
- (a) Under what conditions do the instantaneous centres of rotation and acceleration coincide for the motion of a rigid body?
- (b) When would you prefer to use the methods of instantaneous centre of rotation, graphical construction and vectorial analysis for the analysis of link motions?

- (a) Show that the angular velocity of every point on a link in a general plane motion is the same and hence show that *the angular velocity refers to the entire link rather than for any point on the link.*
- (b) Also show that in a general plane motion of a link, the translational velocity of the link is not uniquely defined.

Tutorial Problems

A simple engine mechanism consists of a crank of radius $OA = 20$ cm and a connecting rod AB of length 1 m. The crank rotates clockwise at a constant rate of 500 revolutions per minutes as shown in Fig. Prob. 7.1. Determine the velocity and acceleration of the piston when the crank makes an angle θ with the inner dead centre line. Determine the positions of the crank and the corresponding values of the velocity and acceleration of the piston when (a) the piston has the maximum velocity and (b) the piston has the maximum acceleration.

(s 10.47 (sin θ + cos θ tan ϕ) (a) 10.68 cm/s; $\theta = 78.7$ $\phi = 11.3$ (b) O ; $\theta = O, \pi$)

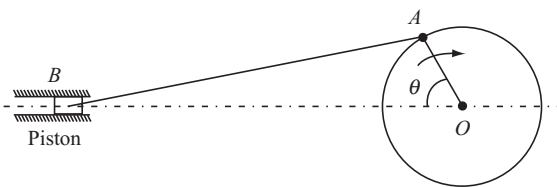


Fig. Prob. 7.1

A disc of radius 10 cm rolls down the inner curve of a circular ring of radius 20 cm as shown in Fig. Prob. 7.2. Find the acceleration of the point A and B on the horizontal diameter of the disc when the centre C of the disc has a velocity of 10 m/s at the instant shown.

(s 1000 m/s², A to C)

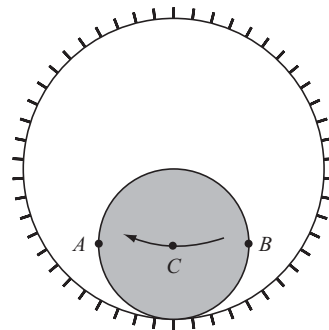


Fig. Prob. 7.2

A circular disc of radius 0.4 m rolls without slip up a 30° incline plane as shown in Fig. Prob. 7.3. The centre of the disc C has a uniform velocity $V = 10$ m/s parallel to the plane. Determine the velocity and acceleration of the point A on the disc vertically above C .

(s 250 m/s²; downward)

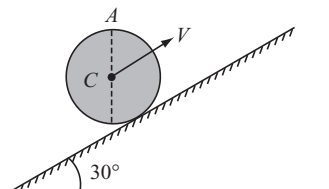


Fig. Prob. 7.3

An oscillating quick-return mechanism consists of a 3 cm crank O_1A rotating clockwise at

5 rad/s. The end A is pinned to a collar C which can slide along an oscillating bar O_2B 15 cm long. At the instant shown in Fig. Prob. 7.4, determine

- (i) the velocity of B
 - (ii) the angular velocity of the bar O_1B
 - (iii) the angular acceleration of the bar O_2B .
- (s (2.27 - 10) m/s; -0.87 rad/s and -95.3 rad/s²)

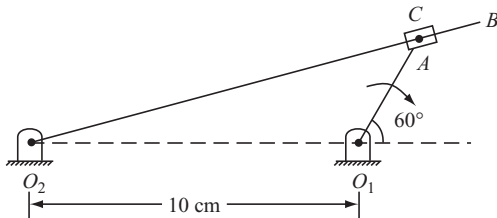
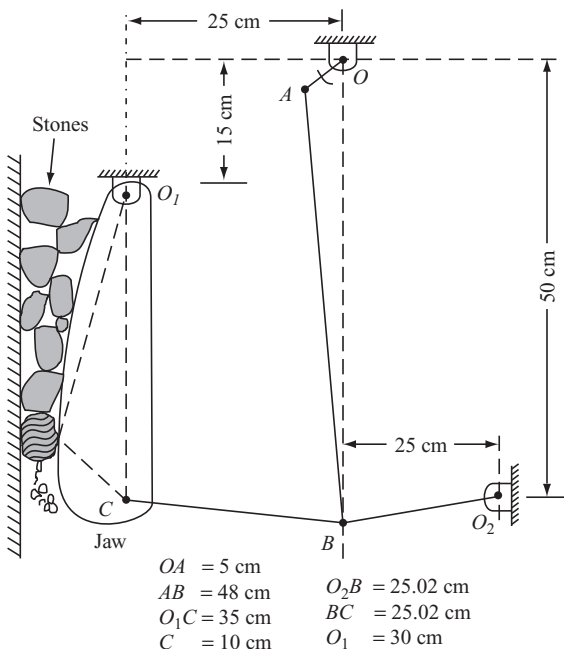


Fig. Prob. 7.4

A stone-crusher mechanism is shown in Fig. Prob. 7.5. The crank OA rotates clockwise at 100 revolutions per minute. For the given configuration, determine the velocity and acceleration of the points marked on the crusher jaw.

(s 0.094 m/s)



- $OA = 5$ cm
- $AB = 48$ cm
- $O_1C = 35$ cm
- $C = 10$ cm
- $O_2B = 25.02$ cm
- $BC = 25.02$ cm
- $O_1 = 30$ cm

Fig. Prob. 7.5

A straight bar OA rotates and accelerates about a fixed axis through O as shown in Fig. Prob. 7.6. It carries a collar C which slides and accelerates with respect to the bar as shown. Find the total absolute acceleration of the collar.

(s (-260 + 180) m/s²)

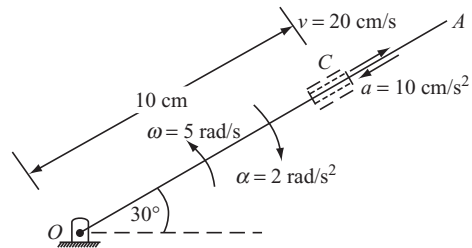


Fig. Prob. 7.6

In a four-bar mechanism $ABCD$ shown in Fig. Prob. 7.7, link AB rotates anticlockwise at 5.25 rad/s and accelerates clockwise at 23 rad/s². Obtain the angular velocity and angular acceleration of the link CD .

(s 4.05 rad/s anticlockwise and 17 rad/s² anticlockwise)

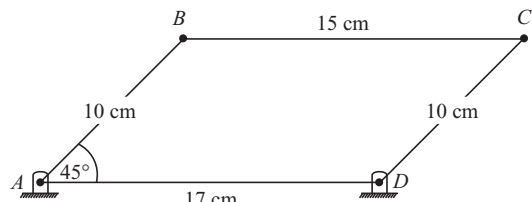


Fig. Prob. 7.7

- 8 A slender bar AB slides down a circular surface and on a horizontal surface as shown in Fig. Prob. 7.8. At an instant when $\theta = 45^\circ$ the velocity of the end A is 2 m/s. Determine the angular velocity of the bar and velocity of the point of contact on the circular surface.

(s 7.07 rad/s; 1.414 m/s)

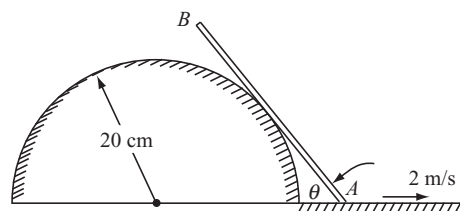


Fig. Prob. 7.8

A wheel of diameter 0.5 m with three equi-spaced spokes rolls without slip on a flat surface and proceeds to the right at 20 m/s as shown in Fig. Prob. 7.9. Determine the angular speed of the wheel and velocity of the point where the top spoke joins the rim when it is vertical.

(s 80 rad/s clockwise; 40 m/s)

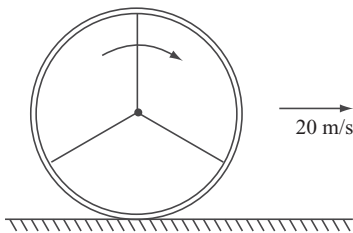


Fig. Prob. 7.9

A slotted link mechanism consists of a curved slotted link AB and a straight link CD with a pin P on it. At the instant of interest, the angular velocity of AB is 3 rad/s clockwise as shown in Fig. Prob. 7.10 and increases at the rate of 2 rad/s each second. At this instant the centre of the link S lie at the line CA produced. Determine the angular velocity of the link CD .

(s 0.89 rad/s clockwise)

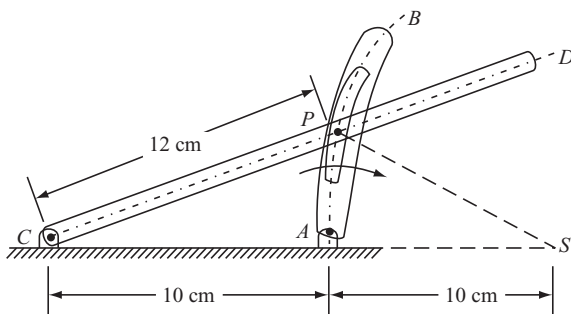


Fig. Prob. 7.10

A link AB 20 cm long oscillates about B while a slider at A slides in a straight slot in

a link CD which is made to oscillate about O . At the instant of interest, $\theta = 30^\circ$ shown in Fig. Prob. 7.11, the link AB rotates at 5 rad/s and accelerates at -25 rad/s^2 . Determine the angular velocity of the slotted member CD at this instant.

(s 9.97 rad/s anticlockwise)

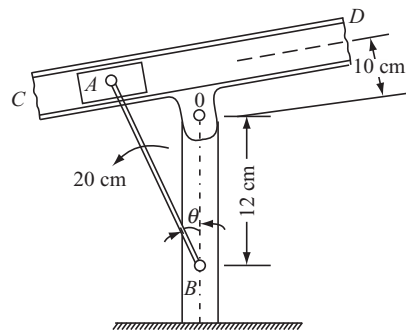


Fig. Prob. 7.11

In Fig. Prob. 7.12, a mass m and a drum of radius a are attached to a string of radius r that winds around the drum as the mass rotates about it. Initially, the distance between the mass and drum is R and the angular velocity of the mass of ω_0 . Express the radius r as a function of time, and find the time at which it reduces to zero. Assume that the system is on a frictionless horizontal table.

$$\left(\text{ s } R \left(1 - \frac{3a\omega_0 t}{R} \right)^{1/3} \text{ and } R/3a\omega_0 \right)$$

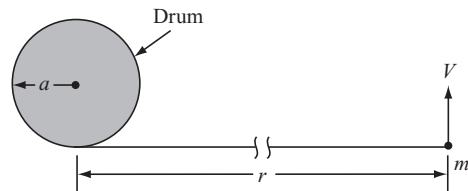


Fig. Prob. 7.12

A cylinder rolls inside a second hollow cylinder as shown in Fig. Prob. 7.13. If a cord wrapped

around the outside of the larger cylinder is pulled at a velocity V and acceleration a , find the magnitudes of the angular velocity and angular acceleration of the inner cylinder.

$$\left(s \frac{r_2}{r_1 r_3} v; \frac{r_2}{r_1 r_3} a \right)$$

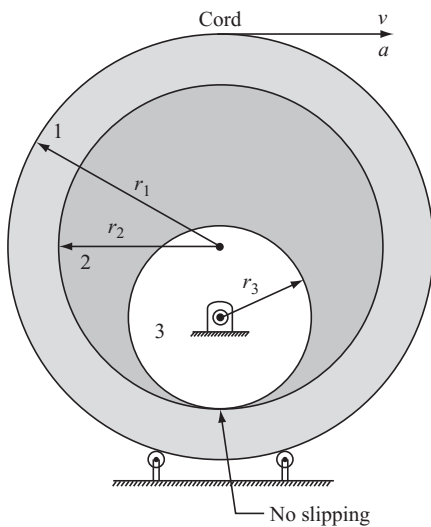


Fig. Prob. 7.13

A circular cylinder of radius R is supported between two horizontal planks as shown in Fig. Prob. 7.14. The planks have horizontal

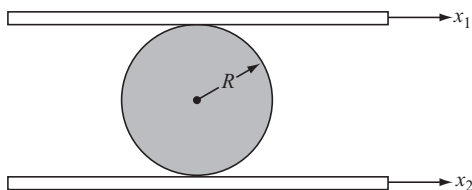


Fig. Prob. 7.14

velocities \dot{x}_1 and \dot{x}_2 as shown. Find the velocity of the centre of the cylinder.

$$\left(s \frac{\dot{x}_1 + \dot{x}_2}{2} \right)$$

The end A of a straight bar moves with a constant tangential velocity v along a semi-cylindrical trough as shown in Fig. Prob. 7.15. Find the velocity of the point B at the point of contact of the bar and the edge of the trough as a function of the angle ϕ between the bar and the horizontal. The bar moves in a plane normal to the axis of the trough.

$$\left(s v/2 \sin \phi \right)$$

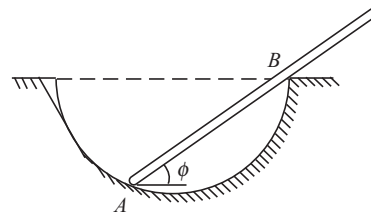


Fig. Prob. 7.15

In Fig. Prob. 7.16, C is a roller fixed to the link OB and sliding in a slot in QD . Determine the velocity of A when $\theta = 30^\circ$ if QD rotates at constant speed 10 rad/s clockwise.

$$\left(s 4.28 \text{ m/s} \right)$$

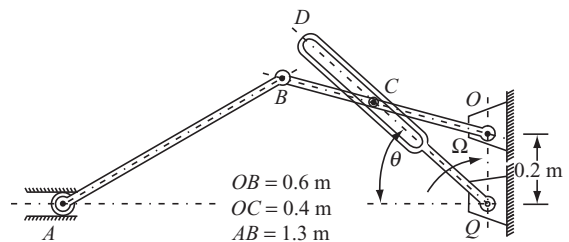


Fig. Prob. 7.16

Figure Prob. 7.17 shows diagrammatically the quick-return mechanism of a machine tool in which the driving crank AB rotates clockwise with a uniform speed of 150 revolutions per minute. What is the velocity and acceleration of the cutting tool at E when the crank AB makes an angle of 60° with the vertical line passing through the pivots A and C ?

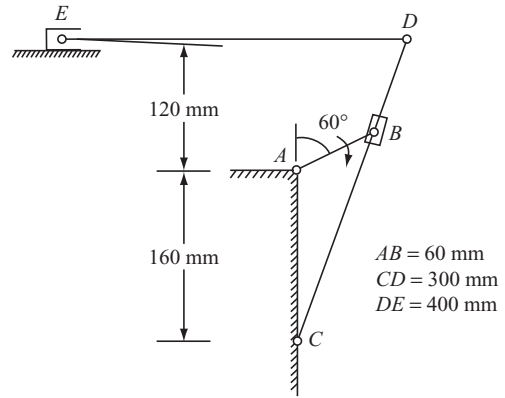


Fig. Prob. 7.17

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

- Degrees of freedom of a rigid body imply the
- (a) angles that it may turn through
 - (b) angular motions the body can have
 - (c) constraints to its motion
 - (d) total number of modes of displacement

- A rigid body, in translation,
- (a) can only move in a straight line
 - (b) may move along a straight or curved path
 - (c) cannot move on a circular path
 - (d) must undergo plane motion only

- The instantaneous centre of rotation
- (a) should also be the instantaneous centre of acceleration
 - (b) is a hypothetical concept to solve problems
 - (c) can exist for any space motion
 - (d) must exist for any plane motion

- For the motion of a rigid link in any mechanism,
- (a) the velocity of one of the ends should be zero
 - (b) the velocity of one end with respect to that of the other should be perpendicular to the link
 - (c) the acceleration of one end with respect to that of the other should be perpendicular to the link
 - (d) the two ends of the link may have different components of velocity along the link

Answers to Multiple-Choice Questions

- 1 (d) 2 (b) 3 (d) 4 (b)



Dynamics of a Particle

and of the Mass Centre of a System

6

6.1 INTRODUCTION

Dynamics refers to the study of motion of bodies in an inertial frame of reference under the application of external forces or moments. The same laws of dynamics are applicable to the motion of a particle and to the centre of mass of any system undergoing translation. This is because the mass of a particle is assumed to be concentrated at a point which is also its centre of mass. This fact is supported by kinematic considerations; the general motion of a rigid body may be considered to comprise the translation of the mass centre and a rotation superimposed upon it. The laws and principles studied in this chapter apply to the translation of any body or a system of bodies. The terms ‘centre of mass’ and ‘particle’ are at times used interchangeably in this chapter and should cause no confusion on the scope of application of the equations.

The dynamics of a mass centre or of a particle is governed by the Newton’s law which states that the external force equals rate of change of momentum:

$$F = \frac{d}{dt}(mv) \quad (6.1(a))$$

This is the fundamental equation of motion which governs the interaction of the applied force with the motion of a particle or a mass centre with variable mass and velocity. Problems in dynamics may be concerned with the determination of the motion, i.e., acceleration, velocity and positions for a prescribed force or vice versa.

It may be stated at the outset that the work-energy principle, impulse-momentum principle and the moment of momentum principle are alternative forms of Newton’s law. One or the other may be preferred under different circumstances. A comparative study of the equivalent dynamical equations is given in Table 6.1. This is indeed a summary of the principles derived and discussed in this chapter.

Table 6.1 Comparative Study of the Equivalent Dynamical Equations for a Particle or a Mass Centre

| | Newton's Law | Work-Energy Equation | Impulse-Momentum Principle | Moment of Momentum Principle |
|---|--|---|---|--|
| <i>General form for a particle</i> | $\mathbf{F} = \frac{d}{dt}(m \mathbf{V}_c) = m \mathbf{a}_c$ | $W_{12} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = KE_2 - KE_1$ | $\int_{t_1}^{t_2} \dot{\mathbf{p}} \, dt = m(\mathbf{v}_2 - \mathbf{v}_1)$ | $\dot{\mathbf{h}}_O = \mathbf{r}_O \times \dot{\mathbf{p}} = \frac{d}{dt}(\mathbf{r}_O \times m \mathbf{v}_c) = \frac{d}{dt} \mathbf{h}_O$ |
| <i>Statement for a particle</i> | Net force acting on a particle equals mass times acceleration of the particle. | Net work done by the external forces acting on a particle equals the change in kinetic energy possessed by the particle. | Impulse of the net force on a particle equals the change of momentum of the particle. | Moment of the resultant force on a particle about a fixed point O equals the rate of change of the moment of momentum referred to the point |
| <i>General form for the mass centre of a body</i> | $\mathbf{F} = \frac{d}{dt}(m \mathbf{V}_c) = m \mathbf{a}_c$ | $W_{12} = \int_{r_1}^{r_2} \mathbf{F}_c \cdot d\mathbf{r}_c = KE_{2,c} - KE_{1,c}$ | $\int_{t_1}^{t_2} \dot{\mathbf{p}}_c \, dt = m(\mathbf{v}_{2,c} - \mathbf{v}_{1,c})$ | $\dot{\mathbf{h}}_O = \mathbf{r}_O \times \dot{\mathbf{p}}_c = \frac{d}{dt}(\mathbf{r}_O \times m \mathbf{v}_{c,c}) = \frac{d}{dt} \mathbf{h}_O$ |
| <i>Statement for the mass centre of a body</i> | Net force acting on a body equals the mass times acceleration of the centre of its mass. | Net work done by the external forces, equals the change in linear kinetic energy. | Impulse of the net force on a body equals the change in linear momentum | Moment of the resultant force of a system about a fixed point O equals the rate of change of moment of momentum about the same point. |
| <i>Special cases Associated Principles</i> | When $\mathbf{F} = 0$; a particle or the mass centre of body must continue to move with constant velocity: $\mathbf{a}_c = \text{Const.}$ | When $\mathbf{F} = -\nabla PE$; in a conservative force field, the mechanical energy is conserved; $KE + PE = \text{Const.}$ | When $\mathbf{F}_c = 0$; the linear momentum is conserved: $\Sigma \mathbf{p}_c = \text{Const.}$ | When $\mathbf{h}_O = 0$; about a fixed point, the moment of momentum relative to the same point is conserved; $\Sigma (\mathbf{r}_O \times m \mathbf{v}_c) = \text{Const.}$ |
| <i>Circumstances under which preferred</i> | When the external force or acceleration is desired to be evaluated | When the work done or one of the velocities is to be evaluated; particularly in a conservative force field. | When the external force or impulse vanishes. | When the force on a particle is directed towards or away from a fixed point, e.g. central force motion. |

6.2 EQUATION OF MOTION

The equation of motion due to Newton Eq. 6.1(a) for the centre of mass of any system or a particle of mass m reduces to

$$m \frac{d^2 s}{dt^2} = F \tag{6.1 (b)}$$

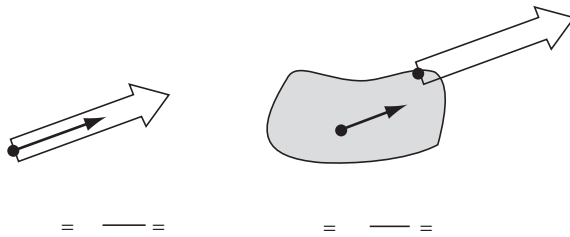


Fig. 6.1 Implication of Newton's law



- Case (a) = Const.
- Case (b) = ()
- Case (c) = ()
- Case (d) = ()

Fig. 6.2 Rectilinear translation

It may be noted that for a particle, the net force, velocity and acceleration refer to the point representation of the particle, while for a rigid body, the net force may be applied anywhere on it but the velocity and acceleration are referred to the mass centre only as shown in Fig. 6.1.

The motion can be determined from a knowledge of the applied force F . Let us first consider some simple cases of rectilinear translation as visualised in Fig. 6.2.

Case (a): Constant Force F along s -direction

Then,

$$m \frac{d^2 s}{dt^2} = F$$

On integration,

$$m \frac{ds}{dt} = Ft + c_1$$

and

$$s = \frac{1}{2} Ft^2 + c_1 t + c_2$$

The constants of integration are determined from the given conditions.

Case (b): Force $F(t)$ is Function of Time along s -direction

Then,

$$m \frac{d^2 s}{dt^2} = F(t)$$

On integration,

$$m \frac{ds}{dt} = \int F(t) dt + c_1$$

and

$$s = \int \left(\int F(t) dt + c_1 \right) dt + c_2$$

where the constants c_1 and c_2 are again determined from the given conditions.

Case (c): Force $F(V)$ is a Function of Speed along s -direction

Then,

$$m \frac{dV}{dt} = F(V)$$

$$\frac{dV}{F(V)} = \frac{1}{m} dt$$

or,

On integration,

$$\int \frac{dV}{F(V)} = \frac{1}{m} t + C_1$$

which provides $V = f(t)$

and on further integration, yields an expression for the displacement s , the constant being determined from the given conditions.

Case (d): Force $F(s)$ is Function of the Rectilinear Displacement s

Then,

$$a = \frac{dV}{dt} = F(s)/m$$

or,

$$\frac{dV}{ds} \frac{ds}{dt} = V \frac{dV}{ds} = F(s)/m$$

or

$$V dV = \frac{1}{m} F(s) ds$$

On integration,

$$\frac{V^2}{2} = \frac{1}{m} \int F(s) ds + C_1$$

and

$$V = \frac{ds}{dt} = \left[\frac{2}{m} \int F(s) ds + C_1 \right]^{1/2}$$

Separating the variables and integrating again provides s as a function of t , the constants of integration being determined from a knowledge of the given conditions.

If the motion of a particle is prescribed, the force required to accomplish it may be determined by employing the equation of motion

$$F = ma$$

and substituting the value of a in it.

Case (e): Force of constant magnitude which is always perpendicular to the velocity of the particle

In this case,

$$\mathbf{F} \cdot \mathbf{v} = 0$$

because \mathbf{F} has no component along the velocity vector.

Since F is constant in magnitude, the particle must have its acceleration constant in magnitude. Directions of \mathbf{v} , \mathbf{a} and \mathbf{F} may vary.

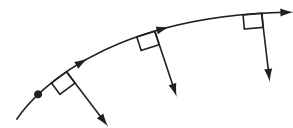


Fig. 6.3

Example 6.1

A 20 kg box is to be slid on a flat horizontal surface with dynamic coefficient of friction $\mu = 0.25$. Determine the acceleration of the box and hence its velocity after 2 seconds of starting from rest in three cases:

- (a) When the force of 200 N is applied horizontally.
- (b) When the force of 200 N is applied to pull up a string at 30° .
- (c) When the force of 200 N is applied to push down the box at 30° .

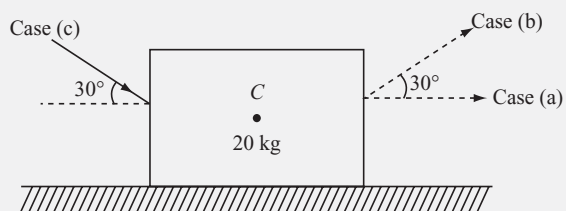


Fig. Ex. 6.1

Solution

(a) Let us draw the *fbd* of the box for case (a)

By equations of motion

$$\Sigma F_x = m a_x$$

$$\Sigma F_y = m a_y$$

In this case,

$$200 - f = 20 \cdot a_x$$

$R - mg = 0$, there being no acceleration in y direction

whence $R = mg = 20 \times 9.81 = 196.2 \text{ N}$

and $f = \mu R = 0.25 \times 196.2 = 49 \text{ N}$

Therefore, $200 - 49 = 20 a_x$

$$a_x = 7.55 \text{ m/s}^2$$

This is a case of constant force and constant acceleration.

Therefore, $V = 0 + 7.55 \times 2 = 15.1 \text{ m/s}$.

(b) With reference to *fbd* for case (b), where the vertical component of the reaction is shown as R , the equation of motion along x -axis

$$\Sigma F_x = m a_x$$

results in

$$200 \cos 30 - f = m a_x$$

and $R - mg + 200 \sin 30 = 0$

From the second equation,

$$R = mg - 200 \times \frac{1}{2}$$

$$= 196.2 - 100 = 96.2 \text{ N}$$

and

$$f = 0.25 \times 96.2 = 24 \text{ N}$$

Substituting in (i),

$$a_x = \frac{200 \times 0.866 - 24}{20} = 7.46 \text{ m/s}^2$$

This is again a case of constant force and constant acceleration.

Therefore, $V = 0 + 7.46 \times 2 = 14.92 \text{ m/s}$.

(c) For case (c), referring to the *fbd*, the equations of motion

$$\Sigma F_x = m a_x$$

$$\Sigma F_y = m a_y$$

provide

$$200 \cos 30 - f = m a_x \tag{i}$$

$$R - mg - 200 \sin 30 = 0 \tag{ii}$$

From the second equation,

$$R = 196.2 + 100 = 296.2 \text{ N}$$

and

$$f = 0.25 \times 296.2 = 74 \text{ N}$$

Substituting in (i)

$$a_x = \frac{200 \times 0.866 - 74}{20} = 4.96 \text{ m/s}^2$$

The velocity after 2 seconds, starting from rest is

$$V = 0 + 4.96 \times 2 = 9.92 \text{ m/s}.$$

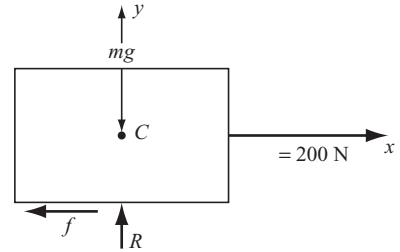


Fig. Ex. 6.1(a) (Solution)

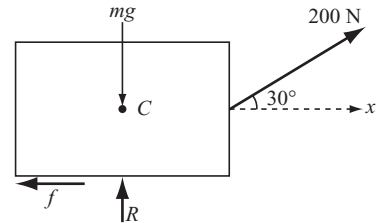


Fig. Ex. 6.1(b) (Solution)

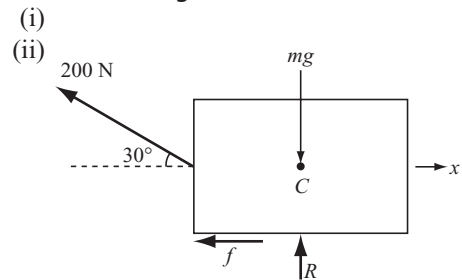


Fig. Ex. 6.1(c) (Solution)

Reflections

By comparing the resulting acceleration in each case with the same force applied at different angles, it is noticed that ‘pulling is better than pushing’! This is because, in pushing, the reaction from the floor and hence the frictional resistance increases! While pulling, the frictional resistance decreases.

Is $\theta = 30^\circ$ the best angle for pulling up? Let us analyse the general case of pulling at an angle θ and discover the value of θ for maximum acceleration.

Consider the *fbd* of the block with the force acting at an arbitrary angle θ , as shown.

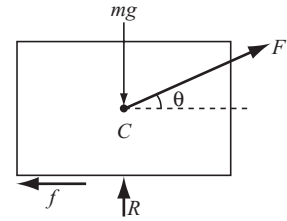


Fig. Ex. 6.1 (d) (Solution)

Then,

$$F \cos \theta - f = m a_x \tag{i}$$

$$F \sin \theta + R - mg = 0 \tag{ii}$$

From (ii),

$$R = mg - F \sin \theta$$

and

$$f = \mu (mg - F \sin \theta)$$

Substituting in (i)

$$F \cos \theta - \mu (mg - F \sin \theta) = m a_x$$

whence

$$a_x = \frac{F}{m} (\cos \theta + \mu \sin \theta) - \mu g$$

For maximum/minimum a_x ,

$$\frac{da_x}{d\theta} = 0; -\sin \theta + \mu \cos \theta = 0$$

$$\sin \theta / \cos \theta = \tan \theta = \mu$$

In this case,

$$\tan \theta = 0.25$$

whence

$$\theta = 14^\circ$$

and

$$\sin 14^\circ = 0.24 \quad \cos 14^\circ = 0.97$$

Therefore,

$$(a_x)_{\max} = \frac{200}{20} (0.97 + 0.25 \times 0.24) - 0.25 \times 9.81$$

$$= 10.3 - 2.45 = 7.85 \text{ m/s}^2.$$

Example 6.2

A particle of mass 1 kg moves in a straight line under the influence of a force which increases linearly with time at the rate of 60 N/s, it being 40 N initially. Determine the position, velocity and acceleration of the particle after a lapse of 5 s if it started from rest at the origin.

Solution From the statement of the problem,

$$F = 40 + 60t$$

which, by Newton’s law should equal mass times acceleration of the particle. Since the mass is 1 kg,

$$a = \frac{d^2x}{dt^2} = 40 + 60t$$

Integrating the terms with respect to time t .

$$v = \frac{dx}{dt} = 40t + 30t^2 + C_1$$

and integrating again,

$$x = 20t^2 + 10t^3 + C_1t + C_2$$

From the initial conditions,

$$v = 0 \quad \text{and} \quad x = 0 \quad \text{at} \quad t = 0$$

the constant C_1 and C_2 vanish.

Hence,

$$v = 40t + 30t^2$$

and

$$x = 20t^2 + 10t^3$$

At the instant,

$$t = 5 \text{ s}$$

$$a = 40 + 60 \times 5 = 340 \text{ m/s}^2$$

$$v = 40 \times 5 + 30 \times 5^2 = 950 \text{ m/s}$$

$$x = 20 \times 5^2 + 10 \times 5^3 = 1750 \text{ m from the origin.}$$

Example 6.3

A particle moving with a velocity v along a straight line is retarded such that the retardation is (a) proportional to velocity and (b) proportional to square of velocity. Determine the expressions for velocity as a function of time and the distance traversed before it comes to rest for both cases.

Solution For case (a), let the retardation be kv

i.e.,
$$a = \frac{dv}{dt} = -kv$$

or
$$\frac{dv}{dt} + kv = 0;$$

On integration
$$\int_u^v \frac{dv}{v} + k \int_0^t dt = 0$$

or
$$\log_e \frac{v}{u} + kt = 0$$

$$v = u^{-kt}$$

Also,
$$a = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds} = -kv$$

or
$$\int_u^v dv + k \int_0^s ds = 0$$

$$v - u + ks = 0$$

$$v = u - ks$$

$$s = \frac{u - v}{k}$$

For case (b), let the retardation be μv^2

i.e.,
$$a = \frac{dv}{dt} = -\mu v^2$$

or
$$\frac{dv}{dt} + \mu v^2 = 0$$

On integration,
$$\int_u^v \frac{dv}{v^2} + \mu \int_0^t dt = 0$$

$$-\frac{1}{v} + \frac{1}{u} + \mu t = 0$$

or
$$v = \frac{u}{1 + \mu u t}$$

Also,
$$a = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds} = -\mu v^2$$

or
$$v \frac{dv}{ds} + \mu v^2 = 0$$

$$\int_u^v \frac{dv}{v} + \mu \int_0^s ds = 0$$

$$\log_e \frac{v}{u} + \mu s = 0$$

$$v = u e^{-\mu s}$$

$$s = \frac{1}{\mu} \log_e \frac{u}{v}$$

It may be noticed that the distance traversed before coming to rest in case (a) is finite whereas that in case (b) is infinite!

Example 6.4

If a body of mass m moves through a liquid at low velocity, the force of resistance due to viscosity is given by

$$F = k v$$

where k is the resisting force at unit velocity. Show that the velocity would decrease exponentially with time and linearly with displacement.

Solution The motion of the centre of mass of the body is given by

$$F = m a = m \frac{dv}{dt} = -k v$$

whence
$$\frac{dv}{dt} = -k / m v$$

or
$$\frac{dv}{v} = -\frac{k}{m} dt$$

On integration,

$$\log_e v = -\frac{k}{m} t + C$$

Using the condition, $v = v_0$ at $t = 0$, $C = \log_e v_0$

$$\log_e \left(\frac{v}{v_0} \right) = -\frac{k}{m} t$$

or
$$\frac{v}{v_0} = e^{-k / m t}$$

which shows that the velocity decreases exponentially with respect to time.

Writing

$$v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = v_0 \cdot e^{-k / m t}$$

or

$$dx = v_0 \cdot e^{-k/mt} dt$$

which, upon integration gives

$$x = \frac{v_0}{k} (1 - e^{-k/mt})$$

Substituting for v_1

$$v = v_0 - kx$$

which shows that the velocity of the body decreases linearly with displacement.

Example 6.5

The angular velocity of a flywheel is observed to decrease by 10% in the first minute. Calculate the decrease in the second minute if the retardation is proportional to the angular velocity.

Solution Given that

$$\frac{d\omega}{dt} = -k\omega$$

Hence,

$$\frac{d\omega}{dt} + k\omega = 0$$

and

$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} + k \int_0^t dt = 0$$

or

$$\log \frac{\omega}{\omega_0} + kt = 0$$

and

$$\frac{\omega}{\omega_0} = e^{-kt}$$

In the first one second, $\frac{\omega}{\omega_0}$ becomes 90%.

In the second, it will become 90% of that, i.e., 81% of the original velocity.

Example 6.6

An object of mass m falls vertically down in a medium with the resistance R proportional to the velocity. Obtain an expression for the velocity at time t if it starts from rest at time $t = 0$. What is the terminal velocity?

Solution The motion of the centre of mass C of the object may be studied by employing the Newton's law,

$$= ma = m \frac{dv}{dt}$$

but the net external force on the object is given by

$$= mg - kv$$

as shown in Fig. Ex. 6.6(a) (Solution).

Therefore

$$m \frac{dv}{dt} = mg - kv$$

or

$$\frac{dv}{mg/k - v} = \frac{k}{m} dt$$

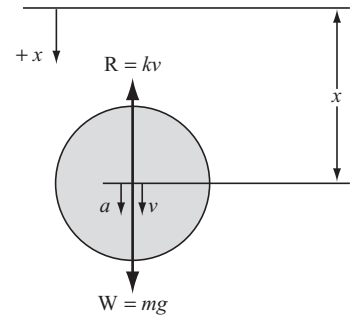


Fig. Ex. 6.6(a) (Solution)

On integrating,

$$-\ln(mg/k - v) = \frac{k}{m} t + C_1$$

Recognising that

$$v = 0 \quad \text{at} \quad t = 0$$

$$C_1 = -\ln(mg/k)$$

Hence

$$-\ln(mg/k - v) = k/m t - \ln(mg/k)$$

or

$$\frac{mg/k - v}{mg/k} = e^{(-k/m)t}$$

from which

$$v = mg/k(1 - e^{(-k/m)t}) \quad \text{(i)}$$

The terminal velocity of the centre of mass occurs at t tending to infinity;

$$V = mg/k \quad \text{(ii)}$$

If it is desired to obtain an expression for the displacement of the centre of mass of the object, the fact that $v = \frac{dx}{dt}$ together with Eq. (i) provides

$$dx = mg/k(1 - e^{(-k/m)t}) dt$$

whence, by integration with the prescribed condition,

$$x = mg/k t + m^2 g/k^2 e^{(-k/m)t} - m^2 g/k^2$$

or

$$x = V t - \frac{V^2}{g} (1 - e^{(-k/m)t})$$

It may also be added that the case of the resistance R being proportional to the velocity of an object is a factual situation for low-speed movements through viscous fluids. At higher speeds, however, the resistance is proportional to the square of velocity of the object.

Another very important comment must be noted. The object was assumed to have a symmetrical shape such that the weight and resistance act along the same line. This assumption was not necessary. In fact, the object chosen could have been unsymmetrical and the line of action of the resistance to motion could have been displaced with respect to that of the weight mg , although parallel as shown in Fig. 6.6(b) (Solution). The equations still apply as far as the motion of the centre of mass is concerned and the results obtained are correct because the net external force is still given by

$$F = m g - k v$$

and the displacement, velocity and acceleration of the centre of mass are implied. The centre of mass drops vertically down. It is quite a different matter whether the object rotates or not. The rotation aspect is not within the purview of the Newton's law of motion.

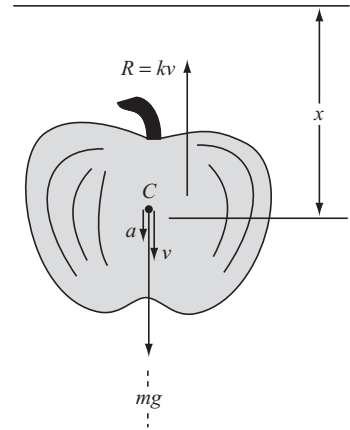


Fig. Ex. 6.6(b) (Solution)

Example 6.7

Two blocks of mass 2.9 kg and 1.9 kg are suspended from a rigid support S by two inextensible wires each of length 1 metre, see Fig. Ex. 6.7. The upper wire has negligible mass and the lower wire has a uniform mass of 0.2 kg/m. The whole system of blocks, wires and support have an upward acceleration of 0.2 m/s^2 . Acceleration due to gravity is 9.8 m/s^2 .

- (i) Find the tension at the mid-point of the lower wire.
- (ii) Find the tension at the mid-point of the upper wire.

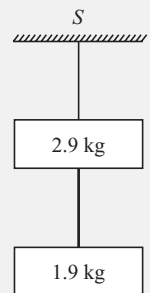


Fig. Ex. 6.7

Solution For the free-body diagram of the lower mass together with 50% of the lower wire, the forces acting are as shown in Fig. Ex. 6.7 (Solution). Applying the Newton's law of motion,

$$- 18.62 - 0.98 = (1.9 + 0.1) \times 0.2$$

whence $\qquad\qquad\qquad = 0.4 + 19.6 = 20 \text{ N}$

In order to find the tension at the mid point of the upper wire, consider the free body diagram of the entire part below it. Then,

$$- 18.62 - 1.96 - 2.9 \times 9.8 = (1.9 + 0.2 + 2.9) \times 0.2$$

whence $\qquad\qquad\qquad = 50 \text{ N}$

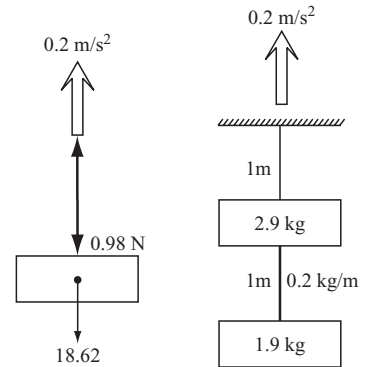


Fig. Ex. 6.7 (Solution)

Example 6.8

A block of mass $m = 5 \text{ kg}$ rests on a smooth inclined surface of a wedge of mass $M = 10 \text{ kg}$. The wedge is resting on a smooth horizontal surface. Assuming the pulley to be weightless, smooth and frictionless and the string to be light and inextensible, find the acceleration of mass m .

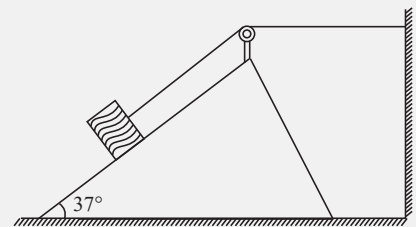


Fig. Ex. 6.8

Solution From the free-body diagram of the mass m , as shown in Figs. Ex. 6.8(a) and (b) (Solution),

$$T = m \cos 37^\circ = 5 \times 9.81 \times 0.8 = 39.2 \text{ N.}$$

$$T = m \sin 37^\circ = 5 \times 9.81 \times 0.6 = 29.4 \text{ N.}$$

Now, let us consider the M of the wedge.

The horizontal force acting on it is $T \sin 37^\circ$

$$= 39.2 \times 0.6 = 23.6 \text{ N}$$

whence $10 \times a = 23.6$

$$a = 2.36 \text{ m/s}^2$$

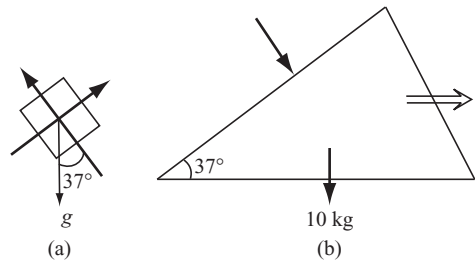


Fig. Ex. 6.8 (a) and (b) (Solution)

Example 6.9

Two blocks $m_1 = 2 \text{ kg}$ and $m_2 = 5 \text{ kg}$ are initially resting on the floor. They are connected by a light and inextensible cord running over a weightless and frictionless pulley as shown in Fig. Ex. 6.9. Find the acceleration of each block and the pulley if an upward force F applied to the pulley is (i) 14 N, (ii) 70 N. Find the force required to lift both the blocks.

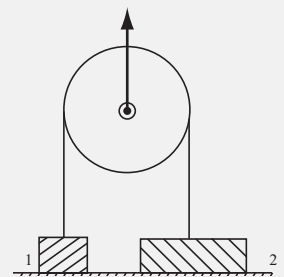


Fig. Ex. 6.9

Solution Weights of the two blocks are $2 \times 9.81 = 19.62$ and $5 \times 9.81 = 49.05$ N respectively.

- (i) With $F = 14$ N, neither of the weights is overcome; none will be lifted. $a_1 = 0 = a_2 = a_p$
- (ii) With $F = 70$ N, tension in each cord = 35 N.

For the smaller block,

$$35 - 19.62 = 0.2 a_1; a_1 = 7.69 \text{ m/s}^2$$

But the bigger block cannot move, $a_2 = 0$

The pulley moves up with the average velocity and acceleration,

$$a_p = (7.69 + 0)/2 = 3.85 \text{ m/s}^2$$

Both the blocks will move up if tension in each string exceeds 49.05 N, the weight of the bigger block. Then $F = 2 \times 49.05 = 98.10$ N.

Example 6.10

Two blocks *A* and *B* are held stationary 10 m apart on a 20° incline as shown in Fig. Ex. 6.10. The coefficient of dynamic friction between the plane and *A* is 0.3 whereas between the plane and *B* is 0.1. If the blocks are released simultaneously, calculate the time taken and distance travelled by each block before they are at the verge of collision.

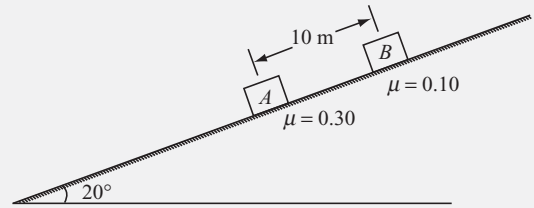


Fig. Ex. 6.10

Solution From the free-body diagrams of the blocks, as shown in Fig. Ex. 6.10 (Solution) for block *A*,

$$m_A g \sin 20^\circ - 0.3 R_A = m_A a_A$$

$$R_A = m_A g \cos 20^\circ$$

or $m_A g \sin 20^\circ - 0.3 m_A g \cos 20^\circ = m_A a_A$

or $9.81 \times 0.342 - 0.3 \times 9.81 \times 0.94 = a_A$

$$a_A = 0.59 \text{ m/s}^2$$

Similarly, for block *B*,

$$m_B g \sin 20^\circ - 0.1 R_B = m_B a_B$$

$$R_B = m_B g \cos 20^\circ$$

whence $a_B = 2.43 \text{ m/s}^2$

If t is the time taken for the blocks to reach at the same elevation to be at the verge of collision, the distance travelled by *A* would be 10 m less than that travelled by *B* over the same time.

$$\frac{1}{2} a_A t^2 + 10 = \frac{1}{2} a_B t^2$$

or $\frac{1}{2} \times 0.59 t^2 + 10 = \frac{1}{2} \times 2.43 t^2$

whence $t = 3.30$ s

The distances travelled by *A* and *B* are

$$S_A = \frac{1}{2} a_A t^2 = \frac{1}{2} \times 0.59 \times 3.30^2 = 3.2 \text{ m}$$

$$S_B = \frac{1}{2} a_B t^2 = \frac{1}{2} \times 2.43 \times 3.30^2 = 13.2 \text{ m}$$

which could have alternatively been determined as

$$S_B = S_A + 10 = 3.2 + 10 = 13.2 \text{ m}$$

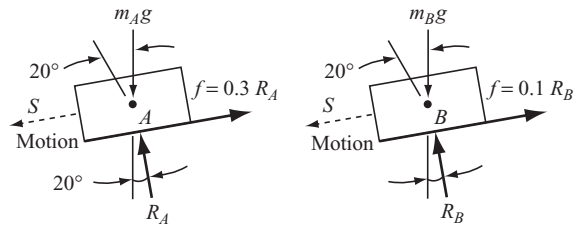


Fig. Ex. 6.10 (Solution)

Example 6.11

A frictionless step-pulley mounted at the top of an incline supports a block *A* restrained to slide over the incline and a block *B* hanging from it as shown in Fig. Ex. 6.11. If the block *A* of mass 3 kg should accelerate at 2 m/s^2 up the incline, determine the mass of block *B*

- (a) if the incline is frictionless and
- (b) if the coefficient of dynamic friction between the incline and block *A* is 0.3.

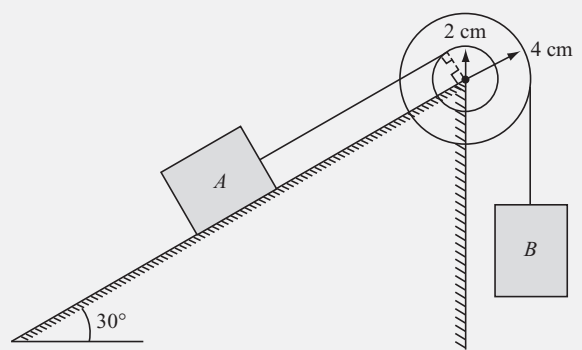


Fig. Ex. 6.11

Solution Assuming the step-pulley to be frictionless as well as massless, the tensions in the cords are such that the summation of their moments about the centre of rotation vanishes

$$2T_A - 4T_B = 0$$

or $T_A = 2T_B$ (i)

From the free-body diagrams of the blocks as shown in Figs. Ex. 6.11(a) and (b) (Solution), the forces along the direction of motion are evaluated and substituted in the equations of motion:

(a) For frictionless incline

For block *A*,

$$T_A - m_A g \sin 30 = m_A a_A \quad \text{(ii)}$$

For block *B*,

$$m_B g - T_B = m_B a_B \quad \text{(iii)}$$

From kinematic considerations for the step-pulley,

$$a_B = 2a_A = 2 \times 2 = 4 \text{ m/s}^2$$

Substituting Eqs. (i) and (iv) into Eqs. (ii) and (iii),

$$\begin{aligned} 2T_B - 3 \times 9.81 \times 0.5 &= 3 \times 2 \\ 9.81 m_B - T_B &= 4 m_B \end{aligned}$$

From the former,

$$T_B = \frac{6 + 14.715}{2} = 10.36 \text{ N}$$

and from the latter,

$$m_B = \frac{10.36}{5.81} = 1.78 \text{ kg}$$

(b) For frictional incline

For block *A*,

$$T_A - m_A g \sin 30 - 0.3 R_A = m_A a_A$$

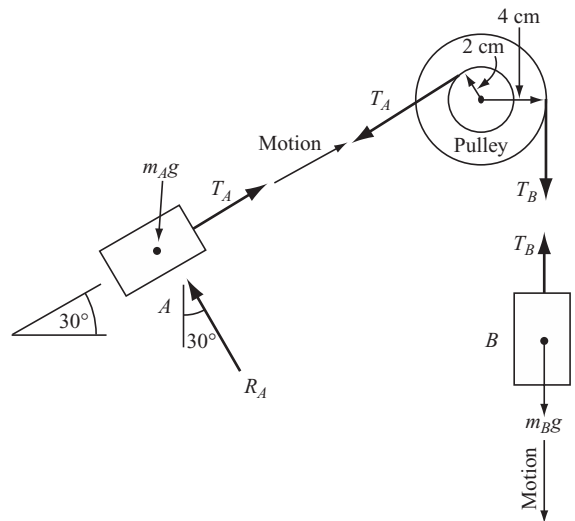


Fig. Ex. 6.11 (a) and (b) (Solution)

(iv)

where

$$R_A = m_A g \times \cos 30$$

$$= 3 \times 9.81 \times 0.866 = 25.49 \text{ N}$$

Then,

$$T_A - 3 \times 9.81 \times 0.5 - 0.3 \times 25.49 = 3 \times 2$$

or

$$T_A - 22.36 = 6$$

whence

$$T_A = 28.36$$

For block B ,

$$m_B g - T_B = m_B a_B$$

Employing the facts that $T_A = 2T_B$ and $a_B = 2a_A = 4 \text{ m/s}^2$

it becomes $9.81m_B - 28.36/2 = 4m_B$

whence $m_B = 2.44 \text{ kg}$

It may be noted that the presence of friction on the incline results in considerably higher tension in the cords and requires a larger mass of the hanging block to cause the same acceleration of the block on the incline. A little reflection will show that if a single pulley was employed instead of a step-pulley, the problem would be lot easier but the mass of block B required for the same purpose would be considerably more!

Example 6.12

A painter of mass 100 kg standing in a jhoola, i.e., suspended cage of mass 25 kg has arranged to pass the rope over a fixed pulley. He pulls the rope with an acceleration in order to rise. At an instant, he exerts an effective weight of 450 N on the jhoola, find the

- (i) acceleration of the painter and
- (ii) tension in the string.

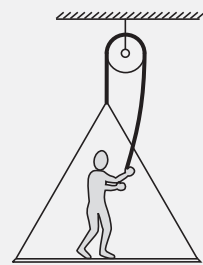


Fig. Ex. 6.12

Solution Let us draw the free body diagram of the jhoola as also of the painter alone. (Ref. Ex. 6.12 (Solution))

For the two free-bodies respectively,

$$T - F - m_1 g = m_1 a \qquad T + F - m_2 g = m_2 a$$

$$T - 450 - 25 \times 9.81 = 25 a \qquad T + 450 - 100 \times 9.81 = 100 a$$

$$T - 25 a = 695.3 \qquad T - 100 a = 531$$

From these two equations, $a = 2.19 \text{ m/s}^2$, $T = 750 \text{ N}$.

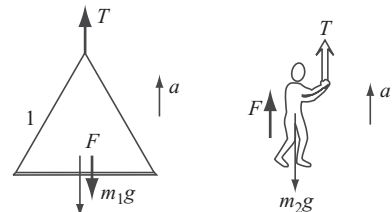


Fig. Ex. 6.12 (Solution)

Example 6.13

A baggage truck pulls two carts A and B as shown in Fig. Ex. 6.13. If the mass of the truck is 500 kg and the carts A and B carry 700 and 300 kg respectively, and the truck develops a tractive force of 2 kN, determine the acceleration of the truck and carts moving together.

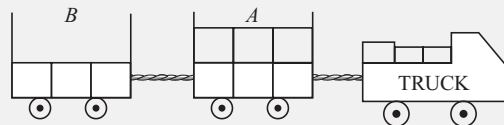


Fig. Ex. 6.13

Find the horizontal forces between the truck and the cart A and between the two carts.

Solution The tractive force is indeed developed by the coupling but its acts on the truck by virtue of the friction at the wheels of the truck. Wheels of carts *A* and *B* merely roll. Therefore, no further frictional force comes into play.

For the truck and the carts moving together,

$$F = m a$$

$$2000 = (500 + 700 + 300) a$$

whence $a = 2000/1500 = 1.33 \text{ m/s}^2$

Now, consider the *fbd* of the truck alone, where $T = 2000 - 665 = 1335 \text{ N}$

This force acts at the coupling between the truck and cart *A*.

Now consider the *fbd* of cart *A* above

$$T - t = 700 \times 1.33$$

whence $t = T - 700 \times 1.33 = 400 \text{ N}$

which is the force on the coupling between the two carts.

We can verify the answer by considering the *fbd* of the cart *B* above.

$$t = 300 \times 1.33 = 400 \text{ N}$$

which is what we have obtained above.

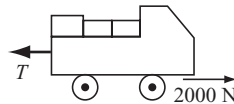


Fig. Ex. 6.13(a) (Solution)

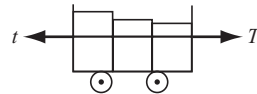


Fig. Ex. 6.13(b) (Solution)

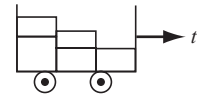


Fig. Ex. 6.13(c) (Solution)

6.3 D'ALEMBERT PRINCIPLE: INERTIA FORCES

If the mass centre of a body or a particle of mass m is subjected to a net force and it acquires an acceleration, then,

$$\Sigma F + (-m a) = 0 \tag{6.2a}$$

This is indeed a restatement of Newton's law but it suggests that the term $(-m a)$ may be considered as a fictitious force, often called *D'Alembert force* or the inertia force as depicted in Fig. 6.4.

According to the D'Alembert principle, the net external force actually acting on the body and the inertia force F_i together keep the body in a state of 'fictitious equilibrium'

$$\Sigma F + F_i = 0 \tag{6.2b}$$

The principle tends to give the solution procedure of a dynamic problem an appearance akin to that of a static problem. The rule of equilibrium for statics, i.e.,

$$\Sigma F = 0$$

may, therefore, be employed for a dynamic problem with the introduction of the concept of *fictitious dynamic equilibrium*.

The significance of D'Alembert principle does not end with the extension of our ability of using the methods of statics in dynamics but goes beyond to be coupled with the principle of virtual work and to lead to an alternative formulation of mechanics on the basis of energy considerations. A glimpse of this formulation is given in Chapter 10 under the heading 'Variational Principles'.

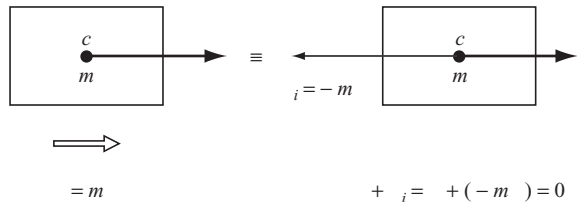


Fig. 6.4 Newton's law – D'Alembert's principle

Let us consider the motion of a particle along a circular path on a smooth plane with reference to Newton's law and the D'Alembert principle. At any instant of time, the velocity must be tangential to the circular path and the acceleration may consist of a tangential component and a radially inward or centripetal component. If the speed of the body is constant, it experiences only the centripetal acceleration, equal in magnitude to v^2/r . The force that must act on the body to enable the body to move in a circular path must be radially inwards at all times such that

$$= m$$

with a magnitude mv^2/r .

This is so shown for two positions of the body in Fig. 6.5(a). The force actually required to be acted on the body is called the centripetal force.

According to the D'Alembert principle, the equation of motion may be written as

$$+ (-m) = 0$$

and the fictitious force $(-m)$ is called the centrifugal force. The equation is interpreted by saying that the body may be considered in a state of 'equilibrium' under the application of two forces: the actual or centripetal force and a fictitious or centrifugal forces CF equal to $(-m)$.

The two forces must be equal and opposite. The centripetal force being radially inward, the centrifugal force must be radially outward at any instant. This is so shown for two positions of the body in Fig. 6.5(b).

Since the concept of centrifugal force replaces that of the 'acceleration' of the body, the body is considered 'non-accelerating' or in 'equilibrium' once the centrifugal force is imagined to be acting upon it.

Let us consider the traditional example of a small stone of mass m tied at the end of a string of length l and whirled at a constant speed v . If whirled in a horizontal plane, the stone is subjected to a vertically downward force equal to its weight mg and a radially outward horizontal centrifugal force equal to mv^2/r . The force due to the string, also in a horizontal plane, can only balance the centrifugal force leaving the vertical force mg unbalanced. The conclusion is that a stone cannot be whirled, by means of a string, keeping the string in a horizontal plane.

Instead, the string must be inclined downward, going outward, to provide a vertical component in the string force to balance the weight as shown in Fig. 6.6. The centrifugal force is still horizontal. It may be observed that the angle of inclination θ is given by

$$\tan \theta = \frac{mg}{CF} = \frac{mg}{mv^2/r} = \frac{gr}{v^2} \tag{6.3}$$

The angle is independent of the mass of the stone but increases as the velocity decreases or the radius of the circular path increases.

If the stone is whirled in a vertical plane, the three forces, i.e., its weight mg acting downward, the centrifugal force CF equal to mv^2/r acting radially outward and the string force T radially inward may be considered to keep it in 'equilibrium' at any instant as shown in Fig. 6.7. At the lowest position,

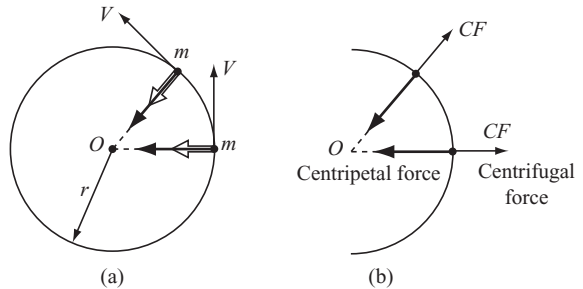


Fig. 6.5 Motion along a circular path

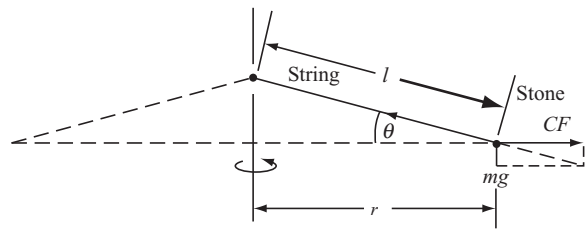


Fig. 6.6 Whirling of a stone in a horizontal plane

$$T = CF + mg = \frac{mv^2}{r} + mg$$

whereas at the uppermost position,

$$CF = T + mg$$

whence

$$T = CF - mg = \frac{mv^2}{r} - mg$$

The tension in the string must be positive at the lowest position for all values of v but it can drop to zero at the uppermost position if

$$\frac{mv^2}{r} - mg \leq 0$$

or

$$v \leq \sqrt{gr}$$

(6.4)

Since a string cannot remain straight without being in tension,

the stone will not reach the uppermost position on the circle if the velocity of whirling drops below \sqrt{gr} .

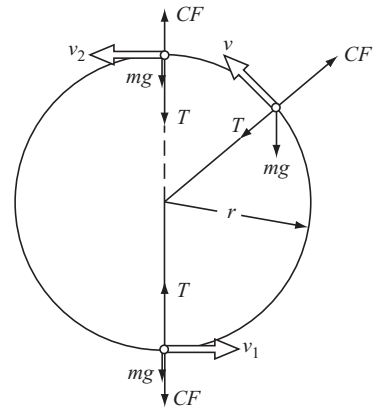
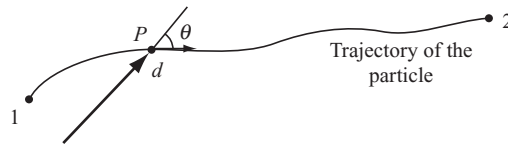


Fig. 6.7 Whirling of a stone in a vertical plane

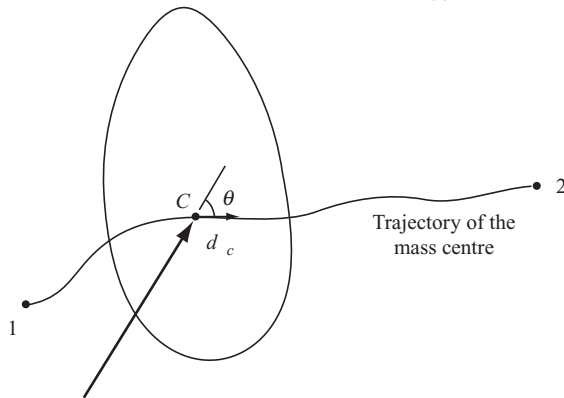
6.4 WORK, POWER AND ENERGY

If a body is subjected to a force and the point of application of the force is displaced by an infinitesimal displacement d as shown in Fig. 6.8(a), the work done by the force is defined as the scalar or dot product of the force and the infinitesimal displacement

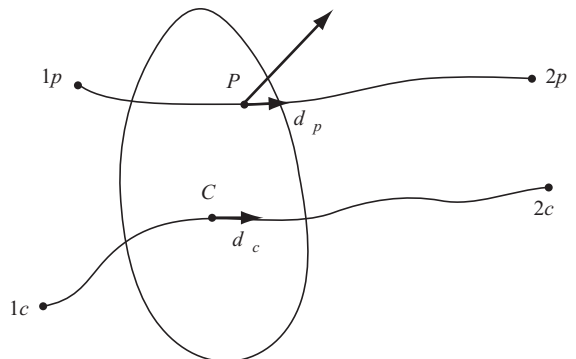
$$dW = \mathbf{F} \cdot d = F dr \cos \theta \quad (6.5)$$



(a) Action of a force on a particle



(b) Action of a force at a mass centre



(c) Action of a force on a rigid body

Fig. 6.8 Action of a force on a particle, mass centre and a rigid body

where θ is the angle between the force and displacement vectors. In other words, the work done by a force is the product of the magnitude of force and the distance $\cos \theta$ moved by the point of application of the force in the direction of the force. Alternatively, the work done may be considered to be the product of the magnitude of the displacement and the component $\cos \theta$ of the force acting in the direction of the displacement of the point of application of the force. The work done by a force may be positive or negative depending upon whether the force component is directed along or opposite to the direction of displacement, i.e., whether the angle between the force and displacement is acute or obtuse.

For a particle, the force acts at the same point for which the displacement is considered. For a rigid body, on the other hand, the displacement of the point of application of the force must be considered.

For a finite displacement of a particle, therefore, the work done due to a force is given by

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{s} \quad (6.6)$$

where s_1 and s_2 refer to the initial and final positions 1 and 2 as shown in Fig. 6.8(a).

The force acting on the particle may be a function of the space coordinates and the displacement may also be in space. The elementary work done can, therefore, be written as

$$dW = (F_x dx + F_y dy + F_z dz) \cdot (ds_x^2 + ds_y^2 + ds_z^2)$$

where F_x , F_y , and F_z may vary with the space coordinates. The total work done to displace the particle from state 1 to state 2 is given by

$$W = \int_1^2 F_x dx + \int_1^2 F_y dy + \int_1^2 F_z dz$$

For a finite displacement of a rigid body, the work done by a force acting at the centre of mass is

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

where s refers to the displacement of G as in Fig. 6.8(b). If the force acted at some other point P such that P did not fall on the line of action of F as in Fig. 6.8(c), then the work done by the force on the body would be

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{s}_P$$

where s_P refers to the displacement of P . It must be noted that the expression

$$\int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

does not mean the work done by the force as it is acting but, instead, the work done by the force

It is interesting to observe the cases where no work is done by a force: (a) when the displacement is zero, e.g., by the force acting on a stationary structure and (b) when the displacement is perpendicular to the force applied, e.g., work by the gravitational force on an object moving horizontally on the surface of the earth.

The rate at which work is done is called power, given by

$$\text{Power} = \frac{dW}{dt} \quad (6.7)$$

Since $dW = \mathbf{F} \cdot d\mathbf{s}$

for a force whose point of application is displaced by an infinitesimal displacement $d\mathbf{s}$,

$$\text{Power} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} \quad (6.8)$$

which shows that power equals the dot product of the force and the velocity of the point of application of the force at that instant.

The term ‘power of a force component’ appears quite often in engineering. For example, the propulsive power of an aircraft implies the power associated with the propulsive force only. An aircraft may be flying at a constant velocity, the net external force on it being zero, yet the propulsive power could be finite. Similarly, the power dissipated in a process is referred to the power associated with the dissipative forces.

The kinetic energy possessed by a particle of mass m moving with a velocity \mathbf{v} is defined as

$$KE = \frac{1}{2} m v^2 \quad (6.9)$$

where v is the speed or magnitude of \mathbf{v} . The kinetic energy of a rigid body of mass m in translation at a velocity \mathbf{v} is also defined by the same expression but the kinetic energy of rigid body in general translational and rotational motion must be determined from a consideration of the velocity of the individual elements dm .

$$KE = \int_{\text{body}} \frac{1}{2} v^2 dm \quad (6.10)$$

where the integration is taken over the entire mass of the body. Similarly, the total kinetic energy for a system of n particles is obtained by the addition of the kinetic energies possessed by the individual particles,

$$KE = \sum_{i=1}^n \frac{1}{2} m_i v_i^2 \quad (6.11)$$

It is also possible to demonstrate that the total kinetic energy of a system can be considered to be composed of two parts: the kinetic energy of the total mass moving with the velocity v_c of the mass centre and the kinetic energy of motion of the elements of the system relative to the mass centre:

$$KE = \frac{1}{2} m v_c^2 + \sum \frac{1}{2} m_i v_{ic}^2$$

It may be remarked that the units of work and energy are the same. In SI units, the unit of work is joule, $J \equiv \text{N m}$ and the unit of energy is $\text{kg m}^2/\text{s}^2 \equiv \text{Nm} \equiv \text{J}$.

A force field is said to be conservative if the force \mathbf{F} in the field is continuous in space and is expressible as a gradient of a scalar function ϕ , i.e.,

$$\mathbf{F} = -\nabla\phi$$

and

$$F_x = -\frac{\partial\phi}{\partial x}, F_y = -\frac{\partial\phi}{\partial y}, F_z = -\frac{\partial\phi}{\partial z}$$

or

$$\mathbf{F} = -\nabla\phi \quad (6.12)$$

The scalar function or force potential ϕ must be such that its partial derivative with respect to a coordinate results in the force component along that direction.

Alternatively, the condition for \mathbf{F} to be equal to the gradient of ϕ requires that

$$\nabla \times \mathbf{F} = 0$$

The work done by a conservative force \mathbf{F} as the particle is displaced from an initial position 1 to a final position 2 is given by

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (F_x dx + F_y dy + F_z dz)$$

$$\begin{aligned}
 &= \int_1^2 (F_x dx + F_y dy + F_z dz) \\
 &= \int_1^2 \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) \\
 &= \int_1^2 d\phi = \phi_2 - \phi_1
 \end{aligned}$$

The change of potential energy PE of a particle between positions 1 and 2 equals the negative of the work done by the conservative force as the particle moves from position 1 to 2,

$$(PE_2 - PE_1) = -W = \phi_1 - \phi_2$$

and it follows that

$$PE = -\phi$$

and

$$\frac{\partial(PE)}{\partial x} = -F_x, \quad \frac{\partial(PE)}{\partial y} = -F_y, \quad \frac{\partial(PE)}{\partial z} = -F_z$$

or simply,

$$\frac{\partial PE}{\partial s} = -F_s \tag{6.13}$$

along a space coordinate s .

It is important to understand that the existence of the potential energy function PE requires the presence of a conservative force field. In other words, potential energy cannot be defined for dissipative and discrete forces. The potential energy of a body in a conservative force field may be defined as the work which the system of forces can do on the body if the body passes from its present configuration to a standard configuration, called the datum position. Further, of a total force acting on a body, a part may be conservative,

$$= \text{Cons.} + \text{Non-cons.}$$

It is possible to identify a potential energy function for the conservative part Cons. and not for the non-conservative part.

It may also be appreciated from the relation

$$W = \phi_2 - \phi_1 = PE_1 - PE_2 \tag{6.14}$$

that the work done between two positions 1 and 2 in a conservative force field depends only upon the end states and *not upon the path followed by the point of application of the force*. If PE_1 and PE_2 are the potential energies, initial and final, respectively, the work done in the field must be $PE_1 - PE_2$ no matter how the particle or mass centre is taken from 1 to 2. It also follows that in a conservative force field, the work done along a closed contour, e.g., 1 – 2 – 1, must be zero

$$W = (PE_1 - PE_2) + (PE_2 - PE_1) = 0$$

These facts are shown in Figs. 6.9(a) and (b).

Consider, for example, the true gravitational field enveloping the earth. Newton's law of gravitation states that

$$= -\frac{GMm}{r^2} \quad r$$

or

$$r = -\frac{GMm}{r^2} = -\frac{\partial(PE)}{\partial r}$$

whence $PE = -\frac{GMm}{r}$

At $r \rightarrow \infty$ $PE \rightarrow 0$

The difference in potential energy between the two points P_1 and P_2 at radii r_1 and r_2 respectively from the centre of the earth, i.e., with an elevation difference

$$h = r_2 - r_1$$

is given by

$$\begin{aligned} \Delta PE &= -\frac{GMm}{r_2} + \frac{GMm}{r_1} = \frac{GMm(r_2 - r_1)}{|r_1 r_2|} \\ &\approx \frac{GMm}{r^2} h = mgh \end{aligned}$$

where $r_1 \approx r_2 = r$, if the points are close together. It may also be appreciated that the gravitational force field of the earth appears to consist of parallel lines of force instead of the radial lines. The expression for a change in potential energy is then given by

$$\Delta PE = mgh$$

as is commonly used for earth-bound objects.

Example 6.14

A vertical lift of total mass 500 kg acquires an upward velocity of 2 m/s over a distance of 3 m of motion with constant acceleration, starting from rest. Calculate the tension in the cable supporting the lift.

If the lift, while stopping, moves with a constant deceleration and comes to rest in 2 s, calculate the force transmitted by a man of mass 75 kg on the floor of the lift during that interval.

Solution The upward acceleration a is obtained by using

$$\begin{aligned} V_2^2 - V_1^2 &= 2as \\ a &= \frac{2^2}{2 \times 3} = 0.67 \text{ m/s}^2 \end{aligned}$$

Examining the forces acting on the lift as shown in Fig. Ex. 6.14(a) (Solution), the net force must equal mass times acceleration in accordance with the Newton's law,

$$T - mg = ma$$

whence

$$\begin{aligned} T &= m(g + a) \\ &= 500(9.81 + 0.67) \\ &= 5240 \text{ N} \end{aligned}$$

Alternately the D'Alembert principle suggests that an inertia force equal to $(-ma)$, i.e., ma downward be imagined acting on the lift together with the external force T and mg as shown in Fig. Ex. 6.14(b) (Solution) and the problem be solved as an equilibrium problem in statics. By this principle,

$$\Sigma = 0$$

$$T - mg + (-ma) = 0$$

or

$$T - 500 \times 9.81 - 500 \times 0.67 = 0$$

whence

$$T = 5240 \text{ N}$$

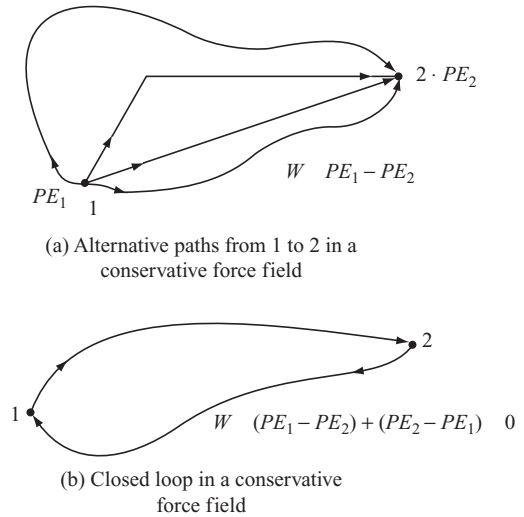
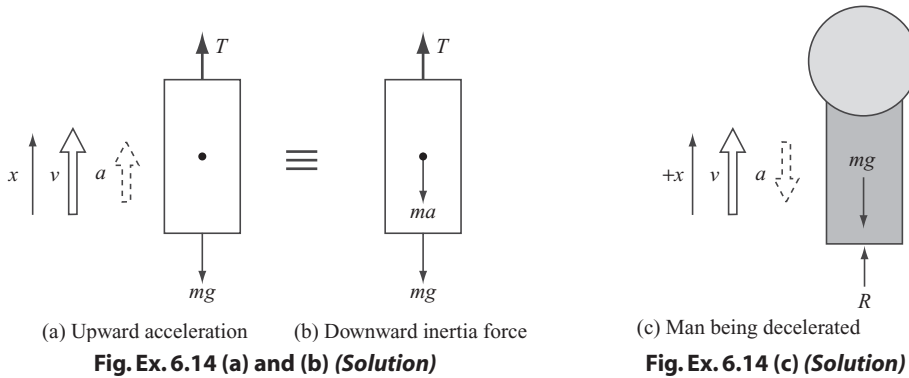


Fig. 6.9 Work done in a conservative force field



While stopping, the acceleration of the lift is

$$a = \frac{0 - 2}{2} = -1 \text{ m/s}^2$$

The negative sign of acceleration implies that the acceleration is opposed to the direction of velocity, as shown in Fig. Ex. 6.14(c) (Solution).

The force transmitted by the man on the floor of the lift equals the reaction R exerted by the lift on the man. With reference to the free-body diagram,

$$\begin{aligned} R - mg &= ma \\ R &= ma + mg \\ &= 75(-1 + 9.81) \\ &= 661 \text{ N} \end{aligned}$$

The force transmitted by the man on the floor of the lift is, therefore, 661 N downwards.

It is interesting to understand that the man in the lift experiences the acceleration imposed upon him and, consequently, experiences a change in his weight W . The reaction R exerted by the lift on the man is equal and opposite to the weight W felt by him. The weight felt by a man is less than mg while accelerating downwards and more than mg while accelerating upwards. In particular, if the lift was to be accelerated downwards at an acceleration g , the man would feel weightless and the reaction by the floor of the lift on the man would become zero. The weight felt by a man in a lift moving at a constant velocity would be mg . The pictorial representation of the weight felt by him in a lift during a round trip is shown in Fig. Ex. 6.14 (d) (Solution).

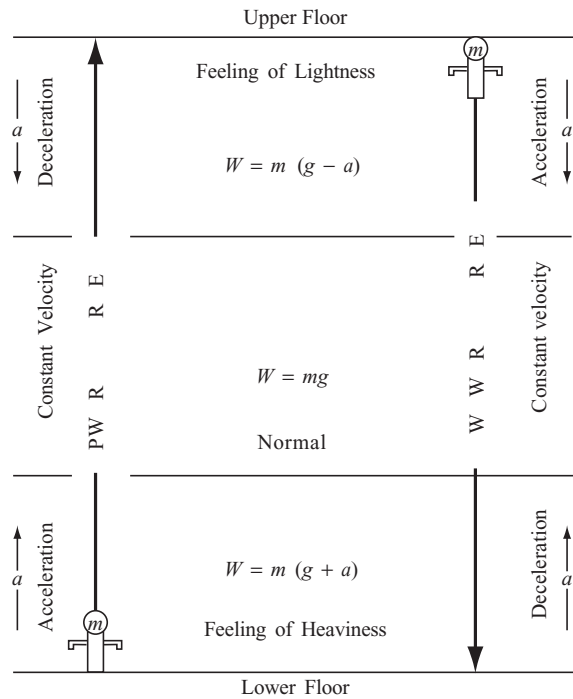


Fig. Ex. 6.14 (d) (Solution) Weight felt in a lift

Example 6.15

A segment of a smooth circular curved road of radius 30 m is located in a zone of 40 km/h speed limit. What should be the angle of banking so that a vehicle may travel on it without any outward side thrust?

If the coefficient of static friction between the road and tyres of a vehicle was 0.3, calculate the maximum possible speed before the vehicle experiences a side-slip.

Solution When a vehicle negotiates a circular curve, it may be considered in 'equilibrium' under the action of three forces, namely the weight mg , centrifugal force CF equal to mv^2/r and reaction R which is normal to the road in the absence of friction.

Resolving the forces along the incline and writing for equilibrium, Fig. Ex. 6.15 (a)

$$mg \sin \theta = F \cos \theta = \frac{mv^2}{r} \cos \theta$$

whence $\tan \theta = \frac{v^2}{gr}$

For the case in hand,

$$\tan \theta = \frac{(40 \times 1000/3600)^2}{9.81 \times 30} = 0.42$$

and $\theta = 22.76^\circ$

In the presence of frictional force which would act down the plane if the velocity exceeds 40 km/h, the limiting velocity may be worked out with reference to the free-body diagram drawn in Fig. Ex. 6.15(b) (Solution).

Along the x -axis,

$$CF \cos \theta - \mu R - mg \sin \theta = 0$$

or

and along the y -axis,

$$\frac{mv^2}{r} \cos \theta - \mu R - mg \sin \theta = 0 \tag{i}$$

$$R - mg \cos \theta - CF \sin \theta = 0$$

or

From Eq. (ii)

$$R - mg \cos \theta - \frac{mv^2}{r} \sin \theta = 0 \tag{ii}$$

$$R = mg \cos \theta + \frac{mv^2}{r} \sin \theta$$

Substituting in Eq. (i) and simplifying it,

$$v^2 = \frac{\mu + \tan \theta}{1 - \mu \tan \theta} \cdot gr \tag{iii}$$

For the case in hand,

$$v^2 = \frac{0.3 + \tan 22.76^\circ}{1 - 0.3 \tan 22.76^\circ} \times 9.81 \times 30$$

whence

$$\begin{aligned} v &= 15.565 \text{ m/s} \\ &= 15.565 \times \frac{3600}{1000} = 56 \text{ km/h} \end{aligned}$$

It is interesting to note that the angle of banking for a circular road calculated on the basis of no-friction is safer because the presence of friction allows even higher maximum velocity without any danger of side-slip. In other words the angle of banking calculated by taking friction into account would have been less than 22.76° as may be seen by substituting the given values in Eq. (iii).

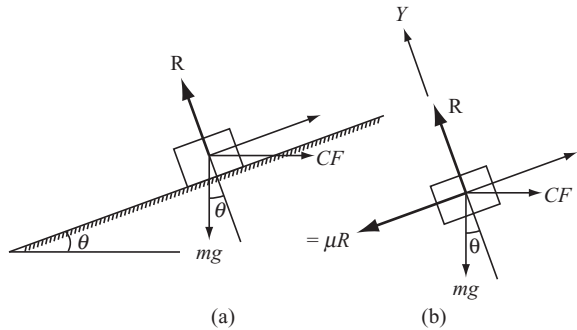


Fig. Ex. 6.15 (a) and (b) (Solution)

Example 6.16 A small spherical object comes rolling down a ramp and leaves the edge horizontally with a velocity V_0 . Describe its trajectory.

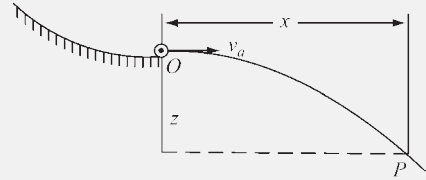


Fig. Ex. 6.16

Solution The spherical object would continue to have its rolling motion at constant angular velocity, in the absence of air resistance, on leaving the ramp.

At any instant of time t ,

$$x = V_0 t \quad z = -1/2 gt^2$$

Eliminating t ,

$$z = -\frac{gx^2}{2V_0^2}$$

which is the equation of the trajectory without air resistance, etc.

How far away would it go horizontally before it strikes the ground below depends upon how far below (z) the ground is and upon the initial velocity V_0 .

If V_0 is zero, $x = 0$ and $z = -1/2 gt^2$.

As V_0 increases, x increases. However, there is no maximum x for a given V_0 since this is a parabolic path.

Example 6.17 A small object slides without friction from a height H and then loops the vertical loop of radius $R = 20$ cm from which a symmetrical section of angle 120° has been removed. Determine the height H such that the object, after leaving contact at A and flying through air reaches point B and continues the loop (Fig. Ex. 6.17).

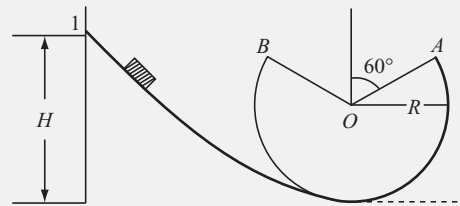


Fig. Ex. 6.17

Solution As shown in Fig. Ex. 6.17 (Solution) time taken for the vertical component of velocity to change from $v \sin 60^\circ$ to $-v \sin 60^\circ$ is given by

$$t = \frac{2v \sin 60^\circ}{g} = \frac{\sqrt{3}v}{g}$$

It must travel the horizontal distance $2R \sin 60^\circ$, i.e., $\sqrt{3}R$ in the same time with the constant horizontal velocity $v \cos 60^\circ$.

i.e.,
$$v \cos 60^\circ \times \frac{\sqrt{3}v}{g} = \sqrt{3}R$$

whence
$$v = 2 \text{ m/s}$$

Equating the potential energy at 1 with the potential plus kinetic energy at A ,

$$mgH = mg(R + R \sin 30^\circ) + \frac{1}{2}mv^2$$

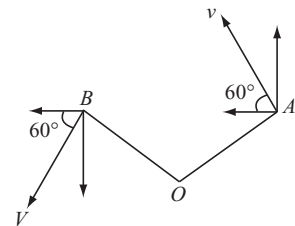


Fig. Ex. 6.17 (Solution)

$$H = 0.2 \left(1 + \frac{1}{2} \right) + \frac{1}{2} \times \frac{2^2}{9.81}$$

$$= 0.5 \text{ m} = 50 \text{ cm.}$$

Example 6.18

A particle of mass 0.05 kg slides down the circular periphery of a horizontal smooth cylinder of radius 12 cm when let go from the top position, as shown in Fig. Ex. 6.18. Determine the reaction of the cylinder on it when it reaches 30 degree position.

Will it leave the surface? If so, where and with what velocity?

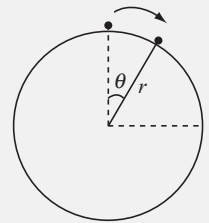


Fig. Ex. 6.18

Solution From the free-body diagram of the mass as shown in Fig. Ex. 6.18 (Solution) at an angle θ ,

$$mg \cos \theta - R = m \frac{v^2}{r} \quad \text{(i)}$$

Also, by conservation of energy,

$$mgh = mg(r - r \cos \theta) = \frac{1}{2} mv^2$$

or

$$g(1 - \cos \theta) = \frac{1}{2} v^2 / r \quad \text{(ii)}$$

Eliminating v^2/r between (i) and (ii)

$$R = mg \cos \theta - 2 mg (1 - \cos \theta)$$

$$= 0.05 \times 9.81 \times 0.866 - 2 \times 0.05 \times 9.81 (1 - 0.866)$$

$$= 0.29 \text{ N.}$$

It must leave the circular periphery, latest when it reaches the extremity of the horizontal diameter when the velocity is zero or the mass is zero. With its finite mass and velocity it will lose contact with the cylinder where $R = 0$, i.e.

Equating $mg \cos \theta$ with $m \cdot 2g \cdot (1 - \cos \theta)$

$$\cos \theta = 2 (1 - \cos \theta)$$

$$\cos \theta = 2 - 2 \cos \theta$$

$$3 \cos \theta = 2$$

$$\theta = \cos^{-1} 2/3 = 48.2$$

At this location,

$$v^2 = 2 gr (1 - \cos \theta) = 2 \times 9.81 \times 0.12 (1 - 2/3) = 0.78$$

$$v = 0.886 \text{ m/s}$$

whence,

The direction of the velocity must be tangent to the circle at that point, i.e., at (-48.2°) with the horizontal direction.

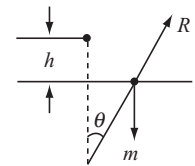


Fig. Ex. 6.18 (Solution)

Example 6.19 Two blocks m_1 and m_2 , of masses 2 and 4 kg respectively are placed one on the other on a horizontal table and connected to a suspended weight through a frictionless pulley as shown in Fig. Ex. 6.19. The coefficient of static friction between m_1 and m_2 is 0.4 and the coefficient of kinetic friction between m_2 and the table is 0.2. Find the maximum mass of the block M in order that m_2 accelerates over the table without m_1 slipping over m_2 .

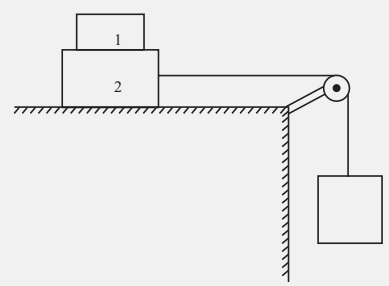


Fig. Ex. 6.19

Solution For the free-body diagram of mass m_1 as shown in Fig. Ex. 6.19(a) (Solution), the frictional force is $0.4 \times 2 \times 9.81$, i.e., 7.85 N. The maximum acceleration of m_1 should be limited to a , such that

$$m_1 \times a = 7.85; \quad a = \frac{7.85}{2} = 3.925 \text{ m/s}^2$$

Now, let us draw the free-body diagram of the two masses m_1 and m_2 together as also for the suspended mass M (Ref. Fig. Ex. 6.19 (b) (Solution)).

By dynamical equations,

$$-11.77 = (2 + 4) \times a; \quad a = 35.77 \text{ N}$$

$$-M \times 9.81 = M \times a; \quad M = 35.77 / (9.81 - 4) = 6.15 \text{ kg}$$

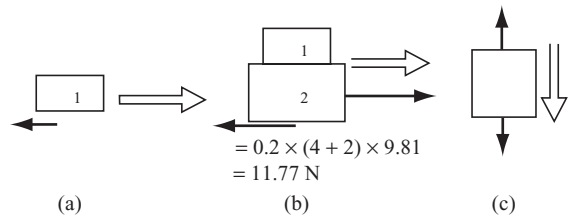


Fig. Ex. 6.19 (a, b, and c) (Solution)

Example 6.20 A bob of mass $m = 20$ grams is attached to a 20 cm long string tied to the apex of a cone with rough surface as shown in Fig. Ex. 6.20. At time $t = 0$, the cone is imparted a constant angular acceleration $\alpha = 0.5 \text{ rad/s}^2$ about its vertical axis. Assuming that the bob has no relative motion w.r.t. the surface of the cone,

- (i) at what time will the bob leave contact with the surface?
- (ii) what would be tension in the string at that instant?

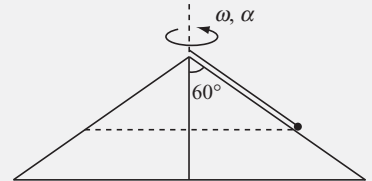


Fig. Ex. 6.20

Solution Consider the free-body diagram of the bob at any instant in rotation as shown in Fig. Ex. 6.20 (Solution).

$$\cos 30^\circ - \cos 60^\circ = \omega^2$$

$$\sin 30^\circ + \sin 60^\circ =$$

whence,

$$\frac{\sqrt{3}}{2} - \frac{1}{2} = \omega^2 \times 0.173$$

and

$$\frac{1}{2} + \frac{\sqrt{3}}{2} = \omega^2 \times 9.81$$

which leads to

$$\omega^2 = (0.173 \omega^2 - \sqrt{3} \times 9.81)$$

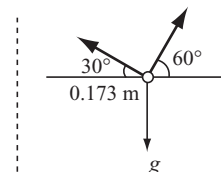


Fig. Ex. 6.20 (Solution)

It leaves contact when $R = 0$; $\omega = 9.9 \text{ rad/s}$

(i) From the relationship,

$$\begin{aligned}\omega &= \omega_0 + \alpha t \\ t &= 9.9/0.5 = 19.8 \text{ seconds}\end{aligned}$$

(ii) Solving for T from the above,

$$T = 0.39 \text{ N}$$

Example 6.21

Two masses m_1 and m_2 are connected by a massless inextensible string which passes over a massless and frictionless pulley, Fig. Ex. 6.21. This constitutes an *Atwood's machine*. Find the acceleration of mass m_1 and the tension in the string as the system moves under gravity.

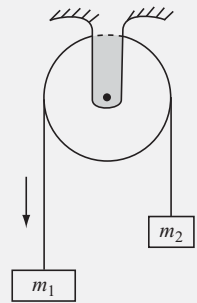


Fig. Ex. 6.21

Solution Considering the free-body diagram of each of the masses (Fig. Ex. 6.21 Solution),

$$\begin{aligned}m_1 g - T &= m_1 a \\ T - m_2 g &= m_2 a\end{aligned}$$

From these equations,

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

and

$$T = \frac{(2m_1 m_2)}{m_1 + m_2} g$$

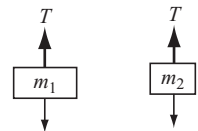


Fig. Ex. 6.21 (Solution)

Atwood used this system to determine the value of g , the acceleration due to gravity. If $m_1 = 2m_2$, which meant $a = g/3$, i.e., the system accelerates at one third the value of g . He used different values of m_1 and m_2 and measured the resulting acceleration whence he computed the value of g ;

$$g = \frac{m_1 + m_2}{m_1 - m_2} a.$$

One can get the same result by employing the work-energy principle or the conservation-of-energy principle in this case. Let us apply the latter.

If the mass m_1 moves down by x the resulting velocity would be given by considering

$$PE + KE = \text{Const.}$$

$$(m_1 - m_2)g x = \frac{1}{2}(m_1 + m_2)v^2$$

or

$$(m_1 - m_2)g x = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Differentiating it w.r.t. time,

$$(m_1 - m_2)g \dot{x} = (m_1 + m_2)\ddot{x} \dot{x}$$

and dividing by \dot{x} , which in general is non-zero,

$$(m_1 - m_2)g = (m_1 + m_2) a$$

whence

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$

Example 6.22

Establish that the gravitational force field close to the surface of the earth and the force field due to a linear spring are conservative force fields.

Solution

(a) *Gravitational force field*

Consider a particle of mass m placed in a parallel gravitational force field close to the surface of the earth is shown in Fig. Ex. 6.22(a) (Solution).

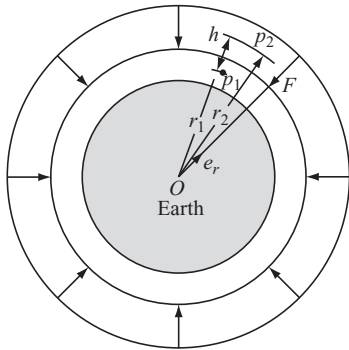


Fig. Ex. 6.22 (a) (Solution)

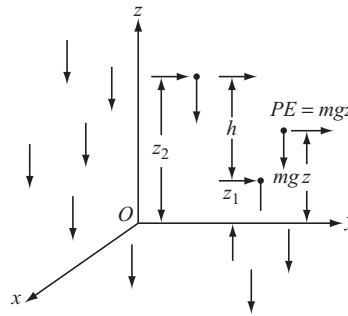


Fig. Ex. 6.22 (b) (Solution) An approximation of the Earth's field

$$F_x = 0, F_y = 0$$

$$F_z = -mg, \text{ the weight of the particle}$$

A potential function ϕ should exist for a conservative force field. For this case,

$$F_x = \frac{\partial \phi}{\partial x} = 0 \tag{i}$$

$$F_y = \frac{\partial \phi}{\partial y} = 0 \tag{ii}$$

$$F_z = \frac{\partial \phi}{\partial z} = -mg \tag{iii}$$

From Eqs. (i) and (ii) it may be seen that the potential function is not a function of x or y . Therefore integrating Eq. (iii),

$$\phi = -mgz + \text{Constant}$$

The existence of ϕ is a guarantee for the field to be conservative. The potential energy for the gravitational force field is

$$PE = -\phi = mgz$$

it being reckoned zero at the origin of the coordinates arbitrarily.

(b) Force field of a linear spring

For a linear spring compressed by x , Fig. Ex. 6.22(b) (Solution) the force field is such that

$$F_x = -kx$$

$$F_y = 0, F_z = 0$$

In an effort to determine the potential function, let

$$F_x = \frac{\partial \phi}{\partial x} = -kx \quad (i)$$

$$F_y = \frac{\partial \phi}{\partial y} = 0 \quad (ii)$$

$$F_z = \frac{\partial \phi}{\partial z} = 0 \quad (iii)$$

Equations (ii) and (iii) in this case suggest that ϕ is neither a function of y nor of z . Integrating Eq. (i), therefore,

$$\phi = -\frac{1}{2}kx^2 + \text{Const.}$$

The existence of the potential function is a sure proof of the conservative nature of the force field due to a linear spring.

The potential energy for the field of a linear spring is

$$PE = -\phi = \frac{1}{2}kx^2$$

referred to zero potential energy in the unstretched position when $x = 0$, i.e., the spring in the equilibrium position.

It may be observed that, in the gravitational field, the change in potential energy from a position 1 at z_1 to a position 2 at z_2 is simply

$$\Delta PE = PE_2 - PE_1 = mg(z_2 - z_1) = mgh$$

where h is the vertical height from position 1 to 2.

In case of a linear spring, the change in potential energy in compression from a position 1 at x_1 to a position 2 at x_2 is

$$\Delta PE = PE_2 - PE_1 = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 = \frac{1}{2}(x_2^2 - x_1^2)$$

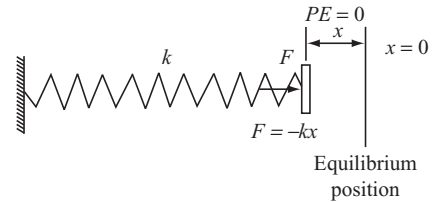


Fig. Ex. 6.22 (c) (Solution)

Example 6.23

Establish whether the force field defined by

$$= (y^2z^3 - 6xz^2) + 2xyz^3 + (3xy^2z^2 - 6x^2z)$$

is a conservative field or not. Determine the potential function, if possible.

Solution A force field is conservative if and only if there exists a scalar function ϕ such that

$$= \text{grad } \phi = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = F_x \quad F_y \quad F_z$$

For the given force field,

$$\frac{\partial \phi}{\partial x} = y^2z^3 - 6xz^2 \quad (i)$$

$$\frac{\partial \phi}{\partial y} = 2xyz^3 \tag{ii}$$

$$\frac{\partial \phi}{\partial z} = 3xy^2z^2 - 6x^2z \tag{iii}$$

Integrating Eqs. (i) with respect to x keeping y and z constant,

$$\phi = y^2z^3x - 3x^2z^2 + f(y, z)$$

Similarly, from Eqs. (ii) and (iii) on partial integration,

$$\phi = xy^2z^3 + g(x, z)$$

and

$$\phi = xy^2z^3 - 3x^2z^2 + h(x, y)$$

From these results,

$$\phi = xy^2z^3 - 3x^2z^2 + C$$

and the potential energy is given by

$$PE = 3x^2z^2 - xy^2z^3 + k$$

Since a scalar potential function for the given force field exists, the field must be conservative.

Alternatively, the condition for the force field to be conservative, i.e., for the existence of ϕ

$$\nabla \times \mathbf{F} = 0$$

could be used. From the data,

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 - 6x^2z^2 & 2xyz^3 & 3xy^2z^2 - 6x^2z \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} (3xy^2z^2 - 6x^2z) - \frac{\partial}{\partial z} (2xyz^3) \right] \\ &\quad + \left[\frac{\partial}{\partial z} (y^2z^3 - 6x^2z^2) - \frac{\partial}{\partial x} (3xy^2z^2 - 6x^2z) \right] \\ &\quad + \left[\frac{\partial}{\partial x} (2xyz^3) - \frac{\partial}{\partial y} (y^2z^3 - 6x^2z^2) \right] \\ &= (6xyz^2 - 6xyz^2) + (3y^2z^2 - 12xz - 3y^2z^2 + 12xz) + (2yz^3 - 2yz^3) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

6.5 WORK-ENERGY PRINCIPLE

Consider a force acting on a particle or on a mass centre which may be displaced from r_1 to r_2 over the course of action. The work done by the force would be

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

Newton's law provides that

$$F = m \frac{d}{dt}$$

Now,

$$\begin{aligned} \int_{t_1}^{t_2} \cdot d &= m \int_1^2 \frac{d}{dt} \cdot d \\ &= m \int_{t_1}^{t_2} \frac{d}{dt} \cdot \frac{d}{dt} dt \\ &= m \int_{t_1}^{t_2} \frac{d}{dt} \cdot dt \\ &= m \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{v^2}{2} \right) dt \\ &= \left[\frac{1}{2} mv^2 \right]_1^2 \\ &= \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2 \end{aligned}$$

or

$$W = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$$

and finally,

$$W = KE_2 - KE_1 \quad (6.15)$$

This is the well-known 'work-energy equation' in mechanics. *In words, the work done on a particle must equal the change in kinetic energy of the particle.* For a rigid body, the work done by the resultant force, as if acting at the mass centre, must equal the change in the translational kinetic energy possessed by the body by virtue of its mass and the velocity of the mass centre.

There are no reservations in using this relation. The work done on the left-hand side is due to the net external force acting on the particle; the force may vary in magnitude and direction as the particle moves, the integration being performed along the path with due regard to the variation of \cdot . The particle may accelerate as it moves. The work energy equation is as general as the Newton's law, it being an alternative form of the law. In fact, there are some points in favour of the work-energy equation.

1. The work-energy equation is a scalar equation relating scalar variables, i.e., work and kinetic energy as against the Newton's law relating vector quantities, i.e., force and acceleration.
2. The work-energy equation is obtained by a path-integration of the equation of motion; it is a step closer to the solution of the second order equation of motion.
3. In certain situations, where the work done may be evaluated readily, the velocities may be computed by employing the work-energy equation.

6.6 CONSERVATION OF MECHANICAL ENERGY

Let us now examine the case where the force is conservative. In a conservative force field, the work done in moving a particle or a mass centre from position 1 to 2 depends upon the potential energy PE of the end states only,

$$W = PE_1 - PE_2$$

Equating this expression for the work done with that provided by the work-energy equation, i.e.

$$W = KE_2 - KE_1;$$

$$PE_1 - PE_2 = KE_2 - KE_1$$

whence $PE_1 + KE_1 = PE_2 + KE_2$

or $(PE + KE)_1 = (PE + KE)_2$ (6.16)

which implies that in a conservative force field, the sum of the potential energy and kinetic energy remains constant for all positions of a particle or a mass centre. The sum of the potential energy and kinetic energy is called mechanical energy. Equation (6.16) is referred to as the *Principle of conservation of mechanical energy*. In other words, *the mechanical energy of a mass centre or a particle is conserved when it moves in a conservative force field*, as also visualised in Fig. 6.10.

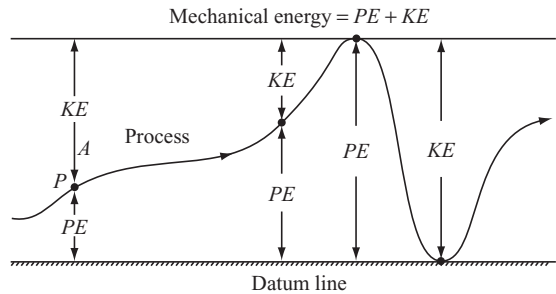


Fig. 6.10 Visualisation of the motion with conservation of mechanical energy ($PE + KE = \text{Const.}$)

If a particle is, instead, made to move in a force field, a part of which is conservative and another part non-conservative, then

$$= \text{Cons.} + \text{Non-cons.}$$

and a potential energy function PE may be determined for the conservative part of the force Cons. . The work-energy principle provides that

$$W = \int_1^2 \cdot d = KE_2 - KE_1$$

Substituting for ,

$$\int \text{Cons.} d + \int \text{Non-cons.} d = (PE_1 - PE_2) + W_{\text{Non-cons.}}$$

$$= KE_2 - KE_1$$

or $W_{\text{Non-cons.}} = (KE_2 + PE_2) - (KE_1 + PE_1)$

or $W_{\text{Non-cons.}} = (KE + PE)_2 - (KE + PE)_1$ (6.17)

In other words, the work done by the non-conservative forces on a particle moving in a mixed force-field must equal the change in mechanical energy of the particle. This is also as general a statement as is the basic work-energy equation or the Newton's law of motion and it may be applied to the motion of a particle or a mass centre in any situation.

Example 6.24

A slider of mass 2 kg attached to a spring of stiffness 400 N/m and unstretched length 0.5 m is released from rest at *A* as shown in Fig. Ex. 6.24. Determine the velocity of the slider as it passes through *B* and *C*. Assume that the slider moves over the bent rod with negligible friction.

Also compute the distance beyond *C* where the slider should come to rest again.

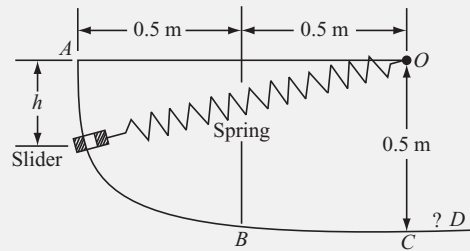


Fig. Ex. 6.24

Solution In the absence of friction between the slider and rod, the reaction on the slider must be normal to the motion and hence it does not contribute to the work done on the slider. The slider may then be considered to be in a conservative force field comprising of gravitational and spring forces. Referring to *A* as the datum,

$$PE = -mgh + \frac{1}{2}kx^2$$

where *h* is the height of the slider below the reference point and *x* is the compression of the spring for the position considered.

$$PE_A = \frac{1}{2} \times 400 \times (1 - 0.5)^2 = 50 \text{ Nm}$$

$$PE_B = -2 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times (0.707 - 0.5)^2 = -1.24 \text{ Nm}$$

$$PE_C = -2 \times 9.81 \times 0.5 = -9.81 \text{ Nm}$$

The kinetic energy at each point is expressed as follows:

$$KE_A = 0$$

$$KE_B = \frac{1}{2}mv_B^2 = \frac{1}{2} \times 0.5v_B^2 = 0.25v_B^2$$

$$KE_C = \frac{1}{2}mv_C^2 = \frac{1}{2} \times 0.5v_C^2 = 0.25v_C^2$$

Employing the principle of conservation of mechanical energy,

$$(PE + KE) = \text{Constant}$$

$$50 + 0 = -1.24 + 0.25v_B^2 = -9.81 + 0.25v_C^2$$

whence

$$v_B = 14.32 \text{ m/s}$$

and

$$v_C = 15.47 \text{ m/s}$$

Let us consider a point *D* where the slider would come to rest again

$$PE_D = -2 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times x^2$$

$$KE_D = 0$$

Again, by conserving mechanical energy,

$$50 + 0 = -2 \times 9.81 \times 0.5 + \frac{1}{2} \times 400 \times x^2$$

$$x = 0.55 \text{ m}$$

The stretch of the spring being 0.55 m, its stretched length should be $0.55 + 0.5 = 1.05$ m which suggests that $CD = (1.05^2 - 0.5^2)^{1/2} = 0.92$ m

Example 6.25

A block of mass 5 kg is released from rest from a position *A* on a 30° incline as shown in Fig. Ex. 6.25. Determine the maximum compression of the spring if the spring constant is 8 N/cm and the coefficient of friction between the block and the incline is 0.2.

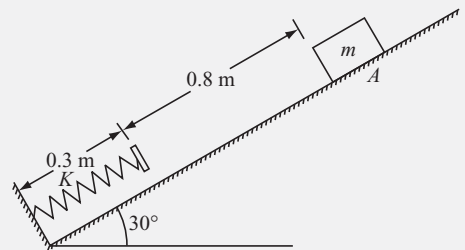


Fig. Ex. 6.25

Solution

Weight of the block = $5 \times 9.81 = 49.05$ N

With reference to the free-body diagram,

$$R = 49.05 \cos 30^\circ = 42.5 \text{ N}$$

$$f = 0.2 \times 42.5 = 8.5 \text{ N}$$

Application of the work energy equation between the initial position (1) of the block and the position (2) when it just touches the spring yields

$$\int_0^{0.8} (49.05 \sin 30^\circ - 8.5) dx = \frac{1}{2} \times 5 \times (v_2^2 - 0)$$

or
$$= [16.025 x]_0^{0.8} = 2.5 v_2^2$$

and
$$v_2 = 2.265 \text{ m/s}$$

From position (2) it just touches the spring to the position (3) of maximum compression δ the work-energy equation provides

$$\int_0^\delta (49.05 \sin 30^\circ - 8.5 - 8 \times 100 x) dx = \frac{1}{2} \times 5 (0 - v_2^2)$$

or
$$16.025 - 400\delta^2 = -12.82$$

$$400\delta^2 - 16.025\delta - 12.82 = 0$$

whence
$$\delta = \frac{16.025 \pm \sqrt{256.8 + 20,512}}{2 \times 400}$$

$$= 0.2 \text{ m}$$

The maximum force developed by the spring is

$$F = 0.2 \times 8 \times 100 = 160 \text{ N}$$

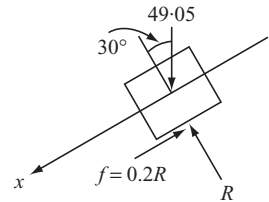


Fig. Ex. 6.25 (Solution)

6.7 IMPULSE OF A FORCE

Impulse of a force acting over a time interval from t_1 to t_2 is defined as the time integral of the force over the interval of time

$$I = \int_{t_1}^{t_2} F dt \quad (6.18)$$

The force may vary or remain constant over a period of time. The impulse of a force be visualised as the area under a force vs time curve.

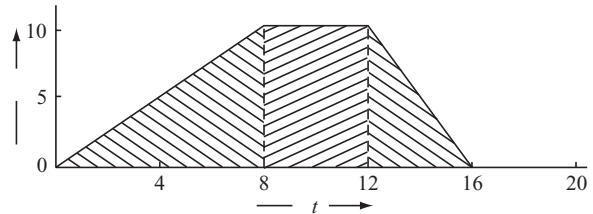


Fig. 6.11 Force vs Time Curve

As an illustration, a force acting over an interval of 20 s causing a particle to move along the direction of the force, as shown in Fig. 6.11, produces an impulse

$$\begin{aligned} \text{Magnitude of } I &= \frac{10 \times 8}{2} + 10 \times (12 - 8) + \frac{10 \times (16 - 12)}{2} + 0 \\ &= 40 + 40 + 20 + 0 = 100 \text{ Ns} \end{aligned}$$

computed as the area under the $F-t$ curve.

The concept of impulse is particularly useful if large forces act over short intervals of time. The action may then be classified as impulsive and expressed in terms of an impulse in its own right without regard to the force associated with it.

The impulse of a force is often called linear impulse in order to differentiate it from angular impulse which may arise due to a moment acting over a period of time.

6.8 IMPULSE-MOMENTUM PRINCIPLE

Newton's law for a particle or a mass centre provides that

$$F = m \frac{dv}{dt}$$

which may be rewritten as

$$F dt = m dv$$

Integrating each side from an initial position at t_1 when the velocity is v_1 to final position at t_2 when the velocity is v_2

$$\int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} m dv = m(v_2 - v_1) \quad (6.19)$$

In words,

$$\text{Impulse} = \text{Change in momentum}$$

$$\text{or} \quad \text{Initial momentum} + \text{Impulse} = \text{Final momentum}$$

This is a vector equation which applies to a particle or mass centre of a rigid body or of a system of bodies. It is the velocity v_c of the mass centre that must be considered to evaluate the linear momentum of a rigid body at any instant, *no matter where the force actually acts on the body.*

The left hand side of Eq. (6.20) can be rewritten as

$$\int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} F_x dt + \int_{t_1}^{t_2} F_y dt + \int_{t_1}^{t_2} F_z dt$$

and similarly the right hand side can also be written as

$$= m(v_{2x} - v_{1x}) + m(v_{2y} - v_{1y}) + m(v_{2z} - v_{1z})$$

Equating the left hand and right hand sides of Eq. (6.20), we get

$$\int_{t_1}^{t_2} F_x dt = m(V_{2x} - V_{1x})$$

$$\int_{t_1}^{t_2} F_y dt = m(V_{2y} - V_{1y})$$

$$\int_{t_1}^{t_2} F_z dt = m(V_{2z} - V_{1z})$$

6.9 CONSERVATION OF MOMENTUM

It may be observed from the impulse-momentum principle that the momentum of a particle or of the centre of mass of a system is conserved in the absence of an external impulse or an external force acting on it. Then,

$$\text{Initial momentum} = \text{Final momentum}$$

Example 6.26

A force given by

$$= 3t^2 + 5t - (8t^3 + 400) \text{ N}$$

acts from $t = 0$ to $t = 10$ s. Determine the impulse of the force. If this impulse acted at the centre of mass of a body of mass 500 kg and brought it to rest, estimate the velocity of the body before it acted.

Solution

$$\begin{aligned} \mathbf{I} &= \int dt \\ &= \int_0^{10} 3t^2 dt + \int_0^{10} 5t dt - \int_0^{10} (8t^3 + 400) dt \\ &= \left| t^3 \right|_0^{10} + \left| 2.5t^2 \right|_0^{10} - \left| 2t^4 + 400t \right|_0^{10} \\ &= 1000 + 250 - 24,000 \text{ Ns} \end{aligned}$$

From the impulse-momentum principle,

$$= m(v_2 - v_1)$$

$$1000 + 250 - 24,000 = 500(0 - v_1)$$

whence v_1 , the velocity before the action of impulse should be

$$(-2 - 0.5 + 48) \text{ m/s}$$

Example 6.27 A railroad coal car of weight W_0 is moving at a constant speed v_0 while being loaded with coal at a constant rate of \dot{m} per second as shown in Fig. Ex. 6.27. Determine the force P necessary to sustain the constant speed, neglecting friction.

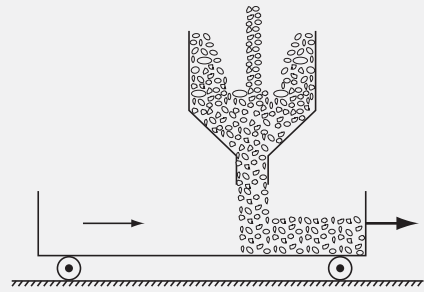


Fig. Ex. 6.27

Solution Considering change of momentum

of the car,

$$|(v_0 + \Delta v)|_2 - |v_0|_1 =$$

because the velocity v_0 is constant.

This must be equal to the impulse

$$=$$

whence

in the direction of motion.

Example 6.28 A 10 kg shell is fired with a velocity of 600 m/s at angle of 30° from an old 2000 kg gun as shown in Fig. Ex. 6.28. Assuming that the barrel of the gun is rigidly attached to the frame and that the frame can recoil freely determine the recoil velocity of the gun and the reaction on the gun if the shell leaves the barrel 10 milliseconds after firing.

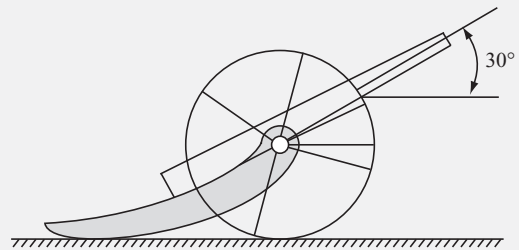


Fig. Ex. 6.28

Solution For the gun-and-shell system the momentum must be conserved in the horizontal direction because there is no external horizontal force.

$$10 \times 600 \times \cos 30^\circ + 2000 \times v = 0$$

whence the recoil velocity $v = -2.6$ m/s in the opposite direction.

Applying the impulse-momentum principle in the vertical direction,

$$(10 \times 600 \times \sin 30^\circ - 0) = \int_0^{0.01} R dt$$

whence

$$= 300,000 \text{ N} = 300 \text{ kN.}$$

This must also be reaction by the ground on the frame in the vertically upward direction.

Example 6.29 A 0.001 kg bullet has a velocity of 1000 m/s as it enters a fixed block of wood. It comes to rest 0.002 s after entering the block. Determine the average force that acted on the bullet and the distance penetrated by it.

Solution From the impulse-momentum principle,

$$\begin{aligned} \mathbf{I} &= \int_1^2 dt = F_{av} \cdot \Delta t = 0.002 F_{av} \\ &= m(v_2 - v_1) = 0.001(0 - 1000) = -1 \text{ N} \\ F_{av} &= -1/0.002 = -500 \text{ N} \end{aligned}$$

whence, which implies that a resistive force of 500 N acts opposite to the direction of motion of the bullet.

The distance penetrated by the bullet into the block may be worked out quite simply by employing the work energy principle or by using Newton's law of motion.

By the work-energy principle,

$$\int_{r_1}^{r_2} \cdot d = \frac{1}{2} m (v_2^2 - v_1^2)$$

or $500 \times \text{distance penetrated} = \frac{1}{2} \times 0.001 \times (0^2 - 1000^2)$

whence the distance penetrated = 1.0 m

Alternatively, the acceleration of the bullet should be

$$\begin{aligned} a &= v/m = (v_2 - v_1)/\Delta t \\ &= \frac{0 - 1000}{0.002} = -500,000 \text{ m/s}^2 \end{aligned}$$

and the distance penetrated is given by

$$v_2^2 - v_1^2 = 2 a s$$

whence $s = \frac{0 - 1000^2}{2 \times 500\,000} = 1.0 \text{ m}$

Example 6.30

A person of mass 60 kg stands at one end of a 6 m long floating boat of mass 240 kg. If the person walks across to the other end at a steady rate of 1.2 m/s, determine (a) the velocity of the boat as observed by an observer on the ground during the process, (b) the distance by which the boat is shifted, (c) the velocity of the boat if the person stops at the other end and (d) the velocity of the boat if the person, while walking, falls out of the boat at the other end.

Solution Considering the person (1) and the boat (2) as a single system, there is no external force on it. The momentum of the person and the boat taken together must, therefore, be conserved. Distances and velocities are referred positive to the right as shown in Fig. Ex. 6.30 (Solution).

(a) Initially, the person and the boat are at rest; their total momentum is zero. As the person moves with a velocity of 1.2 m/s to the right with respect to the boat, the boat may be moving with a velocity V_2 m/s. The absolute velocity of the person is, therefore,

$$V_1 = (-1.2 + V_2) \text{ m/s}$$

Momentum of the system in the process is

$$\begin{aligned} m_1 V_1 + m_2 V_2 &= 60(-1.2 + V_2) + 240 V_2 \\ &= 300 V_2 - 72 \end{aligned}$$

which must be equal to the initial momentum, i.e., zero

$$300 V_2 - 72 = 0$$

whence $V_2 = 0.24 \text{ m/s}$

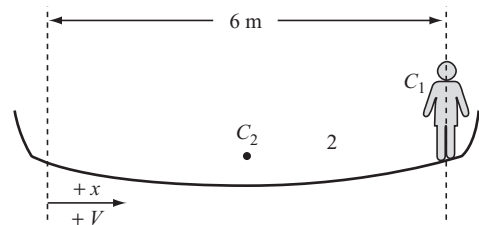


Fig. Ex. 6.30 (Solution)

and $V_1 = -1.2 + 0.24 = -0.96 \text{ m/s}$

An observer on the ground will, therefore, observe the boat to be drifting to the right with a velocity of 0.24 m/s and the person moving to the left with a velocity of 0.96 m/s.

(b) The time taken by the person to walk across the other end is

$$t = 6/1.2 = 5 \text{ s}$$

and the distance the boat travels in 5 seconds is

$$x_2 = 5 \times 0.24 = 1.2 \text{ m}$$

(c) If the person stops at the other end, the boat should also stop because their total momentum must remain zero.

(d) If the person, while walking falls out of the boat at the other end, he would take with him a momentum

$$m_1 V_1 = 60 (-1.2 + 0.24) = -57.6 \text{ Ns}$$

The momentum possessed by the boat is, therefore, +57.6 Ns because the sum of the momenta equals zero all the time. The boat must have a velocity V_2 after the person fell out

$$V_2 = \frac{57.6}{240} = 0.24 \text{ m/s}$$

In the absence of any external force, the boat will thus continue to move to the right at 0.24 m/s.

An alternative approach to the solution of the first two parts of the problem would be as follows:

The person and the boat taken together do not experience any external force. Therefore, the centre of mass of the system must remain unaltered.

Initially
$$x_c = \frac{60 \times 6 + 240 \times 3}{60 + 240} = 3.6 \text{ m}$$

Finally, after the person has moved to be other end,

$$\begin{aligned} 3.6 &= \frac{60 \times (0 + x_2) + 240(3 + x_2)}{60 + 240} \\ &= \frac{300x_2 + 720}{300} \end{aligned}$$

whence,

$$x_2 = 1.2 \text{ m}$$

which shows that the boat must have moved by 1.2 m to the right.

The velocity of the boat during the process is determined from the knowledge of the time taken by the person and the displacement of the boat

$$V_2 = \frac{x_2}{t} = \frac{1.2}{6/1.2} = 0.24 \text{ m/s}$$

Example 6.31

A military truck of mass 4000 kg while developing a tractive force of 12 kN tows a jeep of mass 2000 kg with the help of an inextensible cable up an incline of 1 in 10. A winch mounted on the jeep is operated to approach the truck with a constant acceleration of 0.5 m/s^2 . Before the winch is operated, the truck and jeep were travelling at 12 m/s each. Determine their speed after a lapse of 10 s (refer Fig. Ex. 6.31).

What would the final velocities be if the winch did not operate?

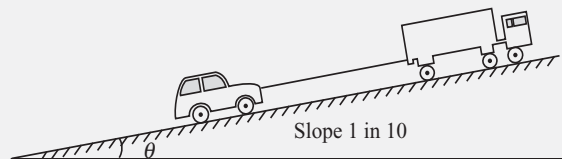


Fig. Ex. 6.31

Solution This problem is best solved by employing the impulse-momentum equation. This preference is in view of the obvious requirement to deal with the action of forces over a prescribed time and to determine the velocities as a result of change of momentum.

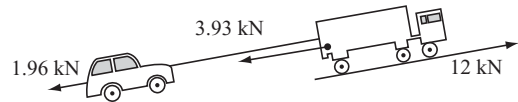


Fig. Ex. 6.31 (Solution)

Taking the truck and jeep together as a single system, the magnitude of the external force acting on it is

$$\begin{aligned} F &= 12 - (4000 + 2000) \times 9.81 \sin \theta / 1000 \\ &= 12 - (4 + 2) \times 9.81 \times 1/10 = 12 - 3.93 - 1.96 \\ &= 6.11 \text{ kN} \end{aligned}$$

The magnitude of the impulse acting on the system is

$$I = F \times 10 = 6.11 \times 10 = 61.1 \text{ kNs}$$

Initial momentum of the system is

$$\begin{aligned} p_1 &= (4000 \times 12 + 2000 \times 12) / 1000 \\ &= 72 \text{ kNs} \end{aligned}$$

Denoting the final velocity of the truck by V_t and that of the jeep by V_j where

$$V_j = V_t + 0.5t$$

the final momentum of the system is

$$\begin{aligned} p_2 &= (4000 \times V_t + 2000 \times V_j) / 1000 \\ &= (4V_t + 2(V_t + 0.5 \times 10)) \\ &= (6V_t + 10) \text{ kNs} \end{aligned}$$

Since

$$\text{Impulse} = \text{Change in momentum}$$

$$61.1 = 6V_t + 10 - 72$$

whence

$$V_t = 20.52 \text{ m/s}$$

and

$$V_j = 20.52 + 0.5 \times 10 = 25.52 \text{ m/s}$$

In the absence of a winch, the final momentum of the system would have been given by

$$\begin{aligned} p_2 &= (4000 \times V + 2000 \times V) / 1000 \\ &= 4V + 2V = 6V \end{aligned}$$

because the velocities of the truck and the jeep would have remained identical;

Then, by the impulse momentum principle,

$$61.1 = 6V - 72$$

whence

$$V = 22.18 \text{ m/s}$$

which would be the final velocity of the truck as well as that of the jeep.

Example 6.32

A body of mass 20 kg slides up an incline of 30° under the action of an applied force 300 N along the incline and in the presence of friction, $\mu = 0.2$. If the body moves from rest determine, after a period of 5 seconds: (a) acceleration of the body, (b) velocity of the body, (c) distance travelled by the body, (d) kinetic energy of the body, (e) work done on the body, (f) momentum possessed by the body and (g) impulse applied on it.

If the body now moves along a level surface with friction, $\mu = 0.2$, and the external force is removed, how long will it take for the body to come to rest and how far will it have travelled?

Solution During the first phase of motion over the incline, the free-body diagram of the body shown in Fig. Ex. 6.32 (solution) that the force acting in the direction of motion is

$$F = 300 - 20 \times 9.81 \sin 30^\circ - 20 \times 9.81 \cos 30^\circ \times 0.2 = 168 \text{ N}$$

(a) From the Newton's law,

$$F = ma$$

$$a = 168/20 = 8.4 \text{ m/s}^2$$

(b) Again, from the Newton's law

$$F = m \frac{dv}{dt}$$

$$dv = \frac{F}{m} dt$$

and $v_2 - v_1 = \frac{F}{m} (t_2 - t_1)$

Taking time $t_1 = 0$ at the beginning when $v_1 = 0$

$$v_2 = \frac{168}{20} \times 5 = 42 \text{ m/s}$$

Alternatively, from

$$v = u + at$$

$$v = 0 + 8.4 \times 5 = 42 \text{ m/s}$$

(c) From the Newton's law,

$$F = m \frac{d^2s}{dt^2}$$

$$s = \frac{F}{m} t^2 / 2 + C_1 t + C_2$$

Noting that both C_1 and C_2 are zero for the given conditions,

$$s = \frac{168}{20} \times 5^2 / 2 = 105 \text{ m}$$

Alternatively, from

$$s = ut + \frac{1}{2} at^2$$

$$s = 0 + \frac{1}{2} \times 8.4 \times 5^2 = 105 \text{ m}$$

(d) The kinetic energy is given by

$$\frac{1}{2} mv^2 = \frac{1}{2} \times 20 \times 42^2 = 17,640$$

(e) The work done on the body is

$$W = 168 \times 105 = 17,640$$

which is indeed equal to the change in kinetic energy of the body.

(f) The momentum possessed is

$$p = 20 \times 42 = 840 \text{ kg m/s or Ns}$$

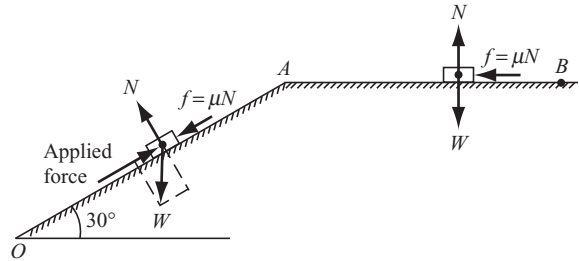


Fig. Ex. 6.32 (Solution)

(g) The impulse imparted to the body is

$$= \int dt = 168 \times 5 = 840 \text{ Ns}$$

which is indeed equal to the change in momentum of the body.

For the second phase of the motion, i.e., along a horizontal surface, the force along the direction of motion is

$$-f = -\mu N = -0.2 \times 196.2 = -39.2 \text{ N}$$

The work done on the body is

$$-39.2 \times \text{distance travelled}$$

Equating it to the change in kinetic energy, $-17,640 \text{ J}$

the distance travelled is $17,640/39.2 = 450 \text{ m}$

The impulse on the body is

$$-39.2 \times \text{time of action}$$

Equating it to the change in momentum,

$$0 - 840 = -840 \text{ Ns}$$

the time of travel is $840/39.2 = 21.43 \text{ s}$

Alternatively, the acceleration of the body can be determined as

$$a = -39.2/20 = -1.96 \text{ m/s}^2$$

and from the Newton's law in the form

$$= m \frac{dv}{dt}$$

the time can be determined and from the form

$$= m \frac{d^2s}{dt^2}$$

the distance travelled can be calculated.

Alternatively, employing the relations

$$v = u + at \quad \text{and} \quad s = ut + 1/2 at^2$$

$$t = \frac{0 - 42}{-1.96} = 21.43 \text{ s}$$

and

$$s = 42 \times 21.43 + 1/2 \times (-1.96) \times 21.43^2 = 450 \text{ m}$$

6.10 MOMENT OF MOMENTUM

The term 'moment of momentum' is defined analogous to the moment of a force. The momentum possessed by a body of mass m moving with a linear velocity v being $m v$, the definition for the moment of momentum is that

$$= r \times m v$$

where r is the position vector of the centre of mass of the body at a given instant with respect to a reference point.

The term 'angular momentum' is used interchangeably with moment of momentum. This term is possibly designed to denote its difference from the linear momentum $m v$, also written simply as momentum. The physical interpretation of Moment of Momentum is not readily clear from its definition but the meaning can be appreciated in the following article, where the rate of change of moment of momentum is related to the external moment about the same reference point for the body.

6.11 MOMENT OF MOMENTUM EQUATION

Starting with the Newton's law in the form

$$\mathbf{F} = \frac{d}{dt}(m \mathbf{v})$$

and taking moment of each term of the equation about a point O in space as shown in Fig. 6.12

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt}(m \mathbf{v})$$

The addition of a zero term, namely

$$\frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) = 0$$

on the right hand side of the above equation yields

$$\mathbf{r} \times \mathbf{F} = \frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) + \frac{d}{dt}(\mathbf{r} \times m \mathbf{v})$$

or
$$\mathbf{M}_0 = \mathbf{r} \times m \mathbf{v} = \frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) \quad (6.20)$$

or
$$\mathbf{M}_0 = \frac{d}{dt} \mathbf{M}_0 = \dot{\mathbf{M}}_0$$

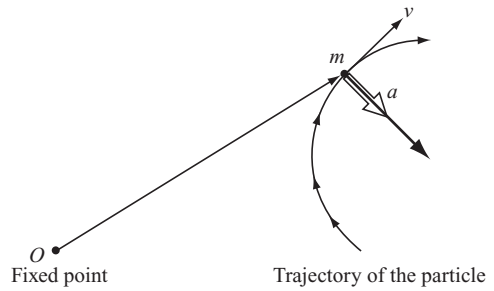


Fig. 6.12 Trajectory of the particle

This is the moment of momentum equation which states that the moment of the resultant force on a particle or at the mass centre of a system about a fixed point O equals the time rate of change of moment of momentum about the same point.

6.12 CONSERVATION OF MOMENT OF MOMENTUM

If the moment of the resultant force acting on a particle or at the centre of mass of a system about a fixed point in space is zero, then

$$\frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) = 0$$

or
$$\mathbf{r} \times m \mathbf{v} = \text{Const.} = \mathbf{M}_0 \quad (6.21)$$

i.e., \mathbf{M}_0 , the moment of momentum is conserved. The moment of the resultant force would be zero if either the force is zero or the force is directed towards or away from the origin O .

Example 6.33

A particle of mass 2 kg tied at the end of an inextensible string is rotated at 20 rad/s along a circle of 1 m radius over a smooth horizontal table top. The string is pulled down through a slot at the centre of the table top at a speed of 5 m/s as shown in Fig. Ex. 6.33. Calculate the speed of the particle when it reaches 0.5 m from the centre. Comment on the variation of tension in the string with time.

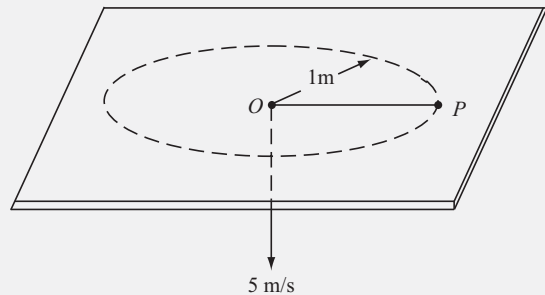


Fig. Ex. 6.33

Solution As the particle is rotated over the table and the string is pulled in, the only force that acts on the particle is the tension in the string which is always radial. Consequently, the fact that

$$\mathbf{v} \times \mathbf{v} = 0$$

prompts the utilisation of the principle of conservation of moment of momentum, i.e.,

$$\mathbf{h} \times \mathbf{m} = \text{Const.} =$$

Since the motion is confined to a plane, the velocity at any instant of time must be given by

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta$$

and

$$\mathbf{h} \times \mathbf{m} = \mathbf{h} \times m(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta)$$

$$= \mathbf{h} \times m v_\theta \mathbf{e}_\theta =$$

or

$$h = \int m r v_\theta =$$

$$= r^2 \omega = \text{Constant} \quad (\text{i})$$

Initially,

$$h_1 = 1^2 \cdot 20 = 20 \text{ m}^2/\text{s}$$

and finally,

$$h_2 = 0.5^2 \cdot \omega_2 = \omega/4 \text{ m}^2/\text{s}$$

From the conservation principle,

$$h_1 = h_2$$

$$20 = \omega/4, \quad \omega_2 = 80 \text{ rad/s}$$

and

$$v_{\theta 2} = 0.5 \times 80 = 40 \text{ m/s}$$

Since

$$v_{r 2} = 5 \text{ m/s}$$

the speed of the particle is

$$v = \sqrt{40^2 + 5^2} = 40.3 \text{ m/s}$$

The tension in the string may be estimated by applying the Newton's law to the particle. There is no force and no acceleration in the tangential direction.

In the radial direction,

$$-F = 2(-r\omega^2)$$

or

$$F = 2r\omega^2 \quad (\text{ii})$$

It is prescribed that

$$r = 1 - 5t$$

and from the moment of momentum conservation Eq. (i)

$$r^2 \omega = 1^2 \cdot 20 = 20$$

$$\omega = \frac{20}{(1 - 5t)^2}$$

which, when substituted in Eq. (ii), gives

$$F = 2(1 - 5t) \frac{20^2}{(1 - 5t)^4}$$

or

$$F = \frac{800}{(1 - 5t)^3} \text{ N}$$

It is interesting to note that the force required to pull the string at a constant speed is not constant; it increases as the particle approaches the centre-slot. The following table should supplement the understanding of the phenomenon.

It is noticed that the particle turns increasingly faster as it approaches the centre and it may, therefore, be expected that the string must break at some position. However, in reality friction and drag come into play and neither the rotational speed nor tension is allowed to increase to such an extent. The fact that

Table 6.2

| | t (s) | r (m) | ω (rad/s) | F (N) |
|--|---------|---------|------------------|----------|
| | 0 | 1 | 20 | 800 |
| | 0.05 | 0.75 | 35.56 | 1,896 |
| | 0.10 | 0.50 | 80 | 6,400 |
| | 0.15 | 0.25 | 320 | 51,200 |
| | 0.20 | 0 | ∞ | ∞ |

the rotational velocity of a mass increases as it is drawn closer to the axis of rotation is used by expert figure skaters on the ice and figure dancers on smooth floors. They break into a whirl with both their arms and one leg extended and then, upon drawing the arms and the leg in, they obtain a greatly increased angular velocity, which is both amusing and amazing!

Example 6.34

A thin circular ring of mass 100 kg and radius 2 m resting flat on a smooth surface is subjected to a sudden application of a force of 300 N at a point on its periphery. Calculate (a) the angular acceleration of the ring and (b) the acceleration of the mass centre.

Solution

(a) Applying the moment of momentum equation with the origin chosen at the mass centre of the ring and realising that \mathbf{r} , \mathbf{v} and $\boldsymbol{\omega}$ are mutually perpendicular,

$$M = \frac{d}{dt}(r m v) = \frac{d}{dt}(r^2 m \omega)$$

$$= r^2 m \dot{\omega}$$

whence

$$\omega = \frac{M}{r^2 m}$$

$$= \frac{300 \times 2}{2^2 \times 100} = 1.5 \text{ rad/s}^2$$

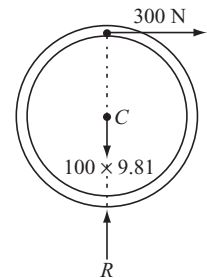


Fig. Ex. 6.34 (Solution)

(b) The acceleration of the mass centre may be determined by applying Newton's law,

$$= m_c$$

Then

$$c = \frac{300}{100} = 3 \text{ m/s}^2$$

The Newton's law is indeed applicable although the force is acting at a point other than the centre of mass C.

Example 6.35

A vertical lift has a cage of total mass of 300 kg raised through a distance of 100 m by a cable. It commences with an acceleration of 1.5 m/s^2 which is held constant until it catches a speed of 10 m/s. During the final stage, it has a constant retardation to bring it to rest. It takes a total of 15 seconds for the 100 m distance. Estimate the tension in the rope during acceleration, at a constant speed and during retardation.

Solution One can solve this problem by writing the equations for different stages of the motion, e.g.,

$$v = 0 + 1.5 t_1 \quad \text{and} \quad s_1 = 0 + \frac{1}{2} \times 1.5 t_1^2$$

$$v = \text{const} \quad s_2 = v t_2$$

$$v = d t_3 \quad s_3 = -\frac{1}{2} d t_3^2, \quad \text{etc.}$$

which produce too many simultaneous equations to be solved.

Alternatively, one may construct a simple velocity-time graph as follows.

The advantage of such a graph is that the area under the v - t graph must be the distance traversed.

For stage 1,

$$10 = 1.5 t_1; \quad t_1 = 6.67 \text{ s}$$

$$s_1 = \frac{1}{2} \times 10 \times 6.67 = 33.35 \text{ m}$$

It follows that $t_2 + t_3 = 15 - 6.67 = 8.33 \text{ s}$ and the remaining distance is $100 - 33.35 = 66.65 \text{ m}$

From the graph,

$$10t_2 + 10t_3/2 = 66.65$$

$$2t_2 + t_3 = 13.33$$

but

$$t_2 + t_3 = 8.33$$

Hence,

$$t_2 = 5 \text{ s}, \quad t_3 = 3.33 \text{ s}$$

The deceleration in the final stage must be given by

$$0 = 10 - d \times 3.33$$

$$d = 10/3.33 = 3 \text{ m/s}^2$$

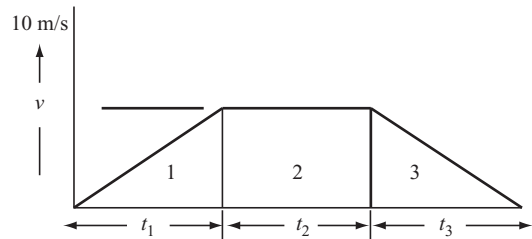


Fig. Ex. 6.35(a) (Solution)

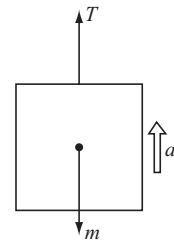


Fig. Ex. 6.35(b) (Solution)

Now, let us draw a free-body diagram of the cage in general, as shown in Fig. Ex. 6.35(b) (Solution). By equation of motion,

$$T - mg = ma$$

or

$$T = m(a + g)$$

For stage 1, $T_1 = 300(1.5 + 9.81) = 3393 \text{ N} = 3.39 \text{ kN}$

For stage 2, $T_2 = 300(9.81) = 2943 \text{ N} = 2.94 \text{ kN}$

For stage 3, $T_3 = 300(-3 + 9.81) = 2043 \text{ N} = 2.04 \text{ kN}$

It may be noted that the 'power' required is also different in the three stages. Given by

Power = $T.v$. It is the maximum in the first stage; equals 33.9 kW.

Example 6.36

A spring of stiffness 20 kN/m is installed between two horizontal plates with an initial compression of 0.025 m. A collar of mass 5 kg is dropped from a height of 200 mm on to the upper plate. Estimate the initial energy stored in the spring and the further compression of the spring. Neglect energy lost during the impact.

Solution The spring is initially under a compressive force of $20 \times 1000 \times 0.025$, i.e., 500 N

Energy stored must be the strain energy given by

$$E_1 = \frac{1}{2} kx^2 = \frac{1}{2} \times 20 \times 1000 \times (0.025)^2 = 6.25 \text{ J}$$

This must also be equal to the triangular area under the F -compression graph;

$$E_1 = \frac{1}{2} F \cdot x = \frac{1}{2} \times 500 \times 0.025 = 6.25 \text{ J}$$

Now, the collar drops by a total distance of $(0.2 + x)$ m where x is the additional compression. Energy released by the collar must equal to additional strain energy in the spring.

$$\begin{aligned} 5 \times 9.81 \times (0.2 + x) &= \frac{1}{2} 20(0.025 + x)^2 - \frac{1}{2} 20x^2 \\ &= 10(0.025 + x)^2 - 10x^2 = 10(0.00625 + 0.05x + x^2) - 10x^2 \\ &= 0.0625 + 0.5x \end{aligned}$$

whence x can be calculated.

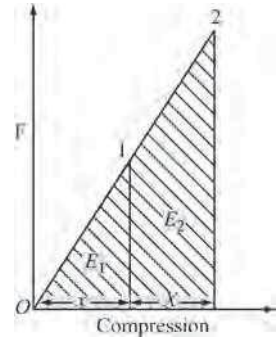


Fig. Ex. 6.36 (Solution)

Concept Review Questions

Recognise and explain the truth in the following statements:

- The acceleration or deceleration for a rectilinear motion is ascertained on the basis of like or unlike signs for the acceleration and velocity vectors rather than on the basis of the acceleration vector alone.
- An object may be travelling at a constant speed but it may have variable acceleration.
- The laws of motion applicable to a particle must also be applicable to the centre of mass of any system whether or not the force acts at the centre of mass.
- The work-energy principle, impulse-momentum principle, moment of momentum principle and D'Alembert principle are alternative manifestations of the Newton's law of motion.

Under what circumstances is Newton's law of motion applicable? When would you prefer the work-energy, impulse-momentum or moment of momentum principle formulations? How does the D'Alembert principle differ from the Newton's law?

Which of the following are conserved in a central-force motion and why?

- Force
 - Linear momentum
 - Moment of momentum
 - Mechanical energy.
4. Match the following definitions correctly with the features of the plots indicated.
- Net applied force
Slope of velocity vs time
Change in kinetic energy
Area of acceleration vs time

Change in velocity
 Area of velocity vs time
 Change in the applied force
 Slope of position vs time
 Change in displacement
 Area of force vs position

Instantaneous acceleration
 Slope of kinetic energy vs position
 Instantaneous velocity
 Slope of linear momentum vs time
 Change in linear momentum
 Area of force vs time

Tutorial Problems

A stone is dropped into a well. If the splash is heard 2.50 seconds later determine the depth of the water surface assuming that the velocity of sound is 330 m/s.

(Hint: $d/330 + \sqrt{2d/9.81} = 2.50$, $d = 28.6$ m)

A particle accelerates as $a = 9\sqrt{x}$ where a is in m/s^2 and x is rectilinear displacement in metres. At $t = 3$ seconds, the displacement is 16 m and the velocity is 27.7 m/s. Determine the displacement, velocity and acceleration of the particle one second later.

(s 67.5 m, 81.6 m/s, 73.9 m/s^2)

A block of mass 2 kg slides down the face of a smooth 45° wedge of mass 10 kg as shown in Fig. Prob. 6.3. The wedge is placed on a frictionless horizontal surface. Determine the acceleration of the wedge.

(s 0.89 m/s^2)

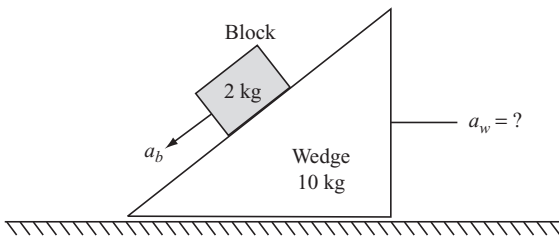


Fig. Prob. 6.3

A vehicle is uniformly accelerated upon an incline of 1 in 20 from rest and attains a velocity of 5 m/s in 15 s. Determine the tractive force if the vehicle has a mass of 2000 kg and the resistance to motion is 200 N.

(s 0.33 m/s^2 , 1848 N)

Three blocks of masses m_1 , m_2 and m_3 are connected by two cords as shown in Fig. Prob. 6.5. Obtain an expression for the acceleration a of the system and determine the tension in the cords.

$$\left(\begin{aligned} s \quad a &= \frac{(m_3 - \mu_1 m_1 - \mu_2 m_2)g}{m_1 + m_2 + m_3} \\ T_1 &= \mu_1 m_1 g + \frac{m_1 g (m_3 - \mu_1 m_1 - \mu_2 m_2)}{m_1 + m_2 + m_3} \\ T_2 &= m_3 g - \frac{m_3 g (m_3 - \mu_1 m_1 - \mu_2 m_2)}{m_1 + m_2 + m_3} \end{aligned} \right)$$

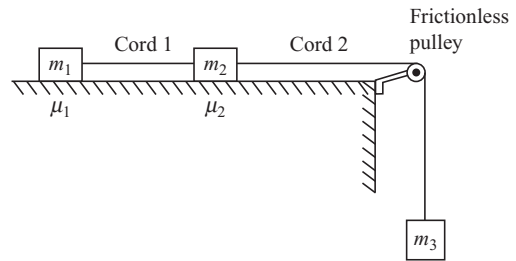


Fig. Prob. 6.5

A block A resting on a smooth floor and carrying a block B upon it is pulled by a horizontal force as shown in Fig. Prob. 6.6. Determine the acceleration of A to cause a slip between A and B if the coefficient of friction between them is μ .

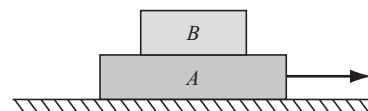


Fig. Prob. 6.6

(s $a = \mu g$)

An object of mass m falls vertically down in a medium with resistance R proportional to the square of the velocity. Obtain an expression for the velocity at time t if it starts from rest at time $t = 0$. What is the terminal velocity?

$$\left(s \quad V = \sqrt{\frac{mg}{k}} \left[\frac{1 - e^{-2t\sqrt{kg/m}}}{1 + e^{-2t\sqrt{kg/m}}} \right]; \sqrt{\frac{mg}{k}} \right)$$

- 8 A particle of mass m is projected vertically upward with a velocity v_0 in a medium whose resistance is kv . Determine the time for the particle to come to rest. What would be the time if the resistance was kv_2 instead of kv ?

$$\left(s \quad m/k \ln(1 + kv_0/mg) \quad \text{and} \quad \sqrt{m/kg} \tan^{-1} v_0 \sqrt{k/mg} \right)$$

A horizontal force of 100 N is exerted on block A of mass 20 kg which is tied by an inclined string to block B of mass 5 kg as shown in Fig. Prob. 6.9. The coefficients of friction between the plane and block surfaces A and B are 0.25 and 0.5 respectively. Calculate the tension in the string and the acceleration of the system.

$$\left(s \quad 28 \text{ N}, 1.11 \text{ m/s}^2 \right)$$

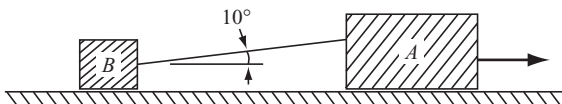


Fig. Prob. 6.9

A particle of mass m rests on the top of a smooth sphere of radius r as shown in Fig. Prob. 6.10. Assuming that the particle starts moving from rest, at what point will it leave the sphere?

$$\text{(Hint: } -N + mg \cos \theta = mr\omega^2 \text{ and } mg \sin \theta = mr\alpha,$$

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}, \quad \int_0^\theta \alpha \, d\theta = \int_0^\omega \omega \, d\omega$$

Substituting $N = 0$, $\cos \theta = 2/3$, $\theta = 48.2^\circ$)

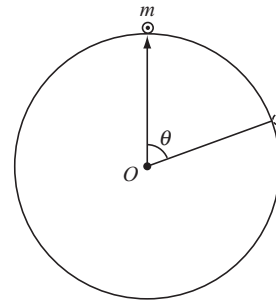


Fig. Prob. 6.10

A conical pendulum consists of a particle of mass m tied to a cord of length l such that it traces a circular path of radius r at an angular velocity ω when the cord makes an angle θ with the axis of rotation as shown in Fig. Prob. 6.11. If its velocity increases, the particle rises and the radius r of the circular path and angle θ increase. Derive a relationship between θ and ω for constant angular velocity and express the frequency in terms of θ .

$$\left(s \quad \omega = \sqrt{g/l \cos \theta}, \quad f = \frac{1}{2\pi} \sqrt{g/l \cos \theta} \right)$$

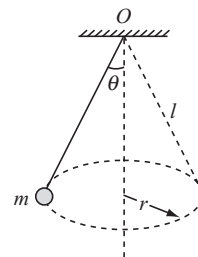


Fig. Prob. 6.11

A small block of mass 2 kg, held by a cord rests on a smooth inclined plane which can turn about the vertical axis $z-z$ as shown in Fig. Prob. 6.12. The cord is 0.6 m long. Determine the tension in the cord when the angular velocity of the plane and block is 10 revolutions per minute.

Also calculate the angular velocity and tension in the cord when the block is at the verge of losing contact with the inclined plane.

$$\left(s \quad 10.75 \text{ N}; 5.72 \text{ rad/s}, 39.24 \text{ N} \right)$$

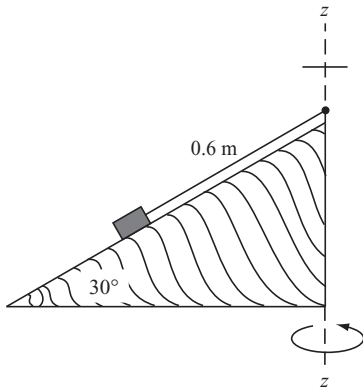


Fig. Prob. 6.12

A wagon of mass 5000 kg is loose shunted to acquire a speed of 10 m/s before it goes and hits a bumper. If the spring constant of the stationary bumper is 200 N/cm and the spring constant of the wagon bumper is 300 N/cm, calculate (a) the maximum compression of the stationary bumper and (b) the time taken to reach that state.

(s 3.87 m; 0.72s)

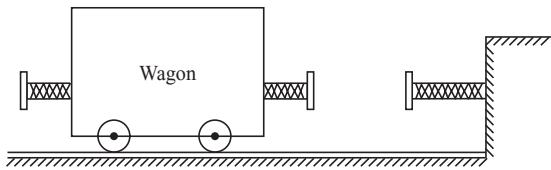


Fig. Prob. 6.13

A photograph of the 7.25 kg shot-put champion in action shows that the initial angle of projection was 45°. The toss on a level ground was 22 m. Calculate the velocity of projection and the time of flight (a) assuming that the height of the champion is negligible in comparison with the toss and (b) assuming that the height of the point of projection is 2 m.

(s 14.7 m/s, 2.15 s; 14.1 m/s, 2.2 s)

A 2 kg ball is suspended by an inextensible string from a ceiling to comprise a pendulum of length 3 m. The ball is released from a position where the string makes an angle of 45° with the vertical. Determine the velocity of the ball when the string makes an angle of 30° with the vertical and when it is at the bottom position.

(s 3.06 m/s, 4.15 m/s)

Determine the minimum velocity a body must have at the top of a vertical circular cylinder of radius r if, when moving circularly, it is to remain in contact with the circular cylinder.

(s $v_{\min} = \sqrt{gr}$)

A bead of mass m moves around a vertical circle of radius r . If the tangential velocity is v_t at the top of the circle, find the tangential velocity v_b at the base of the circle. Assume that the friction is negligible.

(s $v_b = \sqrt{v_t^2 + 4gr}$)

8 A 1 kg stone is whirled at 60 revolutions per minute in a plane vertical circle of radius 1.5 m by means of a string. Determine the tensions in the string at the top and bottom positions. How would the tensions alter if

(a) the mass of the stone were halved?

(b) the whirling speed were halved?

(c) the radius of the circle were halved?

(s 49.4, 69 N; 24.7, 34.5 N; 5.0, 24.6 N; 19.8, 39.4 N)

In a circus, a motor cyclist is moving inside a spherical cage of radius 3 m. The motor cycle and the man together have a mass of 725 kg. Find the least velocity with which the motor cyclist must pass the highest point on the cage without losing contact with the cage.

(Hint: The reaction $\left(mg - \frac{mV^2}{r} \right) \geq 0$ and

$V = \sqrt{gr} = 5.42$ m/s)

A cyclist, riding at 5 m/s, wishes to turn as fast as he can without skidding. If the coefficient of static friction between the tyres and the road is 0.25, estimate the minimum safe radius of his turn.

(Hint: To avoid sliding, $\frac{mV^2}{r} \leq \mu mg$ and

$V \leq \sqrt{\mu gr}; r = 10.2$ m)

A particle of mass m slides down a frictionless track and enters a vertical loop of radius r at A to 'loop the loop' as shown in Fig. Prob. 6.21. What should be the minimum height h at the starting point of the particle in order that it may make a complete circuit in the loop?

(Hint: $v_A = \sqrt{2gh}$ and the top position is the critical point where the reaction $N = 0$ and the velocity v is given by $v^2 = v_A^2 - 4gr$. Employing $mg = mv^2/r$, $h = 2.5r$. If the particle is started from a height less than $2.5r$ it will not reach the top position and if it is started from a height more than $2.5r$, it should loop the loop successfully.)

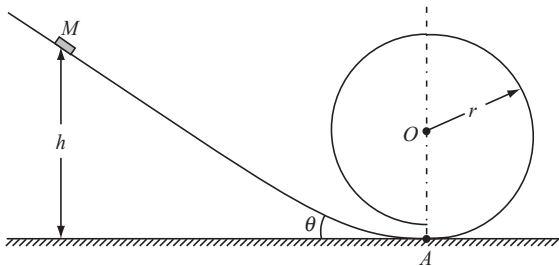


Fig. Prob. 6.21

A vertical shaft rotating at 5 rad/s has a light horizontal arm fixed with it. Two identical collars A and B each of mass 3 kg, slide out with respect to the arm at 2 m/s as shown in Fig. Prob. 6.22. Determine the angular acceleration of the arm.

(Hint: $d/dt (r \times m \dot{\theta}) = 0 = d/dt (r^2 m \dot{\omega})$
 $r^2 d\omega/dt + 2r\dot{r}m\dot{\omega} = 0$
 Hence, $d\omega/dt = -10 \text{ rad/s}^2$)

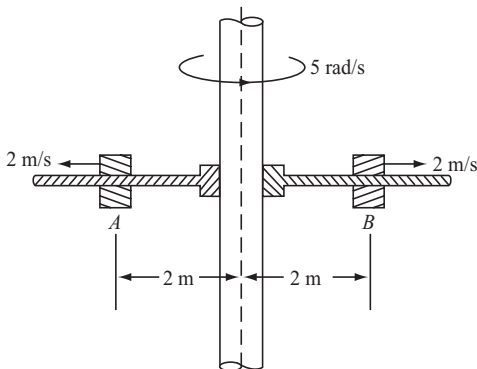


Fig. Prob. 6.22

Determine the work done in winding up a homogeneous cable which hangs from a horizontal drum if its free length is 20 m and weighs 1 kN as shown in Fig. Prob. 6.23.

(s 98.1 kJ)

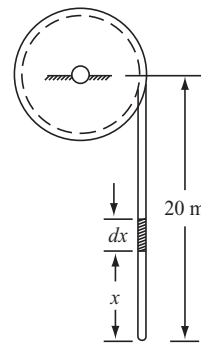


Fig. Prob. 6.23

A bullet enters a 5 cm thick plank with a speed of 600 m/s and leaves with a speed of 240 m/s. Determine the greatest thickness of the plank that can be penetrated by the bullet.

(s 5.95 cm)

A particle weighing 10 N is supported as a simple pendulum by a massless inextensible string, 2 m in length. The pendulum is released from rest at an angle of 30° as shown in Fig. Prob. 6.25. When the string is just vertical, it strikes a rigid support and the particle continues to swing as a pendulum of shorter length (a) Find the maximum value of the angle ϕ . (b) Find the force exerted on the weight by the string when $\phi = 30^\circ$.

(s 43 , 111.2 N)

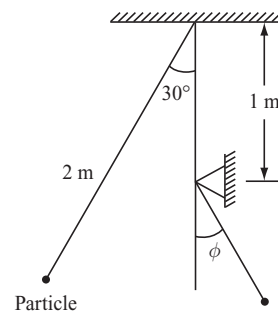


Fig. Prob. 6.25

A particle of mass m rests on a smooth table and is attached to a string which passes through a

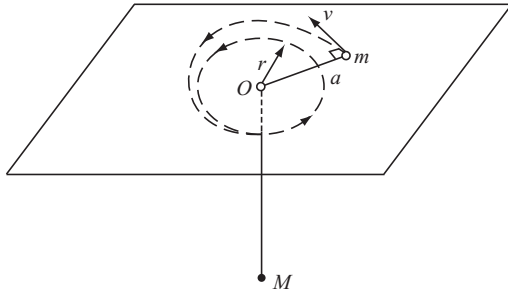


Fig. Prob. 6.26

small hole in the table and supports a mass M in a vertical position as shown in Fig. Prob. 6.26. The mass m is given a velocity v perpendicular to the horizontal portion of the string when the mass is at a distance a from the hole, and the system is then released. After a fall of h the mass M will stop and m will continue to move in a circle of radius r . Determine r .

$$\left[s \frac{mv^2}{3Mg} \left(1 + \frac{2Mga}{mv^2} \right) \right]$$

After a flood, a goat weighing 40 kg finds itself adrift on one end of a 50 kg log, 6 m long. As the other end touches the shore, the goat manoeuvres to that end. When it gets there, how far is it from the shore? Assume the log is at right angle to the shore and the water is almost calm after the storm!

$$(s \ 2.67 \text{ m})$$

- 8 A spring of unstretched length $2a$ and spring constant k is connected to a fixed point O and to a point B on the edge of a wheel as shown in Fig. Prob. 6.28. Find the total work done by the force exerted on the wheel due to the spring as the wheel is rotated from B_1 to B_2 .

$$(s \ 0.472 \ ka^2)$$

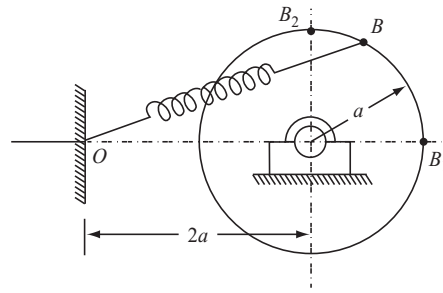


Fig. Prob. 6.28

Look up Hints to Tutorial Problems at the end of the book!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions.

- The motion of a particle, in general, is described by
- the Newton's law and not the work-energy equation
 - the impulse-momentum principle alone if there is no external force
 - the Newton's law, the work-energy equation, impulse momentum principle or the moment-of-momentum principle
 - the principles of conservation of energy and momentum

The D'Alembert principle

- is a hypothetical principle
- provides no special advantage over Newton's law
- is based upon the existence of inertia forces
- allows a dynamical problem to be treated akin to a statical problem

The centrifugal force

- is not an inertia force
- tends to overturn a body outwards on a curved path
- is a fictitious force
- is the real force experienced by a body negotiating a bend

A particle of mass m is projected with a velocity V making an angle of 45° with the horizontal. The magnitude of the angular momentum of the particle about the point of projection when the particle is at its maximum height h is

- (a) zero
 (b) $mV^3/(4\sqrt{2}g)$,
 (c) $mV^3(\sqrt{2}g)$
 (d) $m\sqrt{2gh^2}$

The momentum of a system of two bodies is conserved

- (a) if either body does not exert a force on the other
 (b) when there is no external force acting on either body
 (c) when the external forces act only on one body
 (d) when the external moment on the system is zero.

The impulse-momentum principle is applicable

- (a) if there is no external force acting on the body
 (b) when the impulse is conserved
 (c) when the momentum is conserved

- (d) wherever Newton's law is applicable.

The principle of conservation of mechanical energy requires that

- (a) the acceleration should be zero
 (b) there should be no external forces
 (c) the motion should be restricted to the gravitational field only
 (d) the force-field should be conservative

- 8 A simple pendulum mounted on a lorry moving on a horizontal track with a constant acceleration a will be deflected away from the vertical

- (a) towards the direction of acceleration, $\theta = \sin^{-1}(a/g)$
 (b) against the direction of acceleration, $\theta = \sin^{-1}(a/g)$
 (c) against the direction of acceleration,

- (d) towards the direction of acceleration,

$$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Answers to Multiple-Choice Questions

- 1 (c) 2 (d) 3 (c) 4 (b) 5 (b) 6 (d) 7 (d) 8 (c)



Moment of Inertia: Area and Mass

8

8.1 INTRODUCTION

The area of a surface and mass of a body are important concepts but not less important are the concepts of relative distribution of area and mass over the domains. The shape and orientation of a surface with respect to some reference axes are vital. Quantitative estimates of the relative distribution of area and mass over the regions of interest are made by the concepts of ‘moment of inertia’ and ‘radius of gyration’. The former is the second moment and the latter a length concept emanating from the second moment.

The concept of inertia is provided by Newton’s first law of motion. The property of matter by virtue of which it resists any change in its state of rest or of uniform motion is called . The translatory inertia is identified as whereas the rotational inertia is termed as

In other words, the moment of inertia is the rotational analogue of mass, i.e., it plays the role of resisting a change in rotational motion in quite the same sense as mass plays the role of resisting a change in translatory motion.

The concepts of the moment of inertia and radius of gyration are developed for an area and a mass in quite the same way. The area moment of inertia and mass moment of inertia will, therefore, be dealt with together. It is shown in the text that for thin bodies of uniform thickness and homogeneous density, the area moment of inertia and the mass moment of inertia are directly related.

It may appear, in the first instance, that the same notation for the area-inertia as for the mass-inertia is confusing particularly when the words ‘area’ and ‘mass’ are not used for specifying the moments of inertia. But there is no confusion because the moments of inertia refer to the area or mass in question and it is unnecessary to qualify the moment of inertia once it is known whether it refers to an area or a mass.

8.2 INERTIAL CONCEPTS: AREA

Consider a plane area as shown in Fig. 8.1. Let the reference axes be x and y in the plane of the area as indicated. The moments of inertia of the area about the x and y axes are defined as the second moments of the area about the x and y axes respectively.

$$I_x = \int y^2 \quad (8.1)$$

$$I_y = \int x^2 \quad (8.2)$$

It follows from the definitions that the moments of inertia of an area cannot be negative whether the coordinates of its elements are positive or negative because it is the summation of the product of square quantities, i.e., x^2 or y^2 and $dx dy$. It can also be observed that the moment of inertia for an element farther from the axis is more and the moment of inertia of an element on the axis is zero.

The moments of inertia may also be written as

$$I_x = A k_x^2 \quad (8.3)$$

$$I_y = A k_y^2 \quad (8.4)$$

where k_x and k_y are called the corresponding radii of gyration. The coordinates \bar{x} and \bar{y} locate a point in the area which depend upon the shape of the area and its relationship with the reference axes.

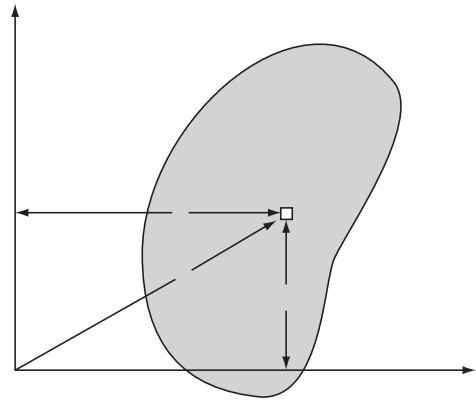


Fig. 8.1 Consideration for inertial concepts

Comparing the forms of writing the moments of inertia,

$$\begin{aligned} I_x &= A k_x^2 = \int y^2 \\ &= \sqrt{I_x / A} = \sqrt{\int (y^2) / A} \\ \text{and} \quad I_y &= \sqrt{I_y / A} = \sqrt{\int (x^2) / A} \end{aligned}$$

relates an area to a set of axes. For example,

$$I_x = \int y^2 \quad (8.5)$$

which may be negative or positive depending upon the location of the area with respect to the reference axes. In particular, the product of inertia is zero if the area is symmetrical about any of the axes.

The product of inertia of an area about an axis normal to the area and passing through a pole

is defined as

$$I_0 = \int x y$$

$$\begin{aligned}
 &= \int_A (x^2 + y^2) dA \\
 &= \int_A y^2 dA + \int_A x^2 dA
 \end{aligned}$$

or $J_0 = I_{xx} + I_{yy}$ (8.6)

It is thus noted that, by definition, the polar moment of inertia about an axis through a pole must be the sum of the moments of inertia about the axes through the pole in the plane of the area. The polar moment of inertia is also denoted by J in literature.

Also,

$$\begin{aligned}
 J_0 &= k_0^2 A \\
 &= I_{xx} + I_{yy} \\
 &= k_x^2 A + k_y^2 A
 \end{aligned}$$

and

$$k_0^2 = k_x^2 + k_y^2$$
 (8.7)

Example 8.1

(a) Determine the moment of inertia of a rectangle with sides b and h about an axis coincident with side b . Determine also the radius of gyration about this axis.

(b) Evaluate the moment inertia about the axis coincident with side h . Also determine the polar moment of inertia about an axis, normal to the plane of the rectangle, passing through the corner O .

Solution Let the x -axis be along the side b and y -axis along h . It is convenient to consider a strip-element of width dy and area.

Part (a)

$$dA = b dy$$

The moment of inertia of the strip-element about the x -axis is

$$dI_{xx} = y^2 dA = y^2 b dy$$

and the moment of inertia of the entire rectangle is

$$I_{xx} = \int_0^h y^2 b dy = \frac{bh^3}{3}$$

Expressing it in terms of the radius of gyration,

$$I_{xx} = k^2 A = k^2 bh$$

and comparing with $bh^3/3$,

$$k = \frac{h}{\sqrt{3}}$$

The point P located by the radius of gyration is, in general, different from C , the centroid of the area because the latter is related to the first moment of area whereas the former depends upon the second moment of area. The point P is also a function of the axes whereas C is not. Moreover, the point P is not unique whereas C is uniquely located for a given area. An axis passing through the centroid and lying in the plane is called a *centroidal axis*.

Part (b)

The moment of inertia about the y -axis, coincident with side h is evaluated, by analogy with the result first obtained.

$$I_{yy} = \frac{hb^3}{3}$$

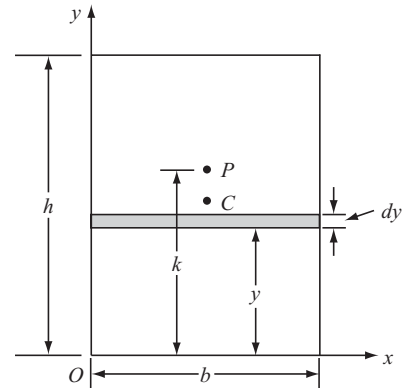


Fig. Ex. 8.1 (Solution)

and the radius of gyration

$$k = \frac{b}{\sqrt{3}}$$

Now, the polar moment of inertia must be the sum of the two about the axes passing through O :

$$\begin{aligned} J_0 &= I_{xx} + I_{yy} \\ &= \frac{bh^3}{3} + \frac{hb^3}{3} = \frac{bh}{3}(h^2 + b^2) \\ &= \frac{A}{3}(h^2 + b^2) \end{aligned}$$

where $A = bh$ is the area of the rectangle.

Example 8.2

Determine the moments of inertia of the area of a circle about the centroidal axes. Determine also the radius of an equivalent cylindrical surface of the same area for the same polar moment of inertia.

Solution Considering a small element of area

$$dA = dr \cdot r d\theta$$

as shown in the figure and noting its moments of inertia,

$$dI_{xx} = (r \sin \theta)^2 dr \cdot r d\theta$$

Integrating over the circular area,

$$\begin{aligned} I_{xx} &= \int_{r=0}^R \int_{\theta=0}^{2\pi} r^3 \sin^2 \theta d\theta dr \\ &= \pi \int_{r=0}^R r^3 dr = \frac{\pi R^4}{4} \end{aligned}$$

Similarly,

$$I_{yy} = \int_{\theta=0}^{2\pi} \int_{r=0}^R r^3 \cos^2 \theta d\theta dr = \frac{\pi R^4}{4}$$

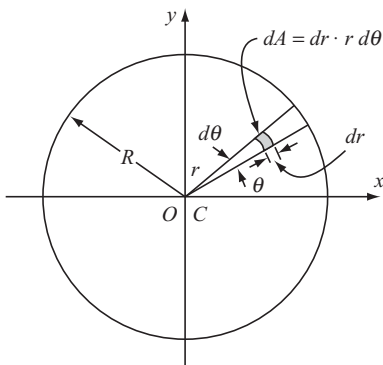


Fig. Ex. 8.2(a) (Solution)

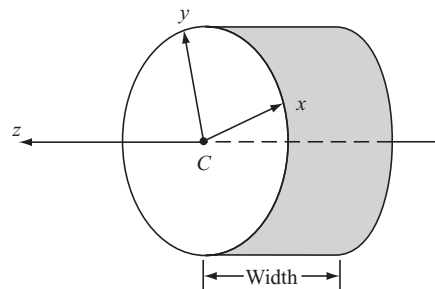


Fig. Ex. 8.2(b) (Solution)

and

$$I_{zz} = J_0 = I_{xx} + I_{yy}$$

$$= \frac{\pi R^4}{4} + \frac{\pi R^4}{4} = \frac{\pi R^4}{2}$$

The radii of gyration about the x , y and r axes are:

$$k_x = \sqrt{\left(\frac{\pi R^4}{4}\right) / \pi R^2} = \sqrt{\frac{R^2}{4}} = \frac{R}{2}$$

$$k_y = k_x = \frac{R}{2}$$

and

$$k_z = \sqrt{\left(\frac{\pi R^4}{2}\right) / \pi R^2} = \sqrt{\frac{R^2}{2}} = \frac{R}{\sqrt{2}}$$

The radius of the equivalent cylindrical surface having the same polar moment of inertia as the circular area is, therefore, $R/\sqrt{2}$. The width of the cylinder is given by

$$2\pi \frac{R}{\sqrt{2}} \times \text{width} = \pi R^2$$

whence $\text{width} = \frac{R}{\sqrt{2}}$

which is the same as the radius of the cylinder.

Example 8.3

Calculate the polar area moment of inertia and the radius of gyration for the area of a flat circular ring of radii R_1 and R_2 .

Solution The polar moment of inertia of a circular area can be determined conveniently by considering concentric ring-elements. For an elementary ring of width dr at a radius r , the elementary area is

$$dA = 2\pi r dr$$

and its polar moment of inertia is

$$dJ_0 = r^2 \cdot 2\pi r dr = 2\pi r^3 dr$$

For the entire ring,

$$J_0 = \int_{R_1}^{R_2} 2\pi r^3 dr \tag{i}$$

$$= \frac{\pi(R_2^4 - R_1^4)}{2} \tag{ii}$$

It may be seen that the polar moment of inertia of a circular area could be obtained if the limits in Eq. (i) were 0 to R or if $R_1 = 0$ and $R_2 = R$ are substituted in Eq. (ii).

For the ring, the area is

$$A = \pi(R_2^2 - R_1^2)$$

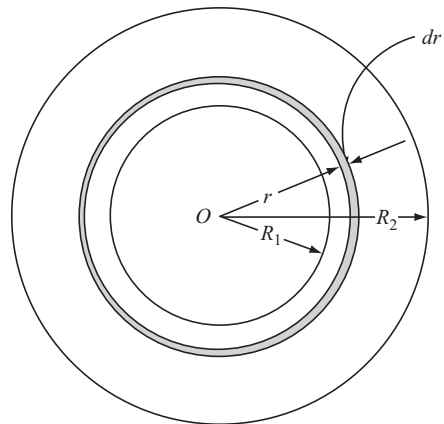


Fig. Ex. 8.3 (Solution)

By expressing the moment of inertia as

$$J_0 = k^2 A$$

and comparing with the result Eq. (ii)

$$k^2 = \frac{R_1^2 + R_2^2}{2}$$

whence
$$k = \sqrt{\frac{R_1^2 + R_2^2}{2}} \quad \text{(iii)}$$

The radius of gyration of a circular area can also be obtained by substituting $R_1 = 0$ and $R_2 = R$ in Eq. (iii).

Alternatively, a ring may be visualised as a circle of radius R_2 with a circle of radius R_1 removed from its centre. Recalling that the polar moment of inertia of a circular area is $\pi R^4/2$, the polar moment of inertia of the composite area is

$$\begin{aligned} J_0 &= \frac{\pi R_2^4}{2} - \frac{\pi R_1^4}{2} \\ &= \frac{\pi(R_2^4 - R_1^4)}{2} \end{aligned}$$

and the corresponding radius of gyration is again obtained as above.

8.3 PARALLEL-AXIS THEOREM: AREA

The parallel-axis theorem, also known as the transfer theorem, permits us to relate the moment of inertia I_{aa} of an area with respect to a given axis aa to the moment of inertia I_{cc} of the area with respect to a centroidal axis cc parallel to aa .

By definition,

$$I_{aa} = \int_A r^2 dA$$

Substituting $r = r_c + s$, as shown in Fig. 8.2,

$$I_{aa} = \int_A r_c^2 dA + \int_A 2r_c s dA + \int_A s^2 dA$$

The first term on the right-hand side represents the moment of inertia of the area about the centroidal axis cc and the second term vanishes because it is

$$\int_A 2sr_c dA = 2s \int_A r_c dA$$

and the first moment of area about the centroid is zero by the definition of centroid. The third term is

$$\int_A s^2 dA = s^2 \int_A dA = s^2 A$$

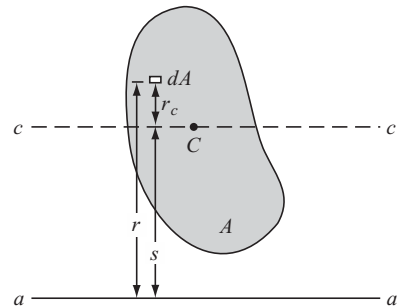


Fig. 8.2 Parallel axes

Finally,

$$I_{aa} = I_{cc} + s^2A \tag{8.8}$$

This is the statement of the parallel-axis theorem. In words, the moment of inertia of an area about an axis aa is in excess of the moment of inertia of the area about a parallel centroidal axis cc by a positive amount s^2A where s is the distance between the axes aa and cc . Obviously, the moment of inertia of an area is the least about an axis passing through the centroid.

The parallel-axis theorem for products of inertia appears in the form

$$\begin{matrix} I_{xy} & = & I_{x'y'} & + & s_1s_2A \\ \text{(for any} & & \text{(for a parallel set of} & & \\ \text{set of axes)} & & \text{centroidal axes)} & & \end{matrix} \tag{8.9}$$

where s_1 and s_2 are the perpendicular distances from the centroid to the x and y axes respectively.

The parallel-axis theorem for the polar moment of inertia of the area states that

$$J_0 = J_c + s^2A \tag{8.10}$$

where s is the distance between O and C .

Let us take an illustration of the application of the Parallel-Axis Theorem:

For a rectangular area $b \times h$ with b along the x -axis, as shown in Fig. Ex. 8.1 (Solution),

I_{xx} about the base $= \frac{bh^3}{3}$ then the moment of inertia about the parallel axis passing through the centroid c located at $h/2$ above the base

$$I_{xxc} = \frac{bh^3}{3} - A\left(\frac{h}{2}\right)^2$$

where

$$A = b \cdot h$$

Hence

$$I_{xxc} = \frac{bh^3}{3} - \frac{bhh^2}{4} = \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12}$$

Generally, the moment of inertia about the centroidal axes are available in the Tables, such as Table 8.1. The application of the parallel-axis theorem enables us to determine the moment of inertia about any desired axes. For a rectangular area,

$$I_{xx} = \frac{bh^3}{12}, \quad I_{yy} = \frac{hb^3}{12}, \quad J_c = \frac{bh}{12}(b^2 + h^2)$$

then, the moment of inertia about the base $x'x'$

$$I_{x'x'} = \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

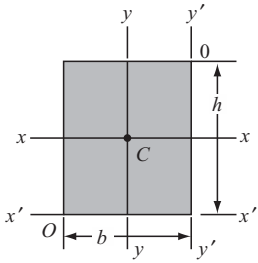
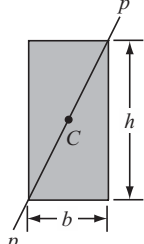
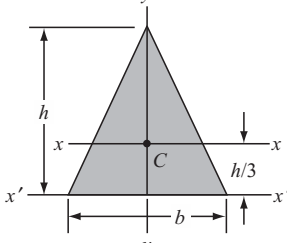
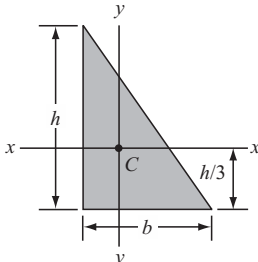
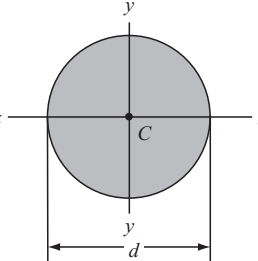
Similarly,

$$I_{y'y'} = \frac{hb^3}{12} + bh\left(\frac{b}{2}\right)^2 = \frac{hb^3}{3}$$

and

$$J_0 = \frac{bh}{12}(b^2 + h^2) + \left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2 = \frac{bh}{3}(b^2 + h^2)$$

Table 8.1 Moment of inertia of plane figures

| Figures | Description | Area | Moments of inertia |
|---|----------------------|---------------------|--|
|  | Rectangle (Upright) | bh | $I_{xx} = \frac{bh^3}{12}; \quad I_{x'x'} = \frac{bh^3}{3}$ $I_{yy} = \frac{hb^3}{12}; \quad I_{y'y'} = \frac{hb^3}{3}$ $J_c = \frac{bh}{12} (b^2 + h^2)$ $J_0 = \frac{bh}{3} (b^2 + h^2)$ |
|  | Rectangle (Upright) | bh | $I_{pp} = \frac{b^3 h^3}{6(b^2 + h^2)}$ |
|  | Triangle | $\frac{bh}{2}$ | $I_{xx} = \frac{bh^3}{36}; \quad I_{x'x'} = \frac{bh^3}{12}$ $I_{yy} = \frac{hb^3}{48} \text{ (for iso. } \Delta \text{)}$ |
|  | Right-Angle Triangle | $\frac{bh}{2}$ | $I_{xx} = \frac{bh^3}{36}$ $I_{yy} = \frac{hb^3}{36}$ $I_{xy} = -\frac{b^2 h^2}{72}$ |
|  | Circle | $\frac{\pi d^2}{4}$ | $I_{xx} = \frac{\pi d^4}{64} = I_{yy}$ $J_c = \frac{\pi d^4}{32}$ |

(Continued)

Table 8.1 (Continued)

| Figures | Description | Area | Moments of inertia |
|---------|---------------------------------|--------------------------------|---|
| | Semicircle | $\frac{\pi d^2}{8}$ | $I_{xx} = 0.00687 d^4$ $I_{yy} = \frac{\pi d^4}{128}$ |
| | Ring | πdt | $I_{xx} = \frac{\pi t d^3}{8} = I_{yy}$ $J_c = \frac{\pi t d^2}{4}$ |
| | Area between Concentric Circles | $\frac{\pi(d_2^2 - d_1^2)}{4}$ | $I_{xx} = \frac{\pi(d_2^4 - d_1^4)}{64} = I_{yy}$ $J_c = \frac{\pi(d_2^4 - d_1^4)}{32}$ |
| | Quarter Circle | $\frac{\pi d^2}{16}$ | $I_{xx} = 0.00344 d^4 = I_{yy}$ $I_{yy} = \frac{\pi d^4}{256}$ |
| | Ellipse | πab | $I_{yx} = \frac{\pi ab^2}{4}$ $I_{yy} = \frac{\pi ba^2}{4}$ $J_c = \frac{\pi ab}{4}(a^2 + b^2)$ |

8.4 MOMENT OF INERTIA OF COMPOSITE SECTIONS

The inertial concepts can be applied to determine the moments and products of inertia of plane sections. The values for simpler geometrical shapes are obtained by integration and remembering these values, the moments and product of inertia for composite sections can be worked out. The moment of inertia of simple plane sections are tabulated in Table 8.1, whereas those for composite sections are computed by subdividing the area into its components. Let a composite area A have its components A_1, A_2, A_3, \dots of which the moments of inertia about the axis in question are obtainable from Table 8.1 and by the application of the parallel-axis theorem. Let these moments of inertia be I_1, I_2, I_3, \dots respectively. The moment of inertia of the entire area A about an axis is the algebraic sum of the moments of inertia of its component areas about the same axis:

$$I = I_1 + I_2 + I_3 + \dots \tag{8.11}$$

It may be mentioned that the composite area can be made up of additive or subtractive component areas. The moment of inertia of a component area may, therefore, be additive or subtractive in the algebraic summation to compute the moment of inertia of the composite area. It should be noted that *the radius of gyration k for the composite area about an axis is not equal to the sum of the radii of gyration of the component areas about the same axis:*

$$k \neq k_1 + k_2 + k_3 + \dots$$

Instead, the radius of gyration of a composite area is determined from the moment of inertia of the composite area:

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{I_1 + I_2 + I_3 + \dots}{A_1 + A_2 + A_3 + \dots}} \tag{8.12}$$

For example, the composite area shown in Fig. 8.3 is made up of the component areas as

$$A = A_1 + A_2 - A_3 + A_4$$

The moment of inertia of the composite area will, therefore, be given by

$$I_{xx} = I_{1xx} + I_{2xx} - I_{3xx} + I_{4xx}$$

and the radius of gyration of the composite area will be given by

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

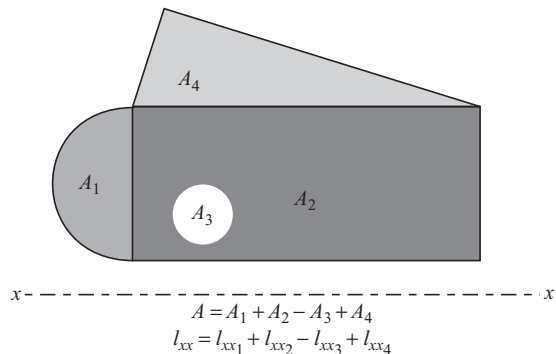


Fig. 8.3 Composite area

Example 8.4

A composite area is made by removing a circle from a rectangle and adding it back to it at top as shown in Fig. Ex. 8.4. Determine: (a) the moment of inertia about the base ox , (b) the product of inertia about the x y axes and (c) the polar moment of inertia about an axis perpendicular to the area, passing through O .

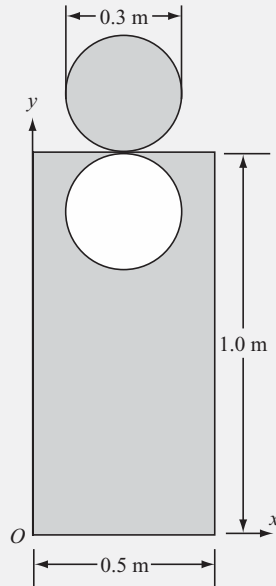


Fig. Ex. 8.4

Solution Since the radius of the circle is 0.15 m, the centre of the circle removed is 0.85 m from the x -axis and the centre of the circle added is 1.15 m from the x -axis. The centroid of the rectangle would have been 0.5 m above the x -axis.

Part (a)

Using the parallel axis theorem,

$$I_{xx} = \frac{1}{12} \times 0.5 \times 1^3 + (0.5 \times 1) \times 0.50^2$$

(for rectangle)

$$= 0.1667 \text{ m}^4$$

$$I_{xx} = \frac{\pi(0.15^4)}{4} + (\pi \times 0.15^2) \times 0.85^2$$

(for circle removed)

$$= 0.0515 \text{ m}^4$$

$$I_{xx} = \frac{\pi \times 0.15^4}{4} + (\pi \times 0.15^2) \times 1.15^2$$

(for circle added)

$$= 0.0939 \text{ m}^4$$

The moment of inertia for the composite area is, therefore,

$$I_{xx} = 0.1667 - 0.0515 + 0.0939 = 0.2091 \text{ m}^4$$

Part (b)

The product of inertia about the base and left side can be calculated as follows:

$$I_{xy} = 0 + (1 \times 0.5) \times 0.5 \times 0.25$$

(for rectangle)

$$= 0.0625 \text{ m}^4$$

$$I_{xy} = 0 + (\pi \times 0.15^2) \times 0.85 \times 0.25$$

(for circle removed)

$$= 0.0150 \text{ m}^4$$

$$I_{xy} = 0 + (\pi \times 0.15^2) \times 1.15 \times 0.25$$

(for circle added)

$$= 0.0203 \text{ m}^4$$

For the composite area,

$$I_{xy} = 0.0625 - 0.0150 + 0.0203 = 0.0678 \text{ m}^4$$

Part (c)

The polar moment of inertia about the pole O can be calculated in a similar way.

$$I_0 = \frac{1}{12} (0.5 \times 1.0) \times (0.5^2 + 1^2) + (0.5 \times 1.0) \times (0.25^2 + 0.5^2)$$

(for rectangle)

$$= 0.2084 \text{ m}^4$$

$$I_0 = \frac{\pi \times 0.15^4}{2} + (\pi \times 0.15^2) \times (0.85^2 + 0.25^2)$$

(for circle removed)

$$= 0.0563 \text{ m}^4$$

$$I_0 = \frac{\pi \times 0.15^4}{2} + (\pi \times 0.15^2) \times (1.15^2 + 0.25^2)$$

(for circle added)

$$= 0.0983 \text{ m}^4$$

For the composite area,

$$I_0 = 0.2084 - 0.0563 + 0.0983 = 0.250 \text{ m}^4$$

Example 8.5

Determine the moments of inertia with respect to the centroidal axes of the wide-flange beam section shown in Fig. Ex. 8.5.

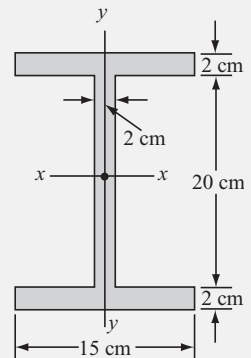


Fig. Ex. 8.5

Solution From the symmetry of the section, the centroid can be located by inspection. The moment of inertia of the composite section can be determined by different choices of the subdivisions. Let us consider some choices for I_{xx} .

Choice I

By subdividing it into three rectangles; 15 cm × 2 cm at top, 15 cm × 2 cm at the bottom and 20 cm × 2 cm in the middle.

$$I_{xx} = 2 \times \left[\frac{15 \times 2^3}{12} + (15 \times 2) \times 11^2 \right]$$

(for the top and bottom rectangles)

$$= 7280 \text{ cm}^4$$

$$I_{xx} = \frac{2 \times 20^3}{12} = 1333 \text{ cm}^4$$

(for the middle rectangle)

$$I_{xx} \text{ of the composite section} = 7280 + 1333 = 8613 \text{ cm}^4$$

Choice II

By subdividing it into five rectangles, 24 cm × 2 cm in the middle and two each above and below of 6.5 cm × 2 cm

$$I_{xx} = 4 \times \left[\frac{6.5 \times 2^2}{12} + (6.5 \times 2) \times 11^2 \right]$$

(for the four rectangles at top and bottom)

$$= 6309 \text{ cm}^4$$

$$I_{xx} = \frac{2 \times 24^3}{12} = 2304 \text{ cm}^4$$

(for the middle rectangle)

$$I_{xx} \text{ for the composite section} = 2304 + 6309 = 8613 \text{ cm}^4$$

Choice III

The easiest choice is perhaps the composition of the section by removing two rectangles 20 cm × 6.5 cm each from a larger rectangle of 24 cm × 15 cm. Then,

$$I_{xx} = \frac{15 \times 24^3}{12} - 2 \times \frac{6.5 \times 20^3}{12} = 8613 \text{ cm}^4$$

Similarly, I_{yy} can be determined by decomposing the area by different choices. It can be seen, by inspection that the quickest method in this case is to consider that section to be made up to three rectangles as per Choice I discussed before.

$$I_{yy} = \frac{20 \times 2^3}{12} = 13.3 \text{ cm}^4$$

(for the middle rectangle)

$$I_{yy} = 2 \times \left(\frac{2 \times 15^3}{12} \right) = 1125 \text{ cm}^4$$

(for the top and bottom rectangles)

$$I_{yy} \text{ for the composite section} = 13.3 + 1125 = 1138.3 \text{ cm}^4$$

The third choice which happened to be the easiest for finding I_{xx} is not the easiest for finding I_{yy} . In view of the facts demonstrated here, it should be noticed that different composite areas may be constituted differently for the purpose required.

8.5 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA: AREA

For an area A and the coordinate axes ox and oy ,

$$I_{xx} = \int_A y^2 dA$$

$$I_{yy} = \int_A x^2 dA$$

$$I_{xy} = \int_A xy dA$$

The coordinates of an area element are (x, y) with respect to the $ox-oy$ axes and

$$x' = x \cos \theta + y \sin \theta$$

$$y' = -x \sin \theta + y \cos \theta$$

with respect to the $ox'-oy'$ axes inclined at an angle θ with ox

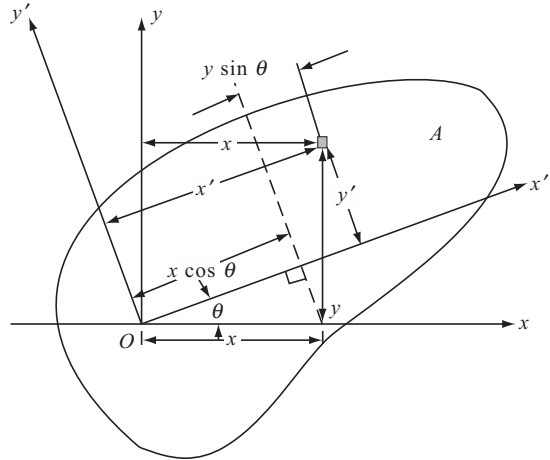


Fig. 8.4 Rotation of axes

$$\begin{aligned} I_{x'y'} &= \int_A (y')^2 dA \\ &= \int_A (-x \sin \theta + y \cos \theta)^2 dA \\ &= I_{yy} \sin^2 \theta + I_{xx} \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta \end{aligned}$$

or, by using the identities,

$$\cos^2 \theta = (1 + \cos 2\theta) / 2$$

$$\sin^2 \theta = (1 - \cos 2\theta) / 2$$

$$2 \sin \theta \cos \theta = \sin 2\theta$$

Similarly,

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad (8.13)$$

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{yy} - I_{xx}}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad (8.14)$$

and

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad (8.15)$$

These equations permit us to determine the moment of inertia and products of inertia of an area about any set of axes with an origin O from the knowledge of I_{xx} , I_{yy} , and I_{xy} about a known set of axes with the same origin.

It can be appreciated that these equations are the parametric equations of a circle. Eliminating θ , by using the identity

$$\sin^2 2\theta + \cos^2 2\theta = 1$$

We obtain,

$$\left(I_{x'x'} - \frac{I_{xx} + I_{yy}}{2} \right)^2 + I_{x'y'}^2 = \left(\frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2$$

Setting the average moment of inertia

$$I_{av} = \frac{I_{xx} + I_{yy}}{2} \tag{8.16}$$

and

$$R = \sqrt{\left(\frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2} \tag{8.17}$$

it can be rewritten as

$$(I_{x'x'} - I_{av})^2 + I_{x'y'}^2 = R^2 \tag{8.18}$$

which is the equation of a circle of radius R centered at a point located by $(I_{av}, 0)$ on the abscissa $I_{x'x'}$ and ordinate $I_{x'y'}$. This circle is often called *Mohr's circle*.

A typical point P on the circle denotes that the moment of inertia about an axis represented by radius vector CP is $I_{x'x'}$ and the product of inertia with reference to the axes is $I_{x'y'}$ as shown in Fig. 8.5. In particular, the product of inertia at A and B is zero. The moment of inertia $I_{x'x'}$ is also an extremum at these points; I_{max} at A and I_{min} at B . These points are located by setting

$$I_{x'y'} = 0$$

whence

$$\tan(2\theta_m) = -\frac{2I_{xy}}{I_{xx} - I_{yy}} \tag{8.19}$$

This equation defines two values of $2\theta_m$ which are 180° apart, i.e., or two values of θ_m at right angles. The two axes thus defined are called *the principal axes*. By definition, therefore, the principal axes of an area about a point O are a mutually perpendicular set of axes, *one about which the moment of principal is a maximum and the other about which the moment of inertia is a minimum*. The maximum and minimum moments of inertia about a point are also called the principal moments of inertia about that point. From the circle diagram shown in Fig. 8.5,

$$I_{max} = I_{av} + R \tag{8.20}$$

and

$$I_{min} = I_{av} - R \tag{8.21}$$

or

$$I_{max-min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2} \tag{8.22}$$

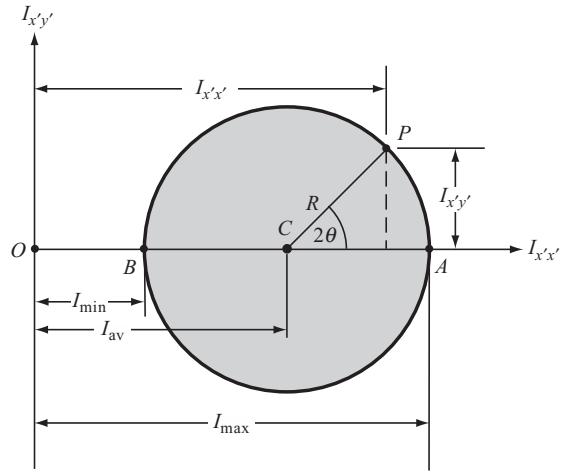


Fig. 8.5 Mohr's circle diagram

The product of inertia of a given area with respect to the principal axes at a point O is zero. It was stated earlier that the product of inertia for a set of axes vanishes if the area is symmetrical about any of the axes. It follows that if an area possesses an axis of symmetry through a point O , this axis must be a principal axis through O . It may also be understood that axes other than the axes of symmetry can be principal axes. An area may or may not possess one or more axes of symmetry but it must have a set of principal axes or inertia about any given point O in the area or outside the area. If the point O is chosen to coincide with the centroid C , the principal axes of the area at this point are called the *principal centroidal axes of the area*.

Example 8.6 Locate the axes of minimum and maximum moments of inertia at the corner of a rectangle and determine the values of the moments of inertia about those axes for a rectangle with $b = 2$ m and $h = 3$ m.

Solution With reference to Fig. Ex. 8.6 (Solution),

$$I_{xx} = \frac{bh^3}{3}$$

$$I_{yy} = \frac{hb^3}{3}$$

$$I_{xy} = \int_A xy \, dA = \int_0^h \int_0^b yx \, dx \, dy$$

$$= \frac{b^2h^2}{4}$$

Inclinations of the axes of maximum and minimum moments of inertia are located by

$$\tan(2\theta_m) = -\frac{2I_{xy}}{I_{xx} - I_{yy}}$$

$$= -\frac{2b^2h^2/4}{bh^3/3 - hb^3/3} = -\frac{3bh}{2(h^2 - b^2)}$$

For the case in hand,

$$b = 2 \text{ m} \quad \text{and} \quad h = 3 \text{ m}$$

$$I_{xx} = 18 \text{ m}^4, \quad I_{yy} = 8 \text{ m}^4, \quad I_{xy} = 9 \text{ m}^4$$

$$\tan(2\theta_m) = -\frac{3 \times 2 \times 3}{2(3^2 - 2^2)} = 1.8$$

or $2\theta_m = 119.05^\circ, 299.05^\circ$
 and $\theta_m = 59.53^\circ, 149.53^\circ$

The principal planes are, therefore, inclined at 59.53° and 149.53° to the x -axis as shown.

The principal moments of inertia are obtained by employing Eq. (8.22).

$$\frac{I_{xx} + I_{yy}}{2} = 13 \text{ m}^4$$

$$\frac{I_{xx} - I_{yy}}{2} = 5 \text{ m}^4$$

$$I_{\max} = 13 + \sqrt{5^2 + 9^2} = 23.3 \text{ m}^4$$

$$I_{\min} = 13 - \sqrt{5^2 + 9^2} = 2.7 \text{ m}^4$$

If the sides of the rectangle were equal, i.e., if it was a square, i.e., $b = h$, then $\tan 2\theta_m$ would be infinite and $\theta_m = 45^\circ$ and 135° , i.e., along the diagonal and perpendicular to it. The principal axes for a square at one of its corners must be along the diagonal through that corner and perpendicular to it. This is, however, not the case for a rectangle, as has been seen above. Let us calculate the moment of inertia about one of its diagonals, say OP . For this diagonal,

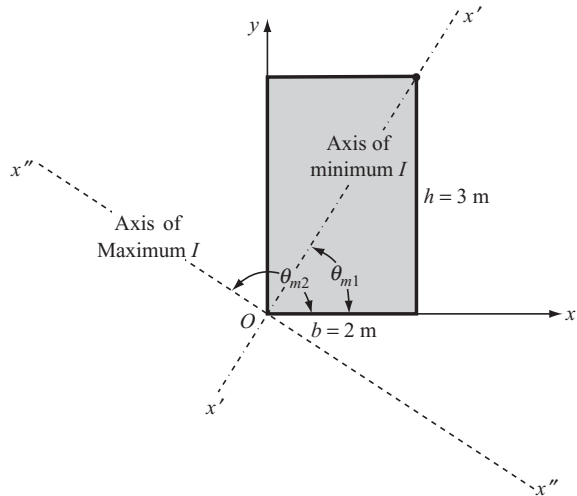


Fig. Ex. 8.6 (Solution)

$$\begin{aligned} \tan \theta &= \frac{h}{b} = 1.5 \\ \theta &= 56.31 \\ 2\theta &= 112.62 \\ \sin(2\theta) &= 0.923 \quad \cos(2\theta) = -0.385 \end{aligned}$$

Employing the expressions for the moments of inertia from Eqs. (8.13) and (8.14),

$$\begin{aligned} I_{x''x''} &= 13 + 5 \times (-0.385) - 9 \times 0.923 \\ &= 2.77 \text{ m}^4 \\ I_{y''y''} &= 13 + 5 \times (0.385) + 9 \times 0.923 \\ &= 19.38 \text{ m}^4 \end{aligned}$$

It can be verified that

$$I_{x''y''} \neq 0$$

8.6 INERTIAL CONCEPTS: MASS

The mass *moment of inertia* is a measure of its inertial behaviour, i.e., resistance to the rotational acceleration of the mass of the body. Consider a body of mass m whose distribution with reference to a Cartesian frame of reference xyz is known.

Let an element of mass dm , also denoted by

$$dm = \rho dV$$

in terms of the mass density and its volume be located by a position vector

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

as shown in Fig. 8.6.

The mass moment of inertia of the element about the x , y and z axes respectively are defined as follows:

$$dI_{xx} = (y^2 + z^2) dm \quad (8.23)$$

$$dI_{yy} = (x^2 + z^2) dm \quad (8.24)$$

$$dI_{zz} = (x^2 + y^2) dm \quad (8.25)$$

The moment of inertia of an element about an axis is given by the product of the mass element and the square of the perpendicular distance from the axis. The moment of inertia about the x -axis is shown in Fig. 8.6.

The mass moments of inertia of that entire body is, therefore,

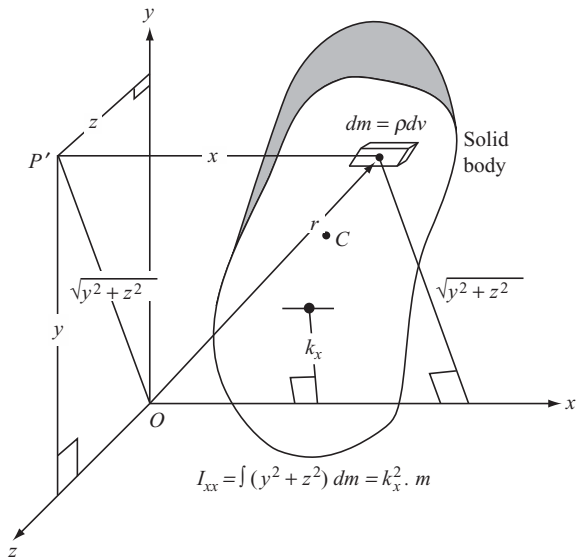


Fig. 8.6 Mass moment of inertia

$$I_{xx} = \int (y^2 + z^2) dm = \int_V (y^2 + z^2) \rho dV \quad (8.26)$$

$$I_{yy} = \int (x^2 + z^2) dm = \int_V (x^2 + z^2) \rho dV \quad (8.27)$$

$$I_{zz} = \int (x^2 + y^2) dm = \int_V (x^2 + y^2) \rho dV \quad (8.28)$$

The mass products of inertia of the body are similarly defined as

$$I_{xy} = \int xy \rho \, dv = I_{yx} \quad (8.29)$$

$$I_{xz} = \int xz \rho \, dv = I_{zx} \quad (8.30)$$

$$I_{yz} = \int yz \rho \, dv = I_{zy} \quad (8.31)$$

about the pairs of the axes specified in the indices.

For a given body, there are nine elements of moments and products of inertia defined above but only six of them are mutually independent; the set depends upon the mass distribution in the body and its relative orientation with respect to the reference axes. These are

$$I_{xx}, I_{yy}, I_{zz} \quad \text{and} \quad I_{xy}, I_{yz}, I_{zx}$$

The sum of the moments of inertia at a point in space for a given body is invariant. This important property can be proved readily by addition and showing that the sum is

$$I_{xx} + I_{yy} + I_{zz} = \int_V 2 |r|^2 \rho \, dv \quad (8.32)$$

which is constant for the chosen point *irrespective of the inclination of the reference axes at that point.*

It can be observed from the definitions that the moments of inertia must be positive quantities whereas the products of inertia can be positive or negative. If two axes of a body form a plane of symmetry for the mass distribution of the body, the products of inertia related to the normal to the plane of symmetry must be zero. The products of inertia may also vanish at a point for a pair of axes other than the axes of symmetry.

The moments of inertia may also be expressed in terms of the corresponding radii of gyration

$$I_{xx} = k_x^2 m \quad (8.33)$$

$$I_{yy} = k_y^2 m \quad (8.34)$$

$$I_{zz} = k_z^2 m \quad (8.35)$$

The radius of gyration of a body is that distance from the axis of rotation where the entire mass of the body may be assumed to be concentrated for the same mass moment of inertia as the body offers.

The mass moments of inertia and the mass products of inertia of a body may be visualised as the nine components of a matrix called the inertia matrix

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

The diagonal components are the moments of inertia about the x , y and z axes whereas the other components are the products of inertia. As already shown,

$$I_{xy} = I_{yx} \quad I_{xz} = I_{zx} \quad \text{and} \quad I_{yz} = I_{zy}$$

The inertia matrix is, therefore, expressed as

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Example 8.7

A uniform slender rod of length l is supported and made to rotate in the plane of the paper, i.e., about the vertical axis through the bar at distance l_1 from one end. Obtain an expression for the moment of inertia about the axis and hence evaluate the moment of inertia if the rod is supported

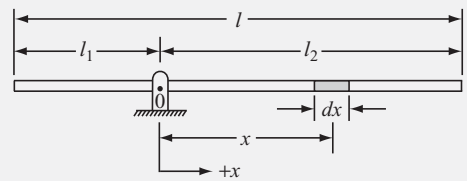


Fig. Ex. 8.7

- (a) at its end
- (b) at its mid-point.

Solution Let us solve this problem by two different methods.

Method 1

Consider an elementary length dx of the rod at a distance x from the axis of rotation. The mass of the element is

$$dm = \rho a dx$$

where ρ is the uniform density and a is the cross-section area

The mass of the rod can be seen to be

$$m = \rho a l$$

The moment of inertia of the element about the axis of rotation is

$$dI_0 = x^2 dm = x^2 \rho a dx \tag{i}$$

The moment of inertia of the entire rod is obtained by integrating it between the end limits of the rod.

$$I_0 = \int_{-l_1}^{+l_2} \rho a x^2 dx \tag{ii}$$

$$= \rho a \left[\frac{x^3}{3} \right]_{-l_1}^{+l_2}$$

$$= \rho a \frac{(l_1^3 + l_2^3)}{3} \tag{iii}$$

where $l_1 + l_2 = l$

is the length of the rod.

In terms of the mass of the rod

$$m = \rho a l$$

the moment of inertia is expressed as

$$I_0 = \frac{1}{3} m \frac{(l_1^3 + l_2^3)}{l} \tag{iv}$$

(a) If the rod is supported at an end, either l_1 or l_2 is zero depending upon at which end it is supported. For support at its left end,

$$l_1 = 0 \quad \text{and} \quad l_2 = l$$

and

$$I_0 = \frac{1}{3} \rho a l^3 = \frac{1}{3} m l^2 \tag{v}$$

(b) If the rod is supported at its mid-point,

$$l_1 = l_2 = \frac{1}{2}l$$

and

$$I_0 = \frac{1}{12} \rho a l^3 = \frac{1}{12} m l^2 \quad (\text{vi})$$

Method 2

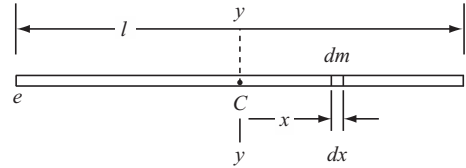
Let us consider the slender rod of length l in isolation and first find its moment inertia about yy axis passing through its centroid C .

Considering an elementary mass dm with length dx at a distance x from the centroid

$$dm = \rho a dx$$

where ρ is the uniform density on a is the cross section area.

$$\begin{aligned} I_{yye} &= 2 \int_0^{l/2} \rho a x \cdot x dx \\ &= 2 \rho a \left[\frac{x^3}{3} \right]_0^{l/2} = \rho a \frac{l^3}{12} \\ &= \frac{m l^2}{12} \end{aligned}$$



where the mass $m = \rho \cdot a \cdot l$.

We already got the answer to Part (b) of the question:

For part (a),
$$I_{yye} = \frac{m l^2}{12} + m \left(\frac{l}{2} \right)^2 = \frac{m l^2}{3}$$

Now, for the given orientation with the hinge at O ,

$$\begin{aligned} I_{yye} &= \frac{m l^2}{12} + m \left(l_2 - \frac{l_1 + l_2}{2} \right)^2 \\ &= \frac{m l^2}{12} + m \left(\frac{l_2 - l_1}{2} \right)^2 \\ &= m \left(\frac{(l_1 + l_2)^2}{12} + \frac{(l_2 - l_1)^2}{4} \right) \\ &= \frac{m}{3} (l_1^2 + l_2^2 - l_1 l_2) \end{aligned}$$

which is the same as

$$I_{yye} = \frac{m}{3} \left(\frac{l_1^3 + l_2^3}{l_1 + l_2} \right) = \frac{m}{3} \frac{(l_1^3 + l_2^3)}{l}$$

Example 8.8

A rectangular prism of cross-section ($a \times b$) and uniform density ρ has a length l . Determine its moments of inertia and the products of inertia about the longitudinal and transverse axes passing through the centre of mass.

Solution For the rectangular prism and coordinate axes through C as shown in Fig. Ex. 8.8 (Solution).

$$\begin{aligned}
 I_{xx} &= \int_V (y^2 + z^2) \rho \, dv \\
 &= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \int_{-a/2}^{a/2} (y^2 + z^2) \rho \, dx \, dy \, dz \\
 &= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} (y^2 + z^2) a \rho \, dy \, dz \\
 &= \int_{-b/2}^{b/2} \left(\frac{l^3}{12} + z^2 l \right) a \rho \, dz \\
 &= \frac{\rho a b l^3}{12} + \frac{\rho a b^3 l}{12} = \rho a b l (l^2 + b^2) / 12 \\
 &= \rho v \frac{l^2 + b^2}{12} = m \frac{l^2 + b^2}{12}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_{yy} &= \int_V (x^2 + z^2) \rho \, dv \\
 &= m \frac{a^2 + b^2}{12} \\
 I_{zz} &= \int_V (x^2 + y^2) \rho \, dv \\
 &= m \frac{a^2 + l^2}{12}
 \end{aligned}$$

The products of inertia are obtained as follows:

$$\begin{aligned}
 I_{xy} &= \int_V xy \rho \, dv \\
 &= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \int_{-a/2}^{a/2} xy \rho \, dx \, dy \, dz \\
 &= \int_{-b/2}^{b/2} \int_{-l/2}^{l/2} \frac{1}{2} y \left(\frac{a^2}{4} - \frac{a^2}{4} \right) dy \, dz \\
 &= 0
 \end{aligned}$$

This result was expected from the symmetry; the above integration is only as an exercise. Similarly, by symmetry,

$$I_{yx} = I_{yz} = I_{xz} = 0$$

It may be verified that if three axes were drawn through a corner instead of the centre of mass C , the results would have been

$$I_{xx} = m \frac{l^2 + b^2}{3}$$

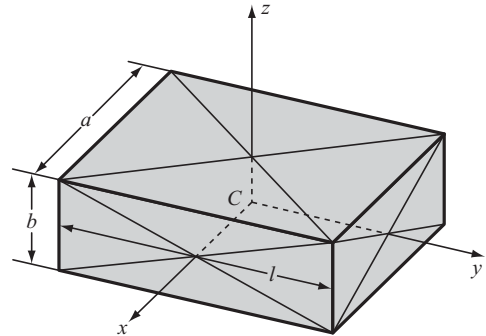


Fig. Ex. 8.8 (Solution)

$$I_{yy} = m \frac{a^2 + b^2}{3}$$

$$I_{zz} = m \frac{a^2 + l^2}{3}$$

and the products of inertia would not vanish due to lack of symmetry

$$I_{xy} = m \frac{al}{4} = I_{yx}$$

$$I_{xz} = m \frac{ab}{4} = I_{zx}$$

$$I_{yz} = m \frac{bl}{4} = I_{zy}$$

8.7 MOMENTS OF INERTIA OF THIN PLATES

Let us now consider the moments of inertia of the uniform flat plates of homogeneous material where the reference axes x and y are contained in the plane and z is normal to the plane of the plate. Consider a flat plate of uniform thickness t and mass density ρ as shown in Fig. 8.7.

$$\begin{aligned} I_{xx} &= \int_V (y^2 + z^2) \rho \, dv \\ \text{(mass)} & \\ &= \rho \int_V (y^2 + z^2) \, dV = \rho \int_A (y^2 + z^2) t \, dA \\ &= \rho t \int_A (y^2 + z^2) \, dA \end{aligned} \tag{8.36}$$

Hence $I_{xx} = \rho t I_{xx}$
 (mass) (area)

Similarly,

$$I_{yy} = \rho t I_{yy}$$

(mass) (area)

$$I_{zz} = \rho t I_{zz}$$

(mass) (area)

It can also be observed that, in the limit when the thickness of the flat plate $t \rightarrow 0$ and the density $\rho \rightarrow \infty$ such that $\rho t \rightarrow 1$, the mass moments of inertia reduce to the area moments of inertia. This is expected because the limiting process was designed to bring the z coordinate to zero, thus reducing the body to a mere area.

The statement that

$$I_{zz} = I_{xx} + I_{yy} \tag{8.37}$$

which was shown to be true for the area moments of inertia is true for the mass moments of inertia only if, the thickness t tends to zero. The relationship is, however, used for plates of small thickness with little error. For thicker plates and for cylindrical bodies, as for other bodies, the mass moments about the three reference axes are mutually independent and must be determined separately.

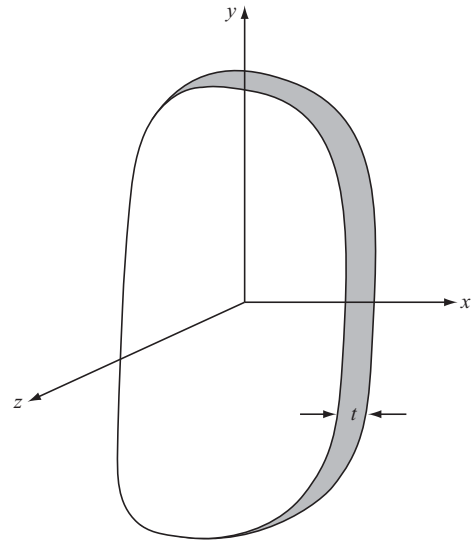


Fig. 8.7 A thin plate of uniform thickness

Example 8.9

A triangular metal sheet of uniform thickness 10 cm and uniform density 5000 kg/m³ is hinged on its base 6 m long as shown in Fig. Ex. 8.9. Determine the moment of inertia about the hinge. The other two sides are 5 m each.

Solution For a thin plate of thickness t and uniform density

$$\frac{I_{xx}}{(\text{mass})} = \rho t \frac{I_{xx}}{(\text{area})}$$

The problem is, therefore, reduced to that of finding the area moment of inertia. Consider an elementary strip of width dy at a height y above the x -axis. From the fact that

$$y = 0 \quad \text{for} \quad x = 3$$

and $y = 4 \quad \text{for} \quad x = 0$

it can be seen that

$$x = -0.75y + 3$$

The length of the strip is thus

$$2x = 6 - 1.5y$$

and the area of the strip is

$$2x \, dy = (6 - 1.5y) \, dy$$

whose moment of inertia is

$$dI_{xx} = (6 - 1.5y) y^2 \, dy = (6y^2 - 1.5y^3) \, dy$$

Therefore, the moment of inertia of the entire area is

$$\begin{aligned} I_{xx} &= \int_A dI_{xx} = \int_0^4 (6y^2 - 1.5y^3) \, dy \\ &= \left[2y^3 - \frac{3}{8}y^4 \right]_0^4 = 32 \, \text{m}^4 \end{aligned}$$

$$I_{xx} = 5000 \times 0.1 \times 32 = 16,000 \, \text{kg m}^2$$

(mass)

$$\text{Mass of the sheet} = \frac{6 \times 4}{2} \times 5000 \times 0.1 = 6000 \, \text{kg}$$

The radius of gyration is, therefore,

$$k = \sqrt{\frac{16,000}{6000}} = 1.63 \, \text{m}$$

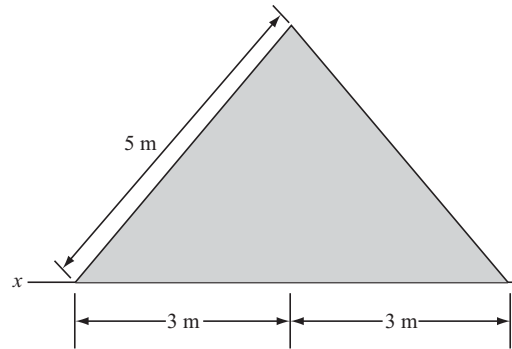


Fig. Ex. 8.9 (Solution)

8.8 PARALLEL-AXIS THEOREM (MASS MOMENT OF INERTIA)

The parallel-axis theorem for the mass moment of inertia states that the mass moment of inertia with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centre of mass plus the product of the mass and the square of the perpendicular distance between the axes. Mathematically,

$$I_{aa} = I_{cc} + s^2 m \tag{8.38}$$

where I_{aa} is the mass moment of inertia about an axis aa and I_{cc} is the mass moment of inertia about an axis parallel to aa and passing through the centre of mass C . The two axes are s apart and the mass of the body is m .

With reference to Fig. 8.8 where a thin strip of mass dm is taken parallel to either of the parallel axes and the vectors r_c and s are measured in a plane perpendicular to the parallel axes and passing through C

$$I_{aa} = \int r^2 dm$$

Substituting

$$\begin{aligned} r &= r_c + s \quad \text{or} \quad r^2 = (r_c + s)^2 = r_c^2 + 2r_c s + s^2 \\ &= r_c^2 + 2r_c s + s^2 \end{aligned}$$

$$I_{aa} = \int r_c^2 dm + \int 2s \cdot r_c dm + \int s^2 dm = I_{cc} + s^2 m$$

because the middle term vanishes by virtue of the definition of the centre of mass, i.e.,

$$\int 2s \cdot r_c dm = 2s \cdot \int r_c dm = 0$$

Similarly, the parallel axes theorem for the mass products of inertia states that

$$I_{xy} = I''_{xy} + s_1 s_2 m \quad (8.39)$$

(for any set of axes) (for a parallel set of axes at the centre of mass)

where s_1 and s_2 are the perpendicular distances from the centre of mass to the x and y axes respectively.

Example 8.10

Determine the mass moments of inertia for a hollow cylinder of radii R_1 and R_2 and axial length l about the longitudinal and transverse axes at the centre of mass.

Solution Consider a thin sliced ring element of radius r having an infinitesimal thickness dz and radial width dr at the distance z from the centre of mass C as shown in Fig. Ex. 8.10 (Solution). The mass of this element is

$$\rho 2\pi r dr dz$$

The mass moments of inertia of this ring is related to its corresponding area moments of inertia as

$$dI_{(\text{mass})} = dI_{(\text{area})} \times \pi t$$

Consequently,

$$\begin{aligned} dI_{zz} &= \rho 2\pi r^2 dr dz \\ &= 2\rho\pi r^3 dr dz \end{aligned}$$

The mass moment of inertia for the entire cylinder is obtained by double integration

$$I_{zz} = \int dI_{zz} = \int_{-l/2}^{l/2} \int_{R_1}^{R_2} 2\rho\pi r^3 dr dz$$

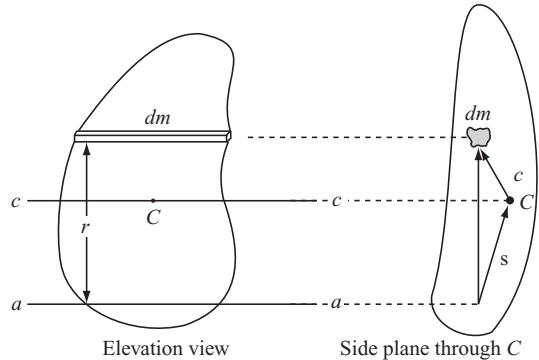


Fig. 8.8 Parallel axes for moment of inertia

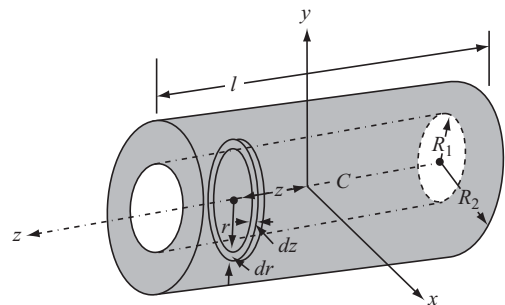


Fig. Ex. 8.10 (Solution)

$$\begin{aligned}
 &= \int_{-l/2}^{l/2} 2\rho\pi \left| \frac{r^4}{4} \right|_{R_1}^{R_2} dz \\
 &= \frac{\rho\pi}{2} \int_{-l/2}^{l/2} (R_2^4 - R_1^4) dz \\
 &= \frac{\rho\pi}{2} (R_2^4 - R_1^4) l
 \end{aligned}$$

Since the mass of the hollow cylinder is given by

$$m = \rho\pi (R_2^2 - R_1^2) l$$

the moment of inertia can be expressed as

$$I_{zz} = \frac{\rho\pi}{2} (R_2^4 - R_1^4) l \times \frac{m}{\rho\pi (R_2^2 - R_1^2) l}$$

or,
$$I_{zz} = \frac{1}{2} m (R_1^2 + R_2^2)$$

In order to determine the moments of inertia about the x and y axes, we again start with the moments of inertia of the ring about the x and y axes. In view of the symmetry about x and y axes

$$\begin{aligned}
 dI_{x'x'} &= dI_{y'y'} = dI_{zz}/2 \\
 &= \rho\pi r^3 dr dz
 \end{aligned}$$

about the axes at the centre of the ring. By the parallel-axis theorem, the moment of inertia of the ring about the axes through the centre of mass C is

$$dI_{xx} = dI_{yy} = \rho\pi r^3 dr dz + \rho 2\pi r dr dz \cdot z^2$$

For the hollow cylinder,

$$\begin{aligned}
 I_{xx} = I_{yy} &= \rho\pi \int_{-l/2}^{l/2} \int_{R_1}^{R_2} (r^3 dr dz + 2r z^2 dr dz) \\
 &= \rho\pi \int_{-l/2}^{l/2} \left[\frac{R_2^4 - R_1^4}{4} + (R_2^2 - R_1^2) z^2 \right] dz \\
 &= \rho\pi \frac{(R_2^4 - R_1^4)}{4} l + \rho\pi \frac{R_2^2 - R_1^2}{12} l^3
 \end{aligned}$$

In terms of the mass of the hollow cylinder,

$$m = \rho\pi (R_1^2 + R_2^2) l$$

$$I_{xx} = I_{yy} = \frac{1}{4} m (R_1^2 + R_2^2) + \frac{ml^2}{12} = \frac{1}{12} m (l^2 + 3(R_1^2 + R_2^2))$$

Let us examine the results for a *solid cylinder* where

$$R_1 = 0 \quad \text{and} \quad R_2 = R$$

$$I_{zz} = \frac{mR^2}{2}; \quad I_{xx} = I_{yy} = \frac{1}{12} m (l^2 + 3R^2)$$

In particular, for a long or *slender rod* where R is very small and

$$R \ll l$$

$$I_{zz} \approx 0; \quad I_{xx} = I_{yy} = \frac{1}{12} ml^2$$

On the other hand, the moment of inertia for a *thin cylindrical shell* is determined by considering mean radius

$$R = \frac{R_1 + R_2}{2}$$

and

$$(R_2 - R_1) \ll R$$

such that

$$R_1^2 + R_2^2 \approx 2R^2$$

$$I_{zz} = \frac{1}{2} m 2R^2 = mR^2$$

$$I_{xx} = I_{yy} = \frac{1}{12} (l^2 + 3 \times 2R^2) = \frac{1}{12} m (l^2 + 6R^2)$$

8.9 MOMENT OF INERTIA OF COMPOSITE BODIES

The moment of inertia of bodies composed of simpler homogeneous bodies can be determined from the knowledge of the corresponding values for the components about the same axes. The steps for computing the moment of inertia for a composite body are, therefore, as follows:

1. decompose the body into its simpler components, positive or negative depending upon the fact that the mass is additive or subtractive
2. look up or recall the moment of inertia of the component bodies about their centroidal axes
3. determine the moment of inertia of the component bodies about the desired axes by the application of the parallel-axis theorem
4. add algebraically the moment of inertia of the component bodies to obtain the moment of inertia of the composite body

$$I = I_1 + I_2 + I_3 + \dots \quad (8.40)$$

The moments of inertia of some simple homogeneous bodies are tabulated in Appendices 2 and 3 for ready reference. It may be noticed that the moments of inertia of these bodies can be deduced from the values for a rectangular prism, hollow circular cylinder and hollow sphere which must be remembered.

It may be added that *the radius of gyration of a composite body cannot be obtained by adding the radii of gyration of the component bodies*

$$k \neq k_1 + k_2 + k_3 + \dots$$

but it may be determined from the moment of inertia of the composite body, i.e.,

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{I_1 + I_2 + I_3 + \dots}{m_1 + m_2 + m_3 + \dots}} \quad (8.41)$$

For example, the composite body shown in Fig. 8.9 consists of a mass m given by

$$m = m_1 + m_2 - m_3$$

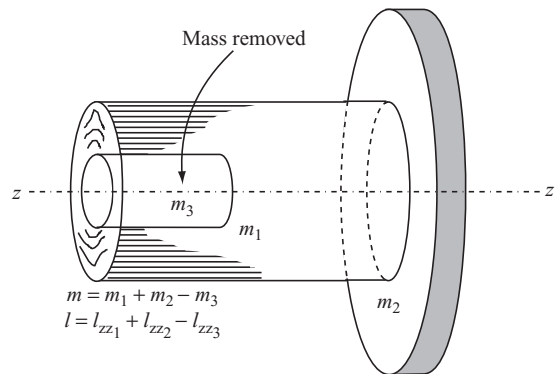


Fig. 8.9 A composite body

Example 8.11

A clock pendulum consists of a slender rod and a circular disc with a hole in it as shown in Fig. Ex. 8.11. The rod has a density of 7000 kg/m^3 and cross sectional area of 50 mm^2 and the disc has a density of 8000 kg/m^3 and a thickness of 5 mm . Compute the moment of inertia of the pendulum about an axis of rotation perpendicular to the plane of oscillations.

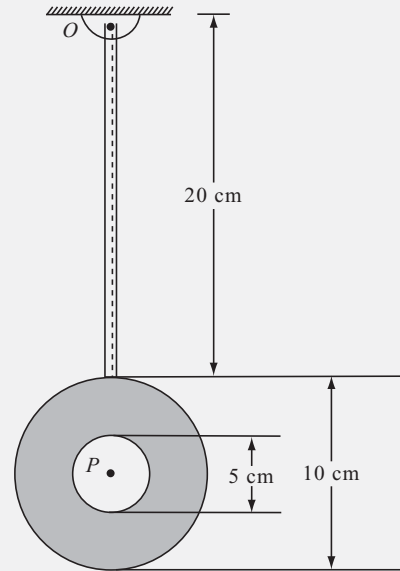


Fig. Ex. 8.11

Solution Let us consider the pendulum to consist of three composite parts:

- (i) a slender rod 20 cm long plus
- (ii) a solid disc 10 cm diameter and minus
- (iii) a solid disc 5 cm diameter

for the sake of computing its moment of inertia.

For the slender rod,

$$I_0 = m \frac{l^2}{3} = (7000 \times 50 \times 10^{-6} \times 0.2^2) / 3$$

$$= 9.33 \times 10^{-4} \text{ kg m}^2$$

For the solid disc of 10 cm diameter,

$$m = 8000 \times 5 \times 10^{-3} \times \pi \times 0.1^2 = 1.257 \text{ kg.}$$

$$I_0 = \frac{1}{2} m R^2 + m (OP)^2$$

$$= \frac{1}{2} \times 1.257 \times 0.1^2 + 1.257 \times 0.25^2$$

$$= 0.085 \text{ kg m}^2$$

For the solid disc 5 cm diameter

$$m = 8000 \times 5 \times 10^{-3} \times \pi \times (0.05)^2 = 0.314 \text{ kg}$$

$$I_0 = \frac{1}{2} m R^2 + m (OP)^2$$

$$= \frac{1}{2} \times 0.314 \times (0.05)^2 + 0.314 \times 0.25^2$$

$$= 0.02 \text{ kg m}^2$$

The moment of inertia of the pendulum about the axis of rotation through O is

$$I_0 = 9.33 \times 10^{-4} + 0.085 - 0.02 = 0.066 \text{ kg m}^2$$

EXPERIMENT E8

MOMENT OF INERTIA OF A BODY

OBJECTIVE TO DETERMINE THE MOMENT OF INERTIA OF A STEPPED PULLEY OR A FLYWHEEL EXPERIMENTALLY.

Apparatus A stepped pulley or flywheel mounted with its axle on bearings as shown in Fig. E8.1. Provision for a cord to be wound round it, to pass over a frictionless fixed pulley, to hold a known mass and to allow its fall through a known height.

Background Information The equations expressing the moment of inertia in terms of the measurable parameters are obtained by employing either the energy principle or Euler's equation of motion.

In terms of the energy principle, a mass m falling through a vertical height h releases its potential energy mgh which is transformed to the following: (a) the translational kinetic energy $1/2 mv^2$ of the falling mass m ; (b) the rotational kinetic energy $1/2 I\omega^2$ of the rotating body with moment of inertia I about its axis of rotation; and (c) the frictional energy loss in the bearings, nF where n is the number of revolutions during the descent of the mass and F is the frictional energy in $J/\text{revolution}$. If the flywheel takes N revolutions to dissipate the entire rotational energy $1/2 I\omega^2$ from the instant the mass reaches the ground,

$$nF = 1/2 I\omega^2$$

whence
$$nF = \frac{n}{N} \cdot \frac{1}{2} I\omega^2$$

By energy balance,

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 + \frac{n}{N} \cdot \frac{1}{2} I\omega^2 \tag{E8.1}$$

Substituting $v = r\omega$ and rearranging the terms,

$$I = \frac{2mgh - mr^2\omega^2}{\omega^2(1 + n/N)} \tag{E8.2}$$

If n is negligible compared to N , the frictional effect would be negligible and the moment of inertia would be given by

$$I = \frac{2mg - mr^2\omega^2}{\omega^2} \tag{E8.3}$$

It may be noted that ω , the rotational speed when the mass has descended to the ground is obtained from

$$\omega = \alpha t$$

as the body starts from rest at the reference time $t = 0$. The angular acceleration α of the rotating body is related to the linear acceleration a of the falling mass as

$$a = r\alpha$$

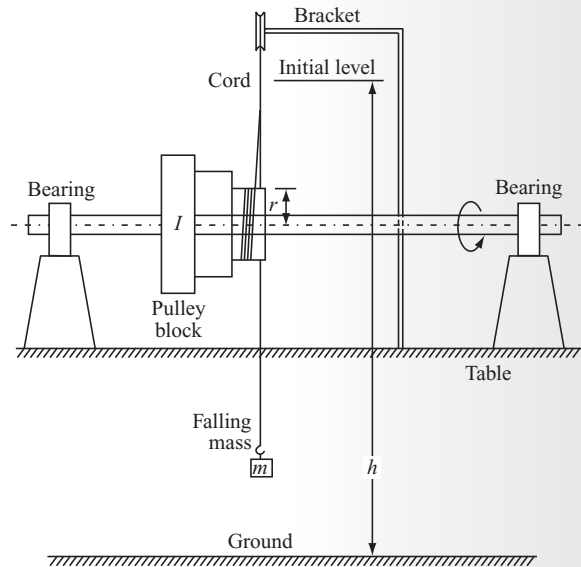


Fig. E8.1 Apparatus for determining moment of inertia

Results and Points for Discussion

Obtain the average moment of inertia from the set of observations and calculations.

Measure the dimensions of the body and estimate its mass. From a knowledge of its moment of inertia and mass, calculate the radius of gyration.

Comment on the accuracy in the measurements of time, length and mass in relation to the accuracy in the measurement of the moment of inertia. Would you recommend the use of a better stop watch, a more accurate scale or a better weighing machine in order to improve the accuracy of the value of the moment of inertia?

From a knowledge of the dimensions of the body, obtain its moment of inertia theoretically and compare it with the experimental result. Account for the difference.

What is the role of the moment of inertia of a body in its rotational motion? Explain why a flywheel should have a large moment of inertia.

Can you suggest some alternative methods for determining the moment of inertia of a body about a given axis if the body is cylindrical or irregular in shape. Examine the method of rolling a body down an inclined plane and the method of oscillating a body about a mean position.

Example 8.12

A spoked flywheel as shown in Fig. Ex. 8.12 consists of four spokes each 0.9 m long and of mass 50 kg which are cast with a rim of inner and outer radii 1 m and 1.5 m respectively and having a mass of 5000 kg. The shaft at the centre of the wheel has a diameter of 0.2 m and a mass of 1500 kg. Calculate the moment of inertia of the flywheel about the axis of rotation and also its radius of gyration.

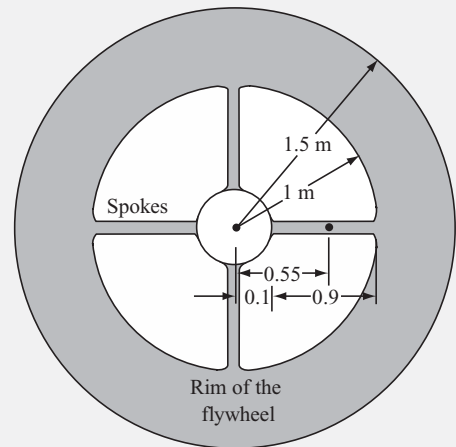


Fig. Ex. 8.12

Solution The moment of inertia of flywheel is the sum of the moments of inertia of the central shaft, spokes and rim. Let the centre of the flywheel be called O.

The greatest contribution is by the rim as is obvious from the figure and the values of the masses. Taking the components one by one,

$$I_{(\text{shaft})} = \frac{1}{2} m r^2 = \frac{1}{2} \times 1500 \times 0.1^2 = 7.50 \text{ kg m}^2$$

The moment of inertia of one spoke about an axis passing through O and parallel to the axis of rotation is

$$I_{(\text{spoke})} = \frac{1}{12} m l^2 = \frac{50 \times 0.9^2}{12} = 3.375 \text{ kg m}^2$$

and by the parallel-axis theorem.

$$I_{oo} = 3.375 + 50 \times 0.55^2 = 18.85 \text{ kg m}^2$$

(spoke)

For four spokes

$$I_{oo} = 18.85 \times 4 = 75.40 \text{ kg m}^2$$

(fourspoke)

The moment of inertia of the rim is

$$I_{oo} = \frac{1}{2} m (R_1^2 + R_2^2)$$

(rim)

$$= \frac{1}{2} \times 5000 \times (1^2 + 1.5^2) = 8125 \text{ kg m}^2$$

The moment of inertia of the flywheel is, therefore,

$$7.50 + 75.40 + 8125 = 8207.9 \text{ kg m}^2$$

It can be seen that the rim alone contributes to the moment of inertia to the extent of

$$\frac{8125}{8207.9} \times 100 = 98.99$$

The radius of gyration of the flywheel can be determined from

$$I_{oo} = mk^2$$

$$k^2 = \frac{8207.9}{4 \times 50 + 5000 + 1500} = 1.225 \text{ m}^2$$

whence
and

$$k = 1.107 \text{ m}$$

Example 8.13

A three-bladed rotor of a helicopter consists of a 0.2 m diameter shaft, 1 m long and 1.5 m long radial blades. The shaft is made of steel, specific gravity 7, and the blades have a mass of 20 kg/m. Assuming the blades as uniform rods, estimate the polar moment of inertia of the rotor about the axis of rotation.

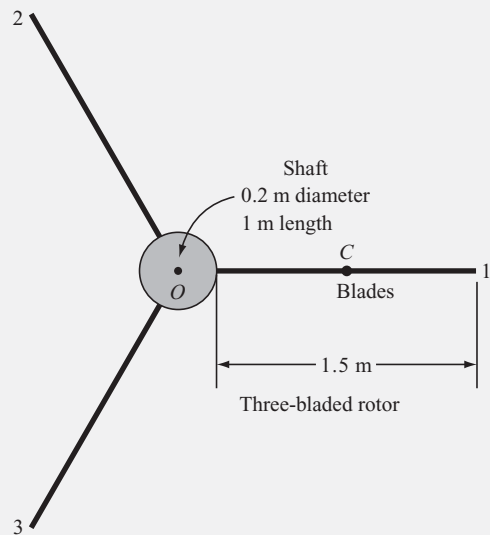


Fig. Ex 8.13

Solution The polar moment of inertia of the shaft alone is

$$\begin{aligned} I_{oo} &= \frac{mR^2}{2} \\ \text{(shaft)} & \\ &= \frac{\pi \times 0.2^2}{4} \times 1 \times 7 \times 1000 \times \frac{0.1^2}{2} \\ &= 1.1 \text{ kg m}^2 \end{aligned}$$

The moment of inertia of one uniform-rod blade about an axis through C parallel to the axis of rotation is

$$\begin{aligned} I_{cc} &= \frac{ml^2}{12} \\ \text{(blade)} & \\ &= (1.5 \times 20) \times \frac{1.5^2}{12} = 5.625 \text{ kg m}^2 \end{aligned}$$

By the parallel-axis theorem, for each blade

$$\begin{aligned} I_{oo} &= 5.625 + (1.5 \times 20) \times (0.75 + 0.1)^2 \\ \text{(blade)} & \\ &= 27.3 \text{ kg m}^2 \end{aligned}$$

For the three blades of the rotor,

$$I_{oo} = 27.3 \times 3 = 81.9 \text{ kg m}^2$$

(three blades)

For the complete rotor, then

$$I_{oo} = 1.1 + 81.9 = 83 \text{ kg m}^2$$

Example 8.14

Determine the moment of inertia of a hollow sphere of radii R_1 and R_2 and hence estimate the moments of inertia of

- (a) a solid sphere of radius 0.5 m and mass 50 kg
- (b) a thin spherical shell of mean radius 0.5 m and mass 20 kg.

Solution Consider a ring element of radius r , radial width dr and axial width dz at a distance z from the centre of the sphere. The mass of this element is

$$\rho \cdot 2\pi r \cdot dr dz$$

and its moment of inertia about the z -axis is

$$dI_{zz} = \rho 2\pi r^3 dr dz \quad \text{(i)}$$

(mass)

The moment of inertia of the hollow sphere is obtained by integration over the domain:

$$\left[\begin{array}{l} r \text{ varying from } 0 \text{ to } \sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } R_1 \text{ to } R_2 \text{ and from } -R_1 \text{ to } -R_2 \end{array} \right.$$

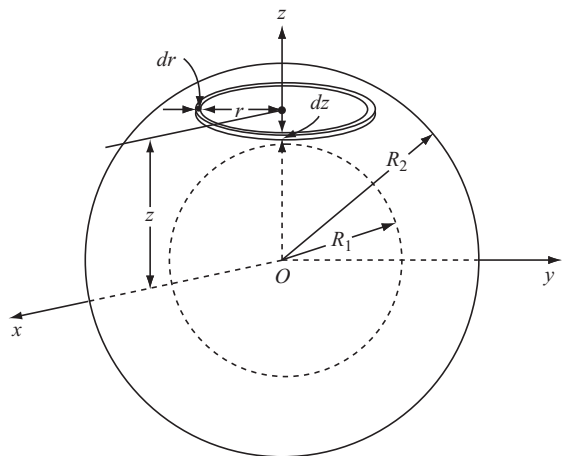


Fig. Ex. 8.14 (Solution)

and

$$\begin{aligned}
 & \left[\begin{array}{l} r \text{ varying from } \sqrt{(R_1^2 - z^2)} \text{ to } \sqrt{(R_2^2 - z^2)} \\ z \text{ varying from } -R_1 \text{ to } R_1 \end{array} \right. \\
 I_{zz} = & \int_{R_1}^{R_2} \int_0^{\sqrt{R_2^2 - z^2}} \rho \cdot 2\pi r^3 \, dr \, dz \\
 & + \int_{R_1}^{-R_2} \int_0^{\sqrt{R_2^2 - z^2}} \rho \cdot 2\pi r^3 \, dr \, dz \\
 & + \int_{-R_1}^{R_1} \int_{\sqrt{R_1^2 - z^2}}^{-\sqrt{R_2^2 - z^2}} \rho \cdot 2\pi r^3 \, dr \, dz = I_1 + I_2 + I_3
 \end{aligned}$$

The sum of the first two integrals is

$$\begin{aligned}
 & 2 \cdot \int_{R_1}^{R_2} \rho \cdot 2\pi \frac{(R_2^2 - z^2)^2}{4} \, dz \\
 = & \rho\pi \left[R_2^4 z + \frac{z^5}{5} - 2R_2^2 \frac{z^3}{3} \right]_{R_1}^{R_2} \\
 = & \rho \cdot \pi \left(R_2^5 + \frac{R_2^5}{5} - \frac{2}{3} R_2^5 - R_2^4 R_1 - \frac{R_1^5}{5} + \frac{2}{3} R_2^2 R_1^3 \right) \\
 = & \rho \cdot \pi \left(\frac{8}{15} R_2^5 - R_2^4 R_1 - \frac{1}{5} R_1^5 + \frac{2}{3} R_2^2 R_1^3 \right)
 \end{aligned}$$

The third integral is now evaluated as

$$\begin{aligned}
 & \int_{-R_1}^{R_1} \rho \cdot 2\pi \left[(R_2^2 - z^2)^2 - (R_1^2 - z^2)^2 \right] / 4 \, dz \\
 = & \frac{\rho\pi}{2} \int_{-R_1}^{R_1} (R_2^4 - R_1^4 + 2(R_1^2 - R_2^2)z^2) \, dz \\
 = & \frac{\rho\pi}{2} \left[(R_2^4 - R_1^4) + 2(R_1^2 - R_2^2) \frac{z^3}{3} \right]_{-R_1}^{R_1} \\
 = & \rho\pi \left(R_2^4 R_1 - \frac{2}{3} R_2^2 R_1^3 - \frac{1}{3} R_1^5 \right)
 \end{aligned}$$

Finally, adding the three integrals,

$$I_{zz} = \rho\pi \left(\frac{8}{15} R_2^5 - R_2^4 R_1 - \frac{1}{5} R_1^5 + \frac{2}{3} R_2^2 R_1^3 + R_2^4 R_1 - \frac{2}{3} R_2^2 R_1^3 - \frac{1}{3} R_1^5 \right)$$

$$= \frac{8}{15} \rho \pi \cdot (R_2^5 - R_1^5) \quad (\text{ii})$$

Since the mass of the hollow sphere is

$$m = \frac{4}{3} \pi \rho (R_2^3 - R_1^3)$$

the moment of inertia can be expressed as

$$I_{zz} \text{ (hollow sphere)} = \frac{2}{5} m \frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)} \quad (\text{iii})$$

By symmetry about the axes and by interchangeability of the axes,

$$I_{xx} = I_{yy} = I_{zz}$$

Let us now observe two special cases as a matter of interest:

(a) Solid Sphere

Setting R_1 to zero and $R_2 = R$

$$I_{zz} \text{ (solid sphere)} = \frac{2}{5} m R^2 \quad (\text{iv})$$

In fact, the moment of inertia of a solid sphere could be determined *ab-initio* by considering elementary rings as for hollow sphere but with simple limits of integration, i.e.

$$r \text{ varying from } 0 \text{ to } \sqrt{R^2 - z^2}$$

and

$$z \text{ varying from } -R \text{ to } R$$

In that case,

$$\begin{aligned} I_{zz} \text{ (sphere)} &= \int_{-R}^R \int_0^{\sqrt{R^2 - z^2}} \rho 2\pi r^3 dr dz \\ &= \int_{-R}^R \rho 2\pi \frac{(R^2 - z^2)^2}{4} dz \\ &= \frac{\rho \pi}{2} \int_{-R}^R (R^4 + z^4 - 2R^2 z^2) dz \\ &= \frac{8}{15} \rho \pi R^2 \end{aligned}$$

In terms of the mass of a sphere,

$$m = \frac{4}{3} \rho \pi R^3$$

$$I_{zz} \text{ (sphere)} = \frac{2}{5} m R^2 \quad (\text{v})$$

For the given sphere;

$$\begin{aligned} R &= 0.5 \text{ m}; \quad m = 50 \text{ kg} \\ I_{zz} &= \frac{2}{5} \times 50 \times 0.5^2 = 5 \text{ kg m}^2 \end{aligned}$$

By symmetry and interchangeability of the axes,

$$I_{xx} = I_{yy} = I_{zz} = 5 \text{ kg m}^2$$

(b) *Thin Spherical Shell*

A thin spherical shell is characterised by a mean radius

$$R = \frac{R_1 + R_2}{2}$$

and the shell thickness

$$\Delta R = (R_2 - R_1) \ll R$$

or

$$\Delta R/R_1 \ll 1 \quad \text{and} \quad R_1 \rightarrow R$$

$$\begin{aligned} I_{zz} \text{ (thin shell)} &= \frac{2}{3} m R_1^2 \times \left[\frac{\left(\frac{R_2}{R_1}\right)^5 - 1}{\left(\frac{R_2}{R_1}\right)^3 - 1} \right] \\ &= \frac{2}{5} m R_1^2 \times \left[\frac{\left(1 + \frac{\Delta R}{R_1}\right)^5 - 1}{\left(1 + \frac{\Delta R}{R_1}\right)^3 - 1} \right] \\ &= \frac{2}{5} m R_1^2 \times \left[\frac{5 \Delta R/R_1}{3 \Delta R/R_1} \right] \\ &= \frac{2}{3} m R_1^2 = \frac{2}{3} m R^2 \end{aligned}$$

For the given shell:

$$R = 0.5 \text{ m} \quad \text{and} \quad m = 20 \text{ kg}$$

$$I_{zz} = \frac{2}{3} \times 20 \times 0.5^2 = 3.33 \text{ kg m}^2$$

By symmetry and interchangeability of the axis,

$$I_{xx} = I_{yy} = I_{zz} = 3.33 \text{ kg m}^2$$

Example 8.15

The moment of inertia of a ‘flywheel and its shaft’ assembly is determined by letting it roll down an incline consisting of two parallel tracks, starting from rest. The incline is kept very low to ensure the shaft to roll without slipping. The experiment is repeated for two different heights in order to eliminate the effect of the frictional resistance.

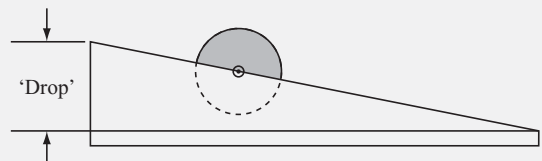


Fig. Ex. 8.15

In a test, the assembly of mass 20 kg rolled 2.0 m down the incline, dropping by 5 cm in 50 seconds. In the second test, it rolled 2.0 m down the incline, dropping by 10 cm in 30 seconds. Estimate the moment of inertia of the assembly, taking the shaft diameter of 5 cm.

Solution Shaft diameter = 0.05 m.

The assembly rolls under constant acceleration; average speed of roll must be half of the maximum speed at the end of the roll.

For test 1: Average speed of roll = $2/50 = 0.04$ m/s
 Maximum speed at end of roll = 0.08 m/s
 Angular velocity at end of roll = $\frac{0.08}{0.025} = 3.2$ rad/s

For test 2: Average speed of roll = $2/30 = 0.067$ m/s
 Maximum speed at end of roll = 0.133 m/s
 Angular velocity at end of roll = $\frac{0.133}{0.025} = 5.33$ rad/s

Energy balance during a roll-down motion requires

$$\text{Loss of potential energy} - \text{Work against frictional resistance } R = \text{Gain of kinetic energy}$$

Hence, for each roll down motion,

$$mgh - R = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2$$

For the two tests,

$$mg \times 0.05 - R = \frac{1}{2} I \times 3.2^2 + \frac{1}{2} \times 20 \times 0.08^2 \tag{i}$$

$$mg \times 0.10 - R = \frac{1}{2} I \times 5.33^2 + \frac{1}{2} \times 20 \times 0.133^2 \tag{ii}$$

By subtraction, R is eliminated and

$$20 \times g(0.1 - 0.05) = \frac{1}{2} I (5.33^2 - 3.2^2) + \frac{1}{2} \times 20 (0.133^2 - 0.08^2)$$

$$9.81 = 9.1 I + 0.113$$

$$I = 0.99 \text{ kg m}^2 \approx 1 \text{ kg m}^2$$

Concept Review Questions

Bring out briefly the conceptual mistakes, if any, in the following statements:

- (a) The area moment of inertia and mass moment of inertia are two entirely unrelated concepts.
- (b) The products of inertia can be obtained by multiplying the moments of inertia about the chosen axes.
- (c) The product of inertia vanishes for an axis of symmetry involved in the product but zero product of inertia provides no indication for an axis of symmetry.
- (d) The radius of gyration for an area can be evaluated if the corresponding moment of inertia and the location of the centroid are known.
- (e) Moments of inertia about non-centroidal axes may be larger or smaller than the centroidal values for parallel axes.
- (a) Comment on the similarity of the concepts of mass and moment of inertia.
- (b) The mass of a body remains the same irrespective of its location or inclination of the axis of rotation but the value of the moment of inertia about an axis of rotation depends upon
 - (i) the position and direction of the axis of rotation
 - (ii) the distribution of the mass of the body with respect to that axis.

Explain this apparent dissimilarity of the concepts of mass and moment of inertia.

Which is more in the following?

- (a) the moment of inertia of a sphere about its diameter or about a tangent.
- (b) the moment of inertia of an equilateral triangle about a side or the moment of inertia of a square of the same area about its side.

Fill in the blanks:

- (a) The principal moments of inertia of area are the _____ values which occur when the product of inertia is _____. The axes along which the principal inertia values occur are called _____. These axes _____ the axes of symmetry at the point under consideration.
- (b) The parallel axis theorem is valid for the transference of the inertia values from one axis to another axis parallel to it if _____.
- (c) Angles on the Mohr's circle representation are _____ the angles on the section or body itself and the centre of the circle represents the _____.
- (d) The sum of the moments of inertia about the axes lying on a section of the body must be the polar moment of inertia, i.e., about an axis normal to those axes for bodies _____.
- (e) The relationship between the area moment of inertia and the mass moment of inertia is _____.
- (f) The sum of the moment of inertia values about a point, regardless of its orientation, is always _____.
- (g) In order that an axis through a point in a rigid body be a principal axis, the _____.

What should be the length to radius ratio, i.e., l/R of a solid cylinder such that the moments of

inertia about the longitudinal and transverse axes are all equal.

$$s \quad l/R = \sqrt{3}$$

From the fact that the moments of inertia for a hollow right circular cylinder are

$$I_{xx} = I_{yy} = \frac{1}{12} m(l^2 + 3(R_1^2 + R_2^2))$$

and

$$I_{zz} = m(R_1^2 + R_2^2)/2$$

Obtain the expression for the moment of inertia of

- (a) a solid cylinder of radius R
- (b) a slender rod of length l
- (c) a thin cylindrical shell of mean radius R
- (d) a thin disc of radius R
- (e) a thin ring of radii R_1 and R_2

You are given two spheres of the same mass, size and appearance but one of them is hollow at the centre and the other is solid throughout. How will you find out which is hollow and which is solid? (*Hint: Consider their moments of inertia and try relating their accelerations when rolled down the same incline. The solid sphere accelerates faster and hence reaches down the incline first.*)

- 8 What is the relative significance of the moment of inertia and the radius of gyration? With what intention are these defined? Under what circumstances is it desirable to have small and large values of the moment of inertia?

Would you imagine that the moment of inertia of the earth around its own axis is a negligible fraction of its moment of inertia about the axis of rotation around the sun?

Take the mean radius of the earth as 6371 km and the mean radius of rotation around the sun as 149.7×10^6 km.

s Yes, it is true

Tutorial Problems

- 8 Determine the moments of inertia of an elliptical disc of mass m and the semi-major and semi-minor axes a and b respectively.

$$\left[s \frac{1}{4} ma^2, \frac{1}{4} mb^2, \frac{1}{4} m(a^2 + b^2) \right]$$

- 8 An isosceles triangle of base b and height h is such that the moment of inertia about the base equals the moment of inertia about a perpendicular axis through the vertex. Determine the b to h ratio. $s \ b = 2h$

- 8 Show that the moment of inertia about a centroidal axis parallel to a side for a cube of mass m is

$$I = \frac{1}{6} ma^2$$

where a is the length of a side.

- 8 Determine the moments of inertia about z axis of a right circular cone of mass m , base radius R and height h , as shown in Fig. Prob. 8.4.

$$\left[s \ I_{zz} = \frac{3}{10} mR^2 \right]$$

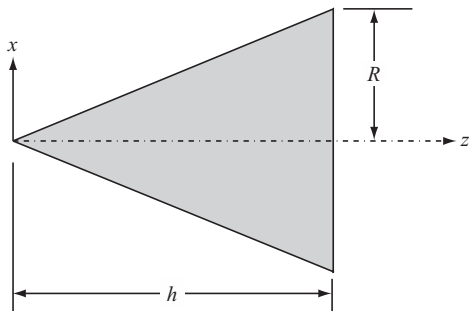


Fig. Prob. 8.4

- 8 Determine the moment of inertia of a hollow sphere of radii R_1 and R_2 and determine the radius of gyration of a solid sphere.

$$\left[s \ 2/5 m \left(\frac{R_2^5 - R_1^5}{R^3 - R^3} \right); 0.63 R \right]$$

- 8 Determine the moments of inertia of an ellipsoid of revolution of mass m , semi-major axis a and semi-minor axis b .

$$\left[s \ \frac{2}{5} mb^2, \frac{2}{5} ma^2, \frac{2}{5} m (a^2 + b^2) \right]$$

- 8 The moment of a thin circular plate of mass 400 kg about an axis AA is 109 kg m^2 . Determine (a) the moment of inertia about an axis BB , (b) the moment of inertia about an axis CC , (c) the radius of the plate, (d) the polar moment

of inertia for pole O , and (e) moment of inertia about an axis DD (Ref. Fig. Prob. 8.7).

$$s \ 265 \text{ kg m}^2, 73 \text{ kg m}^2, 0.3 \text{ m}, 18 \text{ kg m}^2 \text{ and } 109 \text{ kg m}^2$$

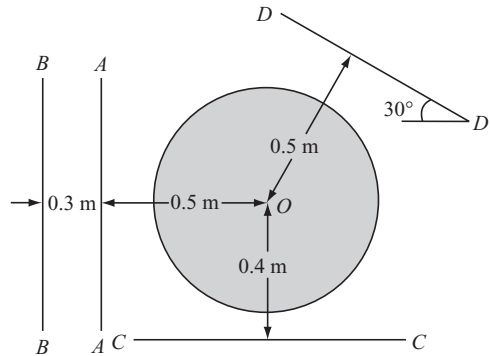


Fig. Prob. 8.7

- 8 8 Calculate the radius of gyration of a 30 cm long aluminum rod pivoted (a) at one end (b) at its mid-span (c) at a point quarter length from one end.

$$s \ 17.3 \text{ cm}, 8.66 \text{ cm and } 11.46 \text{ cm}$$

- 8 A semicircle of radius r has been cut of a circle of radius R as shown in Fig. Prob. 8.9. Calculate the polar moment of inertia about O for the resulting section.

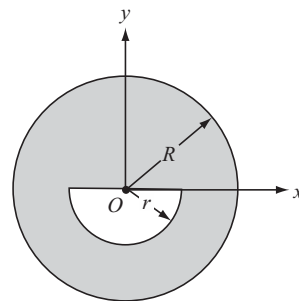


Fig. Prob. 8.9

- 8 A T-section is $15 \text{ cm} \times 15 \text{ cm} \times 2 \text{ cm}$ as shown in Fig. Prob. 8.10. Calculate the moment of inertia of the section about an axis parallel to the base of the T and passing through its centroid.

$$s \ 1160 \text{ cm}^4$$

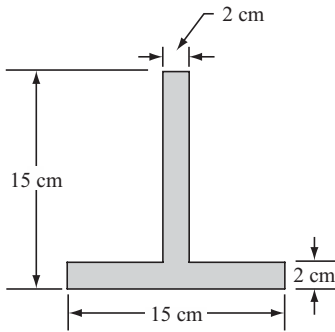


Fig. Prob. 8.10

- 8 Find the moment of inertia of a channel section shown in Fig. Prob. 8.11 about the centroidal axes. s 4558 and 760 cm⁴

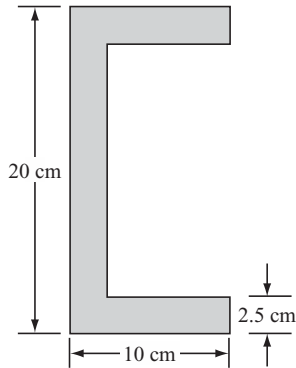


Fig. Prob. 8.11

- 8 The cross-section of a cast iron beam is shown in Fig. Prob. 8.12. Determine the moments of inertia about the centroidal axes. s 2885 and 340 cm⁴

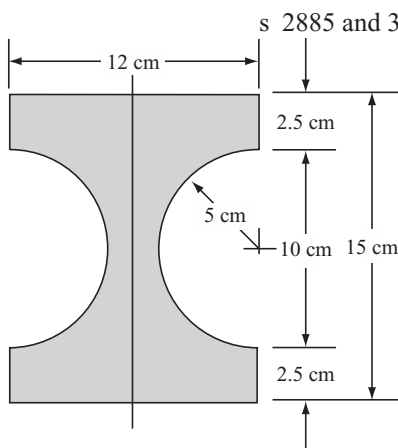


Fig. Prob. 8.12

- 8 Calculate the moment of inertia of a cast iron pulley with respect to its axis of rotation. Mass density of cast iron is 7200 kg/m³.

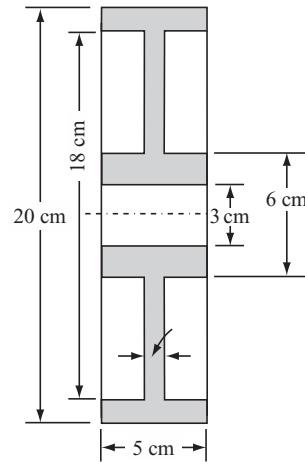


Fig. Prob. 8.13

- 8 A spherical bob of radius R and mass m_b is attached to a slender rod of length l and mass m_r . Calculate the moment of inertia of the assembly about the axis of rotation.

$$\left[s \frac{1}{3} m_r l^2 + \frac{2}{5} m_b R^2 + (l + R/2)^2 m_b \right]$$

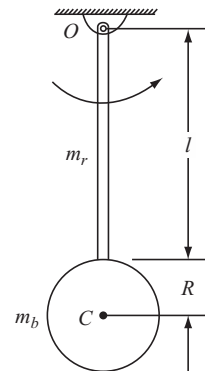


Fig. Prob. 8.14

- 8 A flywheel consist of a 1 m diameter plate 10 cm thick with four holes, each 20 cm in diameter cut at a pitch circle diameter of 40 cm symmetrically. Compute the mass moment of inertia of the flywheel about the axis of rotation. The material of the flywheel is cast iron with specific gravity 7.5. s. 69.4 kg m²

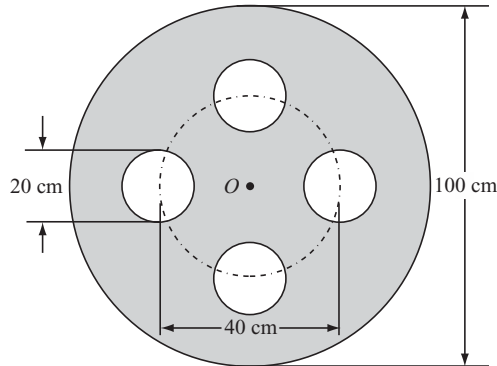


Fig. Prob. 8.15

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

The moment of inertia of a body is

- (a) the moment of its inertia
- (b) the rotational moment acting on the body
- (c) the rotational analogue of mass
- (d) the inertial moment acting on the body

The moment of inertia of a rectangular lamina of sides b and h about its neutral axis parallel to the side b is given by

- (a) $bh^3/12$
- (b) $bh^3/36$
- (c) $hb^3/12$
- (d) $hb^3/36$

The second moment of a plane area about any axis as compared to its second moment about the neutral axis

- (a) is always more
- (b) is always less

- (c) is sometimes more or less depending upon some factors
- (d) is equal

The moments of inertia of a solid circular section of radius r and of a hollow circular section of radii r and R , each about the diametral lines, are equal. Then,

- (a) $R = 2r$
- (b) $R = \sqrt{2}r$
- (c) $R^2 = \sqrt{2}r^2$
- (d) $R = 4r$

A rectangle is divided into two smaller rectangles by drawing the neutral axis parallel to its smaller sides. If the moments of inertia of the smaller rectangles of areas A_1 and A_2 about the neutral axis are I_1 and I_2 , the moment of inertia of the total area of the given rectangle about the neutral axis would be

- (a) $I_1 + I_2$
- (b) $I_1 - I_2$
- (c) $\sqrt{I_1^2 + I_2^2}$
- (d) $|I_1| + |I_2|$

Answers to Multiple-Choice Questions

- 1 (c) 2 (a) 3 (a) 4 (c) 5 (a)

Dynamics of a Rigid Body



9

9.1 INTRODUCTION

A rigid body may be subjected to external forces and moments and kinematic parameters such as velocity and acceleration may be predicted. Conversely, it may be required to evaluate the external action necessary for a desired set of kinematic conditions. This can be achieved by applying Newton's law of motion for the linear motion of the centre of mass of the rigid body and Euler's equation for the rotation of the rigid body. The purpose of this chapter is to present the methodology of applying the laws under different circumstances.

It is chosen, for the sake of simplicity of understanding, to study the dynamics of rigid bodies in steps, i.e., pure translation, fixed-axis rotation, plane motion and finally, space motion. The general form of the Euler's equation obtained for space motion is shown to contract for the special cases of translation, fixed-axis rotation, etc. There is no difficulty, therefore, if it is desired to reverse the approach, i.e., to proceed from general to particular cases.

A rigid-body model is not too hypothetical a model for many engineering situations. Most engineering materials maintain their shape and size and can be considered to be undeformed during their overall motions. It may also be appreciated that most engineering devices can be modelled as undergoing plane motion. More stress is, therefore, laid on the plane motion of rigid bodies than on space motion.

9.2 TRANSLATION OF A RIGID BODY

Recalling the fact that the acceleration and velocity of each element on a rigid body must be the same in a pure translation, the translational motion must be governed by the Newton's law,

$$= m \tag{9.1}$$

where ΣF is the net external force acting on the body and a is the acceleration of any point on the body. The net external force, sometimes written as ΣF , must be the vector sum of all the external forces acting on the body

$$\text{or } \Sigma F = F_1 + F_2 + F_3 + \dots$$

as also shown in Fig. 9.1.

There being no tendency of rotation, the angular velocity and the angular acceleration remain zero. In other words, the relative velocity of any point on the body referred to the centre of mass is zero and the angular momentum of the entire body with respect to the centre of mass is zero. It follows from the Euler's equation that the summation of moments about the centre of mass, C must also vanish:

$$\Sigma M_c = 0 \tag{9.2}$$

These two relations govern the general translatory motion of a rigid body. If the body undergoes a plane motion, the parallel planes of motion being parallel to the $x-y$ plane, the component equations governing the motion reduce to

$$F_x = ma_x$$

$$F_y = ma_y$$

where a_x and a_y are rectangular components of acceleration of the centre of mass, as also of any other point on the body and F_x and F_y are the rectangular components of the net force acting on the body. Also, the moment about the z -axis passing through the centre of mass must vanish:

$$M_z = 0$$

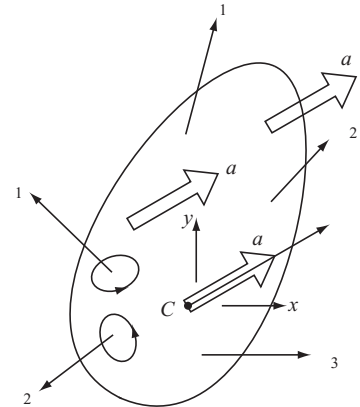


Fig. 9.1 A rigid body in translation

Example 9.1

A motor of mass 8000 kg resting on two supports A and B is pulled along a smooth horizontal surface by a string passing through a hook as shown in Fig. Ex. 9.1.

Calculate the acceleration of the motor and the reactions at the supports for a tension T applied in the string. Calculate the maximum tension in the string for the sliding motion.

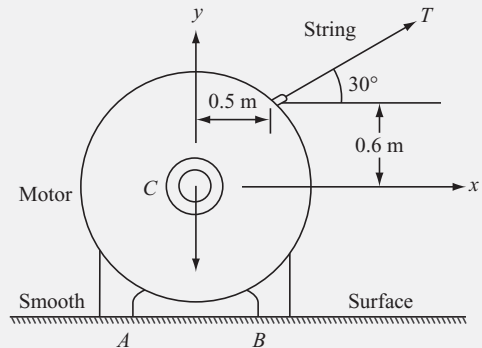


Fig. Ex. 9.1

Solution From the free-body diagram of the motor as shown in Fig. Ex. 9.1 (Solution),

$$F_x = T_x = T \cos 30 = 0.866 T$$

$$F_y = T_y - mg + R_A + R_B = 0.5 T - 78,500 + R_A + R_B$$

$$M_z = 0.5 T \times 0.5 + R_B \times 0.5 - 0.866 T \times 0.6 - R_A \times 0.5$$

$$= 0.5R_B - 0.5R_A - 0.27 T$$

By the equations of motion for a rigid body in translation,

$$F_x = 0.866 T = ma = 8000 a$$

$$F_y = 0.5 T - 78,500 + R_A + R_B = 0$$

$$M_z = 0.5R_B - 0.5R_A - 0.27 T = 0$$

From these equations,

$$a = 0.000\ 108\ T\ \text{m/s}^2$$

$$R_A = 39\ 250 - 0.52\ T$$

$$R_B = 39\ 250 + 0.02\ T$$

It can be observed that the reaction R_A decreases with the tension T increasing. In the limiting case of sliding,

$$R_A = 0 = 39\ 250 - 0.52\ T$$

and

$$T = 39\ 250 / 0.52 = 75\ 480.8\ \text{N} = 75.48\ \text{kN}$$

beyond which the point A will not be restrained to move along the surface. The motor may then overturn forward.

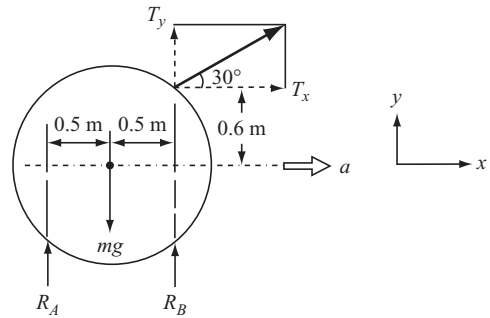


Fig. Ex. 9.1 (Solution)

Example 9.2

A horizontal uniform bar PQ of mass 100 kg and length 30 cm is supported by strings from A and B 30 cm apart and is released from rest when $\theta = 60^\circ$ as shown in Fig. Ex. 9.2.

For a plane curvilinear motion of the bar, determine the tension in the string at the instant when

- (a) it is released from rest
- (b) it crosses the mean position
- (c) $\theta = 30^\circ$.

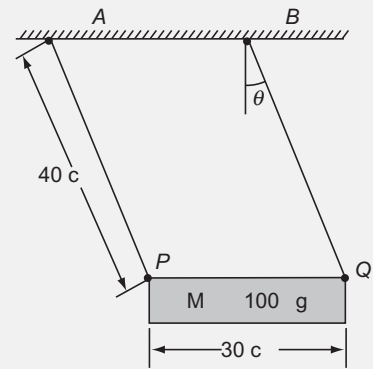


Fig. Ex. 9.2

Solution From the free-body diagram of the bar PQ at any instant as shown in Fig. Ex. 9.2 (Solution),

$$F_r = T_1 - T_2 + mg \cos \theta$$

$$F_\theta = mg \sin \theta$$

$$M_z = -T_1 p_1 + T_2 p_2$$

where p_1 and p_2 are the perpendicular distances from C to the lines of action of T_1 and T_2 respectively.

Since $p_1 = p_2 = p$
from the symmetry and
 $M_z = 0$

for pure translation of the rigid body, it follows that

$$T_1 p - T_2 p = 0$$

or $T_1 = T_2 = T$

i.e., the tensions in the strings must be equal at all times. Then,

$$F_r = -2T + mg \cos \theta$$

$$F_\theta = mg \sin \theta$$

The acceleration of any point on the body is made up of two components; the radially inward component, i.e.,

$$-r \omega^2 = -0.4 \omega^2$$

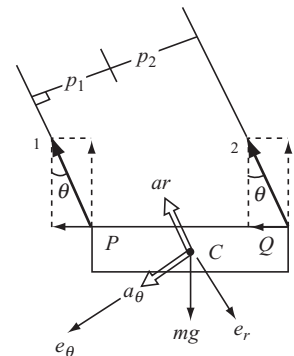


Fig. Ex. 9.2 (Solution)

and the tangentially forward component, i.e.,

$$r\alpha = 0.4\alpha$$

By employing the equations of motion,

$$-2T + mg \cos \theta = 100 \times (-0.4 \omega^2) = -40 \omega^2 \quad (i)$$

or

$$2T - mg \cos \theta = 40 \omega^2$$

and

$$mg \sin \theta = 100 \times 0.4\alpha$$

or

$$mg \sin \theta = 40\alpha \quad (ii)$$

Case (a)

At the instant of release,

$$\theta = 60, \quad \omega = 0$$

From Eq. (i)

$$2T - mg/2 = 0$$

$$T = mg/4$$

$$= 100 \times 9.81/4 = 245.25 \text{ N}$$

and the angular acceleration α is obtained from Eq. (ii) as

$$\alpha = \frac{100 \times 9.81 \times 0.866}{40} = 21.24 \text{ rad/s}^2$$

Cases (b) and (c)

At any instant when the strings are inclined at θ with the vertical, from Eq. (ii),

$$\alpha = \frac{mg \sin \theta}{40} = \frac{d\omega}{dt} = -\omega \frac{d\omega}{d\theta}$$

or

$$\omega/d\omega = -mg/40 \sin \theta d\theta$$

and integrating each side,

$$\omega^2/2 = mg/40 \cos \theta + C$$

Employing the initial condition,

$$\omega = 0 \quad \text{at} \quad \theta = 60$$

$$C = (-mg/40) \cos 60 = -mg/80$$

and

$$\omega^2 = mg/20 \cos \theta - mg/40 \quad (iii)$$

(b) When it crosses the mean position, $\theta = 0$

From Eq. (ii),

$$\alpha = -mg \sin \theta/40 = 0$$

and from Eq. (iii),

$$\omega^2 = mg/20 - mg/40 = mg/40$$

$$\omega^2 = 100 \times 9.81/40 = 24.52$$

or

$$\omega = 4.95 \text{ rad/s}$$

From Eq. (i),

$$2T = mg \cos \theta = 40\omega^2$$

$$2T = 40 \times 24.52 + 100 \times 9.81$$

$$= 1962 \text{ N}$$

$$T = 981 \text{ N}$$

and the tension

(c) At an instant when $\theta = 30$

From (ii),

$$\alpha = -100 \times 9.81/80 = -12.26 \text{ rad/s}^2$$

and from (iii),

$$\omega^2 = 42.5 - 24.5 = 18.0$$

$$\omega = 4.24 \text{ rad/s}$$

From (i),

$$2T - mg \cos \theta = 40 \times 18 = 720$$

$$2T = 720 + 100 \times 9.81 \times 0.866$$

and the tension

$$T = 785 \text{ N}$$

It can be observed that the tension in each string is the least at the extremities of the oscillatory motion. It is the maximum at the mean position. At this position, the weight of the bar acting downward is

$$mg = 981 \text{ N}$$

but the total tension in the two strings upward is 1962 N. The vertically upward unbalanced force (1962–981) N is the one which is responsible for the radially inward (in this position, vertically upward) acceleration, i.e., $0.4\omega^2$. If, on the other hand, the mass was in equilibrium, supported by two equal vertical strings AP and BQ , the tension in each string would only be $mg/2$, i.e., 490.5 N each.

9.3 ROTATION OF A RIGID BODY ABOUT A FIXED PRINCIPAL AXIS

The case of a rigid body rotating about a fixed principal-axis is a special case of plane motion of a rigid body. Let the principal-axis of rotation be the z -axis passing through O such that O and C lie on a plane normal to the axis of rotation. With reference to the coordinate system shown in Fig. 9.2, the rotational velocity and acceleration are ω and α respectively.

An elementary strip of mass dm chosen parallel to the fixed-axis located at a position vector \mathbf{r} as shown is then imparted a velocity and an acceleration given by

$$\begin{aligned} \mathbf{v} &= r\omega \boldsymbol{\theta} \\ \mathbf{a} &= -r\omega^2 \mathbf{r} + r\alpha \boldsymbol{\theta} \end{aligned}$$

The elementary mass must have been acted upon by an elementary force $d\mathbf{F}$ in accordance with Newton's law, i.e.,

$$d\mathbf{F} = dm \mathbf{a} = (-r\omega^2 \mathbf{r} + r\alpha \boldsymbol{\theta}) dm$$

The moment of the elementary force about the axis of rotation passing through O is

$$\begin{aligned} dM_z &= \mathbf{r} \times d\mathbf{F} \\ &= r \mathbf{r} \times (-r\omega^2 \mathbf{r} + r\alpha \boldsymbol{\theta}) dm \\ &= r^2 \alpha dm \end{aligned}$$

This is the moment which must be exerted about O so as to accelerate the elementary mass at an angular acceleration α . The total moment required to rotate and accelerate the rigid body about an axis through O is, therefore,

$$M_z = \int dM_z = \alpha \int r^2 dm$$

or

$$M_z = I_z \alpha \tag{9.3}$$

where I_z , the mass moment of inertia of the body about an axis through O is, by definition, given by

$$I_z = \int r^2 dm$$

and the moment vector M_z is directed perpendicular to the parallel planes of rotation.

It should be recorded here that the moment M_z acting on the rigid body about an axis through O is due to the entire system of forces on the rigid body. For example, the weight of the body, reactions from the supports and other external actions as obtained by drawing a free-body diagram of the rigid body must be taken into account to obtain the net moment M_z .

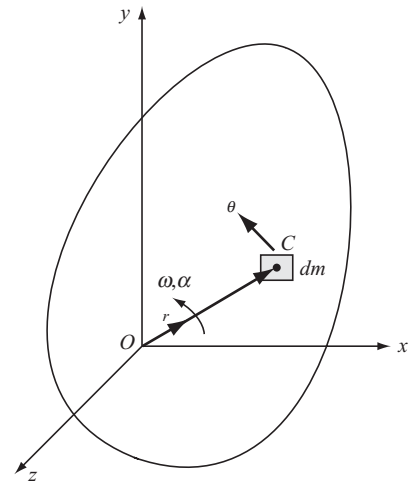


Fig. 9.2 Rotation about a fixed principal axis

The centre of mass C is imparted a velocity and an acceleration given by

$$v_c = \omega \times r_c = r_c \omega_\theta$$

$$a_c = -r_c \omega_r + r_c \alpha_\theta$$

The equation of motion for the centre of mass is given by

$$F = m a_c \tag{9.4}$$

The net external force on the rigid body is due to the weight, reactions from the supports and other actions. This equation is useful if the desired goal is to find the reactions from the supports on a dynamic body.

In particular, if the axis of rotation passes through the centre of mass, i.e., O and C coincide,

$$F = m a_c = 0$$

$$M_z = I_z \alpha$$

It is usual to define a point called *centre of percussion* as a location on a rigid body through which the resultant of the applied forces acts.

Consider, for example, a rigid-body model of a cricket bat free to rotate about an axis through O as shown in Fig. 9.3(a). Its centre of mass is at C but the centre of percussion is at a point P such that if the force due to a ball striking it passes through P , there is no horizontal reaction at O where the bat is held by the batsman and he can receive the ball comfortably, without jarring.

If the moment of inertia of the body about its axis of rotation is I_0 and the corresponding radius of gyration is k_0 , the distance p at which the centre of percussion is located is given by

$$p = \frac{k_0^2}{h}$$

This may be shown by considering the free-body diagram of the bat and by determining the condition of zero horizontal reaction at O . Referring to the free-body diagram shown in Fig. 9.3(b), the equation of motion for the centre of mass is

$$R_x - F = ma \tag{i}$$

and the Euler's equation for rotation about the axis of rotation through O is

$$-F \cdot p = I_0 \alpha = (I_c - mh^2) \alpha$$

$$= mk_0^2 \alpha = m(k_c^2 + h^2) \alpha \tag{ii}$$

Eliminating F between Eqs. (i) and (ii)

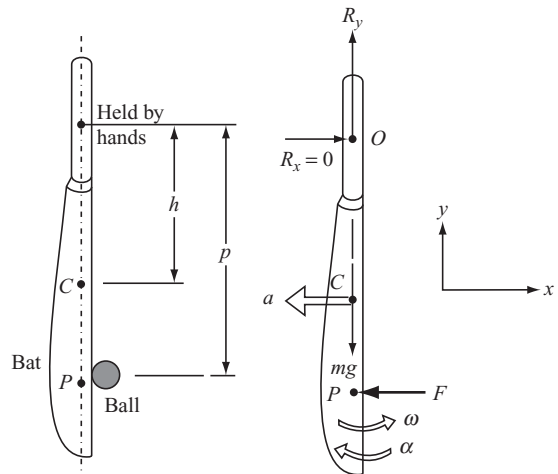
$$R_x = ma - \frac{mk_0^2 \alpha}{p} = ma - \frac{m(k_c^2 + h^2) \alpha}{p}$$

Considering the fact that

$$a = h \alpha$$

the condition for R_x to be zero is

$$p = \frac{k_0^2}{h} = \frac{k_c^2 + h^2}{h} \tag{9.5}$$



(a) Ball hitting the bat held by a player (b) Free-body diagram of the bat at the instant of impact

Fig. 9.3 Concept of the centre of percussion

Example 9.3

A rectangular plate 2 m × 3 m of mass 100 kg/m² is supported by hinges at *A* and *B* as shown in Fig. Ex. 9.3. If the support *A* is removed, determine the reaction at *B*, the angular acceleration of the plate and acceleration of the centre of mass.

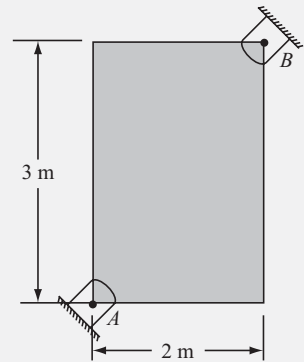


Fig. Ex. 9.3

Solution At the instant when the support *A* is removed, the plate tends to rotate about the axis passing through *B* and normal to the plane of the plate. From the free-body of the plate at that instant, as shown in Fig. Ex. 9.3 (Solution)

$$F_x = R_x$$

$$F_y = R_y - mg = R_y - 600 \times 9.81 = R_y - 5886$$

$$M_B = mg \times 1 = 5886 \text{ N m}$$

$$I_B = \frac{1}{12} \times 600 \times (3^2 + 2^2) + 600 \times (1.5^2 + 1^2)$$

$$= 2600 \text{ kg m}^2$$

By the equation of motion for a rigid-body rotation about a fixed axis,

$$M_B = I_B \alpha$$

or $\alpha = 5886/2600 = 2.26 \text{ rad/s}^2$

Also, $M_c = I_c \alpha$

$$1 \times R_y - 1.5 \times R_x = \frac{1}{12} \times 600 \times (3^2 + 2^2) \times 2.26$$

$$R_y - 1.5 R_x = 1469$$

and $F_x = R_x = ma_x = 600 a_x$
 $F_y = R_y - 5886 = 600 a_y$

At the initial instant, the body starts from rest, $\omega = 0$ although the acceleration α is finite as determined above. The acceleration of the centre of mass *C* is

$$a_c = \alpha \times$$

$$= 2.26 \times (-1 \quad -1.5) = 3.39 \quad -2.26$$

whence, $a_x = 3.39 \text{ m/s}^2, a_y = -2.26 \text{ m/s}^2$

Substituting these values in the equations of motion

$$R_x = 600 \times 3.39 = 2034 \text{ N}$$

$$R_y = 600 \times (-2.26) + 5886 = 4530 \text{ N}$$

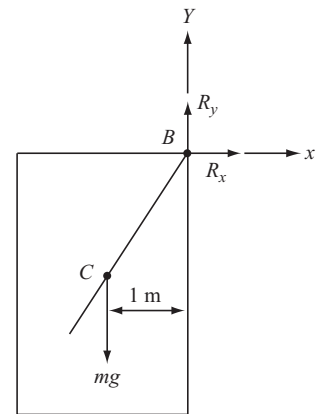


Fig. Ex. 9.3 (Solution)

It can be checked that

$$R_y - 1.5 R_x = 4530 - 1.5 \times 2034 = 1479$$

which is obtained earlier by employing the equation of motion about the centre of mass C .

The answers required are:

Reaction at B consists of

$$R_x = 2034 \text{ N} \quad \text{and} \quad R_y = 4530 \text{ N}$$

The angular acceleration $\alpha = 2.26 \text{ rad/s}^2$, anticlockwise

The acceleration of the centre of mass is given by

$$a_x = 3.39 \text{ m/s}^2 \quad \text{and} \quad a_y = -2.26 \text{ m/s}^2$$

Example 9.4

A disc of radius 0.5 m and uniform thickness weighing 300 N is supported on a shaft in a bearing centrally, the shaft being normal to the plane of the disc as shown in Fig. Ex. 9.4. The disc is made to spin at 1500 revolutions per minute and then let to decelerate due to bearing friction. It comes to rest in 2 minutes. Assuming that the bearing friction is independent of speed, calculate: (a) the bearing torque and (b) the number of revolutions the disc turns before coming to rest.

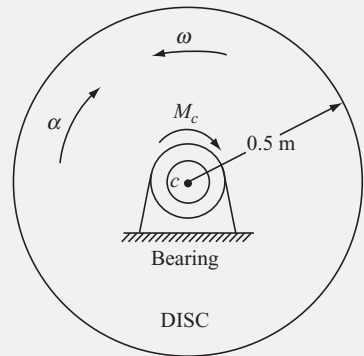


Fig. Ex. 9.4

Solution The disc comes to rest in the face of a constant bearing torque which implies that the acceleration is constant

$$\begin{aligned} \alpha &= \frac{d\omega}{dt} = \frac{\omega_2 - \omega_1}{t} \\ &= \frac{(0 - 1500) \times 2\pi}{2 \times 60 \times 60} = -1.31 \text{ rad/s}^2 \end{aligned}$$

The mass moment of inertia of the disc about the axis of rotation is

$$I_c = \frac{1}{2} \frac{300}{9.81} (0.5)^2 = 3.82 \text{ kg m}^2$$

(a) The torque acting on the disc due to the bearing friction is

$$M_c = 3.82 \times (-1.31) = -5.00 \text{ N m}$$

where the negative sign shows that the bearing friction torque acts to resist the motion and to bring about deceleration of the disc.

This resisting torque is equivalent to a constant resisting force of

$$5.00/0.5 = 10.00 \text{ N}$$

acting over the rim of the disc.

(b) From $\alpha = -1.31 \text{ rad/s}^2$

$$\omega_1 = \frac{1500 \times 2 \times \pi}{60} = 157.1 \text{ rad/s}$$

and $\omega_2 = 0$

Since $\omega_2^2 - \omega_1^2 = 2\alpha\theta$

whence, $\theta = \frac{0 - 157.1^2}{2 \times (-1.31)} = 9420 \text{ rad}$

and $n = 9349/2\pi = 1499.2 \text{ revolutions}$

9.4 PLANE MOTION OF A RIGID BODY

Plane translation, rectilinear or curvilinear, together with rotation about an axis perpendicular to the parallel planes in which translation takes place, constitutes a general plane motion. Conversely, a general plane motion can be regarded equivalent to a combination of translation of the centre of mass and rotation about an axis passing through the centre of mass. Correspondingly, the dynamic behaviour of a rigid body is governed by the set of equations

$$F_x = ma_x \tag{9.6}$$

$$F_y = ma_y$$

for translation of the centre of mass

and

$$M_c = I_c \alpha \tag{9.7}$$

for rotation about an axis passing through the centre of mass of the body.

Example 9.5

A cord passing over a light frictionless pulley carries a weight W_1 suspended vertically at one end and is wrapped around a cylinder of weight W_2 as shown in Fig. Ex. 9.5. Assuming that the cylinder can roll without slip on a horizontal plane, calculate the acceleration of the suspended weight.

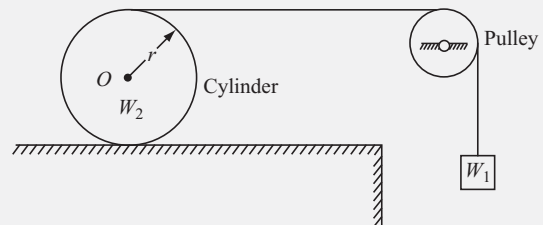


Fig. Ex. 9.5

Solution The free-body diagrams of the cylinder and the suspended mass are shown in Fig. Ex. 9.5 (Solution).

The tensions in the cord on the two sides of the frictionless pulley are equal, each equal to T . The downward acceleration of the suspended weight is not the same as the acceleration of the centre of the cylinder. The equations of motion for the cylinder and the suspended weight are as follows:

$$T - f = \frac{W_2}{g} a_2$$

(i)

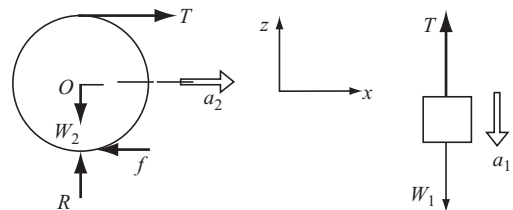


Fig. Ex. 9.5 (Solution)

$$- a_2 = 0 \tag{ii}$$

$$(a_1 + a_2) = \frac{1}{2} \left(\frac{-a_2}{r_2} \right)^2 \alpha \tag{iii}$$

$$a_1 - a_2 = -\frac{1}{2} a_2 \tag{iv}$$

Substituting $\alpha = a_2 / r_2$ for the rolling cylinder, Eq. (iii) reduces to

$$a_1 + a_2 = \frac{1}{2} \frac{a_2}{r_2} \tag{v}$$

From Eqs. (i) and (v)

$$= \frac{3}{4} \frac{a_2}{r_2}$$

which, when substituted in Eq. (iv) yields

$$a_1 = -\frac{3}{4} \frac{a_2}{r_2}$$

From the kinematics of motion, the acceleration of the top point on the cylinder equals the acceleration of the suspended weight

$$a_1 = 2 \alpha = 2 a_2 / r_2$$

Hence,

$$a_1 = \frac{a_2}{1 + \frac{3}{8} \frac{r_2}{r_1}}$$

An observation of this equation shows that a_2 / r_1 is the only factor which governs the acceleration of the suspended weight. The size of the cylinder is immaterial. When a_2 / r_1 is zero, $a_1 = a_2$, the acceleration of the free-fall. As a_2 / r_1 increases, the acceleration of the suspended weight decreases, tending to become zero as a_2 / r_1 tends to infinity.

Example 9.6

A wheel and a differential axle assembly has a mass μ and a radius of gyration k . The radii of the two parts of the axle are r_1 and r_2 . Cords wrapped round these parts carry suspended weights W_1 and W_2 as shown in Fig. Ex. 9.6. Determine the acceleration of the suspended weights when the weight W_1 descends.

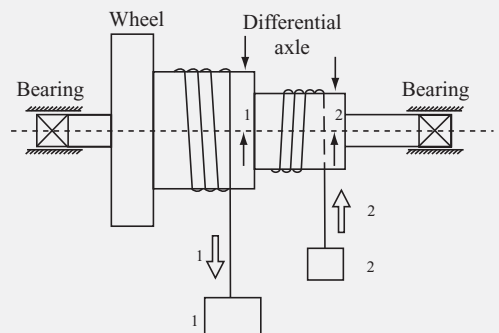


Fig. Ex. 9.6

Solution Let the acceleration of the suspended weights be a_1 and a_2 downwards and upwards respectively. The angular acceleration α of the wheel and differential axle assembly is

$$\alpha = \frac{a_1}{r_1} = \frac{a_2}{r_2}$$

With reference to the free-body diagrams of the suspended weights

$$W_1 - T_1 = \frac{W_1}{g} a_1 \tag{i}$$

$$T_2 - W_2 = \frac{W_2}{g} a_2 \tag{ii}$$

The net external moment on the assembly due to the suspended weight is

$$M = T_1 r_1 - T_2 r_2$$

which must be equal to

$$I\alpha = mk^2\alpha$$

or

$$T_1 r_1 - T_2 r_2 = mk^2\alpha$$

Substituting for T_1 and T_2 from Eqs. (i) and (ii) and solving

$$\alpha = \frac{W_1 r_1 - W_2 r_2}{mgk^2 + W_1 r_1^2 + W_2 r_2^2} g$$

$$a_1 = r_1 \alpha \quad \text{and} \quad a_2 = r_2 \alpha$$

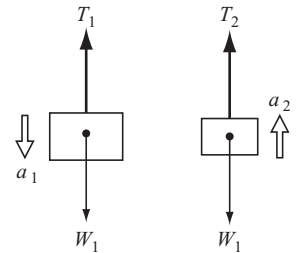


Fig. Ex. 9.6 (Solution)

Example 9.7

A thin uniform bar of mass m and length $2L$ is held at angle θ with the horizontal by means of two vertical inextensible strings, one at each end as shown in Fig. Ex. 9.7. If the string at the right end breaks, leaving the bar to swing, determine the tension in the string at the left end and the angular acceleration of the bar immediately after the string breaks.

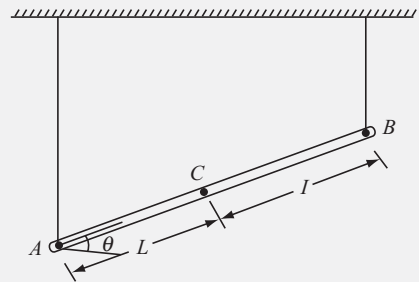


Fig. Ex. 9.7

Solution Let us write down the equations of motion with reference to the free-body diagram of the bar at an instant when the right end string just breaks, as shown in Fig. Ex. 9.7 (Solution).

$$T - mg = ma_y \tag{i}$$

$$0 = ma_x \tag{ii}$$

$$-TL \cos \theta = I_c \alpha \tag{iii}$$

It follows from Eq. (ii) that

$$a_x = 0$$

By kinematic considerations, the acceleration a of the centre of mass is related to the angular acceleration α of the bar by

$$a_y = L \alpha \cos \theta \tag{iv}$$

Substituting Eq. (iv) in Eq. (i)

$$T - mg = mL\alpha \cos \theta$$

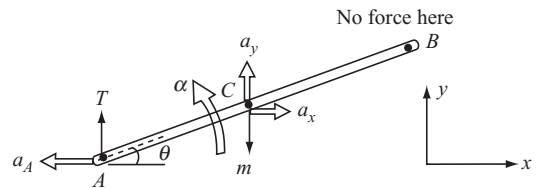


Fig. Ex. 9.7 (Solution)

and solving for T and α simultaneously with Eq. (iii), remembering that

$$I_c = \frac{m(2L)^2}{12} = \frac{mL^2}{3}$$

the expressions for T and α appear as follows:

$$T = \frac{mg}{1 + 3 \cos^2 \theta}$$

and

$$\alpha = \frac{-3g \cos \theta}{L(1 + 3 \cos^2 \theta)}$$

It may be remarked that the positive sign of T shows that the string is indeed in tension and the negative sign of α indicates that the angular acceleration is clockwise and not anticlockwise as assumed in the free-body diagram.

Example 9.8

A uniform bar of length L and weight W rests on smooth surfaces as shown in Fig. Ex. 9.8. Obtain an expression for the angular velocity of the bar and determine the angle θ at which the bar no longer touches the vertical wall.

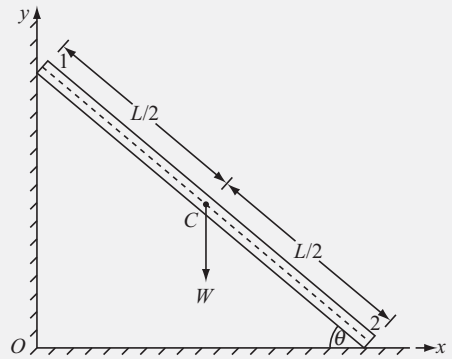


Fig. Ex. 9.8

Solution With reference to the free-body diagram of the bar where the reactions at the two surfaces are taken normal only due to the absence of friction, let us consider the motion of the bar with reference to its centre of mass C . Assuming the motion as a plane motion in the plane of the figure, the equations describing its motion are

$$N_1 = ma_x \tag{i}$$

$$N_2 - W = ma_y \tag{ii}$$

$$N_1 \frac{L}{2} \sin \theta - N_2 \frac{L}{2} \cos \theta = I_c \alpha \tag{iii}$$

Taking

$$I_c = \frac{mL^2}{12}$$

and

$$W = mg$$

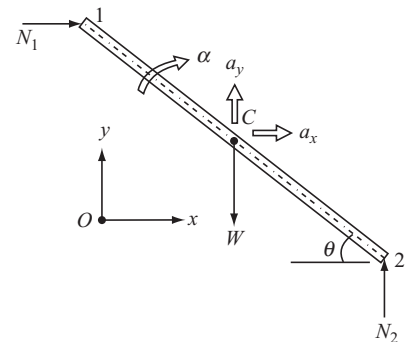


Fig. Ex. 9.8 (Solution)

for the rod, and substituting N_1 and N_2 from Eqs. (i) and (ii) into Eq. (iii),

$$a_x \sin \theta - a_y \cos \theta = g \cos \theta + \frac{L}{6} \alpha \quad (\text{iv})$$

The linear accelerations a_x and a_y of the centre of mass are related to its angular acceleration α by the geometry of the motion. From the coordinates of the centre of mass at any instant,

$$x = \frac{L}{2} \cos \theta \quad \text{and} \quad y = \frac{L}{2} \sin \theta$$

Differentiation with respect to t leads to

$$V_x = \frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = -\frac{L}{2} \sin \theta \frac{d\theta}{dt}$$

$$a_x = \frac{dV_x}{dt} = -\frac{L}{2} \cos \theta \left(\frac{d\theta}{dt} \right)^2 - \frac{L}{2} \sin \theta \frac{d^2\theta}{dt^2}$$

Similarly,

$$a_y = \frac{dV_y}{dt} = -\frac{L}{2} \sin \theta \left(\frac{d\theta}{dt} \right)^2 + \frac{L}{2} \cos \theta \frac{d^2\theta}{dt^2}$$

and

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt} \left(\frac{d\theta}{dt} \right) = \frac{d^2\theta}{dt^2}$$

Substituting these relations in Eq. (iv)

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2L} \cos \theta \quad (\text{v})$$

the angular speed ω is obtained by integration,

$$\omega = \sqrt{\frac{3g}{L}(1 - \sin \theta)} \quad (\text{vi})$$

assuming that the bar started from rest at $\theta = 90^\circ$, i.e., when the bar was vertical.

The condition of it no longer touching the wall means

$$N_1 = 0$$

and from Eq. (i)

$$a_x = 0$$

$$\text{or} \quad -\frac{L}{2} \cos \theta \left(\frac{d\theta}{dt} \right)^2 - \frac{L}{2} \sin \theta \frac{d^2\theta}{dt^2} = 0 \quad (\text{vii})$$

Substituting for the first and second derivatives of θ in Eq. (vii),

$$\sin \theta = \frac{2}{3} = 0.667$$

whence

$$\theta = 41.8$$

Example 9.9

Two homogeneous cylindrical discs, each of diameter 1 m and mass 10 kg connected by an axle of diameter 0.3 m and mass 15 kg, lie on a rough surface as shown in Fig. Ex. 9.9.

A string wrapped round the axle as shown, exerts a force of 50 N at the mid-span of the axle. Analyse the motion to determine the acceleration of the system.

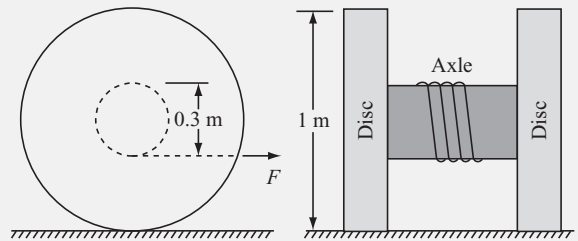


Fig. Ex. 9.9

Solution With reference to the free-body diagram of the system where the motion is assumed to be towards right and the frictional force f is shown to act in a direction to oppose the motion, as shown in Fig. Ex. 9.9 (Solution):

For the reference axes fixed at the centre of mass C , the equations of motion are:

$$F_x = 50 - f = ma_x$$

$$F_y = R - (10 + 10 + 15)g = ma_y$$

$$M_c = 50 \times \frac{0.3}{2} - f \times \frac{1}{2} = I_c \alpha$$

Using the fact that $a_y = 0$
because the body is not being lifted along the y -axis and

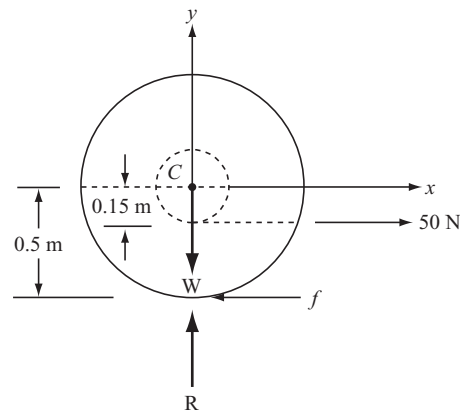


Fig. Ex. 9.9 (Solution)

$$I_c = 2 \times \left(\frac{1}{2} \times 10 \times 0.5 \right) + \frac{1}{2} \times 15 \times 0.15^2 = 2.67 \text{ kg m}^2$$

and $m = 2 \times 10 + 15 = 35 \text{ kg}$

the equations of motion simplify to

$$50 - f = 35a_x \tag{i}$$

$$R - 35g = 0 \tag{ii}$$

$$7.5 - f/2 = 2.67 \alpha \tag{iii}$$

Substituting

$$\alpha = \frac{a_x}{0.5}$$

in Eq. (iii) reduces it to

$$15 - f = 10.68a_x$$

which, together with Eq. (i), provides

$$a_x = 1.44 \text{ m/s}^2$$

EXPERIMENT E9 LOOPING THE LOOP

OBJECTIVE TO DETERMINE THE MINIMUM INITIAL HEIGHT OF A BALL IN ORDER THAT IT MAY SUCCEED IN ‘LOOPING THE LOOP’.

Apparatus A channel-track arranged in the form of a looping-the-loop apparatus as shown in Fig. E9.1, spherical ball and a metre scale.

Background Information When a spherical ball is released from rest from a position on the straight part of the track, it rolls down and gains speed until it reaches the bottom of the loop. Thereafter, it is subjected to a centripetal force and a varying reaction N normal to the track. Considering the free-body diagram of the ball at a point P , as shown in Fig. E9.2, the equations of motion for its centre of mass C are

Tangential: $mg \sin \theta - f = ma$ (E9.1)

Normal: $N - mg \cos \theta = \frac{mv^2}{(R - r)} = \frac{mr\omega^2}{(R - r)}$ (E9.2)

where v is the velocity of the centre of mass of the ball, $(R - r)$ is the radius of the path of the centre of mass as the ball of radius r moves within the circular loop of radius R and ω is the angular velocity of the centre of mass of the ball.

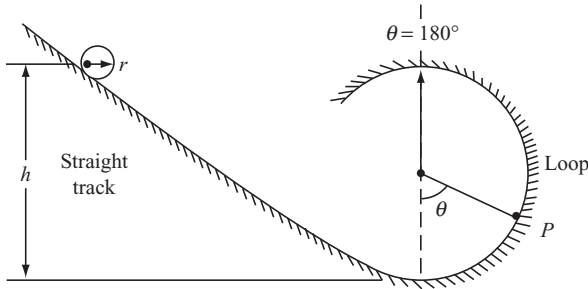


Fig. E9.1 ‘Looping-the-loop’ apparatus

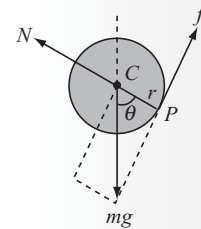


Fig. E9.2 Free-body diagram of the ball

The Euler’s equation for the rotation of the ball must be

$$\begin{aligned} fr &= I_c \alpha = \frac{2}{5} mr^2 \frac{d\omega}{dt} \\ &= \frac{2}{5} mr^2 \omega \frac{d\omega}{d\theta} \end{aligned} \tag{E9.3}$$

Integrating Eq. (E9.3) and substituting for f from Eq. (E9.1) together with the fact that

$$a_1 = r\alpha = r \frac{d\omega}{dt} = r\omega \frac{d\omega}{d\theta}$$

it is seen that

$$\omega = \frac{\log(R - r)}{7r^2} \cos \theta + \frac{\log(h - R + r)}{7r^2} \tag{E9.4}$$

where the initial height h is related to the velocity of the ball by the principle of energy conservation applied during the rolling of the ball down the incline without friction

$$mgh = \frac{1}{2} I_c \omega_0^2 + \frac{1}{2} mv_0^2 \tag{E9.5}$$

where v_0 , the translational velocity of the centre of mass of the ball at $\theta = 0$ position equals r times the rotational velocity ω_0 of the ball at the same position.

Substituting the value of ω in Eq. (E9.2),

$$N = mg \cos \theta + \frac{mr^2}{R-r} \left[\frac{\log(R-r)}{7r^2} \cos \theta + \frac{\log(h-R+r)}{7r^2} \right] \quad (\text{E9.6})$$

It is noticed that the reaction N from the track on the ball varies with θ , the angular position of the ball for a given initial height h , the other parameters, i.e., the radii of the ball and the track remaining the same. The position where the ball loses contact with the track must be such that the reaction N becomes zero. If the ball is to complete the loop, it must be able to go all the way up to the position $\theta = 180^\circ$ because it would then continue on the track thereafter.

Substituting $\theta = 180^\circ$, $\cos \theta = -1$ and $N = 0$ in Eq. (E9.6) and simplifying the result,

$$h_1 = 2.7(R-r) \quad (\text{E9.7})$$

This is the minimum initial height of the ball for it to succeed in looping the loop.

Alternatively, the analysis may be considerably simplified if the moment of inertia of the ball is ignored, i.e., if the ball is assumed to be represented by a mass sliding frictionlessly. In that case only the normal equation of motion, i.e., Eq. (E9.2) together with the energy conservation principle would provide the result

$$N - mg \cos \theta = \frac{mv^2}{R-r} \quad (\text{i})$$

$$mgh = \frac{1}{2} mv^2 + mg(R - R \cos \theta) \quad (\text{ii})$$

Substituting $\theta = 180^\circ$; $\cos \theta = -1$ and $N = 0$ as well as the value of v from Eq. (ii) in Eq. (i),

$$mg = \frac{2mgh - 4mg(R-r)}{(R-r)}$$

whence
$$h_2 = 2.5(R-r) \quad (\text{E9.8})$$

It may be seen that this result is remarkably close to the one obtained above, Eq. (E9.7). *In practice, neither of them may be valid because the ball may roll as well as slide with friction simultaneously.*

Observations and Calculations The experiment is indeed very interesting. *The condition of success of the ball in looping the loop is independent of the mass of the ball, angle of inclination of the straight track and value of g at a particular place.* The minimum initial height h required for a ball to be able to reach the top position and to complete the loop is determined experimentally by hit and trial.

Result State the average of the minimum experimental heights obtained together with the percentage discrepancy in comparison with the theoretical results by the alternative analysis.

| | $R =$ | $r =$ | | |
|------|------------------|--------------------------------|--------------------------------|---|
| S.N. | h_{exp} | $h_1 = 2.7(R-r)$ Eq. (E9.7) | $h_2 = 2.5(R-r)$ Eq. (E9.8) | $\frac{h_{\text{exp}} - h_1}{h} \times 100$ |
| | | | | Differences in $\frac{h_{\text{exp}} - h_2}{h} \times 100$ |

Points for Discussion

Under what circumstances is a ball likely to roll without sliding and under what conditions is it likely to roll as well as slide?

Examine the percentage variation within the experimental results and compare the same with the percentage difference between the experimental and theoretical values. Which of the two theoretical results is closer to the actual result?

The energy conservation principle has been employed in the theoretical analyses. Discuss the validity of the principle for the case of a ball rolling without sliding as assumed in the first analysis. The validity of the principle for frictionless sliding as assumed in the second analysis is quite clear.

If a ball were allowed to complete the loop and made to go up another incline, would it attain a height equal to the initial height h ?

Supposing it was desired to determine the dissipation of energy between the initial position of the ball and its top position in the loop in the limiting case, how would you proceed to estimate it?

If you were to improve the experimental equipment, would you recommend

- (i) the selection of a smoother ball, a lighter ball or a smaller ball?
- (ii) the improvement of the track by minimising friction, by decreasing the radius of the loop or by increasing the inclination of the straight track?
- (iii) the use of a small circular cylinder or a small rectangular block instead of a spherical ball?

Example 9.10

A string is wrapped around the periphery of a thin disc of radius 0.5 m and mass 10 kg as shown in Fig. Ex. 9.10. At an instant when the string is pulled up with a force of 200 N, determine the acceleration of the centre of the disc and the angular acceleration of the disc.

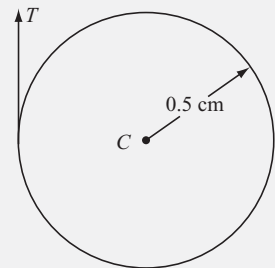


Fig. Ex. 9.10

Solution Considering the free-body diagram of the disc, and taking C as the reference point,

$$T - W = ma_y \tag{i}$$

$$0 = ma_x \tag{ii}$$

$$-Tr = I_c \alpha = mr^2/2 \alpha \tag{iii}$$

From (i), $a_y = \frac{200 - 10 \times 9.81}{10} = 10.2 \text{ m/s}^2$

and from (ii) $a_x = 0$

From (iii) $\alpha = \frac{-200 \times 0.5 \times 2}{10 \times 0.5^2} = -80 \text{ rad/s}^2$

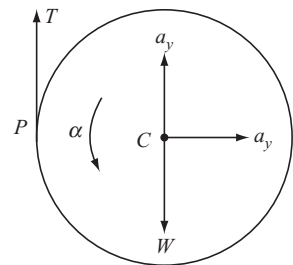


Fig. Ex. 9.10 (Solution)

which means that the angular acceleration is 80 rad/s^2 in the clockwise direction.

One may also determine the acceleration of the string which is the tangential component of acceleration of the disc at point P .

$$a = 10.2 + 0.5 \times 80 = 50.2 \text{ m/s}^2 \text{ upwards.}$$

9.5 WORK-ENERGY FORMULATION FOR PLANE MOTION

Let us recapitulate the definitions of work and energy, obtain expressions for them and formulate the work-energy principle as an alternative principle for studying the dynamics of a rigid body.

The work done due to a force acting at an arbitrary point P on a body equals the dot product of the force with the displacement of the point of application of the force:

$$dW = \mathbf{F} \cdot d\mathbf{p}$$

For a finite displacement of the point of application of force, i.e., from p_1 to p_2 , the work done is given by

$$W = \int_{p_1}^{p_2} \mathbf{F} \cdot d\mathbf{p}$$

If, instead, it is sought to evaluate the integral

$$W' = \int_{p_1}^{p_2} \mathbf{F} \cdot d\mathbf{c} \tag{9.8}$$

as is the case in the work-energy formulation to follow, the integral does not represent the work done by the force on the body, it only represents the work which would have been done by the force on the body if the force acted at its centre of mass.

The action of a moment due to a couple on a rigid body results in an angular displacement θ of the body. The work of the moment is expressed by

$$dW = M \cdot d\theta$$

and for infinite angular displacement of the body from θ_1 to θ_2 the work done is given by

$$W = \int_{\theta_1}^{\theta_2} M \cdot d\theta \tag{9.9}$$

The mechanical energy possessed by a rigid body is partly due to the potential energy by virtue of its position referred to a datum and partly due to the kinetic energy by virtue of its absolute velocity.

The potential energy due to the gravitational field is directly proportional to the height h of the centre of mass above the datum level

$$PE = mgh \tag{9.10}$$

The kinetic energy of a rigid body is due to the motion of the elements of the body. Consider an element of mass dm located at a position P specified by a position vector

\mathbf{r} with respect to C , the mass centre

and \mathbf{R} with respect to O , a fixed point as shown in Fig. 9.4.

For the element,

$$d(KE) = \frac{1}{2} dm(\mathbf{V} \cdot \mathbf{V})$$

by definition.

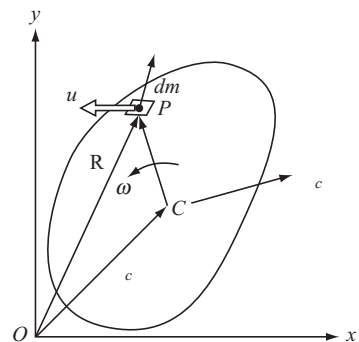


Fig. 9.4 Consideration of kinetic energy

For the entire body

$$\begin{aligned}
 KE &= \frac{1}{2} \int (\dot{r})^2 dm \\
 &= \frac{1}{2} \int (v_c + u) \cdot (v_c + u) dm \\
 &= \frac{1}{2} \int v_c^2 dm + \frac{1}{2} \int u^2 dm \\
 &= \frac{1}{2} m V_c^2 + \frac{1}{2} \int r^2 dm \omega^2
 \end{aligned}$$

or

$$KE = \frac{1}{2} m V_c^2 + \frac{1}{2} I_c \omega^2 \tag{9.11}$$

It may be noted that, for the case of pure translation of a rigid body, i.e., when $\omega = 0$, the kinetic energy is given by

$$KE = \frac{1}{2} m V_c^2$$

and for the case of pure centroidal rotation, i.e., when $V_c = 0$, the kinetic energy is given by

$$KE = \frac{1}{2} I_c \omega^2$$

In view of these facts, it may be stated that the kinetic energy of a rigid body in a general plane motion, as given by Eq. (9.11), may be thought of as the sum of

1. The kinetic energy $(1/2)mV_c^2$ associated with the motion of the centre of mass C of the body as if the total mass were concentrated at that point
2. The kinetic energy $(1/2)I_c \omega^2$ associated with the rotation of the body about an appropriate axis through C .

If a rigid body undergoes fixed axis rotation about an axis which does not pass through the centre of mass C , the expression for the kinetic energy of the body in plane may surely be employed but a simpler expression is more convenient. In order to arrive at it, consider a rigid body rotating about an axis through O as shown in Fig. 9.5. The expression for kinetic energy is

$$KE = \frac{1}{2} m V_c^2 + \frac{1}{2} I_c \omega^2$$

Substituting $V_c = r_c \omega$

$$\begin{aligned}
 KE &= \frac{1}{2} m r_c^2 \omega^2 + \frac{1}{2} I_c \omega^2 \\
 &= \frac{1}{2} (I_c + m r_c^2) \omega^2 \\
 &= \frac{1}{2} I_0 \omega^2
 \end{aligned} \tag{9.12}$$

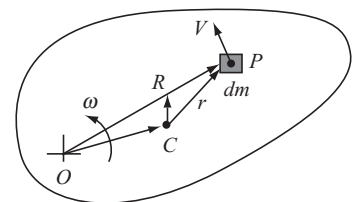


Fig. 9.5 Fixed-axis rotation

which implies that the kinetic energy of a rigid body during a non-centroidal fixed-axis rotation may also be written as $(1/2) I_0 \omega^2$ provided the moment of inertia I_0 is taken about the axis of rotation. The same fact could have been established by considering a small element of mass dm at an arbitrary point P . The kinetic energy of the element is

$$\begin{aligned} KE &= \int \frac{1}{2} V^2 dm \\ &= \int \frac{1}{2} (R\omega)^2 dm \\ &= \int \frac{1}{2} R^2 dm \cdot \omega^2 \end{aligned}$$

or

$$KE = \frac{1}{2} I_0 \omega^2$$

where I_0 , the moment of inertia of the entire body about the axis of rotation through O equals the integral $\int R^2 dm$.

The work-energy equations for a rigid body may be obtained from the Newton's law and Euler's equation. From the Newton's law applied to the centre of mass of the body,

$$= m_c \cdot a_c$$

the work-energy equation which may be derived as

$$W = \int_{r_1}^{r_2} F_c \cdot d_c = \frac{1}{2} m (V_{c_2}^2 - V_{c_1}^2) = KE_2 - KE_1 \quad (9.13)$$

It may again be pointed out that the left-hand side expression does not stand for the actual work done by the net force acting on the body unless the net force happens to act at the centre of mass. The expression indicates that work would have been done if the net force were acted at the centre of mass. Similarly, the right-hand side expression represents the change in translational kinetic energy associated with the total mass m of the body if it were concentrated at the centre of mass. In spite of the 'ifs and buts' in the interpretation of the equation, the equation is called the *work-energy equation applied to the centre of mass* of the body because it is an alternative form of the Newton's law describing the motion of the centre of mass of the body.

From the Euler's equation applied at the centre of mass of the body in a plane motion,

$$M_c = I_c \alpha$$

the work-energy equation which may be derived as

$$W = \int_{\theta_1}^{\theta_2} M_c d\theta = \frac{1}{2} I_c (\omega_2^2 - \omega_1^2) = KE_2 - KE_1 \quad (9.14)$$

The left-hand side expression stands for the work done by the net moment acting on the body about the centre of mass which results in an angular displacement from θ_1 to θ_2 . The right-hand side expression stands for the change in rotational kinetic energy referred to at the centre of mass. The equation is, therefore, referred to as the 'work energy equation for rotation about the centre of mass' and used as an alternative to the Euler's equation applied at the centre of mass of the body.

The total work done by the action of forces and moments acting on a body must be the sum of the work done by the net force, as if it acted at the centre of mass, and the work of the net moment about the centre of mass. The total change in kinetic energy of a rigid body is likewise the sum of the change in translational kinetic energy, as if the entire mass is concentrated at the centre of mass, and the change in rotational kinetic energy due to rotation about the centre of mass.

Symbolically,

$$W_{\text{total}} = \int_{r_1}^{r_2} F_c \cdot d_c + \int_{\theta_1}^{\theta_2} M_c d\theta \quad (9.15)$$

and
$$(KE_2 - KE_1)_{\text{total}} = \frac{1}{2} m(V_{c_2}^2 - V_{c_1}^2) + \frac{1}{2} I_c (\omega_2^2 - \omega_1^2) \tag{9.16}$$

It also follows that

$$W_{\text{total}} = (KE_2 - KE_1)_{\text{total}}$$

$$\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}_c + \int_{\theta_1}^{\theta_2} \mathbf{M}_c d\theta = \frac{1}{2} m(V_{c_2}^2 - V_{c_1}^2) + \frac{1}{2} I_c (\omega_2^2 - \omega_1^2) \tag{9.17}$$

which is termed as the total work-energy equation for the plane motion of a rigid body.

The total work-energy equation may be reduced to a conservation of energy equation in the absence of dissipative external forces and moments acting on a body or a system of bodies. The work done by a conservative force during a displacement of the centre of mass from position 1 to 2 equals the change in potential energy and remembering that the action of a moment about the centre of mass cannot bring about a change in its position, the total work done is

$$W_{\text{total}} = (PE_1 - PE_2)_{\text{total}}$$

Equating it to the total change in kinetic energy,

$$(PE_1 - PE_2)_{\text{total}} = (KE_2 - KE_1)_{\text{total}}$$

Rearranging the terms,

$$(KE_1 + PE_1)_{\text{total}} = (KE_2 + PE_2)_{\text{total}} \tag{9.18}$$

which shows that, under the action of a conservative force field, the sum of the total kinetic energy and potential energy of a rigid body is conserved. This is also referred to as the principle of conservation of mechanical energy for a rigid body.

Example 9.11

In a laboratory, a flywheel of diameter 0.5 m is made to rotate, starting from rest, by means of a suspended mass of 100 kg by an inextensible string wound around a concentric drum of 0.3 m diameter as shown in Fig. Ex. 9.11. If the frictional moment in the bearing is estimated to be 50 N m, determine the moment of inertia of the flywheel if the velocity of the suspended mass after a fall of 1 m is estimated to be 0.3 m/s.

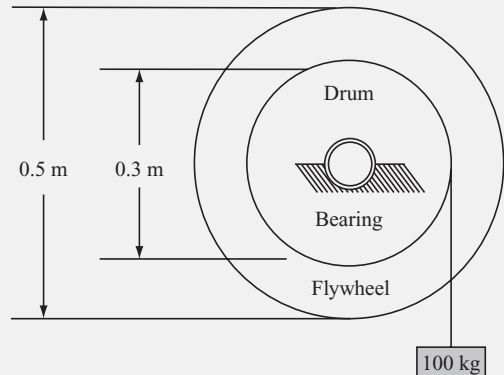


Fig. Ex. 9.11

Solution Let us consider the motion of the system of the flywheel together with the drum and suspended mass. The free-body diagram of the system is shown in Fig. Ex. 9.11 (Solution). Note that the tension in the string does not enter the picture.

The work done by the weight of the suspended mass over a fall of 1.0 m is given by

$$mgh = 100 \times 9.81 \times 1.0 = 981 \text{ N m}$$

and the negative work done by the frictional moment of 50 N m during an angular displacement of

$$\theta = \frac{1.0}{0.15} = 6.67 \text{ rad}$$

is given by

$$M\theta = -50 \times 6.67 = -333.5 \text{ N m}$$

Total work done by the net external force and the net external moment acting on the system is

$$W_{\text{total}} = 981 - 333.5 = 647.5 \text{ N m}$$

The final total kinetic energy of the system must be the sum of the rotational kinetic energy of the flywheel together with the drum and the translational kinetic energy of the suspended mass.

$$\begin{aligned} KE_{2\text{total}} &= \frac{1}{2} \times 100 \times (0.3)^2 + \frac{1}{2} I_c \left(\frac{0.3}{0.15} \right)^2 \\ &= 4.5 + 2 I_c \end{aligned}$$

Since the assembly started from rest, the initial kinetic energy must be zero. The total change in kinetic energy is also given by

$$(KE_2 - KE_1)_{\text{total}} = 4.5 + 2 I_c$$

Using the work energy principle,

$$4.5 + 2 I_c = 647.5$$

whence

$$I_c = 321.5 \text{ kg m}^2$$

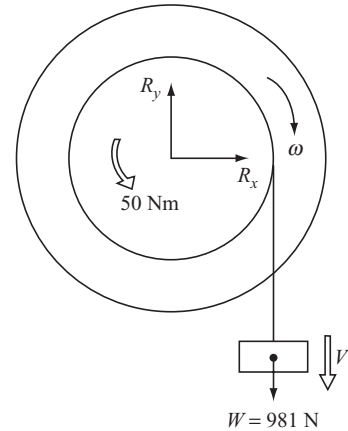


Fig. Ex. 9.11 (Solution)

Example 9.12

The following bodies are released from rest on an incline at the same elevation. In each case, the mass is m and the maximum radius is R . (a) A solid sphere, (b) a hollow sphere of inner radius $R/2$, (c) a solid cylinder, (d) a hollow cylinder of inner radius $R/2$ and (e) a hoop.

Determine the velocity of each body after it has rolled down the incline through the same distance s .

What would have been the velocity of each body if the incline was frictionless and the bodies slid instead of rolling?

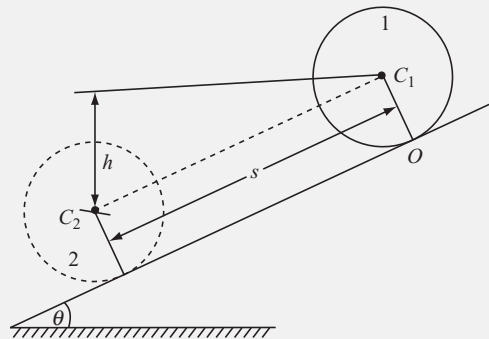


Fig. Ex. 9.12

Solution Consider the *rolling motion of a cylindrical body* in general. The motion takes place in the gravitational field of the earth and the frictional forces do no work in the rolling motion; the mechanical energy of the system is, therefore, conserved.

$$\begin{aligned} (KE + PE)_1 &= (KE + PE)_2 \\ KE_1 &= 0 \end{aligned}$$

$PE_1 = 0$; initial position referred as datum

$$KE_2 = \frac{1}{2} mV_{c_2}^2 + \frac{1}{2} I_c \omega_2^2$$

and

$$PE_2 = -mgh$$

Substituting

$$\begin{aligned} \omega_2 &= V_{c_2}/R \\ h &= s \sin \theta \end{aligned}$$

and from the conservation of mechanical energy,

$$0 = \frac{1}{2} mV_{c_2}^2 + \frac{1}{2} I_c \frac{V_{c_2}^2}{R^2} - mgh$$

whence

$$V_{c_2} = \frac{\sqrt{2gh}}{\sqrt{1 + \frac{I_c}{mR^2}}} = \frac{\sqrt{2gs \sin \theta}}{\sqrt{1 + \frac{I_c}{mR^2}}}$$

Introducing the appropriate values of the moments of inertia for each case,

| Case | Body | I_c | V_{c_2} |
|------|-----------------|---|-------------------------------|
| (a) | Solid sphere | $\frac{2}{5} mR^2$ | $0.845\sqrt{2gs \sin \theta}$ |
| (b) | Hollow sphere | $\frac{2}{5} m \frac{R^5 - (R/2)^5}{R^3 - (R/2)^3}$ | $0.832\sqrt{2gs \sin \theta}$ |
| (c) | Solid cylinder | $\frac{1}{2} mR^2$ | $0.816\sqrt{2gs \sin \theta}$ |
| (d) | Hollow cylinder | $\frac{1}{2} m(R^2 + (R/2)^2)$ | $0.785\sqrt{2gs \sin \theta}$ |
| (e) | Hoop | mR^2 | $0.707\sqrt{2gs \sin \theta}$ |

If the incline was frictionless, the bodies would not be able to roll down. Frictionless sliding through an inclined distance s would be equivalent to a free fall through a vertical distance h

$$h = s \sin \theta$$

The velocity of each body would then be

$$\sqrt{2gh} = \sqrt{2gs \sin \theta}$$

A comparison of the final velocities of the bodies is indeed meaningful. Frictionless fall down the incline results in the maximum possible velocity. The final velocities in the other cases are less *not because of frictional loss of energy but because of the energy stored in rotation of the bodies*. The rotational energy depends on the moment of inertia of the bodies and for this reason their final velocities differ. Of the given bodies of the same mass m and same outer radius R , the solid sphere has the least moment of inertia and hence the greatest final velocity. On the other extreme, in the case of a hoop, the entire mass is concentrated at the radius R which makes its moment of inertia the maximum and its final velocity the least.

The results can be interpreted in another way, i.e., if the bodies in the list were allowed to start from rest at the same level simultaneously, the acceleration of the bodies would be such that the velocity at a later instant would be determined in the analysis. The body which accelerates most reaches first. The solid

sphere will, therefore, reach the finishing point of the race first and then the hollow sphere, solid cylinder and hollow cylinder in that order. The hoop will be the last to reach. It is also interesting to note that the final velocities attained over a given distance along the slope or the accelerations acquired by the bodies are *independent of their mass and size*. In other words, whether a sphere is small or large, light or heavy, it will reach the finishing point before the other bodies do. It also follows that if a number of spheres of different sizes and mass densities are allowed to go down an incline starting from rest simultaneously, they will all acquire the same acceleration and reach the finishing point simultaneously.

Let us also demonstrate the application of the dynamical equations in terms of forces and moments for this problem. With reference to the free-body diagram shown in Fig. Ex. 9.12 (Solution) the dynamical equations may be written as

$$F_x = mg \sin \theta - f = ma_{cx} = ma_c \quad (i)$$

$$F_y = mg \cos \theta - R = ma_{cy} = 0 \quad (ii)$$

$$M_c = -f R = I_c \alpha = I_c \left(-\frac{a_c}{R} \right) \quad (iii)$$

whence, $f = I_c \frac{a_c}{R^2}$

and on substitution in Eq. (i)

$$mg \sin \theta - I_c \frac{a_c}{R^2} = ma_c$$

whence, $a_c = \frac{mg \sin \theta}{(I_c/R^2) + m} = \frac{g \sin \theta}{1 + (I_c/mR^2)}$

In view of the fact that the acceleration of the mass centre C must be constant,

$$V_{c2}^2 - V_{c1}^2 = 2a_c s$$

Since the bodies start from rest,

$$V_{c1} = 0$$

$$V_{c2} = \sqrt{2a_c s}$$

$$= \sqrt{\frac{2gs \sin \theta}{1 + I_c/mR^2}}$$

which is the same expression as that obtained by the conservation of energy principle.

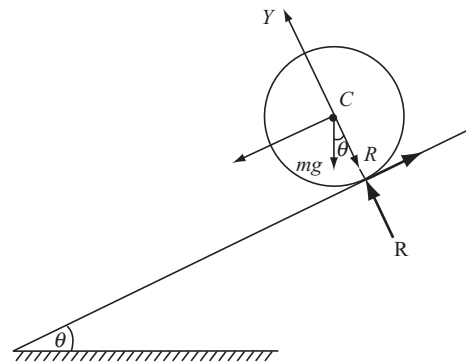


Fig. Ex. 9.12 (Solution)

Example 9.13

A uniform slender bar AB of mass m and length L supported by a frictionless pivot at A is released from rest at its near-vertical position as shown in Fig. Ex. 9.13. Calculate the reaction at the pivot when the bar just acquires the horizontal position shown dotted. If, at this instant, the bar is released from its support gently and allowed to move for t seconds further, estimate its angular speed and the velocity of the centre of mass at that instant.

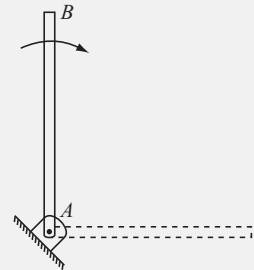


Fig. Ex. 9.13

Solution Applying the energy conservation principle between the vertical position (1) and the horizontal position (2),

$$(KE + PE)_1 = (KE + PE)_2$$

$$(0 + mgL/2) = \left(\frac{1}{2} I_c \omega^2 + \frac{1}{2} mV_c^2 + 0 \right)$$

Using the facts that

$$I_c = \frac{mL^2}{12} \quad \text{and} \quad V_c = \omega L/2,$$

$$\omega^2 = 3g/L \quad \text{or} \quad \omega = \sqrt{3g/L}$$

in the clockwise direction.

Considering the free-body diagram at the instant when it just acquires the horizontal position (Fig. Ex. 9.13 (Solution)).

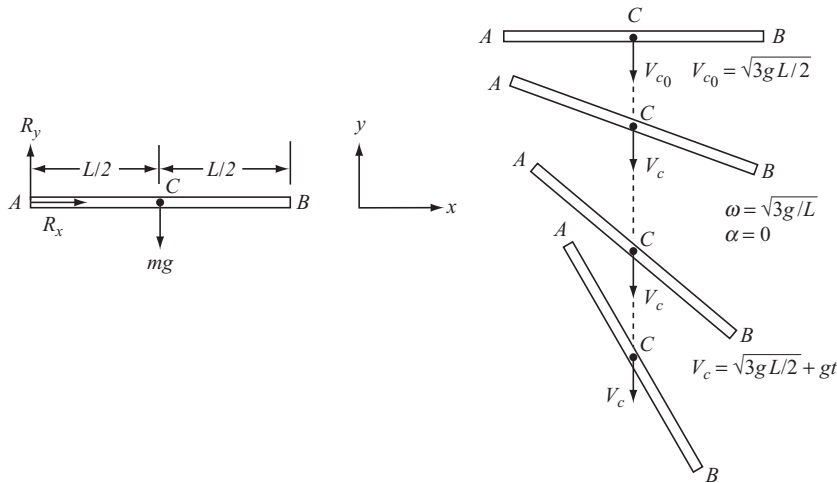


Fig. Ex. 9.13 (Solution)

For the centre of mass of the bar,

$$R_x = ma_{cx} \tag{i}$$

$$R_y - mg = ma_{cy} \tag{ii}$$

and by the Euler's equation applied at A,

$$-mg L/2 = I_A \alpha = \frac{mL^2}{3} \alpha \tag{iii}$$

From Eq. (iii), the angular acceleration

$$\alpha = -3g/2L$$

From kinematics, it is known that

$$a_{cx} = -\omega^2 L/2$$

and

$$a_{cy} = \alpha L/2$$

Substituting these values in Eqs. (i) and (ii)

$$R_x = -3mg/2 \quad \text{and} \quad R_y = mg/4$$

After the bar is gently released from the support, it is left with only one force on it acting at its centre of mass. The angular acceleration vanishes and the angular velocity would remain constant at

$$\omega = \sqrt{3} / l$$

The bar will thus continue to rotate with this angular velocity and its centre of mass will drop vertically down with an acceleration downwards as shown in Fig. Ex. 9.13 (Solution).

The velocity of the centre of mass at the instant it was released is

$$v_0 = \omega \cdot l/2 = \sqrt{3} \cdot l/2 \text{ downwards.}$$

At any time t later, the downward velocity is given by

$$v = v_0 + at$$

$$= \sqrt{3} \cdot l/2 +$$

Example 9.14

A carpet of mass m made of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $R/2$.

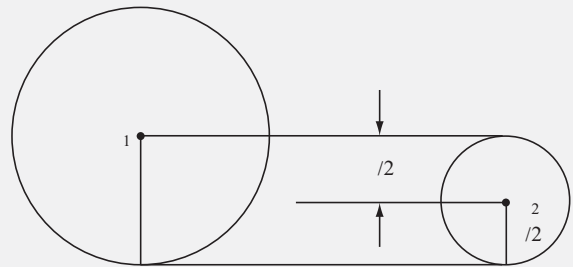


Fig. Ex. 9.14

Solution

The potential energy thus released is converted to the kinetic energy of the rolling part, the flat part being at rest.

Initial potential energy =

Potential energy at the point of interest

$$= \frac{m}{4} \cdot \frac{l}{2} = \frac{m \cdot l}{8}$$

because the rolling mass must be quarter fraction of the initial mass.

Release of potential energy = $7/8$

The kinetic energy at this instant must be given by

$$= \frac{1}{2} \cdot \frac{m}{4} \cdot v^2 + \frac{1}{2} \cdot I \cdot \omega^2$$

$$= \frac{m}{8} v^2 + \frac{1}{2} \cdot \frac{m}{4} \cdot \left(\frac{l}{2}\right)^2 \cdot \left(\frac{v}{l/2}\right)^2 = 3/16 m v^2$$

Hence, $\frac{3}{16} m v^2 = 7/8 m l$

and $v = \sqrt{14/3} \cdot l$

Example 9.15

A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part as shown in Fig. Ex. 9.15. The horizontal part is 1.0 metre above the ground level and the top of the track is 2.4 metres above the ground.

Find the distance on the ground with respect to the point B (which is vertically below the end of the track as shown) where the sphere lands. During its flight as a projectile, does the sphere continue to rotate about its centre of mass? Explain.

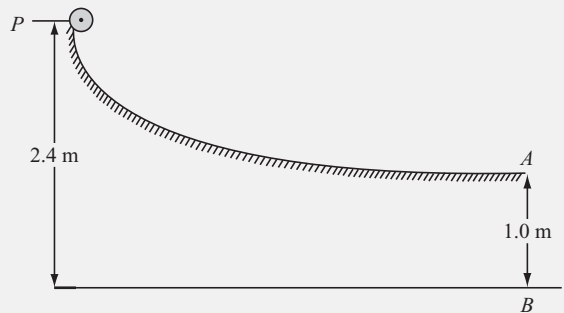


Fig. Ex. 9.15

Solution At the initial point P,

$$KE_1 = 0, PE_1 = 2.4 mg$$

At the end point A,

$$\begin{aligned} KE_2 &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}mV^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\frac{V^2}{R^2} \\ &= \frac{1}{2}mV^2 + \frac{1}{5}mV^2 = \frac{7}{10}mV^2 \end{aligned}$$

$$PE_2 = 1.0 mg$$

By conservation of energy in the absence of dissipation,

$$0 + 2.4 mg = \frac{7}{10}mV^2 + 1.0 mg;$$

whence,

$$V = \sqrt{2g}$$

After leaving A, the sphere would continue to rotate at the same rotational speed in the absence of any external moment acting on it.

Then,

$$z = -\frac{g x^2}{2V_0^2}$$

Therefore,

$$x = \sqrt{2 \times 2g \times 1/g} = 2.0 \text{ m.}$$

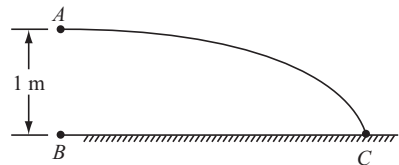


Fig. Ex. 9.15 (Solution)

Example 9.16

State whether the following statement is TRUE or FALSE. Give very brief reasons in support of your answer.

A ring of mass 0.3 kg and radius 0.1 m and a solid cylinder of mass 0.4 kg and of the same radius are given the same kinetic energy and released simultaneously on a flat horizontal surface such that they begin to roll as soon as released towards a wall which is at the same distance from the ring and the cylinder as shown in Fig. Ex. 9.16. The rolling friction in both cases is negligible. The cylinder will reach the wall first.

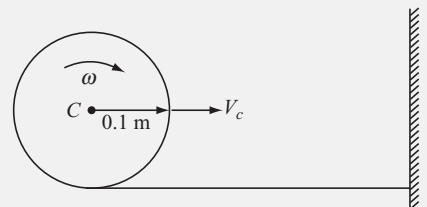


Fig. Ex. 9.16

Solution For the ring,

$$m = 0.3 \text{ kg}$$

$$r = 0.1 \text{ m}$$

$$I_c = 0.3 \times 0.1^2 = 0.003 \text{ kg m}^2$$

For the ring,

$$\begin{aligned} KE &= \frac{1}{2} \times 0.3 \times V_a^2 \\ &+ \frac{1}{2} \times 0.003 \times \left(\frac{V_c}{0.1} \right)^2 \\ &= 0.5 V_c^2 + 0.15 V_c^2 = 0.3 V_c^2 \end{aligned}$$

For the cylinder,

$$m = 0.4 \text{ kg}$$

$$r = 0.1 \text{ m}$$

$$I_c = 0.4 \times \frac{0.1^2}{2} = 0.002 \text{ kg m}^2$$

For the cylinder,

$$\begin{aligned} KE &= \frac{1}{2} \times 0.4 \times V_c^2 \\ &+ \frac{1}{2} \times 0.002 \times \left(\frac{V_c}{0.1} \right)^2 \\ &= 0.2 V_c^2 + 0.1 V_c^2 = 0.3 V_c^2 \end{aligned}$$

Since they have the same kinetic energy at the start, the expressions for KE require that they have the same velocity as well. There being no change in potential energy and negligible friction, they will reach the wall simultaneously.

The given statement is, therefore, FALSE.

Example 9.17

A sphere of radius 0.5 m and mass 10 kg is released gently from rest on a 30° incline as shown in Fig. Ex. 9.17. If it rolls without slipping, determine the minimum coefficient of friction compatible with the rolling motion. What would be the velocity of its centre of mass after it rolled down 5 m distance.

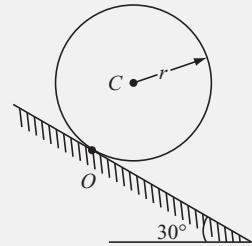


Fig. Ex. 9.17

Solution Considering the free-body diagram of the sphere as shown in Fig. Ex. 9.17 (Solution) rolling without slipping,

$$a = r\alpha$$

$$R = mg \cos \theta$$

$$mg \sin \theta - f = ma$$

$$mg \sin \theta \cdot r = ma \cdot r + I\alpha$$

or $mg \sin \theta r = m \alpha r^2 + 2/5 mr^2 \cdot \alpha$

whence,

$$\alpha = \frac{5g \sin \theta}{7r} = \frac{5 \times 9.81 \times \sin 30^\circ}{7 \times 0.5} = 7 \text{ rad/s}^2$$

and $a = 0.5 \times 7 = 3.5 \text{ m/s}^2$

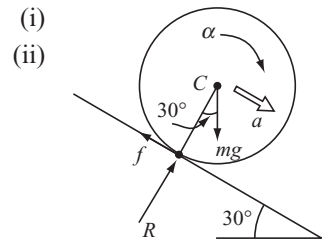


Fig. Ex. 9.17 (Solution)

From (ii), $f = 10 \times 9.81 \times \sin 30 - 10 \times 3.5 = 14.05 \text{ N}$

For $f = \mu R = \mu \cdot mg \cos \theta$

$$\mu = 14.05 / (10 \times 9.81 \times \cos 30) = 0.165$$

Since the acceleration of the centre of mass is constant, 3.5 m/s^2 , the velocity 5 m down the plane is such that

$$v^2 - 0 = 2 \times 3.5 \times 5$$

whence

$$v = 5.91 \text{ m/s}$$

Let us determine the velocity of the sphere if it was sliding all the way on a frictionless plane. Then $f = 0$,

$$mg \sin \theta = ma, \quad \alpha = 0$$

$$a = g \sin \theta$$

$$v^2 - 0 = 2 \times 9.81 \times \sin 30 \times 5$$

$$v = 7 \text{ m/s}$$

This is more than the velocity of the rolling sphere but there is no rotational velocity in this case. Let us compare the kinetic energies in the two cases, i.e., with rotation and frictionless sliding. In the former,

$$KE = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} mv^2 (1 + 2/5) = \frac{7}{10} \times 10 \times 5.91^2 = 245 \text{ J}$$

In the latter,

$$KE = \frac{1}{2} \times 10 \times 7^2 \approx 245 \text{ J}$$

The kinetic energy (as also the potential energy) is the same in both cases. It is indeed so because pure rolling does not entail loss of energy.

Let us do a little reflection. If it was a cylinder instead of a sphere, the moment equation would be

$$mg \sin \theta \cdot r = m \alpha r^2 + m r^2 / 2 \cdot \alpha = 3/2 m r^2 \alpha$$

whence

$$\alpha = \frac{2g \sin \theta}{3r} = \frac{2 \times 9.81 \times \sin 30}{3 \times 0.5} = 6.54 \text{ rad/s}$$

$$a = 0.5 \times 6.54 = 3.27 \text{ m/s}^2$$

Then,

$$f = 10 \times 9.81 \times \sin 30 - 10 \times 3.27 = 16.35 \text{ N}$$

and

$$\mu = 16.35 / (10 \times 9.81 \times \cos 30) = 0.192$$

which means that the surface must be more rough for the rolling of a cylinder than for a sphere. In other words, if an incline is rough enough to cause a cylinder to roll without slip, a sphere of the same mass and radius would surely roll without slip! Extending the same analysis to a hoop of the same mass and radius, $\mu = 0.288$ implies that the hoop is even more likely to slip than a cylinder on the same slope.

9.6 IMPULSE-MOMENTUM FORMULATION FOR PLANE MOTION

The equations of motion due to Newton and Euler may be integrated with respect to time to constitute an alternative set of equations for studying the dynamics of a rigid body.

The statement of the Newton's law

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt}$$

integrated with respect to time t over the limits t_1 to t_2 results in

$$\int_{t_1}^{t_2} F dt = m(v_2 - v_1) \quad (9.19)$$

The left-hand side of Eq. (9.19) is the linear impulse due to the net external force over the period of action whereas the right-hand side is the change in linear momentum of the centre of mass over the same interval of time. The equation is, therefore, called the *linear impulse-momentum equation*, sometimes referred simply as the impulse-momentum equation.

The Euler's equation,

$$M_c = I_c \alpha = I_c \frac{d\omega}{dt}$$

on integration with respect to time t over the limits t_1 and t_2 results in

$$\int_{t_1}^{t_2} M_c dt = I_c (\omega_2 - \omega_1) \quad (9.20)$$

The left-hand side of Eq. (9.20) is the angular impulse due to the net moment M_c acting about the centre of mass C over the period of action, whereas the right-hand side is the change in angular momentum about the centre of mass over the same interval of time. The equation is, therefore, known as the *angular impulse-momentum principle*. It may be noted that for the plane motion of a rigid body, the angular impulse as well the angular momentum about the centre of mass must only be about the axis of rotation through C .

There is no doubt that *the principles of linear impulse-momentum and angular impulse-momentum applied to a rigid body are mutually independent principles*; one not derivable from the other and both of them are required jointly to tackle a problem on rigid-body dynamics in just the same way as Newton's law and Euler's equation are used. Preference for the impulse-momentum principles over the said laws is natural when concerned with the gross effect of the change in linear and angular velocity over a period of time. In particular, when the net external force or net external moment is zero, the corresponding impulse-momentum principle reduces to momentum conservation and may be used more conveniently. A system of two bodies in an encounter, say an impact, have no external force or moment acting upon them; the momentum conservation principle may be applied to advantage to the system of the two bodies. The momentum conservation principles for two bodies may be stated as follows:

Linear-momentum conservation principle:

$$m_1 V_{c_1} + m_2 V_{c_2} = m_1 V'_{c_1} + m_2 V'_{c_2} \quad (9.21)$$

Angular-momentum conservation principle:

$$I_{c_1} \omega_1 + I_{c_2} \omega_2 = I_{c_1} \omega'_1 + I_{c_2} \omega'_2 \quad (9.22)$$

where suffixes 1 and 2 refer to the two bodies; unprimed variables for conditions just before and primed variables for conditions just after the phenomenon.

Example 9.18

A uniform circular cylinder of mass m and radius r is given an initial angular velocity ω_0 and no initial translational velocity as shown in Fig. Ex. 9.18. It is placed in contact with a plane inclined at α to the horizontal and it moves up. Assuming the coefficient of friction μ for sliding between the cylinder and the plane, find the distance the cylinder moves before sliding stops. Assume that μ is greater than $\tan \alpha$.

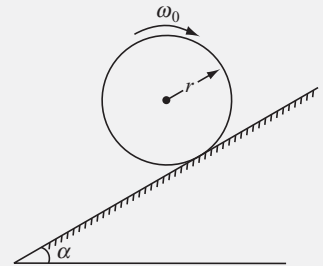


Fig. Ex. 9.18

Solution Consider the free-body diagram of the cylinder as shown in Fig. Ex. 9.18. (Solution). Let us choose the centre of mass O as the reference point and use the angular impulse momentum equation to determine the time t required to decrease the velocity to a value ω .

$$-\mu R r t = I (\omega - \omega_0) = \frac{m r^2}{2} (\omega - \omega_0)$$

Since

$$\begin{aligned} R &= m g \cos \alpha \\ t &= \frac{r (\omega_0 - \omega)}{2 \mu g \cos \alpha} \end{aligned} \quad (i)$$

The velocity of the centre O parallel to the plane is obtained by applying the linear impulse momentum equation.

$$m g (\mu \cos \alpha - \sin \alpha) t = m (v - 0) \quad (ii)$$

with positive direction up the plane.

At the instant sliding stops and pure rolling begins,

$$v = r \omega.$$

From (i) and (ii),

$$\omega = \frac{\mu \cos \alpha - \sin \alpha}{3 \mu \cos \alpha - \sin \alpha} \cdot \omega_0$$

and

$$t = \frac{r \omega_0}{g (3 \mu \cos \alpha - \sin \alpha)}$$

The forces acting on the cylinder remain constant during sliding and hence acceleration is constant. The distance travelled while sliding must be

$$\begin{aligned} d &= 1/2 v t \\ &= \frac{r^2 \omega_0^2 (\mu \cos \alpha - \sin \alpha)}{2 g (3 \mu \cos \alpha - \sin \alpha)^2} \end{aligned}$$

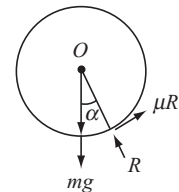


Fig. Ex. 9.18 (Solution)

Example 9.19

Determine the angular momentum of a uniform thin bar of length L and mass m when it is rotating at a constant angular velocity ω

- (a) about its centre of mass
- (b) about one end provided that it stays in the same plane.

Solution The mass per unit length of the bar is m/Lh .

For case (a), the element of mass ($m/L \cdot dr$) shown in Fig. Ex. 9.19 (Solution) rotates at a velocity $r\omega$; its angular momentum is $r m/L r\omega dr$.

Integrating from $-L/2$ to $+L/2$, for the whole bar,

$$H_c = \int_{-L/2}^{+L/2} \frac{m\omega}{L} r^2 dr = \frac{m\omega}{L} \left(\frac{r^3}{3} \right)_{-L/2}^{+L/2} = 1/12 mL^2 \omega$$

$$= 1/12 mL^2 \omega = I_c \omega$$

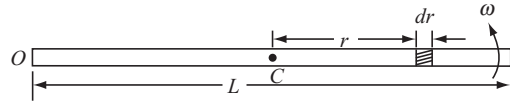


Fig. Ex. 9.19 (Solution)

where I_c is the moment of inertia of the bar about its centroid.

Similarly, for case (b), considering the origin at O and locating the element dr at a distance r for O ,

$$H_c = \int_0^L \frac{m\omega}{L} r^2 dr = 1/3 mL^2 \omega = I_0 \omega.$$

Example 9.20

A uniform bar AD of length 2 m and mass 5 kg hanging freely from a frictionless pivot at A is struck by a 25 g bullet approaching at a velocity of 500 m/s as shown in Fig. Ex. 9.20. The bullet pierces through the bar and emerges with a velocity 40% of its initial value.

Determine the angular velocity of the bar just after the bullet emerges and the maximum angle through which the bar would swing. Comment on the total loss of energy in the process.

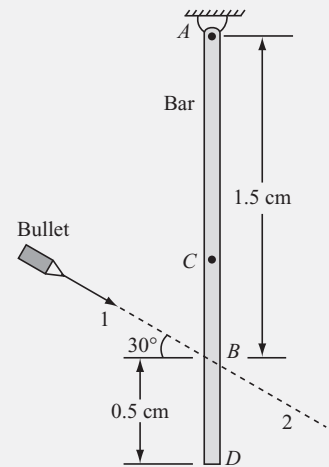


Fig. Ex. 9.20

Solution The free-body diagrams are drawn (Fig. Ex. 9.20 (Solution)) or simply visualised in order to decide whether to apply the linear impulse-momentum principle or the angular impulse-momentum principle.

The fact that there is going to be an unknown impulsive reaction at the pivot A rules out the conservation of linear momentum along any axis. From the fact that the moment of impulsive reaction at A taken about A is zero, the angular impulse momentum principle about A can be employed with advantage. It should be understood that there is no external force acting at B as far as the system of bullet and bar is concerned. The bullet exerts a force on the bar and the bar reacts it undiminished on the bullet.

The net angular impulse must equal the change in the angular momentum. Initially, the angular impulse or the moment of linear momentum taken about A is

$$0.025 \times 500 \cos 30^\circ \times 1.5 = 16.24 \text{ kg m}^2/\text{s}$$

Finally, the angular impulse or the summation of the moments of the linear momentum of the system about A is

$$0.025 \times (500 \times 0.4) \cos 30^\circ \times 1.5 + 5 \times 1^2 \times \omega = (6.5 + 5\omega) \text{ kg m}^2/\text{s}$$

Net angular impulse about A is

$$16.25 - (6.5 + 5\omega) = (9.75 - 5\omega) \text{ kg m}^2/\text{s} \quad (i)$$

Starting from rest, the bar acquires an angular velocity so that the change in the angular momentum is

$$I_c \omega = \frac{1}{2} \times 5 \times 2^2 \omega = 1.67\omega \text{ kg m}^2/\text{s} \quad (ii)$$

Equating Eqs. (i) and (ii)

$$9.75 - 5\omega = 1.67\omega$$

whence $\omega = 1.46 \text{ rad/s}$

Alternatively, the angular impulse-momentum principle applied to the bar can be interpreted as,

Net angular impulse imparted by the bullet on the bar = Change in angular momentum of the bar about its centre of mass

The left-hand side term is

$$0.025 \times (500 - 0.4 \times 500) \cos 30^\circ \times 1.5 = 9.75 \text{ kg m}^2/\text{s}$$

and the right-hand side term is

$$\frac{1}{3} \times 5 \times 2^2 (\omega - 0) = 6.67\omega \text{ kg m}^2/\text{s}$$

which when equated, provide $\omega = 1.46 \text{ rad/s}$, the same result as obtained earlier.

The maximum swing of the bar can be calculated by applying the energy conservation principle because there are no dissipative actions from the instant the bullet emerges out of the bar to the instant of maximum swing. Considering the initial position as the reference position,

$$PE_1 = 0$$

$$KE_1 = \frac{1}{2} \times \left(\frac{1}{3} \times 5 \times 2^2 \right) \times 1.46^2 = 7.1$$

Finally, the bar swings through an angle θ and the centre of gravity is lifted by

$$h = \frac{2}{2} \times (1 - \cos \theta) = (1 - \cos \theta)$$

$$PE_2 = 5 \times 9.81 \times (1 - \cos \theta) \\ = 49.05(1 - \cos \theta)$$

and $KE_2 = 0$

because the bar comes to momentary rest at the position of maximum swing.

By energy conservation,

$$PE_1 + KE_1 = PE_2 + KE_2 \\ 7.1 = 49.05(1 - \cos \theta)$$

whence $\theta = 31.2^\circ$

In order to estimate the total loss of energy in the process, consider the energy in the system just before and after impact.

Just before the impact, only the bullet has kinetic energy given by

$$\frac{1}{2} \times 0.025 \times 500^2 = 3125$$

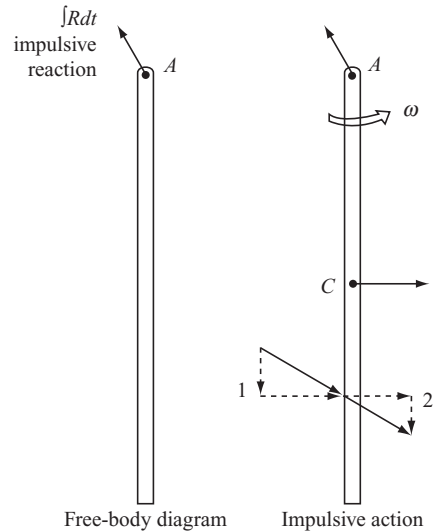


Fig. Ex. 9.20 (Solution)

Just after the impact, the bullet has a part of its kinetic energy and the bar has kinetic energy given by

$$\frac{1}{2} \times 0.025 \times 200^2 + \frac{1}{2} \times \left(\frac{1}{3} \times 5 \times 2^2 \right) \times 1.46^2 = 507.1 \text{ J}$$

Loss in energy of given by

$$3125 - 507.1 = 2617.9 \text{ J}$$

which is

$$\frac{2617.9}{3125} \times 100 = 83.8$$

of the initial energy of the bullet.

The percentage energy imparted to the pendulum in moving the pendulum is

$$\frac{7.1}{3125} \times 100 = 0.23$$

which is indeed a very small fraction of the initial energy of the bullet.

It is the mechanical energy which is lost; in fact, it is converted into heat and sound and thus dissipated.

If it is desired to determine the impulsive reaction at A , it is now possible to do so by applying the linear impulse momentum principle.

$$\begin{aligned} \int R_x dt &= 0.025 (500 - 200) \cos 30^\circ - 5 \times 1 \times 1.46 \\ &= -0.8 \text{ N s} \\ \int R_y dt &= 0.025 (500 - 200) \sin 30^\circ \\ &= 3.75 \text{ N s} \end{aligned}$$

It may be appreciated that the impulsive reaction will not be parallel to the velocity of the bullet because the final momentum of the bar also contributes to the impulsive reaction as seen above.

EXPERIMENT E10

CONSERVATION OF ANGULAR MOMENTUM — A HUMAN EXPERIENCE

OBJECTIVE TO UNDERGO A HUMAN EXPERIENCE OF THE PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM.

Apparatus Two equal handy masses and a turn-table for a person to stand on it, as shown in Fig. E10.1.

Background Information The angular momentum H of a body of mass M rotating in a circle of radius r about an axis is given by

$$H = rMv = Mr^2\omega$$

where v is the linear velocity of the mass and ω its angular velocity about the axis of rotation.

The angular momentum of a body is conserved if the body is not acted upon by an external moment. In that case

$$Mr^2\omega = \text{Const.} \tag{E10.1}$$

If, therefore, a mass M is made to rotate along a circle of radius r_1 at an angular velocity ω_1 is shifted to rotate along a circle of radius r_2 without the action of an external moment, the new angular speed ω_2 is given by

$$Mr_1^2 \omega_1 = Mr_2^2 \omega_2; \quad \omega_2 = \frac{r_1^2 \omega_1}{r_2^2} \tag{E10.2}$$

Observation of a Human Experience If a person holding two equal masses with his/her hands stretched while standing on a frictionless turn-table, as shown in Fig. E10.1 is set into motion with an angular velocity ω_1 about the vertical axis, he/she should continue to rotate at the same angular velocity. If now he/she retracts his/her arms to bring the masses as close to the vertical axis as possible, he/she would experience an increase in the rotational speed ω_2 .

It may be noticed that the rotational speed of the mass increases inversely as the square of the radius of rotation if only the masses were to rotate. When the masses are held by a person whose own moment of inertia about the vertical axis is I_1 , the initial angular momentum is

$$L_1 = (I_1 + 2ml^2)\omega_1$$

and the final angular momentum after the masses are retracted is

$$L_2 = (I_2 + 2mr^2)\omega_2$$

Equating the angular momenta in the absence of any external moment

$$(I_1 + 2ml^2)\omega_1 = (I_2 + 2mr^2)\omega_2$$

whence
$$\frac{\omega_2}{\omega_1} = \frac{I_1 + 2ml^2}{I_2 + 2mr^2} \quad (E10.3)$$

It may be seen from Eq. (E10.3) that the ratio of angular speeds ω_2/ω_1 would be more if the moment of inertia I_2 of the person is minimised.

$$\omega_2 > \omega_1$$

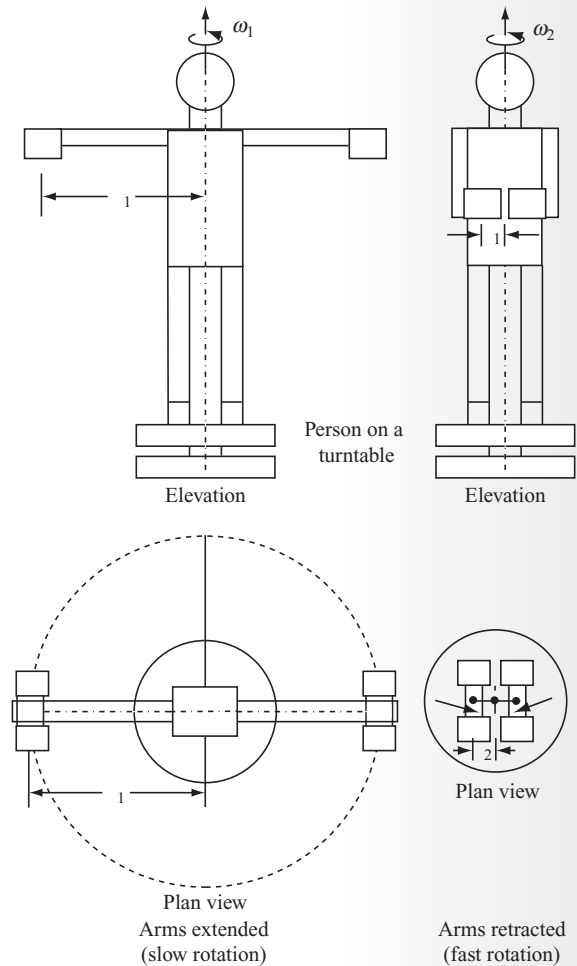


Fig. E10.1 Human experience of conservation of angular momentum

Points for Discussion

From the definition of the angular momentum for the general case of motion of a body, i.e.,

$$L = r \times mv$$

obtain the expression

$$L = I\omega$$

as defined for the present case.

Can you suggest some other experiment to verify the angular momentum conservation principle?

Can you use the apparatus of this experiment to determine the moment of inertia of a person about the vertical axis? If, so determine your own moment of inertia about the vertical axis.

Example 9.21

A cylinder of radius r and mass m rests on a rough horizontal rug. If the rug is pulled from under it with an acceleration A perpendicular to the axis of the cylinder, describe the motion of the centre of mass of the cylinder assuming that it does not slip.

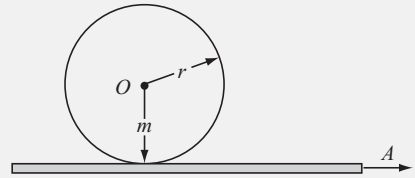


Fig. Ex. 9.21

Solution Consider the free-body diagram of the cylinder as shown in Fig. Ex. 9.21 (Solution) and taking O as the reference point, taking f as the force transmitted by the pull of the rug

$$f = ma \tag{i}$$

$$f \cdot r = I_0 \alpha = \frac{1}{2} mr^2 \cdot \alpha \tag{ii}$$

and for no slip, $a + r\alpha = A$ (iii)

From (i) and (ii),

$$a = \frac{1}{2} r \alpha \tag{iv}$$

Now, from (iii) and (iv),

$$\alpha = 2/3 A/r$$

and from (iii), again,

$$a = A/3$$

These two equations, i.e.,

$$a = \frac{dv}{dt} = A/3$$

and

$$\alpha = \frac{d\omega}{dt} = 2/3 A/r$$

together describe the motion of the centre of mass of the cylinder.

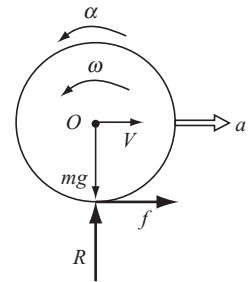


Fig. Ex. 9.21 (Solution)

Example 9.22

A vehicle of mass m negotiates a circular path of radius r with a constant speed v . At what speed does the vehicle tend to overturn and at what speed would the vehicle slip sideways?

How would the safe speed for negotiating a curve change if the road is banked at an angle θ ?

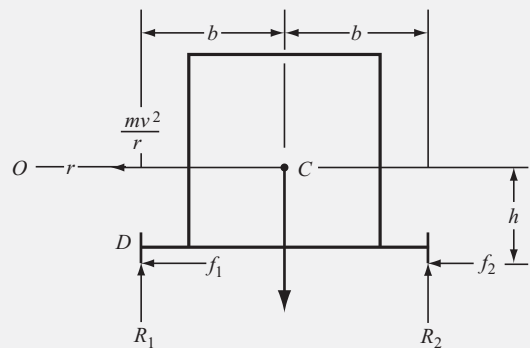


Fig. Ex. 9.22

Solution From the free-body diagram of the vehicle as shown in Fig. Ex. 9.22.

$$R_1 + R_2 - mg = 0$$

$$f_1 + f_2 = m \frac{v^2}{r}$$

and, by taking moments about the contact point P of the inner wheel,

$$R_2 \cdot 2b - mg \cdot b = m \frac{v^2}{r} \cdot h$$

From these equations,

$$R_1 = \frac{mg}{2} - \frac{h}{2b} \cdot m \frac{v^2}{r}$$

$$R_2 = \frac{mg}{2} + \frac{h}{2b} \cdot m \frac{v^2}{r}$$

The vehicle tends to overturn when the reaction at the inner wheel becomes zero and the entire weight is taken by the outer wheel.

$$R_1 = \frac{mg}{2} - \frac{h}{2b} \cdot m \frac{v^2}{r} = 0$$

or

$$v = \sqrt{b/h gr} \quad (i)$$

It shows that the safe velocity limit is higher for

- large b , i.e., wide-base vehicles
- low h , i.e., low centre of mass
- high r , i.e., curve of large radius

and for a location with a higher value of g !

The vehicle would slip side ways when the lateral force equals the frictional resistance.

$$m \frac{v^2}{r} = \mu(R_1 + R_2)$$

$$= \mu mg$$

whence $v = \sqrt{\mu gr} \quad (ii)$

Comparing (i) and (ii), it is observed that on a smooth road, i.e., μ less than b/h , the vehicle will skid rather than overturn. For a rough road, μ greater than b/h , the vehicle tends to overturn than slip sideways.

Let us now consider the free-body diagram of the vehicle turning on a banked road (as shown in Fig. Ex. 9.22 (Solution)).

$$f_1 + f_2 + mg \sin \theta = m \frac{v^2}{r} \cos \theta$$

$$R_1 + R_2 - mg \cos \theta = m \frac{v^2}{r} \sin \theta$$

$$R_2 \cdot 2b - mg(b \cos \theta - h \sin \theta) = m \frac{v^2}{r} (b \sin \theta + h \cos \theta)$$

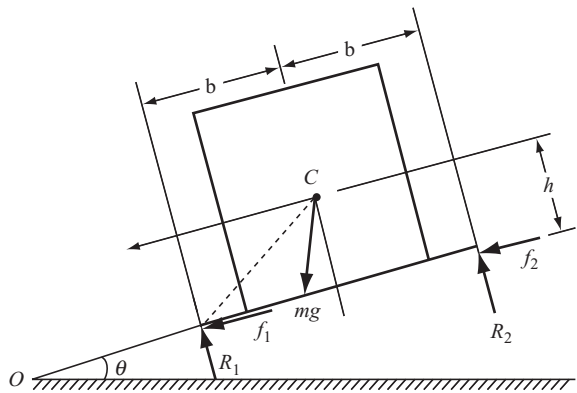


Fig. Ex. 9.22 (Solution)

For the condition of overturning, $R_1 = 0$,

$$v = \sqrt{\frac{1 + h/b \tan \theta}{1 - b/h \tan \theta}} \cdot b/h gr$$

It shows that the 'safe' velocity in respect of overturning increases by a factor of

$$= \sqrt{\frac{1 + h/b \tan \theta}{1 - b/h \tan \theta}} \quad \text{(iii)}$$

The angle of banking θ is usually kept at 15 to 30 degrees depending on the radius of the turn and the intended velocity. *For racing cars, roads are banked at higher angles at the turns.*

Now, a vehicle would slip sideways when

$$f_1 + f_2 = \mu (R_1 + R_2)$$

which, upon substitution of values gives,

$$v = \sqrt{\frac{1 + 1/\mu \tan \theta}{1 - \mu \tan \theta}} \mu gr$$

It shows that the safe velocity in respect of slipping sideways increases by a factor of

$$= \sqrt{\frac{1 + 1/\mu \tan \theta}{1 - \mu \tan \theta}} \quad \text{(iv)}$$

Let us determine the speed at which the side thrust becomes zero, i.e.,

$$f_1 + f_2 = 0$$

This happens if

$$mg \sin \theta = m \frac{v^2}{r} = \cos \theta$$

or

$$\tan \theta = \frac{v^2}{gr}$$

whence

$$v = \sqrt{gr \tan \theta}$$

When the vehicle is stationary, it will overturn if $\tan \theta > b/h$, i.e., when the line of action of its weight falls outside the base. It will slip sideways if $\tan \theta > \mu$, i.e., when the angle of banking exceeds the friction angle. Normally, roads made of asphalt or concrete have $\mu = 0.6$ or less. For wet roads, $\mu = 0.4$ and for snow-covered roads $\mu = 0.2$. High speed vehicles are designed for high b/h , i.e., more width and less height of centre of mass.

It is, therefore, more probable for a vehicle to slip sideways than to overturn.

Do heavier vehicles have a greater tendency to overturn or slip sideways? No, the conditions of overturning and slip are independent of the weight of the vehicle. But then, we do know that



heavy lorries and trucks overturn more often than cars do. The reason is that lorries and trucks are often loaded to make h high which makes b/h low. Hence, overturning. That's why speed limits for heavy vehicles are kept lower than those for light vehicles.

9.7 GENERAL MOTION OF A RIGID BODY

Let us now establish the dynamical equations of the general motion of a rigid body. By general motion, we imply a combination of translation in space and rotation about the coordinate axes.

Consider a rigid body in general motion referred to the fixed reference frame Y as shown in Fig. 9.6. Let a set of body axes xyz be fixed at the point A . Let the body axes rotate with the angular velocity ω and the angular acceleration α whereas the origin A has a general translatory motion with respect to the origin O . For an element of mass dm , the position vector is

$$\mathbf{R} = \mathbf{R}_A + \mathbf{r} \tag{9.23}$$

and the velocity is given by

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_A + \dot{\mathbf{r}} \tag{9.24}$$

The linear momentum of the element is

$$d\mathbf{H} = \dot{\mathbf{R}} dm = (\dot{\mathbf{R}}_A + \dot{\mathbf{r}}) dm \tag{9.25}$$

and the angular momentum or the moment of momentum about A is

$$d\mathbf{H}_A = \mathbf{r} \times (\dot{\mathbf{R}}_A + \dot{\mathbf{r}}) dm \tag{9.26}$$

For the entire body, the angular momentum about A is

$$\begin{aligned} \mathbf{H}_A &= \int d\mathbf{H}_A = \int \mathbf{r} \times \dot{\mathbf{R}}_A dm + \int \mathbf{r} \times \dot{\mathbf{r}} dm \\ &= \left(\int dm \right) \times \dot{\mathbf{R}}_A + \int \mathbf{r} \times \dot{\mathbf{r}} dm \end{aligned}$$

or

$$\mathbf{H}_A = m_{cA} \times \dot{\mathbf{R}}_A + \int \mathbf{r} \times \dot{\mathbf{r}} dm \tag{9.27}$$

The first term on the right-hand side becomes zero under either of the following conditions:

1. A is a point fixed in space such that $\dot{\mathbf{R}}_A$ the velocity of A is zero.
2. A is a point coincident with the centre of mass, for then m_{cA} equals zero.
3. A is a point moving towards or away from the centre of mass, for then m_{cA} and $\dot{\mathbf{R}}_A$ are collinear.

It is usual to choose A as a fixed point on the body or at the centre of mass C so that the angular momentum is given by

$$\mathbf{H} = \int \mathbf{r} \times \dot{\mathbf{r}} dm \tag{9.28}$$

Noting that the velocity of dm relative to point A is given by

$$\dot{\mathbf{r}} = \omega \times \mathbf{r}$$

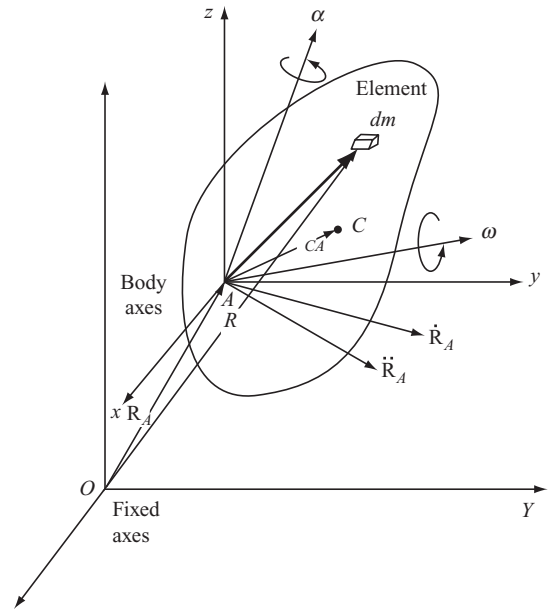


Fig. 9.6 General motion of a rigid body

and the rectangular components of ω and H_A are given by

$$\begin{aligned} \omega &= \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \\ H_A &= H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k} \end{aligned}$$

the expression for the components of angular momentum about A become

$$\begin{aligned} H_x &= \omega_x \int (y^2 + z^2) dm - \omega_y \int xy dm - \omega_z \int xz dm \\ H_y &= \omega_y \int (z^2 + x^2) dm - \omega_z \int yz dm - \omega_x \int yx dm \\ H_z &= \omega_z \int (x^2 + y^2) dm - \omega_x \int zx dm - \omega_y \int zy dm \end{aligned}$$

Recalling the definitions of the moments of inertia, e.g.,

$$\begin{aligned} I_{xx} &= \int (y^2 + z^2) dm \\ I_{xy} &= \int xy dm \end{aligned}$$

the angular momenta can be expressed as

$$\begin{aligned} H_x &= I_{xx} \omega_x - I_{xy} \omega_y - I_{xz} \omega_z \\ H_y &= I_{yy} \omega_y - I_{yz} \omega_z - I_{yx} \omega_x \\ H_z &= I_{zz} \omega_z - I_{zx} \omega_x - I_{zy} \omega_y \end{aligned} \tag{9.29}$$

In matrix form,

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

or

$$H_A = I_A \omega \tag{9.30}$$

The components of the inertia matrix are defined with respect to axes attached to and rotating with the body; these are invariant with respect to the body axes. If the body axes are selected to be the axes of symmetry or if these are the principal axes at the reference point, then the products of inertia vanish i.e.,

$$I_{xy} = I_{yz} = I_{zx} = I_{yx} = I_{xz} = I_{zy} = 0$$

and the principle moments of inertia are

$$I_x = I_{xx}, I_y = I_{yy} \quad \text{and} \quad I_z = I_{zz}$$

Then the angular momentum is expressed as

$$H_A = H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k}$$

where

$$\begin{aligned} H_x &= I_x \omega_x \\ H_y &= I_y \omega_y \\ H_z &= I_z \omega_z \end{aligned} \tag{9.31}$$

If the body axes are the principal axes and, in addition, the angular velocity ω is directed along a principal axis, then

$$H = I\omega \tag{9.32}$$

An example of such a motion is the rotation of a body of revolution about its axis of symmetry; the angular velocity ω and the angular momentum H are then along the axis of rotation.

The Euler's equation

$$\mathbf{M} = \dot{\mathbf{h}} \quad (9.33)$$

can be written in terms of the components of the moment, moment of inertia and angular velocity as follows:

$$= M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k}$$

$$\dot{\mathbf{h}} = \left(\frac{d}{dt} \right)_m \mathbf{h} + \boldsymbol{\omega} \times \mathbf{h} \quad (9.34)$$

$$= \dot{H}_x \mathbf{i} + \dot{H}_y \mathbf{j} + \dot{H}_z \mathbf{k} + \begin{vmatrix} \omega_x & \omega_y & \omega_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= (\dot{H}_x + \omega_y H_z - \omega_z H_y) \mathbf{i} + (\dot{H}_y + \omega_z H_x - \omega_x H_z) \mathbf{j}$$

$$+ (\dot{H}_z + \omega_x H_y - \omega_y H_x) \mathbf{k} \quad (9.35)$$

Substituting for $\dot{\mathbf{h}}$ in Eq. (9.33)

$$M_x = \dot{H}_x + \omega_y H_z - \omega_z H_y$$

$$M_y = \dot{H}_y + \omega_z H_x - \omega_x H_z$$

$$M_z = \dot{H}_z + \omega_x H_y - \omega_y H_x \quad (9.36)$$

It can be appreciated that if the values of H_x , H_y and H_z are substituted from Eq. (9.27), the equations become highly complex.

When the body axes are the principal axes, the values of H_x , H_y and H_z are picked up from Eq. (9.29).

$$M_x = I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z$$

$$M_y = I_y \dot{\omega}_y - (I_z - I_x) \omega_z \omega_x$$

$$M_z = I_z \dot{\omega}_z - (I_x - I_y) \omega_x \omega_y \quad (9.37)$$

This set of equations is generally referred to as the *Euler's equations of motion* for a rigid body. The conditions under which these equations are applicable are as follows:

1. *The reference point where the body axes are fixed is a point fixed in space or is the centre of mass itself. It can also be shown that a point accelerating towards or away from the centre of mass also qualifies to be a reference point.*
2. *The xyz reference frame is fixed on the body and is directed along the principal axes at the reference point.*
3. *The moments of the force are taken about the reference point and also the components of the inertia matrix are determined with respect to the body axes at that point.*

It must be remembered that the dynamics of a rigid body is governed by the Newton's law of motion applied to its centre of mass as far as its translation is concerned. The complete set of equations for the general motion of a rigid body is, therefore, as follows:

$$\left. \begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{aligned} \right\} \text{for the centre of mass}$$

$$\left. \begin{aligned} M_x &= I_x \alpha_x - (I_y - I_z) \omega_y \omega_z \\ M_y &= I_y \alpha_y - (I_z - I_x) \omega_z \omega_x \\ M_z &= I_z \alpha_z - (I_x - I_y) \omega_x \omega_y \end{aligned} \right\} \text{about the centre of mass} \quad (9.38)$$

These equations simplify in size and number for some special case of rigid body motion. Let us consider some such cases.

Case I: Space Motion with Constant Angular Velocity

$$\alpha = 0$$

whence,

$$\alpha_x = \alpha_y = \alpha_z = 0$$

and the set of equations reduce to

$$\left. \begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{aligned} \right\} \text{for the centre of mass}$$

$$\left. \begin{aligned} M_x &= -(I_y - I_z) \omega_y \omega_z \\ M_y &= -(I_z - I_x) \omega_z \omega_x \\ M_z &= -(I_x - I_y) \omega_x \omega_y \end{aligned} \right\} \text{about the centre of mass} \quad (9.39)$$

Case II: Space Motion with Zero Angular Velocity

$$\omega = 0$$

whence,

$$\omega_x = \omega_y = \omega_z = 0$$

and the set of equations reduces to

$$\left. \begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{aligned} \right\} \text{for the centre of mass}$$

$$\left. \begin{aligned} M_x &= I_x \alpha_x \\ M_y &= I_y \alpha_y \\ M_z &= I_z \alpha_z \end{aligned} \right\} \text{about the centre of mass} \quad (9.40)$$

Case III: Torque-Free Motion

$$= 0$$

whence,

$$M_x = M_y = M_z = 0 \quad (9.41)$$

and the Euler's equations are simplified suitably. This, in other words, means the conservation of angular momentum.

Case IV: General Plane Motion

Considering that the different elements of the rigid body are confined in parallel planes parallel to the xy plane,

$$\omega_x = 0 = \omega_y; \quad \dot{\omega}_x = 0 = \dot{\omega}_y \quad \text{and} \quad a_z = 0$$

The set of equations reduces to the following

$$\left. \begin{aligned} F_x &= ma_x \\ F_y &= ma_y \end{aligned} \right\} \text{for the centre of mass.}$$

$$M_z = I_z \alpha_z \text{ about the centre of mass} \quad (9.42)$$

Case V: Fixed-Axis Rotation

Selecting the fixed-axis, which should also be the principal axis, as z -axis, only the ω_z and $\dot{\omega}_z$ are non-zero

$$\omega_x = 0 = \omega_y; \quad \dot{\omega}_x = 0 = \dot{\omega}_y$$

and the linear accelerations also being zero, the set of equations reduces to only one equation,

$$M_z = I_z \alpha_z \quad (9.43)$$

Case VI: Translation of a Rigid Body

$$\omega = 0 \quad \text{and} \quad \dot{\omega} = 0$$

whence,

$$\omega_x = \omega_y = \omega_z = 0$$

and

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

The Euler's equations are then reduced to naught and the dynamical equations for the body are:

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \end{aligned} \quad (9.44)$$

where a_x , a_y and a_z are the acceleration components for any point of the body, as also for the centre of mass.

For a plane translatory motion, therefore,

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \end{aligned}$$

if the parallel planes of motion are parallel to the xy plane.

For a rectilinear motion along the x -axis, the sole equation of motion must be

$$F_x = ma_x$$

9.8 GYROSCOPIC ACTION

If an axisymmetric rigid body such as a plate or wheel spins about its axis of symmetry and this axis is precessing with a uniform angular velocity about an axis perpendicular to that of spin, then a couple called a gyroscopic couple acts on the body which is directed normal to the axes of spin and precession.

If a body spins at a constant angular velocity ω about its x -axis as shown in Fig. 9.7(a) and it has a moment of inertia I about this axis then the angular momentum possessed by it is given by

$$H = I\omega$$

which is shown by a vector oa . In a short interval of time Δt , the axis of spin precesses about the z axis by an angle of $\Delta\phi$. The angular momentum of the body remains the same in magnitude, i.e., $I\omega$ but changes in direction through an angle $\Delta\phi$ as represented by a vector Ob . The change from the initial to the final position over the time Δt is given by $I\omega \Delta\phi$ and the rate of change of angular momentum,

$$\dot{H} = \lim_{\Delta t \rightarrow 0} \frac{I\omega \Delta\phi}{\Delta t} = I\omega \frac{d\phi}{dt} = I\omega \omega_p$$

because the precessional angular velocity is given by

$$\omega_p = \frac{d\phi}{dt}$$

The rate of change of angular momentum, by Euler's equation, must equal the external moment acting on the body

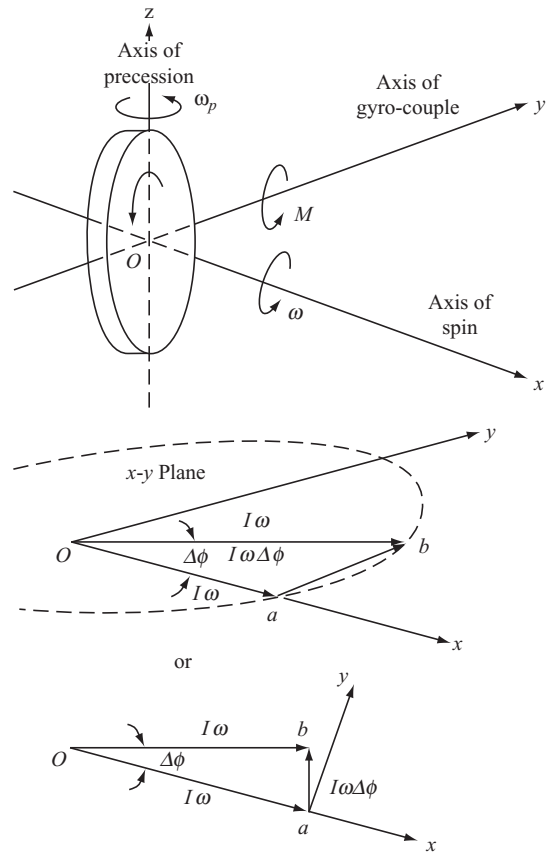
$$M = I\omega \omega_p \tag{9.45}$$

This moment is also termed the gyro-couple.

It may be noted that the gyro-couple acts in accordance with the right-handed triad. The rule to determine the direction of the gyro-couple may be stated as follows:

The spin ω , gyro-couple M and precession ω_p are consistent if all of them are along the positive axes or if two of them are along the negative axes.

A typical gyroscope looks like a suspension of a high-speed gyro rotor as shown in Fig. 9.7(b). The high-speed disc rotor spins about its x axis in the inner gimbal ring. The inner gimbal ring is free to rotate about its y axis relative to the outer gimbal ring. The outer gimbal ring is, in turn, free to rotate about the fixed z axis. The mass centre O of the gyro rotor remains fixed at the same point. The gyroscopic action can result in a space moment if the axis of spin and precession are not mutually perpendicular or if the spin and precession velocities vary with time but the analysis becomes simple and practically important when the axes of spin, precession and gyro-couple are mutually perpendicular as assumed in the analysis.



Vector diagram in the x - y plane for gyroscopic phenomenon

Fig. 9.7 (a) Gyroscope and gyroscopic action

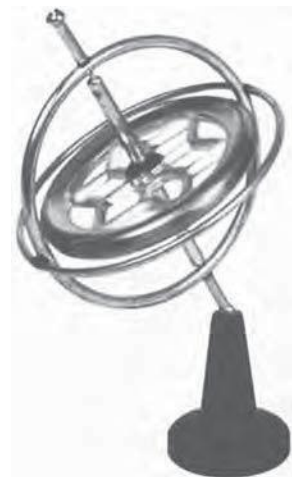


Fig. 9.7 (b) Typical gyroscope

Example 9.23

A circular disc of radius 10 cm and mass 1 kg is made to spin about its axis of symmetry and at the same time the axis rotates in a horizontal plane with a precessional velocity of 5 revolutions per minute anticlockwise as viewed from the top as shown in Fig. Ex. 9.23. Calculate the angular velocity of spin compatible with the steady motion of the system if the length L of the pin attached to the disc is 30 cm.

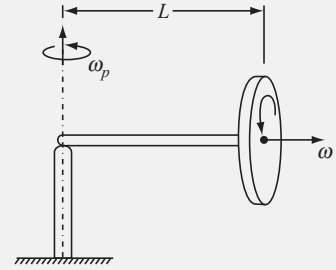


Fig. Ex. 9.23

Solution As shown in the schematic figure, Fig. Ex. 9.23 (Solution)

$$\begin{aligned} \omega_p &= 5 \text{ revolutions per minute} \\ &= 5 \times 2 \pi / 60 = 0.524 \text{ rad/s} \end{aligned}$$

about the z -axis

The moment of inertia about the spin-axis is

$$I_{co} = \frac{1}{2} \times 1 \times 0.10^2 = 0.005 \text{ kg m}^2$$

and the external moment acting about the origin due to the weight of the disc is

$$M = 1 \times 9.81 \times 0.30 = 2.94 \text{ N m}$$

about the y -axis

By the principle of the gyroscope,

$$\begin{aligned} M &= I \omega \omega_p \\ \omega &= \frac{M}{I \omega_p} = \frac{2.94}{0.005 \times 0.524} = 1122 \text{ rad/s} \\ &= 1122 \times \frac{60}{2\pi} = 10,720 \text{ revolutions per minute} \end{aligned}$$

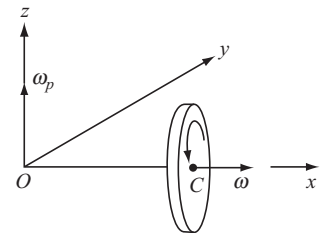


Fig. Ex. 9.23 (Solution)

As is typical of a gyroscopic action, the spinning speed of the gyro-rotor is extremely high. The high speed of rotation of 10,720 revolutions per minute is, hence, not unexpected.

Another fact that emerges from this example is that the gyroscopic couple can be large to be able to precess a rotating body. If the polar moment of inertia of the gyro-rotor is large and the spinning velocity is made extremely high, then the gyro-couple required to precess the rotor by a small angle can be very large. This fact about a gyroscope is exploited by using it to provide stability to vehicles both on the earth and in space. *But for gyroscopic stability, it would have been impossible to drive a two-wheeler vehicle such as a bicycle or motorcycle.* Some other examples of the gyroscope are gyrocompass, rate-of-turn gyro, artificial horizon for aircraft and stabilisation of aero-engines, rockets and ships.

Example 9.24

A horizontal circular disk platform in a children's park can spin about its frictionless vertical axis. The platform has a mass of 50 kg and a radius of 2 m. If it is turned at 1 rad/s and a child of mass 20 kg steps on it, how will the speed of rotation change if he/she moves slowly towards the centre?

Solution The platform is turning freely about its axis; moment of momentum must be conserved!

Moment of inertia of the platform is given by

$$I_p = \frac{1}{2} MR^2 = \frac{1}{2} \times 50 \times 2^2 = 100 \text{ kg m}^2$$

Moment of momentum

$$H = I \omega$$

When the child just steps on it,

$$I_1 = I_p + \frac{1}{2} \times 20 \times 2^2 = 140 \text{ kg m}^2$$

$$H_1 = 140 \times 1 = 140 \text{ kg m}^2/\text{s}$$

When the child is at a radius r ,

$$I_r = 100 + \frac{1}{2} \times 20 \times r^2 = 100 + 10r^2$$

$$H_r = (100 + 10r^2) \times \omega$$

Equating the moments of momenta,

$$H_r = H; \quad \omega = \frac{140}{100 + 10r^2}$$

$$\text{At } r = 1.5 \text{ m } \quad \omega = 1.143 \text{ rad/s}$$

$$\text{At } r = 1.0 \text{ m } \quad \omega = 1.27 \text{ rad/s}$$

$$\text{At } r = 0.5 \text{ m } \quad \omega = 1.366 \text{ rad/s}$$

$$\text{At } r = 0.0 \text{ m } \quad \omega = 1.40 \text{ rad/s}$$

which shows that the rotational speed increases as the child moves slowly towards the centre.

Example 9.25

A force $F = 1.5 t$ newton, where t is in seconds, acts on a mass 200 kg initially at rest, for a period of 20 seconds. Determine

- (a) the impulse on the mass
- (b) the velocity of the mass and
- (c) the average force which would have resulted in the same velocity.

Solution

The force $F = 1.5 t$ N

$$\begin{aligned} \text{Impulse} &= \int_0^T F dt \\ &= \int_0^{20} 1.5 t dt = \left| \frac{1.5 t^2}{2} \right|_0^{20} \\ &= 300 \text{ Ns} \end{aligned}$$

We know that

$$\begin{aligned} \text{Impulse} &= \text{change in momentum} \\ 300 &= m (v_2 - v_1) = 200 (v_2 - 0) \end{aligned}$$

Hence

$$v_2 = 1.5 \text{ m/s}$$

Now,

$$\text{Force} = \text{rate of change of momentum}$$

$$= 200 (1.5 - 0)/20 = 15 \text{ N}$$

This must be the equivalent average force.

Alternatively,
$$\text{Average Force} = \frac{\text{Impulse}}{\text{Time}} = \frac{300}{20} = 15 \text{ N.}$$

Example 9.26

A car of mass 1200 kg descends a hill of 1 in 5 incline. Estimate the average braking force required to bring the car to rest from a speed of 80 km per hour in a distance of 50 m. Take the frictional resistance as 300 N.

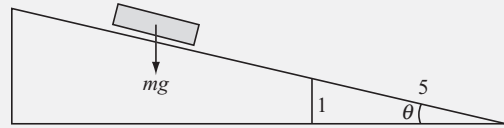


Fig. Ex 9.26

Solution The work-energy equation states that

$$W = \int F \cdot dr_c = KE_2 - KE_1$$

and

$$\sin \theta = 1/5 = 0.2; \quad \theta = 11.54^\circ$$

The weight of the car is $mg = 1200 \times 9.81 = 11,770 \text{ N}$ and the component along the incline is $11,770 \sin \theta = 2354 \text{ N}$.

The frictional resistance is 300 N, in addition to braking force P which makes the net force down the incline as $F = 2354 - 300 - P = 2054 - P$.

$$\text{Work done } W = (2054 - P) \times 50 \text{ J.}$$

$$KE_1 = \frac{1}{2} 1200 \times \left(\frac{80 \times 1000}{3600} \right)^2 = 296,300 \text{ J}$$

$$KE_2 = 0$$

Hence,

$$(2054 - P) 50 = 0 - 296,300$$

$$P = 7980 \text{ N} = 7.98 \text{ kN.}$$

Example 9.27

A lift is being raised by means of a rope-drum assembly with the rope coiled over the drum. The lift accelerates, starting from rest for 5 seconds and rises by 10 m. Data for the lift and rope-drum assembly is as follows:

Mass of the lift 1000 kg

Diameter of the drum 1.5 m

Mass of the drum 750 kg

Radius of gyration of drum 0.6 m

Estimate

- the angular acceleration of the drum,
- the tension in the rope raising the lift,
- the torque on the drum shaft driving the drum and
- the power required

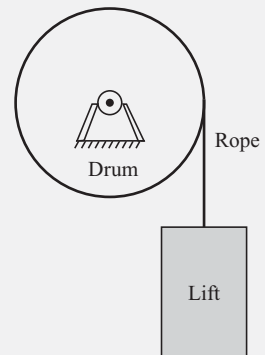


Fig. Ex. 9.27

Solution

From the data

$$M = 1000 \text{ kg}$$

$$r = 0.75 \text{ m}$$

$$m = 750 \text{ kg}$$

$$k = 0.6 \text{ m}$$

For the lift,

$$t = 5 \text{ s}, \quad s = 10 \text{ m}, \quad u = 0$$

the upward acceleration a is given by

$$s = ut + \frac{1}{2}at^2;$$

$$10 = 0 + \frac{1}{2}a \cdot 5^2; \quad a = 0.8 \text{ m/s}^2$$

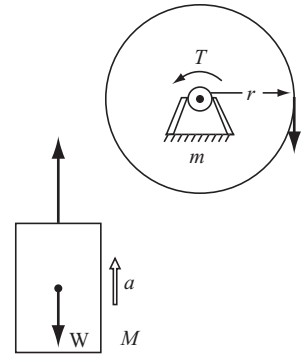


Fig. Ex. 9.27 (Solution)

This is also the acceleration of the rope, which goes over the drum.

(a) Angular acceleration of the drum $\alpha = \frac{a}{r} = \frac{0.8}{0.75} = 1.07 \text{ rad/s}^2$

(b) Considering the free-body diagram of the lift, the equation of motion is

$$F - W = Ma$$

whence

$$\begin{aligned} F &= Ma + Mg = M(a + g) \\ &= 1000(0.8 + 9.81) = 10610 \text{ N} = 10.6 \text{ kN} \end{aligned}$$

(c) Considering the free-body diagram of the drum, the torque on the drum shaft T must be providing the moment with force F acting at the radius r in addition to the angular acceleration,

$$T = F \cdot r + I \alpha$$

Taking the value of F , r , α as above and

$$\begin{aligned} I &= mk^2 = 750 \times 0.6^2 = 270 \text{ kg m}^2 \\ T &= 10610 \times 0.75 + 270 \times 1.07 \\ &= 8246 \text{ Nm} = 8.2 \text{ kNm} \end{aligned}$$

we get

(d) The power required must be

$$P = T \cdot \omega$$

where ω is the angular velocity at the instant when maximum velocity is reached, i.e., at 5 s from the start.

$$\omega = 0 + 1.07 \times 5 = 5.35 \text{ rad/s}$$

Hence

$$P = 8246 \times 5.35 = 44120 \text{ W} = 44.1 \text{ kW.}$$

Example 9.28

- Explain the mechanics of a two-wheeler vehicle (say, bicycle) negotiating a curve, stating clearly the role of frictional forces.
- Calculate the angle of banking on a bend of 100 m radius so that vehicles can travel round the bend at 60 km/hour without side thrust on the tyres.

Solution

Part (a)

Visualising the motion of the two wheeler on a circular track of radius r , as shown, the two wheeler can take a turn only if the wheels provide frictional reactions of F_1 and F_2 . The resultant of F_1 and F_2 must be radial and equal to the centripetal force required to turn the vehicle.

Centripetal force = mass \times radial acceleration towards the centre of the circular track.

$$F = m \cdot \frac{v^2}{r}$$

Vector sum of F_1 and F_2 must, therefore, be equal to $m v^2/r$. It is indeed the frictional forces on the wheels, directed radially inwards, as shown which permit negotiation of a curve. If the surface is smooth or frictionless like deep frozen ice or if the tyres are worn out, etc., the frictional forces F_1 and F_2 would not be provided; hence the vehicle would skid, i.e., appear to go outward although it would actually go along a straight line (in accordance with the Newton's First law of Motion!)

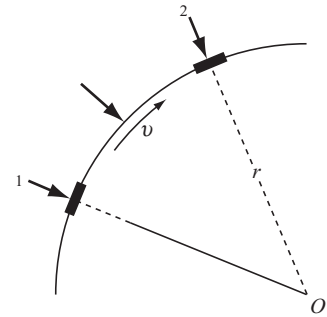


Fig. Ex 9.28(a) (Solution)

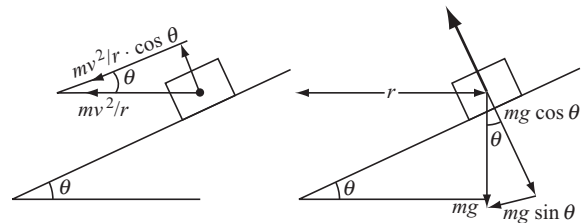


Fig. Ex. 9.28(b) (Solution)

Part (b)

For a correct angle of banking θ , there is no side thrust on the tyres; the only real forces are the weight mg acting downwards and the normal reaction N , normal to the inclined surface.

The normal reaction is balanced by the $mg \cos \theta$ component of the weight,

$$N = mg \cos \theta$$

The $mg \sin \theta$ component of the weight along the incline should provide for the centripetal force directed towards the centre of the curved path, i.e., mv^2/r .

However, it is the component along the incline, i.e., $mv^2/r \cdot \cos \theta$. which is the one provided by the force $mg \sin \theta$.

Hence,
$$mg \sin \theta = \frac{mv^2}{r} \cos \theta$$

or,
$$\tan \theta = \frac{v^2}{gr}$$

Given that
$$v = 60 \text{ km/hour} = \frac{60 \times 1000}{3600} = 16.67 \text{ m/s}$$

$$r = 100 \text{ m}$$

so,
$$\tan \theta = \frac{16.67^2}{9.81 \times 100} = 0.283$$

and
$$\theta = 15.8^\circ.$$

EXPERIMENT E11 PRINCIPLE AND ACTION OF A GYROSCOPE — A HUMAN EXPERIENCE

OBJECTIVE TO UNDERGO A HUMAN EXPERIENCE OF THE PRINCIPLE OF A GYROSCOPE AND TO STUDY ITS ACTION.

Apparatus A bicycle wheel mounted on handles and a turntable for a person to stand on it as shown in Fig. E11.1.

Background Information The angular momentum of an axisymmetric rigid body spinning about its axis at a constant angular velocity ω is given by

$$H = I\omega$$

where I is the moment of inertia of the body about the axis of rotation. If the spinning body such as a disc, as shown in Fig. E11.2, is to precess about an axis perpendicular to the axis of spin at a constant angular velocity ω then a moment, also called a gyroscopic couple, acts on the disc such that the magnitude of the moment is ωH , the direction of action is perpendicular to the plane containing the axes of spin and precession and the sense is governed by the right-handed screw rule such that the axes of the spin couple and precession constitute a triad as also shown in the same figure.

Observation of a Human Experience If a person holds the wheel of a bicycle mounted on handles and, while standing on a turntable, sets the wheel spinning at a high angular velocity about a transverse axis, he should be able to stand upright and stationary as shown in Fig. E11.1. If now he precesses the spinning body by way of quickly turning the hands and the wheel about the fore-and-aft axis, he should experience a gyroscopic moment about the vertical axis. The direction of moment would be in accordance with the right-handed screw rule applied to the axes of spin, couple and precession. Referring to Fig. E11.1, if the wheel is spun about its axis in a counter-clockwise direction as seen from the left-hand side of the person and it is precessed counterclockwise as seen by the person, the moment would be in a counterclockwise direction as experienced by the person looking towards the ground.

It is interesting to observe this rule and to verify what happens to the direction of the moment if either the spin or precession or both of them are reversed.

The gyroscopic properties of spinning rotors have been put to many practical uses. The property of keeping its direction fixed when mounted freely suggests its use as a gyro-compass instead of a magnetic compass. The property of precession on the application of an external moment has been used in auto-pilots for ships and aircrafts, in steering torpedoes, in stabilising rifle bullets and in many other ways including the stabilisation of land, air and sea vehicles. A vehicle

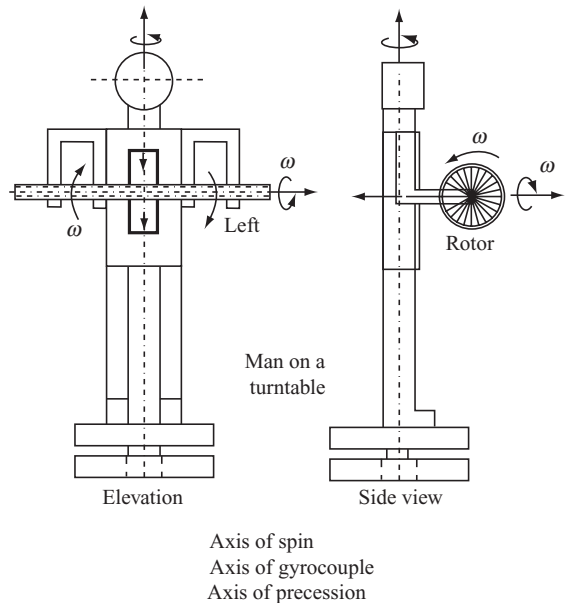


Fig. E11.1 Human experience of the principle and action of a gyroscope

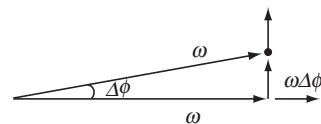


Fig. E11.2 Angular momentum diagram. The principle of a gyroscope

may thus be equipped with more than one gyro. A ship, for example, may be fitted with a number of stabiliser gyros, a control gyro as well as a gyro-compass!

Points for Discussion

Estimate the speed at which the cycle wheel spins. Repeat the experience at two or three different speeds and arrive at the conclusion that the rotor of a practical gyroscope should spin at a very high speed. It may be noted that rotors are run by continuous-rating motors at speeds in the range 20,000 to 30,000 revolutions per minute.

Explain, with reference to your observations with a spinning-cycle wheel gyroscope, the gyroscopic action used by a cyclist or motorcyclist.

Sketch an arrangement constituting a gyroscope for the purpose of a gyro-compass.

Concept Review Questions

State, giving reasons, why

- the reference point on a rigid body is usually chosen as the centre of mass or a point fixed in space.
- the reference axes are attached with a rigid body oriented to coincide with the principal axes.
- Newton's law and Euler's equation together govern the motion of rigid bodies.
- the kinetic energy of a rigid body in general motion is the sum of the translational kinetic energy associated with the centre of mass and the rotational kinetic energy of the mass rotating about the centre of mass.

Under what conditions and for what states of motion is the simple form

$$M = I\alpha$$

adequate to describe the motion of a rigid body?

Write down the set of equations governing the dynamic behaviour of

- a rigid body in rectilinear translation, plane translation, and space translation.
- a rigid body in general plane motion.
- a rigid body rotating about a fixed axis which is also its principal axis.

Why is it that

- Euler's equations do not contain the products-of-inertia terms?

- the components of the inertia matrix do not change with time during the motion of a rigid body?

- the reference point has to be chosen with restrictions for the application of Euler's equations?

State the assumptions made in the development of Euler's equations and trace the origin of these assumptions.

State the form of the following principles as applied to the dynamics of a rigid body.

- Impulse-momentum principle
- Work-energy principle
- Energy conservation principle.

Explain how the gyroscopic action helps in stabilising the motion of a bicycle, a two-wheeler scooter and an aeroplane.

- 8 The following are shown in the figure: (a) homogeneous semicylinder rolling without slip on a convex surface, (b) a composite cylinder rolling without slip on an incline, (c) a uniform ladder sliding between the wall and the ground, (d) a homogeneous cylinder rolling without slip on a concave surface and (e) a homogeneous cylinder rolling without slip on a flat accelerating surface. A number of points are marked on each figure. For each point, state, *by the side of the figures*, whether the Euler's equation in the form $M = I\alpha$ is applicable or not giving reasons.

Answer

| Given figures | Yes | No | Reason for the answer |
|---------------|---------------------|--|--|
| | C | O A | Centre of mass [Neither fixed nor accelerating towards or away from C nor centre of mass of the body. |
| | | C ₁ C ₂ O A | [Neither fixed points nor centre of mass of the composite body nor accelerating towards or away from the centre of mass of the composite body |
| | C I | A B O | Centre of mass Instantaneous centre of rotation; accelerating towards C [Neither fixed points nor accelerating towards or away from C nor centres of mass [Not identified with the body |
| | O ₂ A | O ₁ B | [Centre of mass Instantaneous centre of rotation; accelerating towards C [Not identified with the body |
| | O | A | Centre of mass Accelerating in a direction other than towards or away from the centre of mass |

Tutorial Problems

A filing cabinet of mass 50 kg rests flat on a horizontal floor. A man wishing to slide it sideways applies a force of 200 N as shown in Fig. Prob. 9.1.

Determine the minimum height h for which the application of the force would result in tipping about the corner B . Assume that the coefficient of static friction between the cabinet base and the floor is 0.3. Determine also the sideways acceleration of the cabinet if the force were applied below the critical height.

$$\left(\text{ s } h = 0.8 \text{ m}, a = 1.06 \text{ m/s}^2 \right)$$

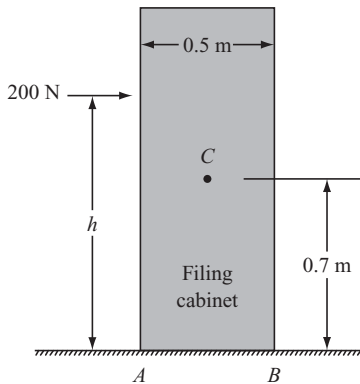


Fig. Prob. 9.1

A heavy spherical ball is constrained in a frame as shown in Fig. Prob. 9.2.

Considering the inclined surface as smooth determine the maximum acceleration with which the frame can move horizontally without causing the ball to leave the frame.

$$\left(\text{ s } a = 5.66 \text{ m/s}^2 \text{ to right} \right)$$

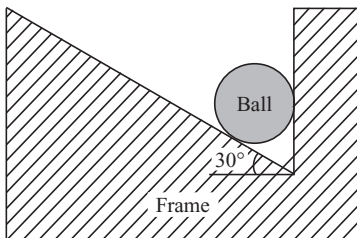


Fig. Prob. 9.2

A simple wheel and axle has a mass m and a radius of gyration k (Fig. Prob. 9.3).

The system is accelerated by a descending weight W attached to a cord wound round the axle of radius r . Determine the acceleration of the descending weight and tension in the cord.

$$\left(\text{ s } a = \frac{W}{W + mgk^2/r^2} g, T = \frac{mgWk^2}{Wr^2 + mgk^2} \right)$$

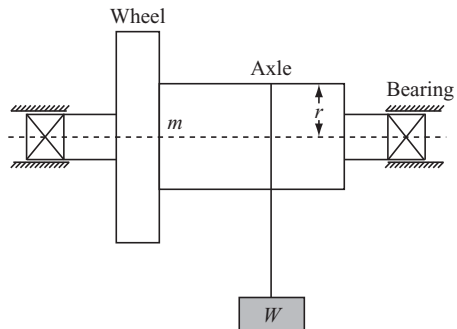


Fig. Prob. 9.3

A wheel-and-axle is made to roll without slip on a horizontal surface by means of a 10 kg mass suspended by a light and inextensible string wrapped around the axle as shown in Fig. Prob. 9.4. If the mass of the wheel-and-axle is 50 kg and the radius of gyration about the centroidal axis cc is 0.2 m, calculate the angular speed acquired by it when it has rolled a distance of 2 m, starting from rest.

$$\left(\text{ s } 6.1 \text{ rad/s} \right)$$

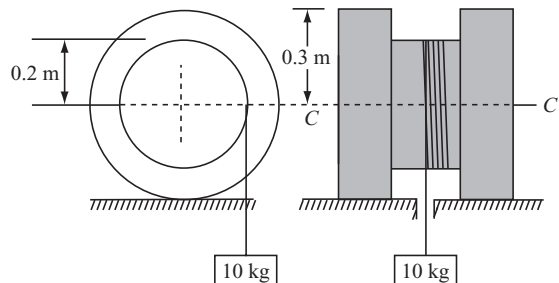


Fig. Prob. 9.4

A cord passes over a pulley of mass m , radius r and radius of gyration k and carries suspended weights W_1 and W_2 at its ends. Assuming that $W_1 > W_2$ and that the cord does not slip over the pulley, determine the acceleration of the cord and the difference in the tension on the two sides of the cord.

$$\left(\begin{array}{l} \text{s } a = \frac{W_1 - W_2}{W_1 + W_2 + mgk^2/r^2} g, \\ T_1 - T_2 = \left(\frac{(W_1 - W_2) mgk^2/r^2}{W_1 + W_2 + mgk^2/r^2} \right) \end{array} \right)$$

A circular disc (2) is driven through frictional contact with disc (1) as shown in Fig. Prob. 9.6. External torque T_1 acts on disc (1). Determine the resultant angular acceleration of disc (2).

$$\left(\begin{array}{l} \text{s } \alpha = \frac{r_1 r_2 T_1}{r_2^2 I_1 + r_1^2 I_2} \end{array} \right)$$

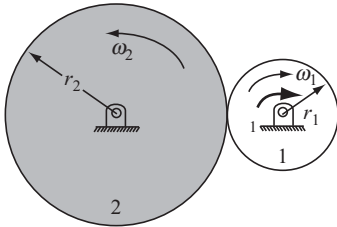


Fig. Prob. 9.6

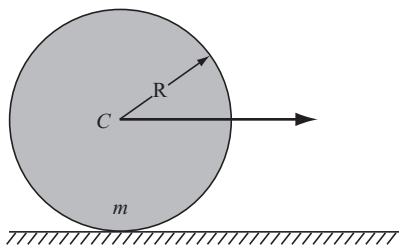


Fig. Prob. 9.7

A horizontal force F applied to the centre of mass of a sphere of radius R and mass m causes it to roll without slip (Fig. Prob. 9.7).

- Find the acceleration of the sphere.
- If there was a cylinder of the same radius and mass instead of a sphere and the force was applied centrally on it, would it have accelerated more or less and why?

Assume the friction coefficient as m in each case.

$$\left(\begin{array}{l} \text{s } (5F/7) m \text{ and } (2F/3) m \text{ less} \end{array} \right)$$

- At what distance p should the horizontal force F be applied to the homogeneous bar, homogeneous cylinder and homogeneous sphere so that the horizontal component of the reaction at the point of suspension is zero (Ref. Fig. Prob. 9.8).

$$\left(\begin{array}{l} \text{s } \frac{2}{3} d, \frac{3}{4} d \text{ and } \frac{7}{10} d \end{array} \right)$$

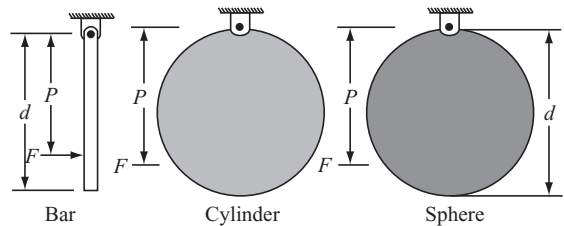


Fig. Prob. 9.8

An aeroplane is flying at 500 km/hr. Find the angle with the horizontal at which it must bank in order to turn without slip sideways, in a circular path of radius 5 km. Assume that the resultant air pressure on it acts through its centre of gravity at right angles to the angle of banking.

$$\left(\begin{array}{l} \text{s } 21.5^\circ \end{array} \right)$$

A uniform rigid cylinder of mass m and radius r rolls without slip on a horizontal surface. What is its kinetic energy when its centre has a speed V_c ? Would a sphere of the same radius, same mass and same velocity of the centre of mass have less or more kinetic energy?

$$\left(\begin{array}{l} \text{s } \frac{3}{4} mV_c^2; \text{ less, } \frac{7}{10} mV_c^2 \end{array} \right)$$

A bar AB of mass M and length L is pinned to a disc of mass m and radius r and the two are suspended at point a to hang freely. If a horizontal force F is applied at the centre of the bar, determine the angular accelerations of the bar and disc at the instant as shown in Fig. Prob. 9.11.

$$\left(\begin{array}{l} \text{s } \frac{24f}{13 mL}, \frac{12f}{13 mr} \end{array} \right)$$

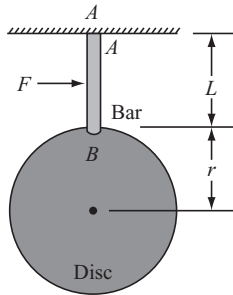


Fig. Prob. 9.11

A thin bar of steel of length L and weight W , bent through a right angle is suspended from a fulcrum O . If it is released from rest when one limb of the bar is vertical as shown in Fig. Prob. 9.12, determine the maximum angular velocity acquired by it.

$$\left(\text{ s } \sqrt{\frac{3g}{L}(\sqrt{2}-1)} \right)$$

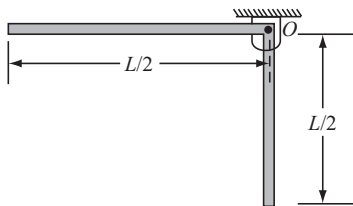


Fig. Prob. 9.12

A box containing a bar of length L and mass m suspended from its ceiling at C as shown in Fig. Prob. 9.13 is accelerated horizontally at an acceleration a . Determine the inclination of the bar with the vertical in the steady state. What difference will it make if the acceleration was imparted at an angle θ with the horizontal?

$$\left(\text{ s } \tan \theta = a/g \text{ and } a_x/(a_y + g) \right)$$

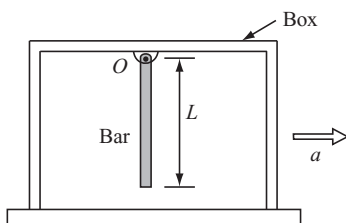


Fig. Prob. 9.13

A uniform bar of length L and weight W rests on rough surface as shown in Fig. Prob. 9.14. Assuming that the coefficient of friction on both surfaces is 0.2, determine the reaction by the surfaces at an instant when the bar is released from rest at $\theta = 30^\circ$.

$$\left(\text{ s } 0.98 W, 1.23 W \right)$$

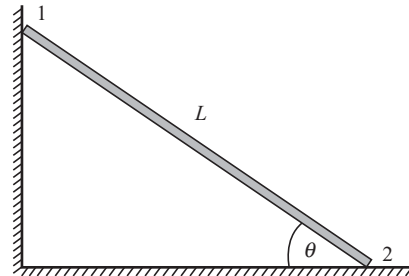


Fig. Prob. 9.14

Find the angular acceleration of the thin homogeneous lever of length 0.5 m and mass 2 kg when it is in the horizontal position (Fig. Prob. 9.15).

$$\left(\text{ s } -12.4 \text{ rad/s}^2 \right)$$

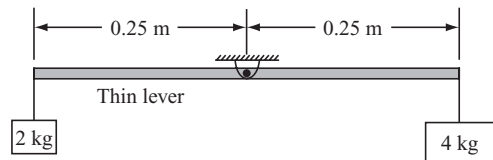


Fig. Prob. 9.15

A slender rod of length L and mass m is released from rest in the position shown in Fig. Prob. 9.16. Determine the normal reaction exerted by the horizontal surface on the roller at the base of the rod at the instant after release.

$$\left(\text{ s } mg(1 - 3/4 \sin^2 \theta) \right)$$

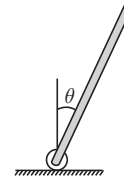


Fig. Prob. 9.16

A thin homogeneous bar of length L and mass m is placed on the horizontal top of a table with its centre overhanging by a distance a from the edge as shown in Fig. Prob. 9.17. The bar is

released from rest when it is horizontal and begins to rotate about the edge of the table. Assuming the static coefficient of friction, compute the angle with the horizontal the bar makes after which the bar slips over the edge.

$$\left(s \tan^{-1} \frac{L^2}{(L^2 + 36a^2)} \right)$$

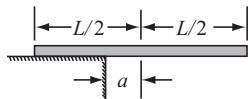


Fig. Prob. 9.17

- 8 A uniform bar of length 1 m and mass 20 kg hangs freely below a 0.5 m long thin rigid bar firmly attached to it as shown in Fig. Prob. 9.18. A bullet of mass 25 g approaching horizontally at 500 m/s pierces the bar and emerges horizontally with a velocity of 200 m/s. Determine the velocity of the lower end of the bar just after the bullet leaves it and the maximum angle through which the bar may swing.

(s 0.52 m/s; 6.6)

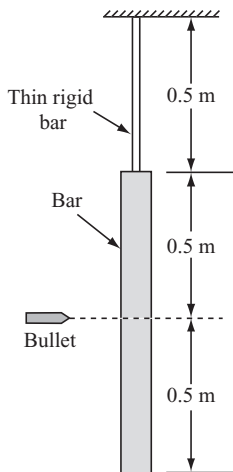


Fig. Prob. 9.18

An inextensible cord going round a homogeneous cylinder *A* of mass 100 kg holds a massless plate *B*. The collar *C* of mass 30 kg is released from rest in the position shown in Fig. Prob. 9.19 and drops upon the plate. Determine the velocity of the collar when it

has descended an additional 0.5 m after striking the plate. Assume that there is no rebound. i.e., *C* and *B* move downwards locked together and that the cord remains taut.

(s 2.5 m/s)

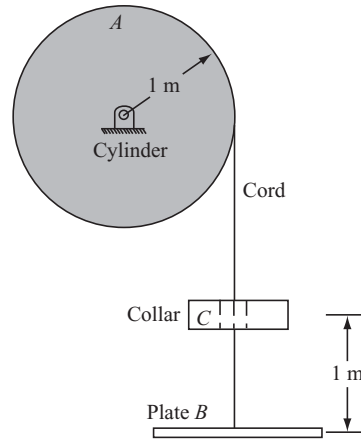


Fig. Prob. 9.19

In Fig. Prob. 9.20, a homogeneous cylinder and a homogeneous sphere of equal weights $W = 200$ N and equal radii R are harnessed together by a light frame and are free to roll without slip down a plane inclined 28° with the horizontal. Determine the force in the frame. Assume that the bearings are frictionless

(s 3.25 N)

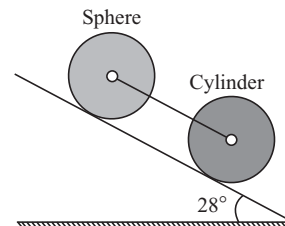


Fig. Prob. 9.20

A thin uniform hemispherical shell has a weight W and radius r . The shell is pulled along a horizontal surface by a constant horizontal force F . The coefficient of sliding friction between the shell and the surface is μ which may be assumed to be constant. Find the distance above the plane that the force F should be applied so that there is no tipping of

the shell. What is the acceleration of the shell under these conditions?

$$\left(s \ x = \frac{r}{2} \left(1 - \frac{\mu W}{F} \right), a = \left(\frac{F - \mu W}{W} \right) g \right)$$

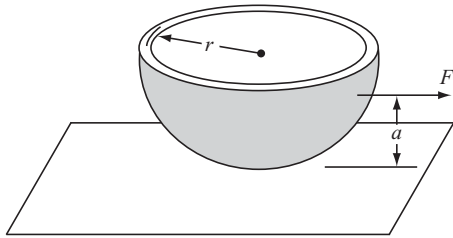


Fig. Prob. 9.21

A homogeneous bar of length L is pivoted about a point which is at a distance a from one end as shown in Fig. Prob. 9.22. If the bar is released from rest in the 30° position, what will be the angular speed when the bar is vertical?

$$\left(s \ \omega^2 = \frac{4g(L-2a)}{L^2 - 3La + 3a^2} \right)$$

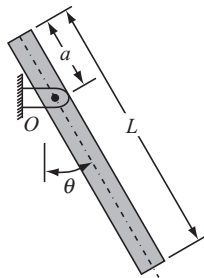


Fig. Prob. 9.22

A 5 kg homogeneous disc 20 cm in diameter is attached rigidly to a 20 kg homogeneous disc 40 cm in diameter. If the assembly is released from rest as shown in Fig. Prob. 9.23, what will be the angular speed when the small disc is at the bottom of its travel? (s 3.9 rad/s)

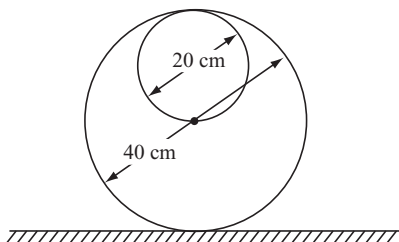


Fig. Prob. 9.23

A sphere rolling with an initial velocity of 9 m/s starts up an inclined plane of 30° with the horizontal. How far would it roll up the incline, assuming rolling without slip?

$$(\ s \ 11.56 \text{ m})$$

Find the acceleration of the falling weight and of the cylinder weighing 200 N connected by an inextensible string passing over a frictionless pulley as shown in Fig. Prob. 9.25.

$$(\ s \ 5.61 \text{ m/s}^2 \quad ac = 2.8 \text{ m/s}^2, \alpha = 28 \text{ rad/s}^2)$$

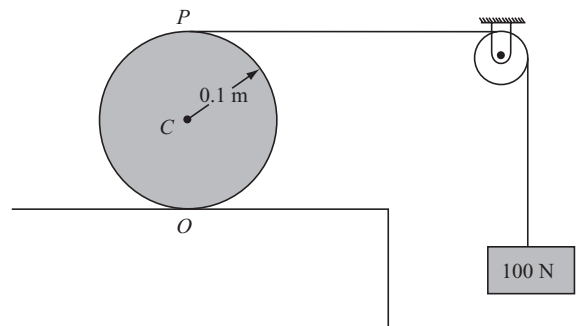


Fig. Prob. 9.25

A ball of weight 100 N is supported in a vertical plane by a string AB and a bar BC hinged at C , as shown in Fig. Prob. 9.26. Find the force in the bar BC while it is in equilibrium. If now the string AB is gently cut, find the force in the bar just after the string is cut. Neglect the weight of the bar.

$$(\ s \ 73.21 \text{ N}; 50 \text{ N})$$

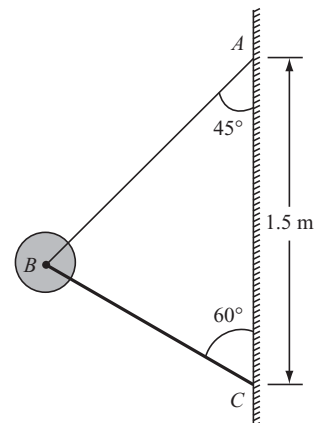


Fig. Prob. 9.26

A slender prismatic bar of length L and weight W is supported in a horizontal position as shown in Fig. Prob. 9.27. Find the instantaneous reaction at the hinge support A just after the string attached at B is gently cut.

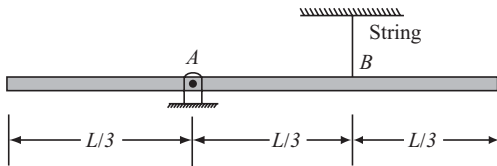


Fig. Prob. 9.27

- 8 A solid circular cylinder and a solid sphere of the same mass and same diameter are started from rest at the top of an inclined plane at the same time and both roll without sliding down the plane.
- Which one would reach the bottom of the plane first and why?
 - If one leads the other by 0.12 m at the instant of reaching the bottom of the plane, find the total length of the plane.
- (s (a) Solid sphere; less I_c (b) 3.48 m)

A homogeneous plate of mass 20 kg having the shape of an equilateral triangle with each side 40 cm long hangs in a vertical plane by pins at two of its corners A and B as shown in Fig. Prob. 9.29. The pin B is removed gently. Determine the reaction at A just after that instant.

(s $R_{Ax} = -68 \text{ N}$, $R_{Ay} = +78.4 \text{ N}$)

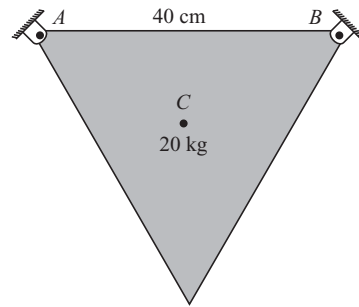


Fig. Prob. 9.29

A homogeneous rectangular block with a mass of 200 kg rotates at 10 rad/s about a main diagonal held by bearings at A and B mounted on a vehicle moving at a speed of 10 m/s as shown in Fig. Prob. 9.30. Calculate the kinetic energy of the block.

(s KE translation 10 kNm; KE rotation 2700 Nm)

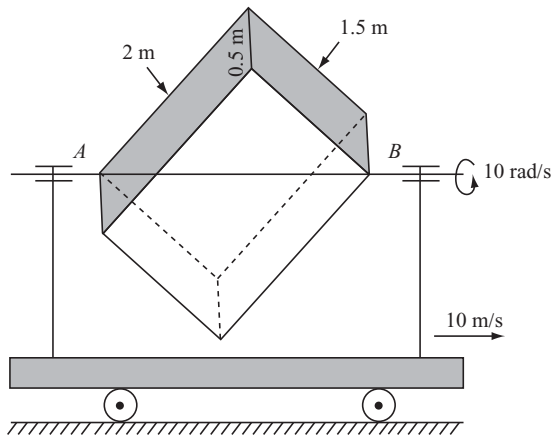


Fig. Prob. 9.30

Look up Hints to Tutorial Problems at the End!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

The centre of percussion is a point on a rigid body

- through which the resultant of the applied forces acts
- where the impact is made
- where the largest external force acts
- about which the body may rotate

Euler's equation of motion should *not* be applied about

- (a) the centre of mass of the body
- (b) the fixed point about which the body may rotate
- (c) a point accelerating towards the centre of mass
- (d) a point fixed in space.

The kinetic energy of a rigid body of mass m moving such that the velocity of its centre of mass is Vc and the rotational velocity is ω , is given by

- (a) $1/2 mVc^2$
- (b) $1/2 Ic\omega^2$
- (c) $1/2 mVc^2 + 1/2 Ic \omega^2$

A solid cylinder, solid sphere and a loop, each of radius R and mass μ , are released simultaneously from rest on an incline at the same elevation. The order in which they will pass through a common mark at the same level will be

- (a) cylinder, sphere and hoop
- (b) sphere, cylinder and hoop
- (c) hoop, sphere and cylinder
- (d) hoop, cylinder and sphere

The principle of conservation of angular momentum for a rigid body is applicable when

- (a) the external forces add to zero
- (b) the internal forces add to zero
- (c) the external forces and moments are absent
- (d) the resultant external moment is zero

Answers to Multiple-Choice Questions

- 1 (a) 2 (d) 3 (c) 4 (b) 5 (d)

Applications in Dynamics

This section consists of some salient applications in dynamics under the following two chapters:

-
- **D1 IMPACT OF TWO BODIES**
 - **D2 CENTRAL FORCE MOTION**
-

Impact of Two Bodies



D1

D1.1 INTRODUCTORY CONCEPTS

An impact is a sudden short-term encounter between two bodies. The common form of impact is the collision between two bodies. The bodies experience large variable forces over short intervals of time with various dissipative phenomena, such as wave propagation, partial plastic deformation, relative slipping under friction and radiation of heat, sound and light. In view of these complications, exact solutions for the collision problems do not exist but simplified treatments based upon some postulates and empiricism have been developed.

When two bodies collide, it is assumed that they come in contact over an infinitesimal plane area surrounding the point of contact. The plane of the area is known as the *plane of contact* as shown in Fig. D1.1.

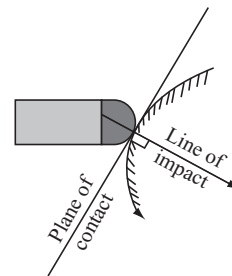


Fig. D1.1 Impact: collision

The *line of impact* is defined as a line, normal to plane of contact at the point of contact. The plane of contact and line of impact are shown in Fig. D1.1. The direction along the line of impact is called the normal direction whereas the direction along the plane of contact is called the tangential direction. The velocity of a body just before and just after the impact may be resolved into a normal component and a tangential component.

D1.2 TYPES OF IMPACT

An impact, i.e., collision of two bodies may be quite general, with two arbitrary shaped bodies with initial velocities v_1, ω_1 , and v_2, ω_2 colliding at a point P as shown in Fig. D1.2 (a). Such a general case of ‘non-central’ and ‘indirect impact’ is beyond our scope.

Some other types of impact are shown in Figs D1.2 (b, c, d and e).

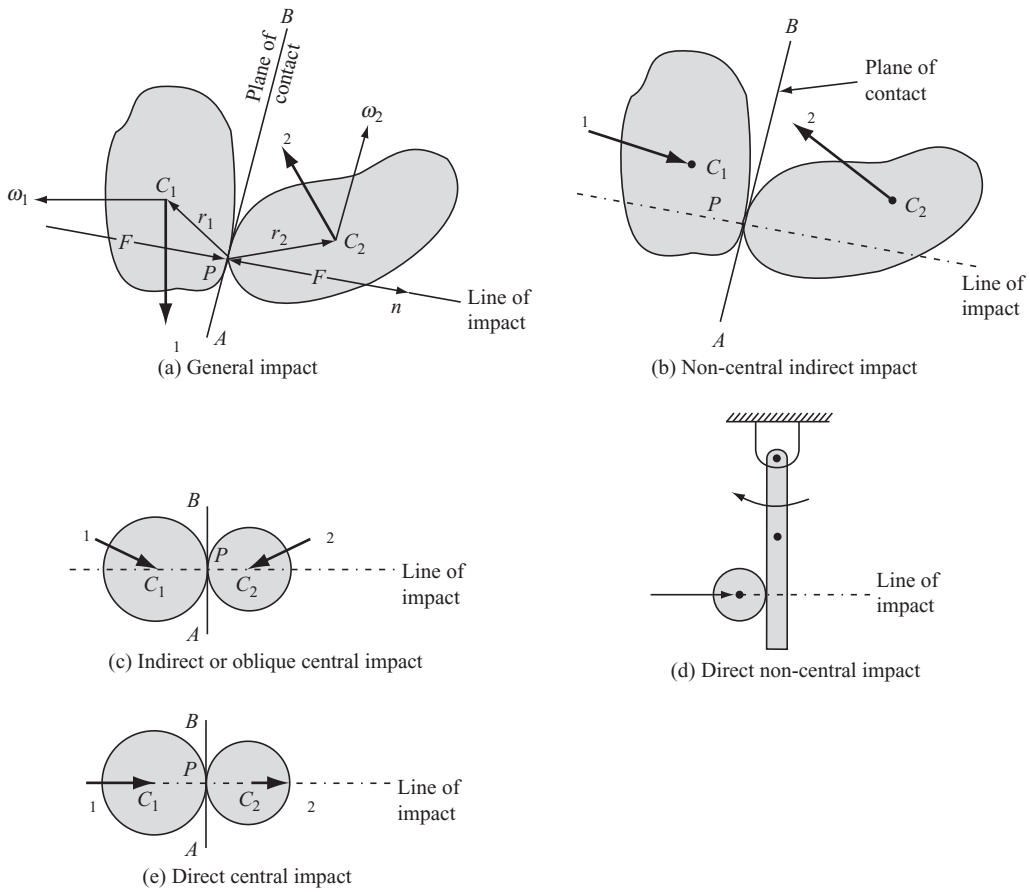


Fig.D1.2 Classification of impacts

If the line of impact passes through the centres C_1 and C_2 of masses m_1 and m_2 respectively, it is called *central impact* and if the line of impact does not pass through the centres of masses, it is called *non-central* or *eccentric impact*. If the velocities of the bodies before collision are collinear with the line of impact, it is said to be *direct impact*, otherwise *indirect* or *oblique impact*.

W s s
s . P s s
s s.

It is postulated that the total period of impact Δt consists of two components Δt_1 and Δt_2 characterised as the deformation and recovery periods. Over these short intervals of time, the contact forces are large and the impulse of each force is assumed to be finite. In comparison with the large contact forces, other finite forces such as that due to gravity are usually neglected.

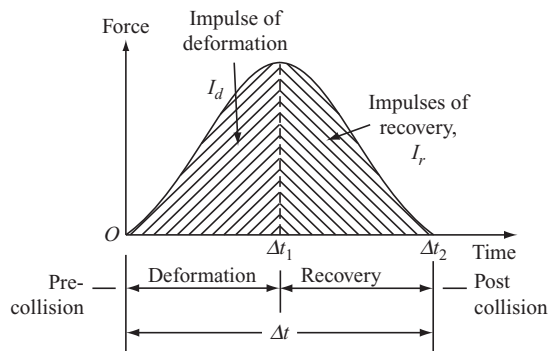


Fig.D1.3 Postulated contact force-time history

The impulses of deformation and of recovery are finite even though the time interval of impact is extremely small. This implies that there is a *step change in the velocities* of the two bodies without any change of coordinates.

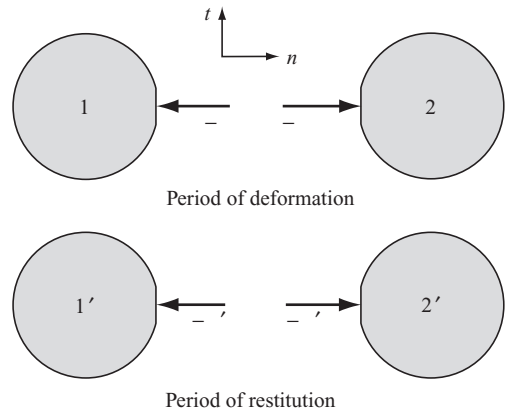
From the fact that the contact force acts wholly along the line of impact,

$$\dot{p}_n = F, \text{ the magnitude of} \quad (D1.1)$$

where n is the unit vector along the line of impact. similarly, the impulse is wholly along the line of impact,

$$\int dt \cdot \dot{p}_n = \int F dt = I \quad (D1.2)$$

If a collision takes place without any external force, the internal forces, i.e., the forces exerted by either body on the other be equal and opposite. It follows that the impulses of the forces on the two bodies must also be equal and opposite. The contact-force time history of each body must be the same. The force-time history drawn in Fig. D1.3, therefore, applies to either body with due care to the sign. The contact force develops from zero value when the collision starts and increases to a maximum value at the end of the period of deformation. The period of recovery is characterised by a diminishing contact force until it reaches a zero value again at the end of the collision. Free-body diagrams of the bodies during impact are shown in Fig. D1.4.



(a) Free-body diagrams of the bodies during collision

Fig. D1.4 Direct central impact

D1.3 COEFFICIENT OF RESTITUTION

The energy dissipation effects are assumed to be described by a single scalar parameter e , the *coefficient of restitution* defined as the ratio of the impulses of recovery and deformation for either body. Considering the equation of motion for each body, the impulse of contact force equals the change in momentum for the body along the line of impact in the vicinity of the contact point.

With reference to an indirect or oblique central impact, as shown in Fig. D1.5.

$$e = \frac{\text{Impulse of recovery}}{\text{Impulse of deformation}} = \frac{I_r}{I_d}$$

$$= \frac{\int_{\Delta t_1}^{\Delta t_2} F dt}{\int_0^{\Delta t_1} F dt}$$

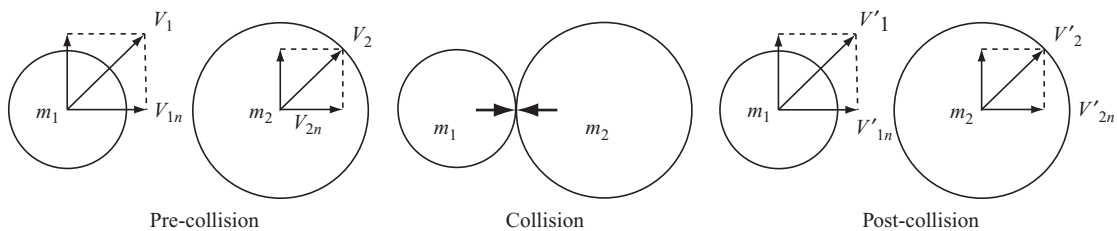


Fig. D1.5 Indirect central impact

For body 1,

$$e = \frac{m_1 (V_1' - V_c) \cdot n}{m_1 (V_c - V_1) \cdot n} \quad (i)$$

For body 2,

$$e = \frac{m_2 (V_2' - V_c) \cdot n}{m_2 (V_c - V_2) \cdot n} \quad (ii)$$

Combining Eqs. (i) and (ii)

$$e = \frac{(V_2' - V_1')}{(V_1 - V_2)} = -\frac{V_{sn}}{V_{an}} \quad (D1.3)$$

$$e = -\frac{\text{Velocity of separation along the line of impact}}{\text{Velocity of approach along the line of impact}}$$

The coefficient of restitution e may now be related to the dissipation of energy during a direct central impact. Assuming that the potential energy of each body remains the same during the infinitesimal time of impact, the kinetic energy just before the impact is

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \quad (D1.4)$$

and the kinetic energy just after the impact is

$$KE' = \frac{1}{2} m_1 V_1'^2 + \frac{1}{2} m_2 V_2'^2 \quad (D1.5)$$

Assuming that the second body remains at rest,

$$V_2 = 0 = V_2'$$

such as the case of a ball striking the floor, the initial and the final kinetic energies are

$$\frac{1}{2} m V_1^2 \quad \text{and} \quad \frac{1}{2} m V_1'^2$$

respectively.

The ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{KE'}{KE} = \frac{V_1'^2}{V_1^2} = \left(\frac{\text{Velocity of separation}}{\text{Velocity of approach}} \right)^2 = e^2 \quad (D1.6)$$

Hence, e^2 can be thought of as the ratio of the final kinetic energy to the initial kinetic energy in a direct central impact of a body against a massive body practically at rest.

As an example, consider a body freely dropped from a height H to a flat plate on the surface of the earth. The kinetic energy just before the impact is equal to the initial potential mgH . The height of rebound h should be such that the kinetic energy just after the impact which is also equal to the final potential energy mgh , is related to the initial kinetic energy by Eq. (D1.6), i.e.,

$$\frac{mgh}{mgH} = e^2$$

or

$$h = e^2 H$$

The value of e for a pair of bodies depends upon the properties of the surfaces of the materials to a large extent. The geometry of bodies and size, centrality or non-centrality of impact, velocity of approach and some other factors have also but a marginal influence on the value of e .

EXPERIMENT E12 COEFFICIENT OF RESTITUTION

OBJECTIVE TO DETERMINE THE COEFFICIENT OF RESTITUTION FOR THE GIVEN PAIRS OF MATERIALS.

Apparatus A trolley with an attachment for a block of material on one end and a cord on the other, a trolley track with an attachment for the other material on a bumper and a pulley-in-slot arrangement for hanging a weight as shown in Fig. E12.1, different materials and a metre rod.

Background Information The trolley is placed on the bumper end of the track. The cord and weight are tied to it at the other end and released. As soon as the weight drops to the ground the trolley is let go freely on the track. It decelerates and comes to rest after a distance S_1 given by

$$u_1^2 = 2aS_1 \tag{E12.1}$$

where u_1 is the speed of the trolley at the instant the weight just touches the ground.

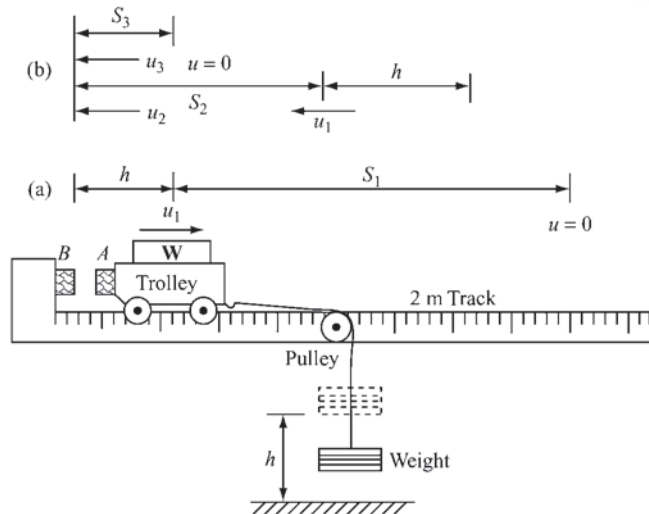


Fig. E12.1 Apparatus for coefficient of restitution

If the trolley is now placed on the other side of the pulley and the weight, attached to the same length of the cord, is dropped by the same height h , the speed acquired by the trolley would be u_1 and it would go on for a distance S_1 . However, after the trolley travels a distance S_2 , an impact takes place, a fraction of energy is dissipated and the trolley retracts by a distance S_3 instead of the distance $(S_1 - S_2)$ as expected in the absence of an impact.

The speed of the trolley just before the impact is given by

$$u_1^2 - u_2^2 = 2aS_2$$

whence

$$u_2^2 = u_1^2 - 2aS_2 = 2aS_1 - 2aS_2 = 2a(S_1 - S_2)$$

and

$$u_2 = \sqrt{2a(S_1 - S_2)} \tag{E12.2}$$

The speed of the trolley just after the impact would be such that

$$u_3^2 = 2aS_3; \quad u_3 = \sqrt{2aS_3} \tag{E12.3}$$

D1.4 PLANE CENTRAL COLLISION

Let us now analyse a plane central collision of two bodies with reference to Fig. D1.6

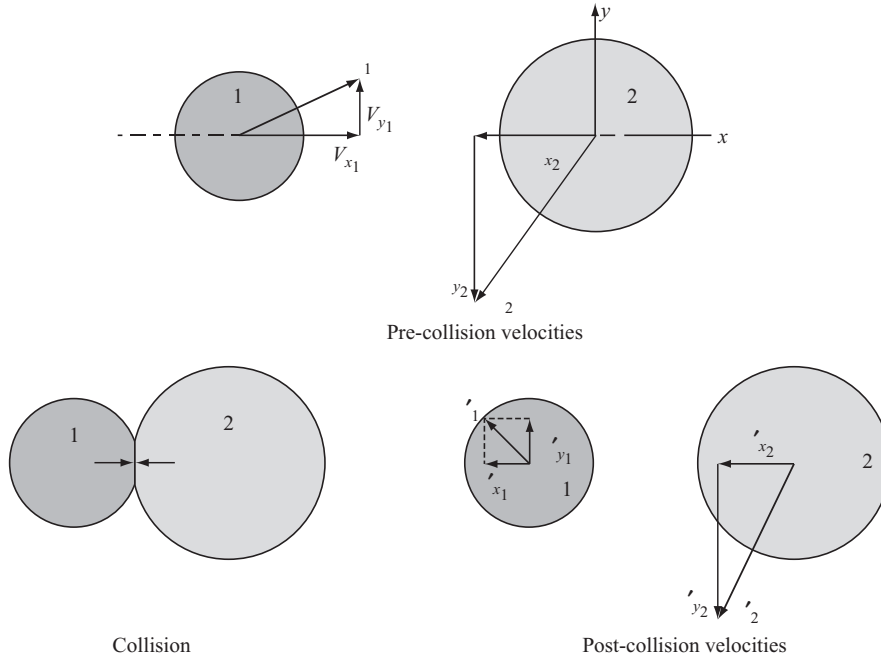


Fig. D1.6 Central collision of two bodies

| | Body 1 | Body 2 |
|-------------------------|--------|--------|
| Pre-collision velocity | v_1 | v_2 |
| Post-collision velocity | v'_1 | v'_2 |

With the knowledge of the initial conditions, computation of post-collision velocities implies the evaluation of four velocity components. Four scalar equations must, therefore, be set up:

- 1, 2. Conservation of momentum for the pair of bodies during the impact would provide two scalar equations. It may be noted that the momentum of the pair of bodies taken together is conserved because no external force acts on them taken together as a system.
3. The definition of the coefficient of restitution e provides one more equation. It expresses empirically the gross effect of the energy loss.
4. One more equation must come from the friction characteristics of the surfaces or the constraints to motion. The absence of friction at the surfaces implies that *the velocity components in the tangential direction remain unaltered.*

By the conservation of momentum principle,

$$m_1 v_{1n} + m_2 v_{2n} = m_1 v'_{1n} + m_2 v'_{2n} \tag{D1.7}$$

In the scalar form,

$$m_1 V_{1n} + m_2 V_{2n} = m_1 V'_{1n} + m_2 V'_{2n} \tag{D1.8}$$

$$m_1V_{1t} + m_2V_{2t} = m_1V'_{1t} + m_2V'_{2t} \quad (D1.9)$$

From the definition of e ,

$$e = -\frac{V_{sn}}{V_{an}} = -\frac{V'_{1n} - V'_{2n}}{V_{1n} - V_{2n}} \quad (D1.10)$$

From the frictionless constraints, one gets

$$V'_{1t} = V_{1t} \quad \text{and} \quad V'_{2t} = V_{2t} \quad (D1.11)$$

However,

$$V'_{1t} = V'_{2t} \quad (D1.12)$$

for rough surfaces of bodies satisfying the 'stick condition'.

It can also be seen that a direct central impact is even easier to analyse. The velocities are then along the line of impact only. Let us consider a direct central impact when the precollision velocities of the bodies are in the same direction and sense as shown in Fig. D1.7.

By the conservation of momentum principle,

$$m_1V_1 + m_2V_2 = m_1V'_1 + m_2V'_2 \quad (D1.13)$$

and by definition of e ,

$$e = -\frac{V_{sn}}{V_{an}} = -\frac{V'_1 - V'_2}{V_1 - V_2} \quad (D1.14)$$

This is a set of equations for the two unknowns, say the final velocities after the impact.

It is interesting to see the change of kinetic energy during the impact:

Before the impact

$$KE = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$$

After the impact

$$KE' = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2$$

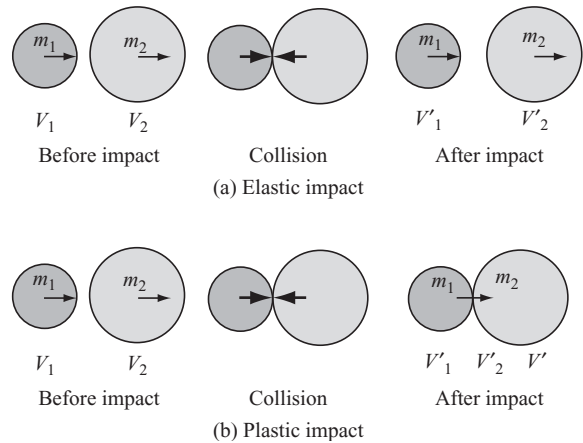


Fig. D1.7 Direct central impact

By employing Eqs. (D1.13) and (D1.14), the expression for KE' can be reduced in terms of V_1 and V_2 . The amount of kinetic energy dissipated is given by

$$KE - KE' = \frac{1 - e^2}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \quad (D1.15)$$

It may be noted that the dissipation of kinetic energy depends upon the coefficient of restitution e , masses m_1 and m_2 and the initial difference of velocities of the bodies. The dissipation is zero if $e = 1$, i.e., when the impact is elastic.

Perfectly Elastic Direct Central Impact

For a perfectly elastic direct central impact, as shown in Fig. D1.7(a), the coefficient of restitution is unity

$$e = 1 = -\frac{V_{sn}}{V_{an}} = -\frac{V'_1 - V'_2}{V_1 - V_2} \quad (D1.16)$$

It follows that

$$V'_1 - V'_2 = V_2 - V_1 = -(V_1 - V_2) \quad (D1.17)$$

which means that the velocity of separation equals the velocity of approach in magnitude but is opposed to it in direction.

From Eq. (D1.15), for $e = 1$, the kinetic energy dissipated is

$$KE - KE' = 0 \quad (D1.18)$$

Since the potential energy remains unaltered during a collision, Eq. (D1.18) may be interpreted to imply that the mechanical energy of the system of bodies is unaltered during a direct central elastic collision. If the surfaces are frictionless, the mechanical energy will also be conserved in an indirect central elastic collision because the tangential velocity component of each body will then remain unaltered during the collision.

$$KE_1 + KE_2 = KE'_1 + KE'_2$$

or
$$\frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_1 V_1'^2 + \frac{1}{2} m_2 V_2'^2 \quad (D1.19)$$

Perfectly Plastic Direct Central Impact

For a perfectly plastic direct central impact, as shown in Fig. D1.7(b).

$$e = 0 = -\frac{V_{sn}}{V_{an}} = -\frac{V_1' - V_2'}{V_1 - V_2} \quad (D1.20)$$

It follows that

$$V_1' - V_2' = 0; \quad V_1' = V_2' = V' \quad (D1.21)$$

which means that the two bodies move together at a common post-collision velocity V' following a direct central plastic impact.

From Eq. (D1.15), for $e = 0$, the kinetic energy dissipated is

$$KE - KE' = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \quad (D1.22)$$

This is incidentally not the entire kinetic energy possessed by the bodies before the collision. The kinetic energy KE before the collision was

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$$

and that after the collision is

$$KE' = \frac{1}{2} (m_1 + m_2) V'^2$$

The percentage dissipation of energy is given by

$$\frac{KE - KE'}{KE} \times 100$$

D1.5 COLLISION OF A SMALL BODY WITH A MASSIVE BODY

If a small body collides with a massive rigid body of a flat or curved surface, as shown in Fig. D1.8, the velocity of the massive body remains unaltered because of its large mass and the large momentum required to produce a small change in its velocity. If a normal is drawn at the point of impact, the velocity component of the small body in the direction of the normal must obey the restitution hypothesis

$$e = -\frac{V_n'}{V_n}$$

or
$$V_n' = e V_n \quad (D1.23)$$

which implies that V_n' is opposed to V_n and is reduced in magnitude by a factor e . The velocity component tangential to the surface remains unaltered in the absence of frictional forces,

$$V_t' = V_t \tag{D1.24}$$

The angle of incidence θ_1 is given by

$$\tan \theta_1 = \frac{V_t}{V_n} \tag{D1.25}$$

and the angle of deflection θ' is given by

$$\tan \theta_1' = \frac{V_t'}{V_n'} = -\frac{V_t}{V_n e} = -\tan \theta_1 / e \tag{D1.26}$$

This implies that θ_1' must be reverse in sign to that of θ_1 , i.e., the angles of incidence and deflection must be subtended on either side of the normal direction. Since e may lie within 0 and 1 for different materials, numerically

$$\tan \theta_1' \geq \tan \theta_1$$

or

$$\theta_1' \geq \theta_1$$

In particular, for an elastic impact, $e = 1$

$$\tan \theta_1' = \tan \theta_1; \quad \theta_1' = \theta_1$$

the angles of incidence and deflection must be equal.

For a plastic impact, $e = 0$

$$\tan \theta_1' \rightarrow \infty; \quad \theta_1' = 90^\circ$$

The small body must be deflected tangential to the massive body at the point of contact.

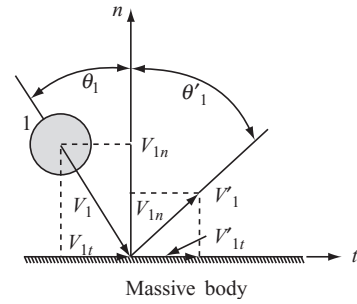


Fig. D1.8 Collision of a small body with a massive rigid body

Example D1.1

(a) A small steel ball is dropped on to a plane surface from a height h and it rebounds to a height h_1 after impact. If the ball is allowed to drop and rebound repeatedly, determine the height of rebound after n impacts. (b) If the ball bounces down a flight of stairs of step height d , determine the height h for which the ball will bounce to the same height above each step.

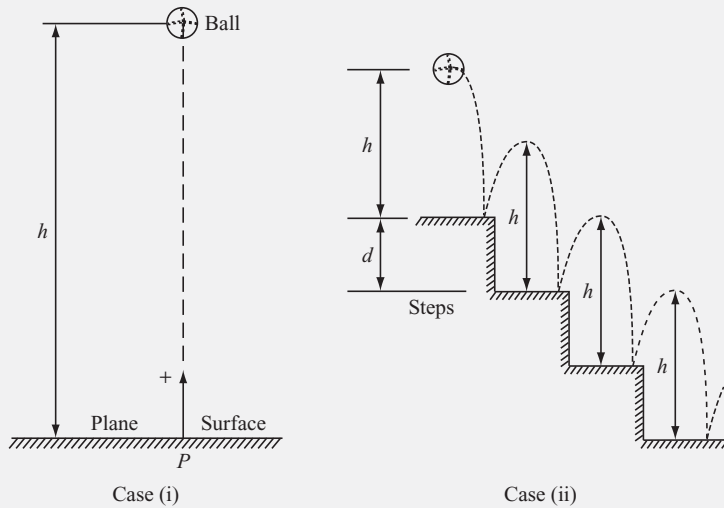


Fig. Ex. D1.1

Solution Assigning suffix 1 to the ball and 2 to the stationary plane surface, the pre-collision velocities are

$$V_1 = -\sqrt{2gh} \quad \text{and} \quad V_2 = 0$$

Since the ball rebounds to a height h_1 , the post-collision velocities must be

$$V_1' = \sqrt{2gh_1} \quad \text{and} \quad V_2' = 0$$

Employing the definition of the coefficient of restitution,

$$e = -\frac{V_{2n}}{V_{1n}} = \frac{\sqrt{2gh_1}}{\sqrt{2gh}} = \frac{\sqrt{h_1}}{h}$$

The height of rebound after the second impact is

$$h \times \left(\frac{h_1}{h}\right)^2 = \frac{h_1^2}{h}$$

and after n impacts, the height of rebound is

$$h \times \left(\frac{h_1}{h}\right)^n = \frac{h_1^n}{h^{n-1}}$$

For the second case, the height of rebound above a datum on any step is $(h - d)$ for a height h of fall. Hence,

$$e = \sqrt{\frac{h-d}{h}}$$

or

$$e^2 h = h - d$$

whence

$$h = \frac{d}{1 - e^2}$$

It has been assumed that the ball drops approximately vertically on each step and that the step width is small.

Example D1.2

A bullet of mass 20 g moving with a velocity of 100 m/s hits a 2 kg bob of a simple pendulum horizontally as shown in Fig. Ex. D1.2. Determine the maximum angle through which the pendulum string 0.5 m long may swing if

- (a) the bullet gets embedded in the bob
- (b) the bullet escapes from the other end at 20 m/s
- (c) the bullet is rebounded from the surface of the bob at 20 m/s

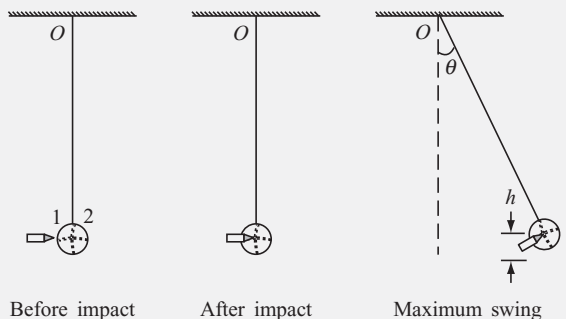


Fig. Ex. D1.2

Solution Just before the impact, for the bullet,

$$V_1 = 100 \text{ m/s} \quad \text{and} \quad m_1 = 0.02 \text{ kg},$$

$$m_1 V_1 = 2 \text{ kg m/s}$$

and for the bob

$$V_2 = 0 \quad \text{and} \quad m_2 = 2 \text{ kg}; \quad m_2 V_2 = 0$$

Now,

$$m_1 V_1 + m_2 V_2 = 2 \text{ kg m/s}$$

Case (a)

Just after the impact, the bullet and the bob become united to travel at a velocity of V' m/s and their momentum is

$$(2 + 0.02)V' = 2.02V' \text{ kg m/s}$$

Equating the initial and final momenta,

$$2.02 V' = 2, \quad V' = 0.99 \text{ m/s}$$

After the impact, the bob must rise by a distance h according to the principle of energy conservation

$$\frac{mV'^2}{2} = mgh$$

whence,

$$h = \frac{0.99^2}{2 \times 9.81} = 0.05 \text{ m}$$

and

$$\theta = \cos^{-1} \left(\frac{0.5 - 0.05}{0.5} \right) = 25.85^\circ$$

Case (b)

If the bullet escaped from the other end of the bob at a velocity of 20 m/s, the final momentum would have given by

$$0.02 \times 20 + 2 \times V'_2$$

which when equated to the initial momentum 2 kg m/s provides,

$$V'_2 = 0.8 \text{ m/s}$$

The height through which the bob is lifted becomes,

$$h = \frac{0.8^2}{2 \times 9.81} = 0.0326 \text{ m}$$

and

$$\theta = \cos^{-1} \left(\frac{0.5 - 0.0326}{0.5} \right) = 20.8^\circ$$

Case (c)

If, instead the bullet rebounded at 20 m/s, the final momentum would be given by

$$-0.02 \times 20 + 2 \times V'_2$$

which when equated to the initial momentum of 2 kg m/s provides,

$$V'_2 = 1.2 \text{ m/s}$$

$$h = \frac{1.2^2}{2 \times 9.81} = 0.0734$$

and

$$\theta = \cos^{-1} \left(\frac{0.5 - 0.0734}{0.5} \right) = 31.4^\circ$$

Example D1.3

Two homogeneous spheres A and B of masses 10 kg and 2 kg and radii 30 cm and 20 cm respectively are moving initially along parallel paths 10 cm apart as shown in Fig. Ex. D1.3. If the coefficient of restitution between the two surfaces is 0.4 and friction is negligible, determine the velocities of the spheres immediately after the collision. Also estimate the energy lost during the collision.

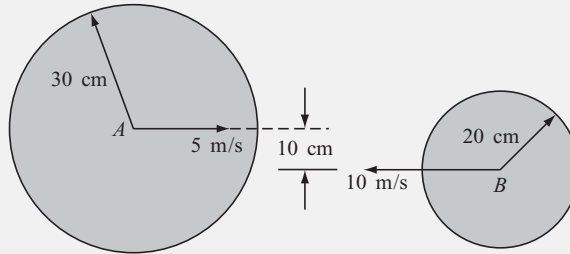


Fig. Ex. D1.3

Solution From the geometry at the scene of collision,

$$\sin \theta = \frac{10}{50} = 0.2$$

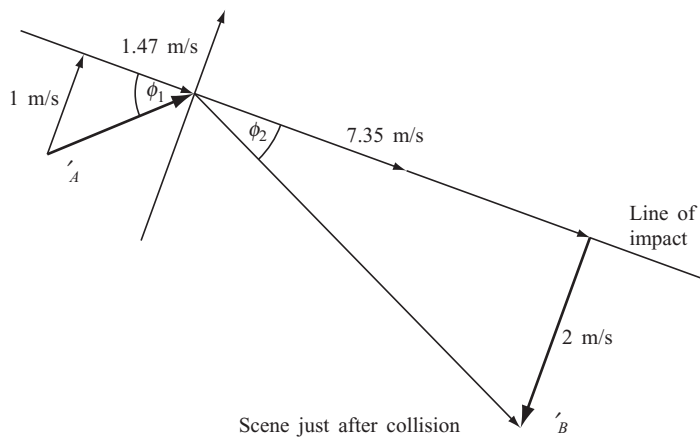
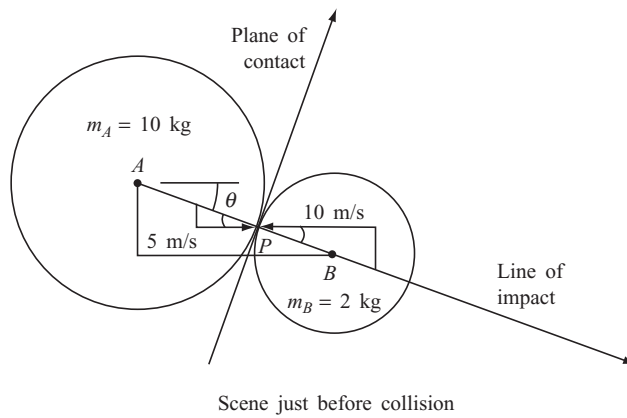


Fig. Ex. D1.3 (Solution)

and $\theta = 11.54^\circ$

Just before the collision,

$$\begin{aligned} V_A &= 5 \cos \theta + 5 \sin \theta = (4.9 + 1) \text{ m/s} \\ V_B &= -10 \cos \theta - 10 \sin \theta = (-9.8 - 2) \text{ m/s} \end{aligned}$$

During the collision process, no external force acts on the pair of spheres. The forces of impact are internal forces for the pair of bodies and frictional forces are negligible. The momentum must, therefore, be conserved during the collision. Conserving the momentum along the line of impact,

$$\begin{aligned} 10V'_{An} + 2V'_{Bn} &= 10V_{An} + 2V_{Bn} \\ &= 10 \times 4.9 - 2 \times 9.8 = 29.4 \text{ m/s} \end{aligned}$$

or $10V'_{An} + 2V'_{Bn} = 29.4 \text{ m/s}$ (i)

The additional information that there is no frictional retardation on either body in the tangential direction enables us to write that

$$\begin{aligned} V'_{At} &= V_{At} \\ V'_{Bt} &= V_{Bt} \end{aligned} \quad \text{(ii)}$$

From the definition of the coefficient of restitution,

$$\begin{aligned} e &= -\frac{V'_{An} - V'_{Bn}}{V_{An} - V_{Bn}} \\ V'_{An} - V'_{Bn} &= -0.4 \times (4.9 - (-9.8)) = -5.88 \text{ m/s} \end{aligned} \quad \text{(iii)}$$

or $2V'_{An} - 2V'_{Bn} = -11.76$

and from (i), $10V'_{An} + 2V'_{Bn} = 29.40$

On addition, $12V'_{An} = 17.64$

whence, $V'_{An} = 1.47 \text{ m/s}$

and $V'_{Bn} = 7.35 \text{ m/s}$

Finally,

$$\vec{V}'_A = (1.47 \quad 1) \text{ m/s}$$

$$\vec{V}'_B = (7.35 \quad -2) \text{ m/s}$$

With reference to the figure showing the components of velocity after the collision,

$$V'_A = 1.76 \text{ m/s}; \quad \phi_1 = \tan^{-1}(1/1.47) = 34.2$$

$$V'_B = 7.62 \text{ m/s}; \quad \phi_2 = \tan^{-1}(2/7.35) = 15.2$$

$$\begin{aligned} \text{The kinetic energy before the collision} &= \frac{1}{2} \times 10 \times 5^2 + \frac{1}{2} \times 10 \times 10^2 \\ &= 625 \text{ J} \end{aligned}$$

$$\begin{aligned} \text{and kinetic energy after the collision} &= \frac{1}{2} \times 10 \times (1.76)^2 + \frac{1}{2} \times 10 \times (7.62)^2 \\ &= 301.30 \text{ J} \end{aligned}$$

Since there is no change in the potential energy, the energy loss is

$$= 625 - 301.30$$

$$= 323.70 \text{ J}$$

Example D1.4 State whether the following statement is true or false. Give reasons for your response:

Two particles of masses 1 kg and 3 kg move towards each other under their mutual force of attraction. No other force acts on them. When the relative velocity of approach of the two particles is 2 m/s, their centre of mass has a velocity of 0.5 m/s. When the relative velocity of approach becomes 3 m/s, the velocity of the centre of mass is 0.75 m/s.



Fig. Ex. D1.4

Solution From the fact that there is no external force acting on the two masses taken together, their momentum must be conserved. This is equivalent to stating that their centre of mass must continue to move with the same velocity because

$$\text{Momentum} = m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_C = m_1 v_1' + m_2 v_2'$$

From the given statement of the problem, the velocity of the centre of mass changes from 0.5 m/s to 0.75 m/s. This is just not possible as the momentum should be conserved. Hence, the statement is false.

Example D1.5 A shell is fired from a cannon with a velocity V at an angle θ with the horizontal direction as shown in Fig. Ex. D1.5. At the highest point in its path it explodes into two pieces of equal mass. One of the pieces retraces its path to the cannon and the speed of the other piece immediately after the explosion is:

- (A) $3V \cos \theta$ (B) $2V \cos \theta$
 (C) $\frac{3}{2} V \cos \theta$ (D) $\frac{\sqrt{3}}{2} V \cos \theta$

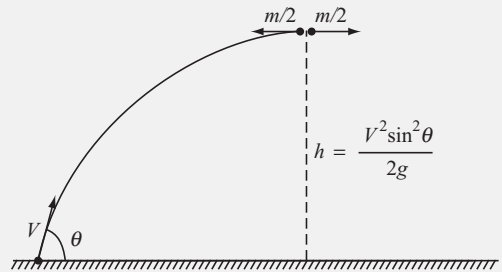


Fig. Ex. D1.5

Solution The velocity of the shell as it reaches the highest point is given by

$$V_x = V \cos \theta$$

Now, it explodes into two pieces. The momentum possessed by the total shell, i.e., $mV \cos \theta$ must be preserved.

$$mV \cos \theta = -\frac{m}{2} V \cos \theta + \frac{m}{2} V'$$

whence
$$V' = 3 V \cos \theta$$

Therefore (A) is correct. All others are incorrect.

Example D1.6 A mass explodes, breaking into three pieces. Two pieces 2 kg and 4 kg fly off in perpendicular directions with velocities 6 m/s and 4 m/s respectively. Find the mass of the third piece if it flies off at 40 m/s.

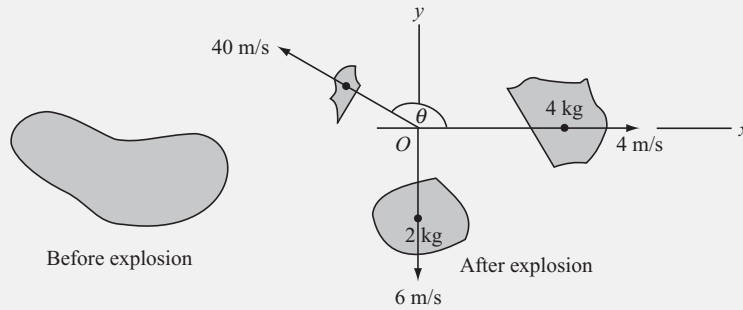


Fig. Ex. D1.6

Solution During the process of explosion, there is no external force; hence the momentum of the system is conserved.

$$4 \times 4 + 40 m \cos \theta = 0 \quad \text{along } x\text{-axis}$$

$$-2 \times 6 + 40 m \sin \theta = 0 \quad \text{along } y\text{-axis}$$

whence

$$m \cos \theta = -2/5$$

$$m \sin \theta = 3/10$$

and

$$\tan \theta = -\frac{3}{10} \times \frac{5}{2} = -3/4$$

$$\theta = -36.9^\circ, 143.1^\circ$$

$$m = \frac{3}{10} \sin 143.1^\circ = 0.5 \text{ kg}$$

Example D1.7

An object of mass 5 kg is projected with a velocity of 20 m/s at an angle of 60° to the horizontal. At the highest point of its path the projectile explodes and breaks up into two fragments of masses 1 kg and 4 kg. The fragments separate horizontally after the explosion. The explosion releases internal energy such that the kinetic energy of the system at the highest point is doubled. Calculate the separation between the two fragments when they reach the ground.

Solution For the highest position, (see Fig. Ex. D1.7 (Solution))

$$h = \frac{20^2 \sin^2 60^\circ}{2g} = \frac{400 \times 0.75}{2 \times 9.81} = 15.29 \text{ m}$$

During explosion, the momentum of the system must be conserved,

$$m_1 v_1' + m_2 v_2' = m v$$

$$= 4 v_1' + 1 v_2' = 5 \times 10 = 50 \quad \text{(i)}$$

Before the explosion, the kinetic energy is

$$\frac{1}{2} m V^2 = \frac{1}{2} \times 5 \times 10^2 = 250 \text{ J}$$

and the potential energy is

$$mgh = 5 \times 9.81 \times 15.29 = 750 \text{ J.}$$

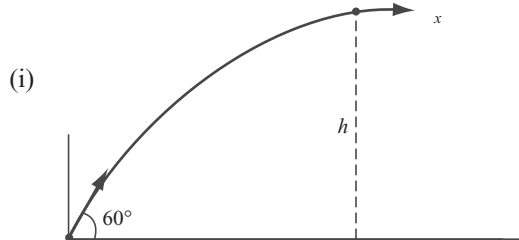


Fig. Ex. D1.7 (Solution)

After the explosion, the kinetic energy becomes

$$\frac{1}{2} \cdot 4 \cdot V_1'^2 + \frac{1}{2} \cdot 1 \cdot V_2'^2 = 2 \times 250 = 500$$

$$4V_1'^2 + V_2'^2 = 1000 \tag{ii}$$

From (i) and (ii)

$$V_1' = 5 \text{ or } 15 \text{ m/s} \quad \text{and} \quad V_2' = -10 \text{ and } 15 \text{ m/s}$$

Feasible sets of answers are

$$V_1' = 5 \quad \text{and} \quad V_2' = 30 \text{ m/s} \quad \text{Set I}$$

and

$$V_1' = 15 \quad \text{and} \quad V_2' = -10 \text{ m/s} \quad \text{Set II}$$

The horizontal separation between the two on the ground equals the sum of their ranges.

Time taken to reach the ground is given by

$$0 + \frac{1}{2} g t^2 = 15.29; \quad t = 1.766 \text{ seconds}$$

$$D = 15 \times 1.766 + 10 \times 1.766 = 44.15 \text{ m} \quad \text{for set I}$$

Alternatively,

$$D = 30 \times 1.766 - 5 \times 1.766 = 44.15 \text{ m} \quad \text{for set II}$$

The two answers happen to be identical

Example D1.8

A freely falling body from a given height H hits an inclined plane in its path at a height h as shown in Fig. Ex. D1.8. As a result of the impact, the direction of the velocity becomes horizontal. For what value of h/H will the body take maximum time to reach the ground?

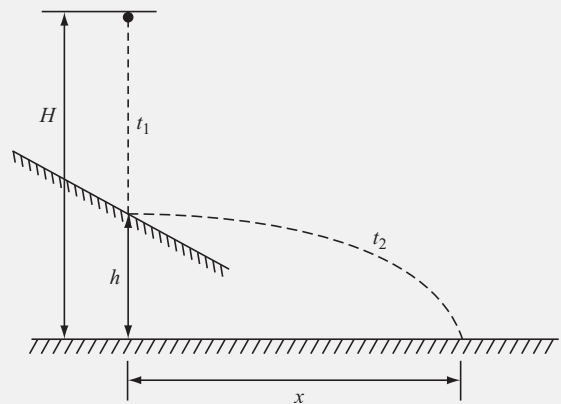


Fig. Ex. D1.8

Solution Time taken for the body to fall until it hits the incline t_1 , is given by

$$H - h = \frac{1}{2} g t_1^2$$

whence,

$$t_1 = \sqrt{\frac{2(H - h)}{g}}$$

Further time,

$$t_2 = \frac{\sqrt{2h}}{g}$$

Total time
$$t = t_1 + t_2 = \sqrt{\frac{2H}{g}} \left[\sqrt{\frac{h}{H}} + \sqrt{\left(1 - \frac{h}{H}\right)} \right]$$

Let
$$h/H = r \quad \text{and} \quad \sqrt{\frac{2H}{g}} = K, \text{ a constant}$$

$$t = K (\sqrt{r} + \sqrt{1-r})$$

For maximum time,
$$\frac{dt}{dr} = K \left(\frac{1}{2\sqrt{r}} - \frac{1}{2\sqrt{1-r}} \right) = 0$$

and
$$r = h/H = 1/2 = 0.5$$

It should hit the incline half way above the ground.

Example D1.9

A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position and released (see Fig. Ex. D1.9). The ball hits the wall, the coefficient of restitution being $2/\sqrt{5}$. What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60 degrees?

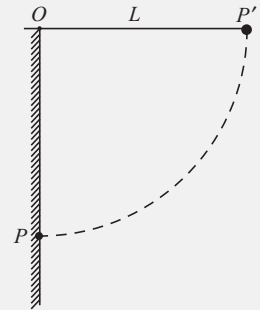


Fig. Ex. D1.9

Solution In the initial state, with P as datum,

$$PE = mgL, KE = 0$$

Just before hitting the wall,

$$PE = 0, KE = mV^2$$

By energy conservation,

$$\frac{1}{2}mV^2 = mgL; V = \sqrt{2gL}$$

Now,

$$e = \frac{\text{Velocity of separation}}{\text{Velocity of approach}} = \frac{V'}{V} = \frac{2}{\sqrt{5}}$$

It also implies that the kinetic energy becomes e^2 , i.e., $4/5$ times i.e., 80% with every impact.

At the position of 60° oscillation,

$$PE = mgL(1 - \sin 30) = mgL/2$$

which corresponds to a kinetic energy 50% of the original value. The kinetic energy would be less than 50% after 3 collisions since

$$80\% \times 80\% \times 80\% \text{ is just less than } 50\%$$

The minimum number of collisions is, therefore, three.

Example D1.10

A block of mass 2 is placed on another block of mass 4 which in turn is placed on a fixed table. The two blocks have the same length 4 and they are placed as shown in Fig. Ex. D1.10. The coefficient of friction (both static and kinetic) between the block and the table is μ . There is no friction between the two blocks. A small object of mass moving horizontally along a line passing through the centre of mass (, see figure) of the block and perpendicular to its face with a speed collides elastically with the block at a height above the table.

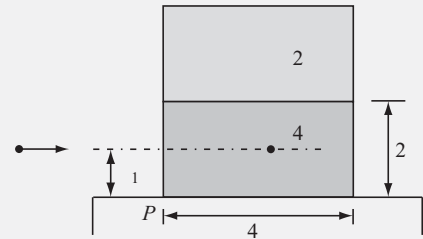


Fig. Ex. D1.10

- (a) What is the minimum value of (call it v_0) required to make the block topple?
- (b) If $h = 2v_0$, find the distance (from the point P in the figure) at which the mass falls on the table after collision. Ignore the role of friction during the collision.

Solution The block would topple if the block below it slides past it half way, i.e., a distance of 2 to the right. The frictional force between the table and the block at the state of impending motion and thereafter is given by

$$F_f = 6 \mu$$

opposing the direction of motion of the block. The energy required for the block to move a distance of 2 is, therefore,

$$W = 6 \mu g \times 2 = 12 \mu$$

- (a) The minimum value of velocity of the block just after impact should be such that

$$\frac{1}{2} (4) \cdot v^2 = 12 \mu$$

whence

$$v = \sqrt{6\mu}$$

During the impact, the momentum is conserved,

$$4v + 0 = 0$$

$$4v + 0 = 0$$

and, by the definition of restitution,

$$e = -\left(\frac{v_2 - v_1}{u_2 - u_1}\right) = 1$$

From these two equations

$$v = \frac{2}{5} \sqrt{6\mu}; \quad v_1 = -3/5 v_0$$

or

$$v_0 = \frac{5}{2} \sqrt{6\mu}$$

- (b) In this case,

$$v = 2v_0 = 5 \sqrt{6\mu}$$

For momentum conservation during the impact,

$$4v + 0 = 0$$

and by virtue of elastic impact

$$e = -\left(\frac{V_{Bf} - V_{Af}}{0 - V}\right) = 1; V_{Bf} - V_{Af} = V$$

whence,

$$\begin{aligned} V_{Af} &= -\frac{3}{5}V \\ &= -\frac{3}{5}(2V_0) = -\frac{3}{5} \times 2 \times \frac{5}{2} \sqrt{6\mu g d} \\ &= -3\sqrt{6\mu g d} \end{aligned}$$

Example D1.11

A drop hammer 1 with a mass of 6 Mg falls from rest 0.8 m onto a forged anvil 2 mounted on springs which have a composite stiffness 2 MN/m as shown in Fig. Ex. D1.11. Find the maximum compression of the springs after the impact if the anvil has a mass of 4 Mg and the coefficient of restitution between the hammer and the anvil is $e = 0.5$. Neglect the friction along the vertical guide posts.

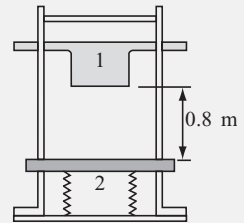


Fig. Ex. D1.11

Solution The velocity of the hammer just before the impact is given by

$$\frac{1}{2} m_1 V_1^2 = m_1 gh$$

$$V_1 = \sqrt{2 \times 9.81 \times 0.8} = 3.96 \text{ m/s}$$

During the impact, momentum is conserved,

$$m_1 V_1 + m_2 V_2 = m_1 V_1' + m_2 V_2'$$

whence

$$6 \times 10^3 \times 3.96 + 0 = 6 \times 10^3 V_1' + 4 \times 10^3 V_2'$$

or

$$3V_1' + 2V_2' = 11.88$$

(i)

(Note: 1 Mg = 1×10^6 g = 1000 kg)

Also, from the definition of coefficient of restitution,

$$e = -\frac{V_s}{V_a} = \frac{V_2' - V_1'}{V_1 - V_2}$$

$$0.5 = \frac{V_2' - V_1'}{3.96 - 0}$$

or

$$V_2' - V_1' = 1.98$$

(ii)

From (i) and (ii)

$$V_2' = 3.564 \text{ m/s}; V_1' = 1.584 \text{ m/s.}$$

The kinetic energy possessed by the anvil is used up in creating the potential energy of the springs:

$$\frac{1}{2} \times 4000 \times 3.564^2 = \frac{1}{2} \times 2 \times 10^6 x^2$$

The spring compression is given by

$$x = 0.16 \text{ m}$$

provided that the hammer does not hit the anvil again.

Example D1.12 A ball of radius r hits the ground with an initial velocity V_1 and top-spin, i.e., angular velocity ω_1 . Determine an expression for its linear velocity V_2 just after the impact (see Fig. Ex. D1.12).

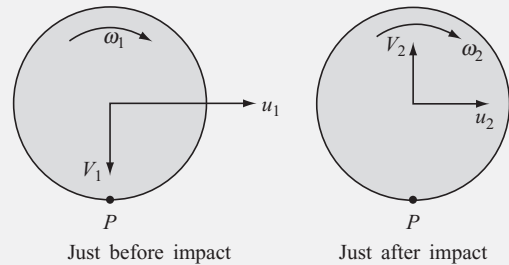


Fig. Ex. D1.12

Solution Just before the impact, the ball may have a horizontal component and a vertical component of velocity, say u_1 and V_1 respectively.

The vertical rebound and change in the vertical component of velocity from V_1 to V_2 takes place depending upon the coefficient of restitution between the ball and the ground. It is not effected by the spin of the ball.

The horizontal component of velocity changes from u_1 to u_2 and the spin changes from ω_1 to ω_2 . It is reasonable to assume that the slipping at the point of contact will cease, the earth being fairly rough and sticky. Consequently, the angular velocity of the ball just after the impact will only be u_2/r ;

$$\omega_2 = u_2/r$$

During impact, there is no change in the moment of momentum of the ball about the point of contact P .

$$m(u_1 r + k^2 \omega_1) = m(u_2 r + k^2 u_2/r)$$

It is noticed that the change in the vertical velocity component does not figure in the moments of momenta.

From the above,

$$u_2 = \frac{r^2 u_1 + k^2 r \omega_1}{r^2 + k^2}$$

For a hollow sphere,

$$k^2 = 2/3 r^2$$

and

$$u_2 = \frac{3}{5} u_1 + \frac{2}{5} r \omega_1$$

For a solid sphere,

$$k^2 = 2/5 r^2$$

and

$$u_2 = \frac{5}{7} u_1 + \frac{2}{7} r \omega_1$$

Golf balls are solid as also small and rough. For a golf ball to stop dead on touching the ground,

$$u_2 = \frac{5}{7} u_1 + \frac{2}{7} r \omega_1 = 0$$

whence

$$\omega_1 = -2.5 r u_1$$

which means that the golf ball should have an initial under-spin of 2.5 r times initial velocity in order to stop immediately on hitting the ground.

Example D1.13 A thin uniform bar lies on a frictionless horizontal surface and is free to move in any way on the surface. Its mass is 0.16 kg and length is $\sqrt{3}$ metres. Two particles, each of mass 0.08 kg, are moving on the same surface and towards the bar in a direction perpendicular to the bar, one with a velocity of 10 m/s and the other with 6 m/s, as shown in Fig. Ex. D1.13. The first particle strikes the bar at point A and the other at point B. Points A and B are at a distance of 0.5 m from the centre of the bar. The particles strike the bar at the same instant of time and stick to the bar on collision. Calculate the loss of kinetic energy of the system in the above collision process.

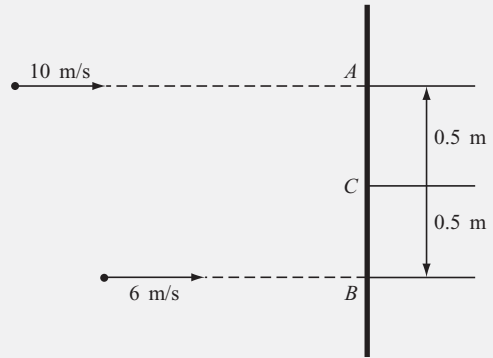


Fig. Ex. D1.13

Solution The initial kinetic energy is given as

$$\frac{1}{2} \times 0.08 \times (10^2 + 6^2) = 5.44 \text{ J}$$

There is no change of potential energy in the entire process.

Momentum of the upper particle = $0.08 \times 10 = 0.8$ kg m/s and for the lower particle, $0.08 \times 6 = 0.48$ kg m/s.

For the system, in the absence of external forces, the linear momentum is conserved.

$$0.8 + 0.48 = (0.16 + 0.08 + 0.08) V_c$$

whence

$$V_c = 4 \text{ m/s}$$

Also, in the absence of external moments, the moment of momentum of the system must be conserved.

$$0.8 \times 0.5 - 0.48 \times 0.5 = 0.16 \times (\sqrt{3})^2 / 12 \times \omega + 2 \times 0.08 \times (0.5)^2$$

Therefore,

$$\omega = 2 \text{ rad/s}$$

The final kinetic energy is given by

$$\begin{aligned} \frac{1}{2} (0.16 + 0.08 + 0.08) \times 4^2 + \frac{1}{2} \times 0.08 \times (2)^2 \\ = 2.56 + 0.16 = 2.72 \text{ J} \end{aligned}$$

Loss of kinetic energy in the process is, therefore, $5.44 - 2.72$, i.e., 2.72 J.

Example D1.14 A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speeds $2V$ and V , respectively, strike the bar (as shown in Fig. Ex. D1.14) and stick to the bar after collision. Denoting angular velocity (about the centre of mass), total energy and centre of mass velocity by ω , E and V_c respectively, which of the following is (are) true after collision?

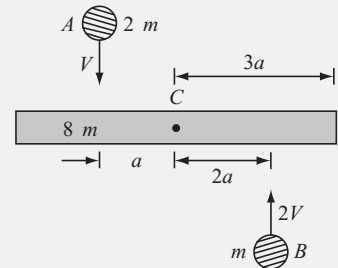


Fig. Ex. D1.14

(A) $V_c = 0$

(C) $\omega = \frac{V}{5a}$

(B) $\omega = \frac{3V}{5a}$

(D) $E = \frac{3mV^2}{5}$

Solution In the state before striking the bar, the linear momentum of masses A and B are

$$p_A = -2 mV \quad \text{and} \quad p_B = 2 mV \text{ respectively.}$$

The bar being at rest, the total initial momentum is

$$-2 mV + 2 mV + 0 = 0$$

In the absence of any external force on the system of three bodies, the linear momentum is conserved. The final momentum of the system must be zero. Hence $V_c = 0$ which means that (A) is correct.

The initial angular moments of the masses A, B and the bar are $2 mVa, 4 mVa$ and zero respectively which add to $6 mVa$. The final angular momenta of the masses rotating at ω are

$$H_A = 2 m (a\omega) a = 2 ma^2 \omega$$

$$H_B = 4 m (2 a\omega) a = 8 ma^2 \omega$$

and for the bar, it is $1/12 (8 m), (6a)^2 \omega = 24 ma^2 \omega$

In the absence of any external moment, the moment of momentum is conserved.

Hence,

$$6 mVa = (2 ma^2 + 4 ma^2 + 24 ma^2) \omega$$

$$\omega = \frac{V}{5a} \quad \text{which means that (C) is correct}$$

whence

and (B) is incorrect.

Finally, the total energy of the system after the collision must be

$$E = \frac{1}{2} m V_A^2 + \frac{1}{2} m V_B^2 + \frac{1}{2} I_C \omega^2$$

$$= \frac{1}{2} 30 ma^2 \cdot \left(\frac{V}{5a}\right)^2 = \frac{3}{5} mV^2$$

which means that (D) is also correct.

Example D1.15

A bar of length L and mass m lies on a frictionless horizontal plane. A pellet, also of mass m , moving with a speed v_0 strikes perpendicular to it at $L/4$ from one end. If the collision is elastic, determine the final velocity of the pellet and the angular velocity of the bar after collision (ref. Fig. Ex. D1.15).

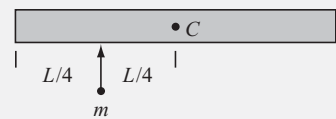


Fig. Ex. D1.15

Solution

Let the velocity of the pellet after the collision be v and that of the centre of mass of the bar be v_c . By conservation of momentum,

$$m v + m v_c = m v_0 \tag{i}$$

By conservation of angular momentum,

$$m v L/4 + 1/12 m L^2 \omega = m v_0 L/4 \tag{ii}$$

By conservation of mechanical energy for elastic collision,

$$1/2 m v^2 + 1/2 m v_c^2 + 1/2 \cdot 1/12 m L^2 \omega^2 = 1/2 m v_0^2 \tag{iii}$$

Solving these equations,

$$v = 3/11 v_0, \quad v_c = 8/11 v_0 \quad \text{and} \quad \omega = 24/11 v_0/L$$

Example D1.16

A big vehicle moving at 20 km per hour hits a smaller vehicle, half its mass, while coasting in front of it, in neutral at 10 km/hr along the same straight line on a level road. Assuming the coefficient of restitution as 0.3, estimate

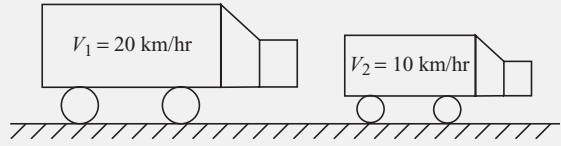


Fig. Ex. D1.16

- (a) their speeds after the collision,
- (b) loss of energy, if any, and
- (c) the maximum force of impact over one tenth of a second during which it increases and then decreases linearly

Solution This is a case of simple collinear plane central impact without any external force, i.e., braking or driving force.

$$V_1 = \frac{20 \times 1000}{3600} = 5.55 \text{ m/s}; \quad V_2 = \frac{10 \times 1000}{3600} = 2.78 \text{ m/s}$$

$$m_1 = 2m \qquad m_2 = m$$

By conservation of momentum,

$$m_1 V_1 + m_2 V_2 = m_1 V_1' + m_2 V_2'$$

$$2 \times 5.55 m + 2.78 m = 2m V_1' + m V_2'$$

$$11.10 + 2.78 = 2V_1' + V_2' = 13.88 \text{ m/s}$$

or

Now,

$$e = \frac{V_2' - V_1'}{V_1 - V_2} = 0.3 = \frac{V_2' - V_1'}{5.55 - 2.78} = \frac{V_2' - V_1'}{2.77}$$

whence,

$$V_2' - V_1' = 0.83$$

- (a) Solution for V_1' and V_2'

$$V_1' = 4.35 \text{ m/s} \quad \text{and} \quad V_2' = 5.18 \text{ m/s}$$

- (b) The collision is on a level surface; the energy before and after is only kinetic energy. Before the collision,

$$E = \frac{1}{2} 2m \times 5.55^2 + \frac{1}{2} m \times 2.78^2$$

$$= 34.66 m \text{ Nm}$$

After the collision,

$$E' = \frac{1}{2} 2m \times 4.35^2 + \frac{1}{2} m \times 5.18^2$$

$$= 32.34 m \text{ Nm}$$

Loss of energy = $(34.66 - 32.34) m = 2.32 m \text{ Nm}$.

The energy is lost because the impact is not elastic; part of it goes in disfiguring the vehicles and part in sound and heat!

Momentum transfer has taken place between the two vehicles. It can be determined by considering the change in momentum of either vehicle. For the smaller vehicle,

$$\text{Gain in momentum} = m \times 5.18 - m \times 2.78 = 2.4 \text{ m}$$

(For the larger vehicle, loss of momentum is $2 \text{ m} \times (5.55 - 4.35) = 2.4 \text{ m}$; the same)

This must equal the impulse = Fdt

or
$$\frac{F \times 0.5}{2} \times 2 = 2.4 \text{ m}; \quad F = 4.8 \text{ m}$$

Concept Review Questions

Justify the truth in the following statements, if true:

- The contact force which acts on a body during a collision must be the same as that acting on the other body.
- The impulse of deformation equals the impulse of recovery for an elastic impact.
- The entire kinetic energy is dissipated during a plastic impact.
- The momentum is conserved for all central, non-central, direct and indirect impacts of two bodies.
- Two bodies going together after an impact indicate that they must have had a plastic impact.
- An elastic impact must be noiseless.

A spherical mass approaching with a velocity v strikes an identical spherical mass at rest. What would their final velocities be if the impact was (a) perfectly elastic and (b) perfectly plastic.

Define the term 'coefficient of restitution' and state what factors influence its value for a pair of

two materials. Can you suggest a simple experiment to estimate the coefficient of restitution for a given pair of materials?

Define the terms 'impulse of deformation' and 'impulse of recovery'. Why is the concept of impulse preferred to that of force for impact problems?

A body freely dropped from a height H on to a flat surface rises to a height h after rebound. What should be the value of the coefficient of restitution?

If the coefficient of restitution is 0.5, to what fraction of its original height of drop would a body rise after the rebound?

A steel ball falling freely strikes an inclined surface and, after impact, proceeds horizontally. Express the angle of the incline in terms of the coefficient of restitution e and also relate the final velocity just after the impact to the initial velocity just before the impact.

Tutorial Problems

A spherical object drops from the ceiling of a room and after rebounding twice from the floor it reaches a height half as that of the ceiling. Show that the coefficient of restitution is 0.841.

A ball of mass 0.2 kg moving with a velocity of 3 m/s impinges on a ball of mass 0.4 kg moving

with a velocity of 1 m/s. The velocities of the balls are parallel and inclined at 30° to the line joining their centres at the instant of the impact. If the coefficient of restitution is 0.5, determine the velocities of the balls after the impact.

(s 2.52 and 1.26 m/s)

A billiards ball 1 is hit at position (a) to reach at position (b) where it impacts with a stationary ball 2 and goes off into the pocket as shown in Fig. Prob. D1.3. Determine the line of impact, i.e., angle θ if the coefficient of restitution between the balls is 0.8. (s $\theta = 34^\circ$)

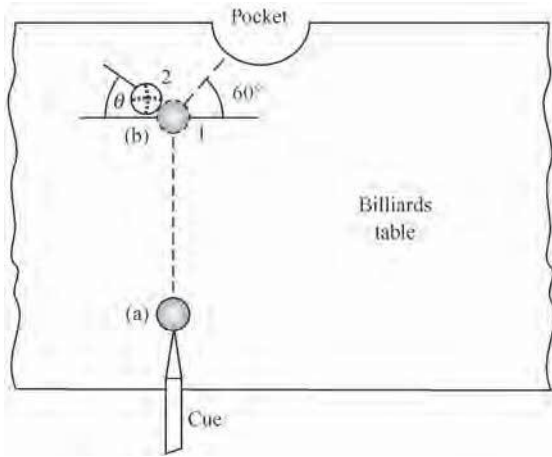


Fig. Prob. D1.3

A ball of mass 0.15 kg approaching with a velocity of 30 m/s hits a stationary block of mass 0.2 kg centrally as shown in Fig. Prob. D1.4. If the coefficient of restitution is 0.7. Find the distance travelled by the block on a flat horizontal rough surface with dynamic friction 0.3. (s 81.1 m)

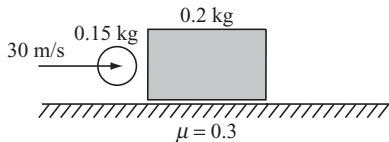


Fig. Prob. D1.4

Two identical simple pendulums hang side by side when the bobs just touch each other. If one of them is displaced by 30° as shown in Fig. Prob. D1.5 and released to collide with the bob of the other centrally and directly, determine the maximum angular displacement of the second pendulum if the coefficient of restitution between the materials of the bobs is 0.8. Also determine the kinetic energy lost in the collision. (s 27° and 0.236 J/kg)

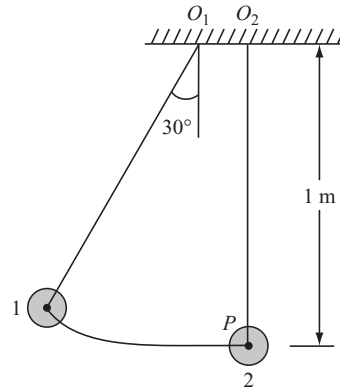


Fig. Prob. D1.5

Two spherical balls of masses 0.5 kg and 0.2 kg approach each other to collide obliquely as shown in Fig. Prob. D1.6. Determine their velocities of separation just after the impact if

(a) the impact is elastic

(b) the impact is plastic.

(s 11.24 and 27.35 m/s; 14.47 m/s)

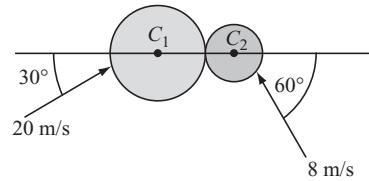


Fig. Prob. D1.6

A uniform slender bar of mass 2 kg and 1.2 m long hangs freely about a hinge O to be able to swing in a plane as shown in Fig. Prob. D1.7. When the bar is at rest in the vertical position, a small object of mass 0.02 kg hits it at P , as shown, with a velocity of 400 m/s. Determine the maximum angle through which the bar may swing if

(a) the collision is perfectly elastic, i.e., there is no loss of energy

(b) the collision is perfectly plastic, i.e., the object is embedded in the bar

(c) if the object pierces through the bar and appears with a velocity of 200 m/s along the original direction of flight.

- (s 11.5 rad/s, rotation $\omega = 5.77$ rad/s, 111 ;
2.95 rad/s, 49.5)

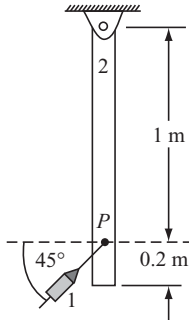


Fig. Prob. D1.7

- 8 A tennis ball moving with a velocity of 25 m/s strikes the court at an angle of 70° to the vertical. If the coefficient of restitution is 0.75, find the velocity and direction of motion of the ball after the impact.
(s 24.35 m/s, 74.75° to the vertical)

A bullet of mass 0.018 kg strikes and becomes embedded in a stationary block of mass 5 kg which is free to move horizontally. If the bullet is travelling horizontally at 600 m/s, find the velocity of the block after the impact. What percentage of the energy is lost in the impact?
(s 2.155 m/s, 99.65%)

A head-on collision takes place between a car and a truck. The car has a mass of 1000 kg and was travelling at 60 km/h and the truck has a mass of 15 000 kg and was travelling at 20 km/h. After the collision, the two vehicles remain locked together. Determine the magnitude and direction of the velocity after collision. Calculate the total kinetic energy before and after impact and account for the difference.

- (s. 15 km/h in the direction of the initial velocity of the truck; 370 kJ and 138.5 kJ; plastic deformation, heat and sound!)

A bullet of mass m is fired horizontally with velocity v into a block of mass M that rests on a horizontal surface as shown in

Fig. Prob. D1.11. Determine the distance to which the block slides on the surface before it comes to rest. Assume that M is much greater than m .

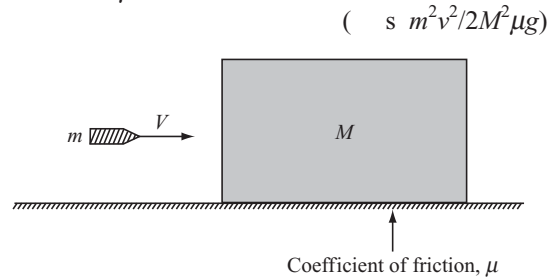


Fig. Prob. D1.11

Four identical bodies each of mass m are set up in a straight line on a smooth horizontal plane with distance d between them. A fifth body, identical to the other four, approaches with a velocity v and makes a perfectly elastic impact with the first body, as shown in Fig. Prob. D1.12.

- (a) Describe the motion of the bodies.
(b) If the distances d approach zero, what is the resulting motion?

- (s Each body moving with velocity v in turn; $d = 0, v'_1 = 3/5 v; v'_2 = 2/5 v$)

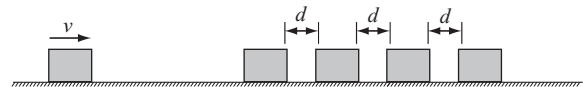


Fig. Prob. D1.12

A glass ball is dropped on to a smooth horizontal floor from which it bounces to a height of 9 m. On the second bounce it attains a height of 6 m. What is the coefficient of restitution between the glass and the floor?

- (s $e = 0.82$)

A ball thrown from position A against a smooth vertical circular wall rebounds and hits position B at the other end of the diameter through A as shown in Fig. Prob. D1.14. Show that the coefficient of restitution is equal to the square of the tangent of angle θ .

$e = \tan^2 \theta$

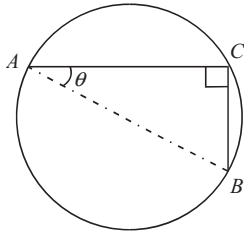


Fig. Prob. D1.14

A 30 kg block is dropped from a height of 2 m onto the 10 kg pan of a spring scale as shown in Fig. Prob. D1.15. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The spring has a stiffness of $k = 20 \text{ kN/m}$.

(s 0.23 m)

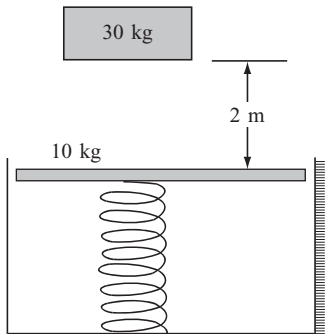


Fig. Prob. D1.15

A ball is dropped from a height of 2.4 m upon a 15° incline and it rebounds. If the coefficient of restitution e is 0.8, find the distance s along the incline where the ball would next strike the incline.

(s 2.04 m down the plane)

Two simple pendulums with their bobs weighing W and $3W$ hang side by side as shown in Fig. Prob. D1.17. The pendulum of the smaller bob is released from rest from a position of 60° and strikes the bigger bob centrally. Assuming a purely elastic impact, calculate the angle through which the other pendulum would swing? What would have been the angle of swing if the bobs of the pendulums weighed the same?

(s 29° and 60°)

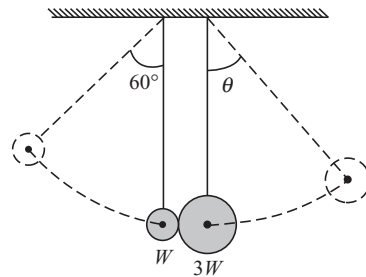


Fig. Prob. D1.17

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or most appropriate response from among the available alternatives in the following multiple-choice questions:

Central impact of two bodies

- (a) also implies direct impact
- (b) requires that the bodies should not rotate at all
- (c) must always be elastic impact in nature
- (d) may either be direct or indirect.

The coefficient of restitution is defined on the basis of

- (a) velocity components along the line of impact only
- (b) velocity component normal to the line of impact
- (c) velocity vectors before and after the collision
- (d) energies of bodies before and after the impact.

For a perfectly plastic central impact

- (a) the entire kinetic energy of the two bodies must be lost
- (b) the two bodies must move stuck together, whether the impact is direct or indirect

- (c) the two bodies must move stuck together only if the impact is direct
- (d) a body, initially at rest, should stay at rest.

When two bodies collide without the presence of any other forces or force fields,

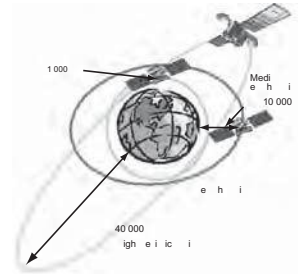
- (a) their total momentum must be conserved
- (b) their total kinetic energy must be conserved
- (c) the collision must be direct
- (d) the collision must be central.

Answers to Multiple-Choice Questions

- 1 (d) 2 (a) 3 (c) 4 (a)

Central Force Motion and Space Mechanics

D2



D2.1 INTRODUCTORY CONCEPTS

Central force motion of a body is referred to a fixed point in space such that the force on the body is directed towards or away from that fixed point. For example, the centre of the earth is assumed to be the fixed point towards which the force of gravity acts on all earth-bound objects. The motion of a freely falling body is, therefore, a central force motion. Likewise, repulsive force between a massive and a small like-charged bodies may also result in a central force motion of the small body. These are, however, simple examples of radial or rectilinear motion. In general, central force motion may take place along a trajectory, e.g., throwing of a stone at an angle inclined to the horizontal direction or a periodic motion of a satellite.

The motion of a particle under the action of a force directed towards or away from a fixed point is termed as . The force which is directed towards or away from a fixed point , as shown in Fig. D2.1, is called a central force and the force field

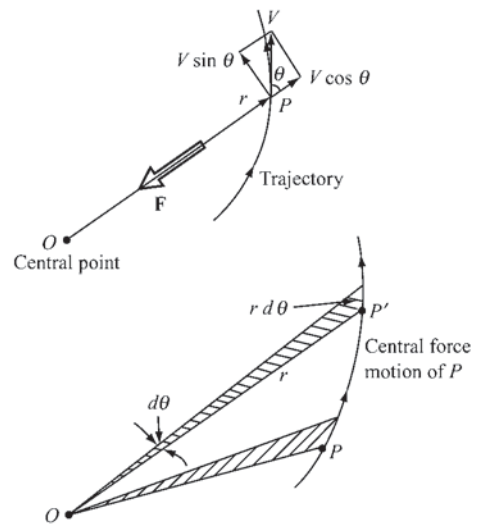


Fig. D2.1 Central force motion

is referred to as $\mathbf{F} = F(\rho)\mathbf{e}_\rho$. A central force is generally a function of the radial distance from the fixed point

$$\mathbf{F} = F(\rho)\mathbf{e}_\rho \tag{D2.1}$$

where \mathbf{e}_ρ is a unit vector along the outward radial direction. The central force is said to be one of attraction if directed towards O and of repulsion if directed away from O . For example, in studying the motion of a satellite in relation to the gravitational field of the earth, the central force is attractive and varies according to the inverse square law with the radial distance from the centre of the earth

$$\mathbf{F} = -\frac{\mu}{\rho^2}\mathbf{e}_\rho \tag{D2.2}$$

In a central-force motion, the moment

$$\mathbf{h}_O = \mathbf{r} \times \mathbf{v} = 0$$

since \mathbf{r} and \mathbf{v} are collinear at all times.

In the absence of external moment, the rate of change of angular momentum must also vanish

$$\dot{\mathbf{h}}_O = \dot{\mathbf{r}} \times \mathbf{v} = 0$$

and

$$\mathbf{h}_O = \mathbf{r} \times \mathbf{v} = \text{a constant vector}$$

or

$$\mathbf{r} \times \mathbf{v} = \text{a constant vector}, \tag{D2.3}$$

This result leads us to a very important conclusion:

$\sin \theta$ is constant, i.e.,

$$\rho_1 \sin \theta_1 = \rho_2 \sin \theta_2$$

during any two positions of the particle. The direction of \mathbf{h}_O is perpendicular to the plane containing \mathbf{r} and \mathbf{v} or the plane of motion; constant direction of \mathbf{h}_O implies that

\mathbf{h}_O is constant. In other words, a central-force motion is a plane motion in which ρ , $\dot{\rho}$ and θ are related by the constancy of $\rho \sin \theta$.

Realising that

$$\sin \theta = \frac{h_O}{\rho v} = \frac{h_O}{\rho \omega \rho} = \frac{h_O}{\rho^2 \omega}$$

it can be observed that

$$\rho^2 \omega = \text{Const.} = \frac{h_O}{\sin \theta} = \frac{h_O}{1} = h_O \tag{D2.4}$$

It may be seen from Fig. D2.1 that as the particle P generates a trajectory, the radius vector \mathbf{r} sweeps an area; the area swept by it over an infinitesimal time δt is

$$\delta A = \frac{1}{2} \rho^2 \delta \theta = \frac{1}{2} \rho^2 \omega \delta t$$

The areal velocity of the particle is

$$\frac{dA}{dt} = \frac{1}{2} \rho^2 \omega \tag{D2.5}$$

which is indeed constant in view of Eq. (D2.4). The conclusion, therefore, is that when a particle moves under a central force, its areal velocity is constant. In other words, the radius vector is such that

$\frac{dA}{dt} = \frac{1}{2} \rho^2 \omega = \text{const.}$. Another point which follows from the above discussion is that the tangential velocity v_θ varies inversely with ρ . At small ρ , v_θ is large and vice versa.

D2.2 ACCELERATION DUE TO GRAVITY (Above and below the surface of the earth)

The gravitation intensity or acceleration due to gravity at a point is given by equating the force to inertial mass times acceleration $\mathbf{F} = m\mathbf{a}$.

Hence,
$$g_r = -\frac{GM}{r^2}$$

At the surface of the earth, $r = R$

$$g = -\frac{GM}{R^2}$$

directed towards the centre of the earth.

It follows that

$$g_r/g = (R/r)^2$$

which means that, above the surface of the earth, the acceleration due to gravity varies inversely as the square of the distance from the centre of the earth. This is shown plotted in Fig. D2.2.

At an altitude, i.e., height above the earth's surface h , $r = R + h$

$$g_r/g = \left(\frac{R}{R+h}\right)^2 = \left(1 + \frac{h}{R}\right)^{-2}$$

which, approximated by the binomial theorem, becomes

$$g_r/g = 1 - 2h/R \tag{D2.6}$$

or
$$g_r = (1 - 2h/R)g = g - 2h/Rg$$

or
$$g_r - g = -2h/Rg \tag{D2.7}$$

It implies the reduction in acceleration due to gravity at a point h above the surface of the earth in comparison with that on the surface is $2h/Rg$, for small height h .

Let us now look at the variation of the acceleration due to gravity below, i.e., inside the surface of the earth at a point P as shown in Fig. D2.3. In this case, the mass of the earth inside it is

$$M' = (b/R)^3 M$$

where M is the mass of the earth.

This is because the mass is proportional to volume for constant density and the volume is proportional to the cube of the radius of a sphere.

We are neglecting the gravitational effect of hollow sphere of outer radius R and inner radius b or assuming that the net effect of that is zero.

Then, the gravitational force is given by

$$F = -\frac{GmM'}{b^2} r$$

and

$$g_b = -\frac{GM'}{b^2} r = -\frac{GM}{R^2} b/R r$$

whence

$$g_b = g b/R \quad \text{or} \quad g_b/g = b/R \tag{D2.8}$$

which means that, below the surface of the earth, the acceleration due to gravity varies directly as the radial distance from the centre of the earth as also shown plotted in Fig. D2.2.

From this simple analysis, we also conclude that the acceleration due to gravity g_{max} is the *maximum at the surface of the earth* itself ignoring the variation in density of the earth.

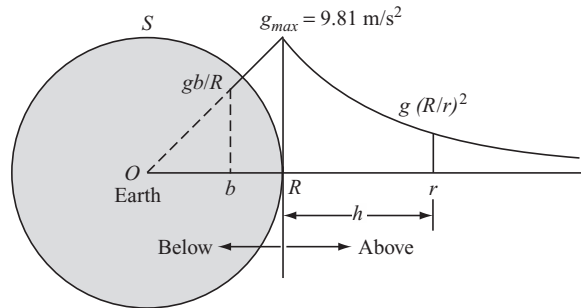


Fig. D2.2 Variation of acceleration due to gravity above and below the surface of the earth

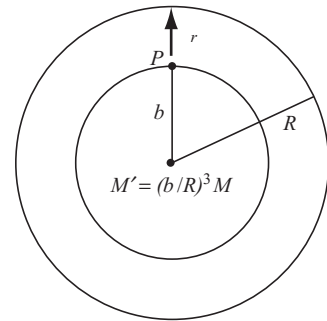


Fig. D2.3

It is also interesting to consider the variation of the acceleration due to gravity with latitude θ . The variation is due to the differential contribution of acceleration by virtue of the spinning motion of the earth.

The point P located by R and θ as shown in Fig. D2.4 is in a circular motion of radius $r (= R \cos \theta)$ with a rotational speed ω . A body placed at P experiences a real acceleration $g = GM/R^2$ towards the centre of the earth and a pseudo acceleration $g_s = \omega^2 r$ as shown. The resultant acceleration g' is given by

$$g' = \left[g^2 + g_s^2 - 2g g_s \cos \theta \right]^{\frac{1}{2}}$$

$$= g \left[1 + (R\omega^2/g)^2 \cos^2 \theta - 2 R\omega^2/g \cos^2 \theta \right]^{\frac{1}{2}}$$

At the equator, $\theta = 0$, $\cos \theta = 1$, $r = R$

$$g' = g - R \omega^2 \tag{D2.9}$$

which is the least anywhere on the earth.

At the pole, $\theta = 90^\circ$, $\cos \theta = 0$, $r = 0$

$$g' = g$$

since there is no effect of spin of the earth at the poles.

Non-spherical shape of the earth may also be taken into account by considering the actual radial distance of the point from the centre of the earth. Since the earth is flattened near the poles, the acceleration due to gravity at the poles (9.83 m/s^2) is more than that at the equator (9.78 m/s^2). For a mean radius of the earth the value of g at the surface has been standardised as 9.80665 m/s^2 .

Non-homogeneity of the material inside the earth, i.e., variation of density does not have any effect on the value of the acceleration due to gravity at points at and above the surface of the earth. The effect of non-homogeneity on the value of acceleration due to gravity below the surface of the earth is non-linear and complicated.

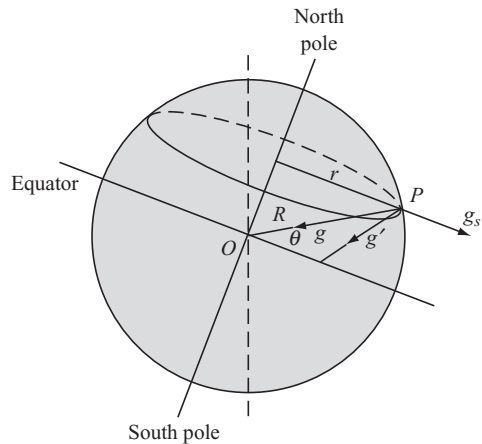


Fig. D2.4

Example D2.1

A particle undergoing central force motion has a tangential velocity of 20 m/s while at a distance of 300 m from the central point. Using the fact that the areal velocity of the particle must be constant, find its tangential velocity when it is 400 m away from the central point.

Solution Given that

$$V_{\theta_1} = 20 \text{ m/s} \quad \text{at} \quad r_1 = 300 \text{ m}$$

The areal velocity, i.e., rate of change of the area swept by the particle must be

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = k$$

Noting that

$$\frac{d\theta}{dt} = \omega = \frac{V_{\theta}}{r}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{V_{\theta}}{r} = \frac{1}{2} r V_{\theta} = k$$

implying that

$$r V_{\theta} = \text{Constant}$$

Therefore, V_{θ_2} at 400 m distance must be

$$V_{\theta_2} = \frac{20 \times 300}{400} = 15 \text{ m/s.}$$

D2.3 TRAJECTORIES FOR CENTRAL FORCE MOTION

From the Newton's law of motion and the law of gravitation

$$\mathbf{F} = \frac{-GM}{r^2} m \quad r = m$$

it follows that

$$= -\frac{GM}{r^2} \quad r \tag{D2.10}$$

The interpretation of this equation is that, in a central-force motion, the acceleration is always directed towards the central point O . Of course, the force and acceleration must be collinear. For a satellite motion, the acceleration is independent of the mass of the satellite but it is in direct proportion to the mass of the earth or, in general, to the mass of the planet in question.

Substituting the value of acceleration, the equation of motion becomes

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = \frac{F}{m} = \frac{-GM}{r^2}$$

Substituting

$$u = 1/r$$

and

$$\frac{d\theta}{dt} = \frac{C}{r^2} = Cu^2$$

The equation of motion is transformed to

$$\frac{d^2 u}{dt^2} + u = \frac{F}{mC^2 u^2} = \frac{GM}{C^2} \tag{D2.11}$$

The solution of the differential equation is obtained by adding the particular solution

$$u = \frac{GM}{C^2}$$

to the general solution

$$u = D \cos (\theta - \theta_0)$$

where D is constant, i.e.,

$$\frac{1}{r} = u = \frac{GM}{C^2} + D \cos \theta \tag{D2.12}$$

if the polar axis is chosen so that $\theta_0 = 0$. This is the equation of a conic section in the cylindrical coordinates with the origin O located at the centre of the earth. Comparing it with the standard form of a conic section

$$\frac{1}{r} = u = \frac{1}{ep} + \frac{1}{p} \cos \theta \tag{D2.13}$$

it is seen that

$$D = \frac{1}{p}$$

$$ep = \frac{C^2}{GM}$$

or

$$e = \frac{DC^2}{GM} \tag{D2.14}$$

which is the eccentricity of the conic section.

Four cases are given in Table D2.1.

Table D2.1 Trajectories under central force

| Case | Value of e | Features | Type of conic section |
|------|--------------|---|-----------------------|
| 1 | $e = 0$ | $D = 0, r = \frac{C^2}{GM}$ | Circle |
| 2 | $e < 1$ | Finite r for all θ | Ellipse |
| 3 | $e = 1$ | $r \rightarrow \infty$ at $\theta = \pi$ | Parabola |
| 4 | $e > 1$ | $r \rightarrow \infty$ for two values of θ | Hyperbola |

The eccentricity of the orbit

$$e = \frac{DC^2}{GM}$$

depends upon the constant quantities G and M as well as the constants of motion, C and of the conic section, D . The state of a satellite can be predicted with the help of Eq. (D2.13) if the conditions at any one state, say the moment of burnout, are known.

Assuming that the burnout occurs at the end of a powered flight at a position P in space such that the velocity of the rocket is parallel to the surface of the earth, as shown in Fig. D2.5, the satellite or space vehicle is said to

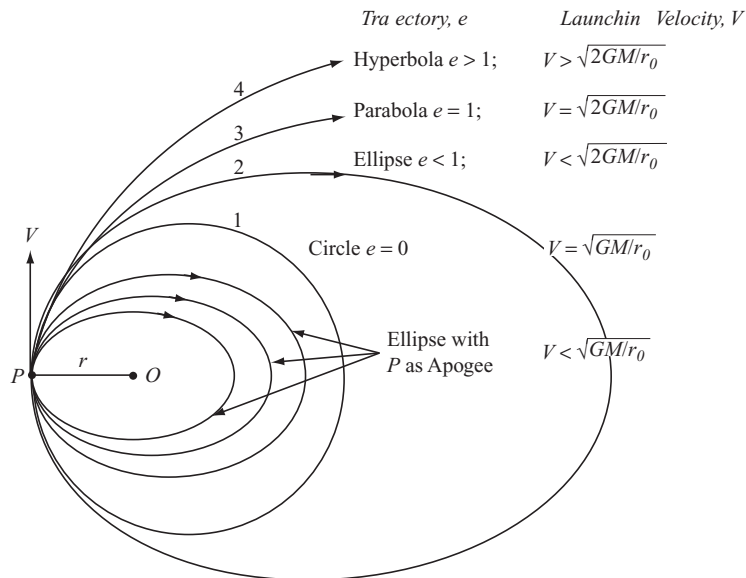


Fig. D2.5 Possible satellite trajectories

begin its free-flight at the vertex P . If the satellite, at this instant, is located at a distance r_0 from the centre of the earth and has a velocity V_0 parallel to the surface of the earth

$$C = r_0 V_0$$

and

$$\frac{1}{r_0} = \frac{GM}{C^2} + D \quad (\text{D2.15})$$

whence

$$\begin{aligned} D &= \frac{1}{r_0} - \frac{GM}{C^2} = \frac{1}{r_0} - \frac{GM}{r_0^2 V_0^2} \\ &= \frac{1}{r_0} - \frac{gR^2}{r_0^2 V_0^2} \end{aligned}$$

since g the acceleration at the surface of the earth is given as

$$g = \frac{GM}{R^2} \quad (\text{D2.16})$$

where R is the radius of the earth.

Let us again discuss the four possible cases in terms of the launching parameters. Reference is made in Figs. D2.2 and D2.3.

Circular Orbit

$$e = 0 = \frac{DC^2}{GM}; \quad D = 0$$

The path is given by

$$\frac{1}{r} = \frac{GM}{C^2} = \frac{GM}{(r_0 V_{\theta_0})^2} = \frac{1}{r_0} = \text{Constant}$$

whence

$$V_{\theta_0} = \sqrt{\frac{GM}{r_0}} = \sqrt{g_0 r_0} \quad (\text{D2.17})$$

This is the velocity required to launch the satellite for a circular orbit. It can be seen that the velocity of a satellite in a circular orbit is more if it is closer to the earth and less if away from it.

Modern-day communication satellites are geo-stationary or synchronous i.e., remain fixed in location relative to the earth spinning about their own axes; *the period of revolution of a communication satellite should also be 24 hours*. The orbit of a communication satellite must be circular and it must be in the equatorial plane of the earth. The rotational speed of the earth is

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = 0.000\,073 \text{ rad/s}$$

The rotational speed of a geo-stationary satellite must also be the same.

$$\omega = V_{\theta}/r = 0.000\,073 \text{ rad/s}$$

$$V_{\theta} = 0.000\,073 \, r \text{ m/s} \quad (\text{i})$$

But, for a circular orbit

$$V_{\theta} = \sqrt{GM/r} = \sqrt{3.9860 \times 10^{14} / r} \quad (\text{ii})$$

From Eqs. (i) and (ii)

$$r = 42,180,000 \text{ m} = 42,180 \text{ km}$$

and $V_{\theta} = 3080 \text{ m/s} = 11,100 \text{ km/h}$

Geo-stationary satellites must, therefore, be located at a radial distance of 42,180 km or at an altitude h given by

$$h = 42,180 - 6370 = 35,810 \text{ km}$$

above the mean surface of the earth and must move

with a velocity of 11,100 km/h in circular orbits in an equatorial plane of the earth and in the same sense as shown in Fig. D2.6. The period of rotation of a communication satellite must be 24 hours.

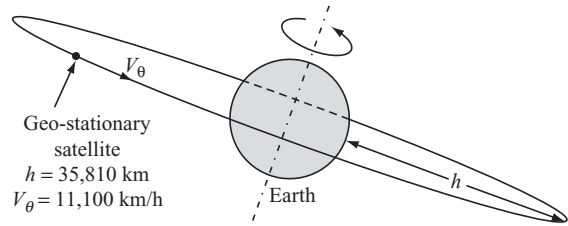


Fig. D2.6 A communication satellite

Parabolic Trajectory

$$e = 1 = \frac{DC^2}{GM}, \quad D = \frac{GM}{C^2}$$

The path is given by

$$\frac{1}{r} = \frac{GM}{C^2} + D \cos \theta$$

For launching at $\theta = 0$ and $r = r_0$

$$D = \frac{1}{r_0} - \frac{GM}{C^2} = \frac{GM}{C^2}$$

Substituting $C = r_0 V_{\theta_e}$ in this equation

$$V_{\theta_e} = \sqrt{\frac{2GM}{r_0}} = \sqrt{2g_0 r_0} = \sqrt{2} V_{\theta_c} \quad (D2.18)$$

= $\sqrt{2}$ times the launching velocity for a circular orbit

If this velocity is imparted to a particle, it will follow a parabolic path, i.e., escape from the gravitational field of the earth. For such a path

$$\theta \rightarrow \pi, \frac{1}{r} \rightarrow 0, r \rightarrow \infty$$

Since GM is a constant for the earth, the escape velocity V_{θ_e} is maximum at the surface of the earth and it decreases if the satellite is launched from higher altitudes. This fact explains why satellites are launched after reaching very high altitudes by powered flights!

Elliptical Orbit

$$0 < e < 1, \quad \frac{DC^2}{GM} < 1$$

The launching velocity is obviously between the values

$$V_{\theta_c} = \sqrt{\frac{GM}{r_0}} \quad \text{for a circular orbit}$$

and

$$V_{\theta_e} = \sqrt{\frac{2GM}{r_0}} \quad \text{for a parabolic escape trajectory}$$

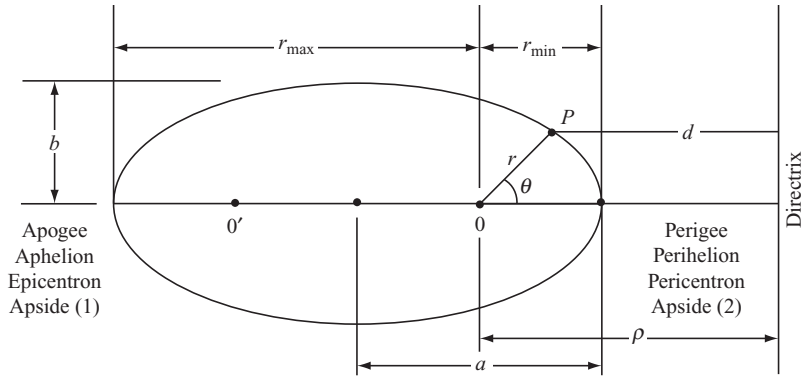


Fig. D2.7 Features of an ellipse

If, however the launching velocity is less than that for a circular orbit, i.e.,

$$0 < V_{\theta} < \sqrt{\frac{GM}{r_0}}$$

the satellite still goes on an elliptical orbit with the centre of the earth as the second focus instead of the first. It may also be appreciated that the launching velocity can be adjusted to touch the surface of the earth at the diametrical opposite point.

If the launching velocity is less than this value, the object will hit the surface of the earth while tending to complete the elliptical orbit. A bullet fired from a gun or stone thrown by hand are examples of this case. In the first instance, it may appear contrary to our belief that the trajectory of an earth-bound object, such as a stone, bullet or jet of water should be parabolic. The paradox is resolved by remembering that the value of g is taken as constant in magnitude and direction for the motion of earth-bound objects. Analysis with constant g yields a parabolic trajectory which is an approximation to the elliptical path obtained by taking variable g directed towards the centre of the earth.

Although it is admissible to have the eccentricity e between 0 and 1, it is usual to keep it very low, i.e., the orbit close to a circular orbit. The orbital eccentricities of the planets are also very low. For example,

Orbital eccentricity of the earth 0.017

Orbital eccentricity of the moon 0.055

Some salient features of an elliptic conic section are shown in Fig. D2.7.

Eccentricity $e = \frac{r}{d} (< 1)$

$$= \frac{r_{\min}}{p - r_{\min}}$$

$$= \frac{r_{\max}}{p + r_{\min}}$$

Semimajor axis $a = \frac{r_{\max} + r_{\min}}{2}$

Semiminor axis $b = \sqrt{r_{\min} \cdot r_{\max}} = a\sqrt{1 - e^2}$

Distance between the foci $= 2\sqrt{(a^2 - b^2)}$

Area of the ellipse $= \pi ab$

Equation of the orbit is given by

$$r = a(1 - e \cos \theta) \tag{D2.19}$$

whence

$$r_{\max} = a(1 + e)$$

and

$$r_{\min} = a(1 - e)$$

$$ep = a(1 - e^2)$$

Hyperbolic Trajectory

$$e > 1, \frac{DC^2}{GM} > 1$$

$$V_{\theta_0} > \sqrt{\frac{2GM}{r_0}}$$

From the equation of the trajectory,

$$\frac{1}{r} = \frac{GM}{C^2} + D \cos \theta$$

it may be observed that when $r \rightarrow \infty$

$$0 = \frac{GM}{C^2} + D \cos \theta$$

$$\cos \theta = -\frac{GM}{DC^2} = -\frac{1}{e}$$

and

$$\theta = \cos^{-1}\left(-\frac{1}{e}\right)$$

describes the asymptote for the final direction of the particle.

It is observed that if the launching velocity equals or exceeds the escape value, i.e.,

$$V_{\theta_0} \geq \sqrt{\frac{2GM}{r_0}}$$

the object escapes the gravitational field of the earth. This is indeed the case of a mission to reach the Moon or Mars or any other planet in the solar system.

The time taken for a particle in a central-force motion to travel from a position θ_0 to a position θ can be computed by starting from

$$\frac{r^2 d\theta}{dt} = C$$

or

$$dt = \frac{r^2 d\theta}{C}$$

The time taken T is given by

$$T = \int dt = \frac{1}{C} \int_{\theta_0}^{\theta} r^2 d\theta$$

$$= \frac{1}{C} \int_{\theta_0}^{\theta} \frac{d\theta}{\left(\frac{GM}{C^2} + D \cos \theta\right)^2} \tag{D2.20}$$

For an elliptical orbit, the area traversed in one revolution is πab and the areal velocity is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{C}{2}$$

The time period for one revolution is, therefore,

$$T = 2\pi ab/C \quad (D2.21)$$

For a circular orbit,

$$a = b = r_0$$

and the time period is

$$T = 2\pi r_0^2/C = 2\pi r_0/V_{\theta_0} \quad (D2.22)$$

It can again be observed that a synchronous or geo-stationary satellite would have a time period of 24 hours.

$$T = 2\pi r_0/V_{\theta_0} = 24 \times 60 \times 60 \text{ s}$$

or

$$V_{\theta_0} = 0.000\ 073 \text{ } r \text{ m/s}$$

which is the same as determined earlier in Eq. (i) under 'Circular Orbit'.

D2.4 ENERGY EXPENDED FOR DIFFERENT TRAJECTORIES

Energy methods may be used to advantage for studying some aspects of the central-force motion. For the gravitational field,

$$F_r = -\frac{GMm}{r^2}, \quad F_{\theta} = 0$$

The potential energy PE is given by

$$\frac{d(PE)}{dr} = -F_r = \frac{GMm}{r^2}$$

or

$$PE = -\frac{GMm}{r} \quad (D2.23)$$

which implies that the gravitational force field is a conservative force field. The law of conservation of mechanical energy is, therefore, applicable in a central-force field:

$$KE + PE = \text{Constant}$$

or

$$\frac{1}{2} m V^2 - \frac{GMm}{r} = \text{Constant} \quad (D2.24)$$

For a body at the surface of the earth of radius R

$$PE = -\frac{GMm}{R}$$

If a satellite is launched from a radial position at a distance r_0 from the centre of the earth,

$$KE + PE = \frac{1}{2} m V_0^2 - \frac{GMm}{r_0}$$

Conservation of mechanical energy

$$\frac{1}{2} m V^2 - \frac{GMm}{r} = \frac{1}{2} m V_0^2 - \frac{GMm}{r_0} \quad (D2.25)$$

provides a relation to relate the velocity with the radial position of a satellite in terms of the launching conditions.

For a circular orbit, the radial distance r is constant and equal to r_0 at all times; so is the velocity V equal to V_0 at all times.

The escape velocity V_e of a body may be determined from the fact that the body should continue indefinitely and reach

$$r \rightarrow \infty \quad \text{when} \quad V = 0$$

where

$$KE + PE = \frac{1}{2}mV^2 - \frac{GMm}{r} = 0 - 0 = 0$$

From the conservation of mechanical energy, the conditions at the launching position should be such that,

$$(KE + PE)_{\text{launching}} = 0$$

or

$$\frac{1}{2}mV_e^2 - \frac{GMm}{r_0} = 0$$

whence

$$V_e = \sqrt{\frac{2GM}{r_0}}$$

Since by definition,

$$\frac{GMm}{r_0^2} = mg_0$$

$$V_e = \sqrt{\frac{2GM}{r_0}} = \sqrt{2g_0 r_0} \tag{D2.26}$$

Table D2.2 Energy expended for different trajectories

| Trajectory | Velocity | Energy expended |
|---|---|--|
| Circular orbit | $V = \sqrt{\frac{GM}{r_0}}$ | $E = \frac{GMm}{2r_0}$ |
| Elliptic orbits | (i) $V < \sqrt{\frac{GM}{r_0}}$ | $E < \frac{GMm}{r_0}$ |
| | (ii) $\sqrt{\frac{GM}{r_0}} < V < \sqrt{\frac{2GM}{r_0}}$ | $\frac{GMm}{2r_0} < E < \frac{GMm}{r_0}$ (Continued) |
| Escape trajectory (Parabolic-minimum values) | $V = \sqrt{\frac{2GM}{r_0}}$ | $E = \frac{GMm}{r_0}$ |
| Hyperbolic escape | $V > \sqrt{\frac{2GM}{r_0}}$ | $E > \frac{GMm}{r_0}$ |

If a satellite is launched from a position r_0 , the energy required to be expended to set the satellite into a trajectory may be determined if the intended velocity of the satellite at any position is known

$$\begin{aligned}
 E &= \left(\frac{1}{2} m V^2 - \frac{G M m}{r} \right) - \left(-\frac{G M m}{r_0} \right) \\
 &= \frac{1}{2} m V^2 + G M m \left(\frac{1}{r_0} - \frac{1}{r} \right) \\
 &= \frac{1}{2} m V_0^2 \qquad \qquad \qquad (D2.27)
 \end{aligned}$$

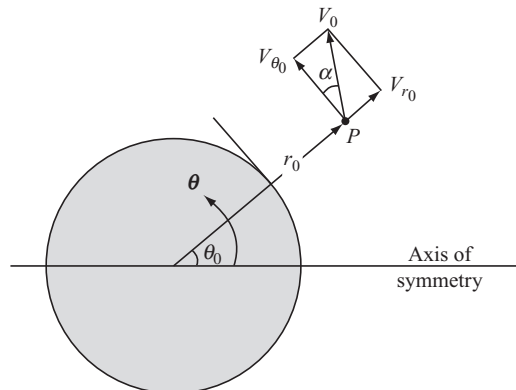
Energy requirements for different trajectories are given in Table D2.2.

D2.5 LAUNCHING OF SATELLITES AT AN ANGLE

Consider a general case of launching at $r = r_0$ with an angle of launch α , as shown in Fig. D2.8. Let the launching velocity be V_0 such that,

$$V_{\theta_0} = V_0 \cos \alpha$$

$$V_{r_0} = V_0 \sin \alpha$$



Launching at P

Fig. D2.8(a) Launching of a Satellite at an angle

Let θ_0 be the angle between the axis of symmetry of the conic and the radius vector r_0 .

$$\frac{1}{r_0} = \frac{GM}{C^2} + D \cos \theta_0$$

for the position of launching.

In general,

$$\frac{1}{r} = \frac{GM}{C^2} + D \cos \theta$$

and

$$V_r = DC \sin \theta; \quad V_{\theta_0} = DC \sin \theta_0$$

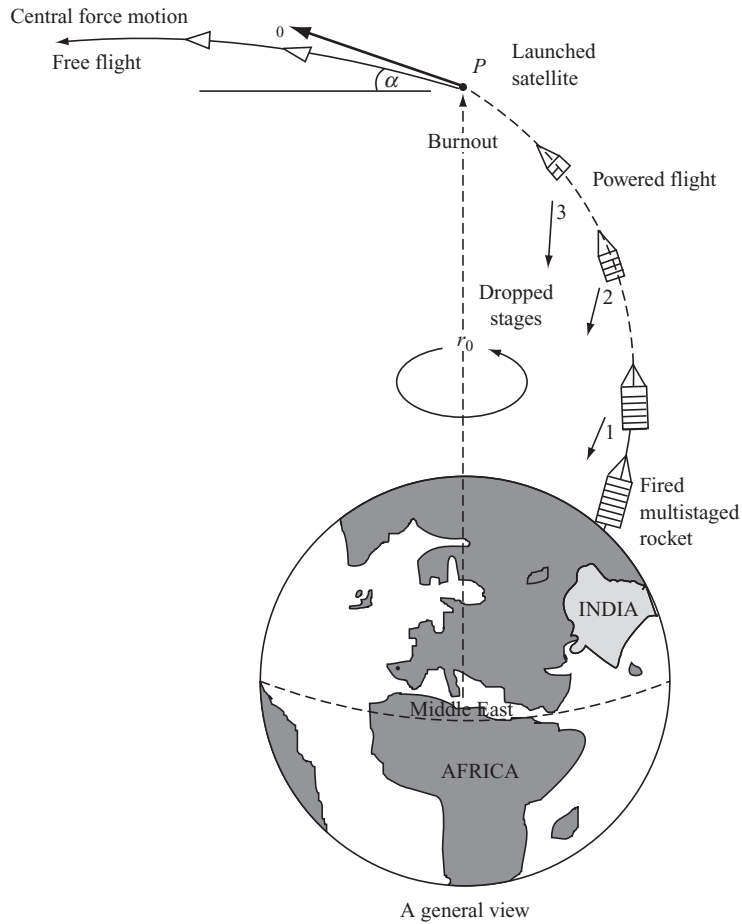


Fig. D2.8(b)

Since

$$D \cos \theta_0 = \frac{1}{r_0} - \frac{GM}{C^2}$$

and

$$D \sin \theta_0 = \frac{V_{r_0}}{C}$$

$$D = \left[\left(\frac{1}{r_0} - \frac{GM}{C^2} \right)^2 + \left(\frac{V_{r_0}}{C} \right)^2 \right]^{1/2} \tag{D2.28}$$

$$e = \frac{DC^2}{GM}$$

Consider the condition for the escape of the satellite, $e = 1$. We can obtain that

$$r_0 V_0^2 \cos^2 \alpha (r_0 V_0^2 - 2GM) = 0$$

whence, the escape velocity

$$V_0 = \sqrt{\frac{2GM}{r_0}}$$

This is independent of the angle of launching may it be 0, 90 or any other. A satellite projected vertically or radially outwards from the surface of the earth, as shown in Fig. D2.9, can also escape the earth's gravitational field at the same speed. It may, however, be added that the V_0 for escape is measured with respect to the centre of earth considered as an inertial frame. The motion of the earth's surface adds to the final velocity of the satellite if the blast off takes place on the equator and along the rotation of the earth and subtracts from the final velocity if the blast off opposes the rotation. No such 'gain' or 'loss' is experienced in the velocity if the satellite is fired from the poles.

Another interesting fact is that if a space vehicle is launched in order to escape the gravitational field of the earth, it may so happen that the space vehicle escapes the entire solar system. This is due to the fact that the earth is rotating around the Sun at approximately 30 km/s and this may add to the absolute velocity of launching of the space vehicle (if launched along the rotation of the earth around the Sun), thus attaining even the escape velocity for the Sun.

Let us now consider the vertical projection of a particle in some detail. Let V be the speed and $v_r = V \cos \theta$ the velocity at any time t . By Newton's law of motion,

$$-\frac{GMm}{r^2} = m \frac{dv_r}{dt}$$

or

$$-\frac{GM}{r^2} = \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = V \frac{dV}{dr} = \frac{d}{dr} \left(\frac{V^2}{2} \right)$$

By integrating with respect to r on each side

$$\frac{GM}{r} = \frac{V^2}{2} + K$$

The constant K can be evaluated from the boundary condition. If $V = V_0$ at $r = R$ at the point of projection,

$$K = V_0^2/2 - \frac{GM}{R}$$

Hence,

$$V^2 = V_0^2 + 2GM \left(\frac{1}{r} - \frac{1}{R} \right)$$

Let

$$r = R + h,$$

$$V^2 = V_0^2 - \frac{2GMh}{R(R+h)}$$

whence

$$V = \sqrt{V_0^2 - \frac{2GMh}{R(R+h)}} \tag{D2.29}$$

It may be observed from here that if h must be made to approach infinity,

$$V \rightarrow \sqrt{V_0^2 - \frac{2GM}{R}} \tag{D2.30}$$

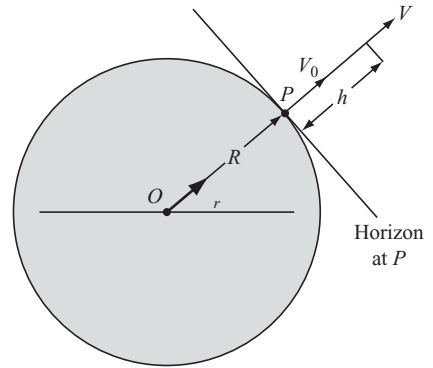


Fig. D2.9 Vertical projection

If, in addition, the velocity V at $h \rightarrow \infty$ becomes negligibly small or $V \rightarrow 0$, then the launching velocity V_0 equals $\sqrt{\frac{2GM}{R}}$ which is the escape velocity of the particle as is expected from the earlier analysis of launching at any arbitrary angle.

Alternatively, the equation

$$V^2 = V_0^2 - \frac{2GMh}{R(R+h)}$$

shows that the maximum height attained by a particle would correspond with the minimum residual velocity squared V^2 , i.e., $V^2 = 0$

Hence, h_{\max} is given by

$$V_0^2 - \frac{2GMh_{\max}}{R(R+h_{\max})} = 0$$

whence

$$h_{\max} = \frac{V_0^2 R}{\left(2\frac{GM}{R} - V_0^2\right)} = \frac{V_0^2 R}{(2gR - V_0^2)} \quad (\text{D2.31})$$

It is obvious from this expression that if

$$V_0^2 = 2\frac{GM}{R}$$

or

$$V_0 = \sqrt{2gR} = \sqrt{2\frac{GM}{R}}$$

h_{\max} would become infinite, i.e., the particle would escape the gravitational field of the earth. On the contrary, if h_{\max} is small, let $h_{\max} = h$ and h/R is negligible,

$$(2gR - V_0^2) \frac{h}{R} = \frac{V_0^2 R}{R}$$

$$2gh - V_0^2 \frac{h}{R} = V_0^2$$

$$V_0 = \sqrt{2gh} \quad (\text{D2.32})$$

which is a familiar expression for distance close to the surface of the earth.

D2.6 ASTRONOMICAL FACTS AND LAWS OF KEPLER

A system consisting of a star, planets and satellites is called a solar system. A star is a source of light, planets only reflect light and orbit around the star and the satellites revolve about the planets. Satellites may be natural or artificial. Man-made artificial satellites may be launched to revolve around the planets.

In our solar system, the sun is the star; the nine planets are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto in order of their mean distance from the sun. Thirty satellite bodies orbit around these planets. The moon is the satellite of the earth. A number of artificial satellites are orbiting the earth with a view to weather forecasting, intercontinental television relaying and spying missions.

A study of the central-force motion would be incomplete without discussing the contribution of Kepler who laid the foundation of orbital mechanics as also of Newtonian mechanics.

The three laws of Kepler for planetary motion are:

1. Every planet moves in an orbit which is an ellipse with the sun at one focus.
2. The radius vector drawn from the sun to any planet sweeps out equal areas in equal times.
3. The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.

These laws were enunciated by Kepler (1571–1630) from an analysis of the data recorded by Tycho Brahe. This was long before the enunciation of the laws of motion by Newton and the development of mathematical calculus. Newton (1643–1727) published his work in 1687 and set the stage for the development of classical mechanics.

Kepler's laws related to planetary motion can be derived from the laws of Newton and Newton's law can also be derived from Kepler's laws.

The first law refers to a conclusion derivable from the study of central-force motion; the motion in a central-force field must be plane and for the eccentricity $e < 1$, the conic section traced is an ellipse for an inverse square force field due to the Newton's law of gravitation.

The second law refers to the conclusion that the areal velocity is constant for a particle in a central-force field. It means that the areas swept by the particle are equal in equal time intervals. When the particle passes through the perigee, it must move faster than when it passes through the apogee. *Planets move faster when they are nearer the sun*

In order to prove the third law of Kepler, we proceed as follows:

The time period for an elliptical orbit is

$$T = 2\pi ab/C$$

From the geometry of the ellipse,

$$b = a\sqrt{1 - e^2}$$

and by comparing with the standard form of the equation for an ellipse with that for the planetary motion,

$$ep = a(1 - e^2) = \frac{C^2}{GM}$$

From these relations,

$$T^2 = 4\pi^2 a^3 / GM \quad \text{or} \quad T^2 \propto a^3 \quad \text{(D2.33)}$$

which shows that the squares of time periods of planets are proportional to the cubes of the semi-major axes of their elliptical orbits.

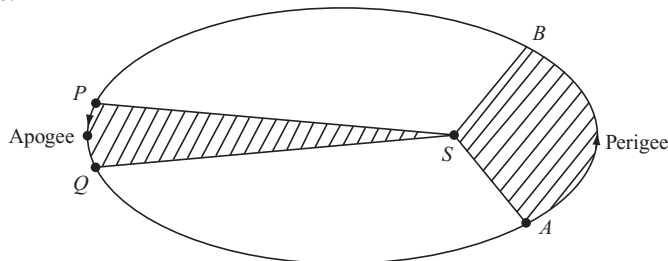


Fig. D2.10

Example D2.2

The Aryabhata was launched by a Soviet launch pad and set into an orbit by being imparted a velocity of 7600 m/s at a distance of 600 km above the surface of the earth and parallel to it at that point.

Comment on its orbit and its salient features. It is given that GM for earth = $3.9860 \times 10^{14} \text{ m}^3/\text{s}^2$ and the radius of the earth is 6371 km.

Table D2.3 Astronomical data for our solar system

| Body | E uatorial radius R/R_e | Mass M/M_e | Density ρ/ρ_e | Distance in astronomical units from the sun | Gravity g/g_e | Eccentricity e | Number of its satellites | Orbital time (Solar days) | Spin time (Solar days) | Escape velocity km/h |
|---------|-----------------------------|--------------------|-----------------------|---|-----------------|------------------|--------------------------|---------------------------|------------------------|----------------------|
| Sun | 109.1 | 3.33×10^5 | 0.255 | | 28 | — | — | — | 25.38 days | 2.22×10^6 |
| Moon | 0.273 | 0.0123 | 0.607 | 1.000 | 0.1645 | 0.055 | — | 27.32 days | 27.32 days | 8550 |
| Mercury | 0.38 | 0.056 | 0.96 | 0.387 | 0.36 | 0.206 | 0 | 87.97 days | 59.0 days | 15,460 |
| Venus | 0.96 | 0.81 | 0.90 | 0.723 | 0.87 | 0.007 | 0 | 224.7 days | 243.0 days | 37,000 |
| Earth | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.017 | 1 | 365.256 days | 1 day | 40,270 |
| Mars | 0.53 | 0.107 | 0.715 | 1.523 | 0.38 | 0.093 | 2 | 687.0 days | 24.6 hours | 18,095 |
| Jupiter | 11.20 | 318.00 | 0.241 | 5.203 | 2.65 | 0.048 | 12 | 11.86 years | 9.9 hours | 214,580 |
| Saturn | 9.47 | 95.1 | 0.13 | 9.539 | 1.14 | 0.056 | 10 | 29.46 years | 10.2 hours | 127,600 |
| Uranus | 3.75 | 14.5 | 0.25 | 19.19 | 1.1 | 0.047 | 5 | 84.02 years | 10.7 hours | 250,410 |
| Neptune | 3.50 | 17.5 | 0.41 | 30.07 | 1.00 | 0.009 | 2 | 164.8 years | 15.8 hours | 90,050 |
| Pluto | ≈ 1 | ≈ 1 | ≈ 1 | 39.52 | ≈ 1 | 0.249 | 0 | 248.0 years | 6.3 days | 40,270 |

Rate of energy production in the sun = 3.90×10^{16} W

Subscript e , for the earth (Universal $G = 6.673 \times 10^{-11}$ Nm²/kg²)

For the Earth

$$M = 5.97 \times 10^{24} \text{ kg}$$

$$GM = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2$$

$$g = 9.8065 \text{ m/s}^2$$

$$R = 6.371 \times 10^6 \text{ m} = 6371 \text{ km}$$

Solution The satellite Aryabhata had the following initial conditions:

$$r_0 = 600 + 6371 = 6,971 \text{ km} = 6,971,000 \text{ m}$$

$$V_{\theta_0} = 7600 \text{ m/s} \quad \text{and} \quad \theta = 0$$

$$C = r_0 V_{\theta_0} = 5.30 \times 10^{10} \text{ m}^2/\text{s}$$

The orbit is described by the equation

$$\frac{1}{r} = \frac{GM}{C^2} + D \cos \theta$$

and

$$\begin{aligned} D &= \frac{1}{r_0} - \frac{GM}{C^2} \\ &= \frac{1}{6,971,000} - \frac{3.9860 \times 10^4}{(5.30 \times 10^{10})^2} \\ &= 1.50 \times 10^{-9} \text{ m}^{-1} \end{aligned}$$

The eccentricity is given by

$$e = \frac{DC^2}{GM} = \frac{1.50 \times 10^{-9} \times (5.30 \times 10^{10})^2}{3.9860 \times 10^{14}} = 0.01$$

The Aryabhata has, therefore, gone into an elliptic orbit described by

$$\frac{1}{r} = \frac{3.986 \times 10^{14}}{(5.30 \times 10^{10})^2} + 1.50 \times 10^{-9} \cos \theta$$

or

$$\frac{1}{r} = (142 + 1.5 \cos \theta) \times 10^{-9}$$

The coordinates of the satellite at salient locations are:

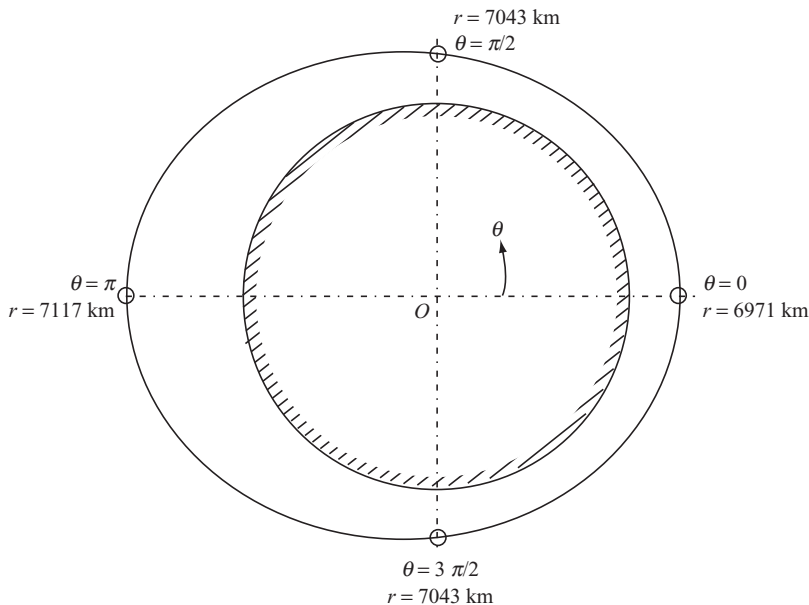


Fig. Ex. D2.2 (Solution a, b)

| | |
|-------------------|-----------------------|
| $\theta = 0$ | $r = 6971 \text{ km}$ |
| $\theta = \pi/2$ | $r = 7043 \text{ km}$ |
| $\theta = \pi$ | $r = 7117 \text{ km}$ |
| $\theta = 3\pi/2$ | $r = 7043 \text{ km}$ |

The maximum distance of the satellite from the surface of the earth is

$$7117 - 6371 = 746 \text{ km}$$

It can be seen that the velocity that would have been required to set the satellite into circular orbit is

$$V_c = \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{3.986 \times 10^{14}}{6,971,000}}$$

$$= 7562 \text{ m/s}$$

The velocity actually imparted to the Aryabhata is 7600 m/s which is slightly more than this value. From this or from the fact that the eccentricity is 0.01, it can be concluded that the orbit is nearly circular as shown in Fig. Ex. D2.2(b)

The periodic time of the satellite is

$$T = \frac{2\pi ab}{C}$$

For the elliptical orbit,

$$a = \frac{6971 + 7117}{2} = 7044 \text{ km}$$

$$b = \sqrt{6971 \times 7117} = 7043.6 \text{ km}$$

$$C = 530 \times 10^{10} \text{ m}^2/\text{s}$$

Hence,

$$T = \frac{2\pi \times 7044 \times 10^3 \times 7043.6 \times 10^3}{5.30 \times 10^{10}}$$

$$= 5880 \text{ s} = 1.63 \text{ h}$$

It makes 15 revolutions per day around the earth. It is far from being geo-stationary.

Example D2.3

A satellite is given a perigee launch from an altitude of 300 km with a velocity of 30,000 km/h. Obtain the following information: (a) the apogee altitude and (b) the time elapsed and position when a quarter-orbit has been completed.

Solution At the perigee, (Ref. Fig. D2.3 (Solution))

$$r_0 = 300 + 6371 = 6671 \text{ km} = 6,671,000 \text{ m}$$

$$V_0 = 30,000 \times 1000/3600 = 8333 \text{ m/s}$$

$$C = 6,671,000 \times 8333 = 5.56 \times 10^{10} \text{ m}^2/\text{s}$$

$$GM = 3.9860 \times 10^{14} \text{ m}^3/\text{s}^2$$

$$D = \frac{1}{r_0} - \frac{GM}{C^2} = 21 \times 10^{-9} \text{ m}^{-1}$$

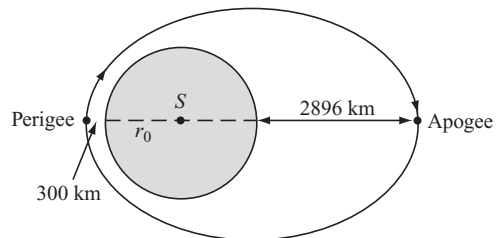


Fig. Ex. D2.3 (Solution)

$$e = \frac{DC^2}{GM} = \frac{21 \times 10^{-9} \times (5.56 \times 10^{10})^2}{3.9860 \times 10^{14}} = 0.163$$

The orbit is, therefore, elliptic with the equation

$$\begin{aligned} \frac{1}{r} &= \frac{GM}{C^2} + D \cos \theta \\ &= \frac{3.9860 \times 10^{14}}{(5.56 \times 10^{10})^2} + 21 \times 10^{-9} \cos \theta \end{aligned}$$

or
$$\frac{1}{r} = (1.289 + 0.21 \cos \theta) \times 10^{-7}$$

(a) At the apogee, $\theta = \pi$

$$r = 9,267,000 \text{ m} = 9267 \text{ km}$$

and the altitude at the apogee is

$$9267 - 6371 = 2896 \text{ km}$$

(b) Time elapsed for one complete revolution is

$$T = \frac{2\pi ab}{C}$$

and the time for one-quarter orbit is

$$t = \frac{\pi ab}{2C}$$

In this case,

$$a = \frac{6671 + 9267}{2} = 7969 \text{ km}$$

$$b = \sqrt{6671 \times 9267} = 7862 \text{ km}$$

$$\begin{aligned} t &= \pi \times 7969 \times 7862 \times 10^6 / 2 \times 5.56 \times 10^{10} \\ &= 1770 \text{ s} = 0.492 \text{ h} \end{aligned}$$

Example D2.4

Determine the escape velocity of a satellite to escape from the gravitational field of the earth when launched from

- (a) the surface of the earth
- (b) any altitude above the surface of the earth and comment on the most favourable mode of escape.

Solution

The escape velocity is given by

$$\begin{aligned} V_e &= \sqrt{\frac{2GM}{r_0}} = \sqrt{2 \times 3.9860 \times 10^{14}} / \sqrt{r_0} \\ &= 28.2 \times 10^6 / \sqrt{r_0} \text{ m/s} \end{aligned}$$

if r_0 is expressed in m.

- (a) At the surface of the earth,
 $r_0 = 6371 \text{ km}$
 $V_e = 28.2 \times 10^6 \sqrt{6371 \times 10^3}$
 $= 11,170 \text{ m/s} = 11.17 \text{ km/s}$
- (b) At any altitude h km above the earth.
 $r_0 = 6371 + h$
 $V_e = 28.2 \times 10^6 / \sqrt{6371 + h}$

| h (km) | V_e (m/s) |
|----------|-------------|
| 1000 | 10,390 |
| 3000 | 9210 |
| 5000 | 8360 |
| 10,000 | 6970 |

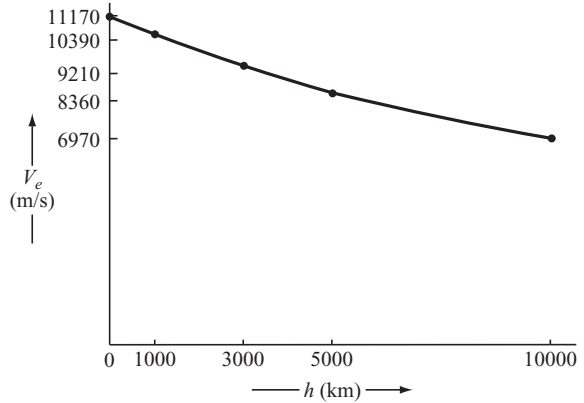


Fig. Ex. D2.4 (Solution)

It may be seen from the curve plotted between V_e and h that the velocity of escape is less if the satellite is launched from higher altitudes as shown in Fig. Ex. D2.4 (Solution). On the other hand, it is a problem to reach a high altitude of the order of thousands of kilometres. If we were to escape, we would attach the satellite to a multistage rocket motor which would take it up into the thin air and when all the stages drop off, the satellite would be on its own. We would also like to take advantage of the spinning motion of the earth and launch the satellite so as to gain a component of the velocity from the spin of the earth before leaving it.

Example D2.5

A body is released at a distance far away from the surface of the earth. Calculate its velocity when it is near the surface of the earth, ignoring air resistance and all other forces except gravity.

Solution For the body, gain in kinetic energy equals loss in potential energy

$$\frac{1}{2} mV^2 = -GM \left(\frac{1}{L} - \frac{1}{R} \right) m = \frac{GMm}{R}$$

taking $1/L$ as zero for L to be very large.

Then,
$$V^2 = \frac{2GM}{R} = 2gR$$

$$= 2 \times 9.81 \times 6.37 \times 10^6$$

and
$$V = 11.17 \times 10^3 \text{ m/s}$$

Example D2.6

A satellite of mass 200 kg, initially at rest on the earth, is to be launched in a circular orbit at a height equal to the radius of the earth, i.e., $6.37 \times 10^6 \text{ m}$. Calculate the minimum energy required.

Solution Radius of the circular orbit is $2R$.

Initial mechanical energy of the satellite at the surface of the earth is

$$E_1 = -GMm/R$$

Mechanical energy of the satellite in the circular orbit of radius $2R$ should be

$$\frac{1}{2} m V_0^2 + \left(-\frac{GMm}{2R} \right)$$

Since

$$V_0 = \sqrt{\frac{GM}{2R}}$$

the mechanical energy in rotation becomes

$$E_2 = \frac{1}{2} m \frac{GM}{2R} - \frac{GMm}{2R} = -\frac{1}{4} \frac{GMm}{R}$$

The minimum energy required to launch the satellite should be

$$\begin{aligned} E_2 - E_1 &= -\frac{1}{4} \frac{GMm}{R} - \left(-\frac{GMm}{R} \right) \\ &= \frac{3}{4} \frac{GMm}{R} = \frac{3}{4} gR \\ &= \frac{3}{4} \times 9.81 \times 6.37 \times 10^6 = 9.365 \times 10^9 \text{ J} \end{aligned}$$

Example D2.7

The Mars and the earth have their masses in proportion to 0.107 and their radii in the ratio of 0.53 (Mass being smaller), compare their (i) densities; (ii) gravitational intensities; (iii) escape velocities; and (iv) periods of their satellites.

Solution

$$\frac{M_m}{M_e} = 0.107, \quad \frac{R_m}{R_e} = 0.53$$

(i) Density is mass divided by volume, i.e., $M / \frac{4}{3} \pi R^3$

$$\frac{d_m}{d_e} = \frac{M_m}{M_e} \cdot \left(\frac{R_e}{R_m} \right)^3 = 0.107 \times \frac{1}{0.53^3} = 0.718$$

(ii) Gravitational intensity $g = \frac{GM}{R^2}$

$$\frac{g_m}{g_e} = \frac{M_m}{M_e} \left(\frac{R_e}{R_m} \right)^2 = 0.107 \times \frac{1}{0.53^2} = 0.381$$

(iii) Escape velocity equals $\sqrt{\frac{2GM}{R}}$

$$\frac{V_{em}}{V_{ee}} = \sqrt{\frac{M_m}{M_e} \cdot \frac{R_e}{R_m}} = \sqrt{0.107 \times \frac{1}{0.53}} = 0.45$$

(iv) Period of a satellite = $\sqrt{\left(\frac{4\pi^2}{GM} R^3\right)}$

$$\frac{T_m}{T_e} = \sqrt{\frac{M_e}{M_m} \cdot \left(\frac{R_m}{R_e}\right)^3} = \sqrt{0.107} \times 0.53^3 = 1.18$$

Example D2.8

A sky laboratory of mass 2000 kg has to be lifted from a circular orbit of radius $2R$ to another circular orbit of radius $3R$. Calculate the minimum energy required to do so.

Solution Mechanical energy of the satellite in a circular orbit of radius r is

$$\frac{1}{2} m V^2 - \frac{GMm}{r}$$

where

$$V = \sqrt{\frac{GM}{r}}$$

At a radius $2R$, the mechanical energy

$$E_1 = \frac{1}{2} m \frac{GM}{2R} - \frac{GMm}{2R} = -\frac{1}{2} \frac{GMm}{2R}$$

At a radius $3R$, the mechanical energy

$$E_2 = \frac{1}{2} m \frac{GM}{3R} - \frac{GMm}{3R} = -\frac{1}{2} \frac{GMm}{3R}$$

The minimum energy required

$$\begin{aligned} E_2 - E_1 &= -\frac{1}{2} \frac{GMm}{3R} - \left(-\frac{1}{2} \frac{GMm}{2R}\right) = \frac{1}{12} \frac{GMm}{R} \\ &= \frac{1}{12} gR = \frac{1}{12} \times 9.81 \times 6.37 \times 10^6 \times 2000 \\ &= 10.4 \times 10^9 \text{ J} \end{aligned}$$

Example D2.9

An artificial satellite revolves about the earth (radius 6380 km) at a height h above the surface. Determine the orbital speed and orbital period so that a person in the satellite will be in a state of weightlessness.

Solution ‘Weightlessness’ of a person refers to a state of ‘dynamical equilibrium’ when the centrifugal force balances the gravitational force of the earth

$$\frac{mV^2}{r} = \frac{GMm}{r^2} = \frac{gR^2 m}{r^2} \tag{i}$$

where R is the radius of the earth, g the gravitational acceleration and r the radial distance of the satellite from the centre of the earth. In terms of the altitude h above the surface of the earth

$$r = R + h$$

which, when substituted in Eq. (i) results in

$$V = \sqrt{\frac{g}{R+h}} R$$

The orbital time t may be computed from a knowledge of the distance $2\pi r$ travelled in one revolution.

$$t = \frac{2\pi r}{V} = \frac{2\pi(R+h)}{R} \sqrt{\frac{R+h}{g}} = \frac{2\pi(R+h)^{3/2}}{R\sqrt{g}} \quad (\text{ii})$$

In the special case when the satellite is close to the surface of the earth,

$$h \ll R$$

$$V = \sqrt{gR}$$

and

$$t = 2\pi\sqrt{R/g}$$

If

$$R = 6380 \text{ km} \quad \text{and} \quad g = 9.81 \text{ m/s}^2$$

the orbital speed

$$V = \sqrt{6380 \times 1000 \times 9.81} = 7910 \text{ m/s} \quad 7.9 \text{ km/s}$$

and the orbital time

$$t = 2\pi\sqrt{6380 \times 1000/9.81} = 5070 \text{ s} \quad 84.5 \text{ minutes}$$

Example D2.10

A space shuttle releases a 400 kg communications satellite while in an orbit 250 km above the surface of the earth. A rocket engine on the satellite boosts it into a geosynchronous orbit. Determine the energy provided by the engine.

Solution A geosynchronous orbit is that in which the satellite stays directly over a single location on the earth. For example, a satellite which is always first above New Delhi is geostationary! Such a satellite should also be spinning with the earth with a period of 24 hours or 86,400 seconds with an angular velocity.

$$\omega = \frac{2\pi}{86\,400} = 0.000073 \text{ rad/s} = V_\theta / r$$

Also,

$$V_\theta = \sqrt{GM/r} \text{ Taking } GM = 3.9860 \times 10^{14} \text{ m}^3/\text{s}^2$$

$$r = 42.18 \times 10^6 \text{ m and } V_\theta = 3080 \text{ m/s}$$

Now, let us look at satellite ready to be released by the space shuttle.

$$R = 250\,000 + 6370\,000$$

$$= 6.62 \times 10^6 \text{ m}$$

Energy associated with a circular orbit of radius r_0 is expressed as

$$E = \frac{-GMm}{2r_0}$$

$$E_1, \text{ energy before release} = \frac{-GMm}{2} \left(\frac{1}{6.62 \times 10^6} \right) \text{J}$$

$$E_2, \text{ energy in geosynchronous orbit} = \frac{-GMm}{2} \left(\frac{1}{42.18 \times 10^6} \right) \text{J}.$$

For $m = 400$ kg, the energy provided by the engine

$$E = E_2 - E_1 = 1.01 \times 10^{10} \text{ J}.$$

Example D2.11

The Indian satellite ROHINI launched on July 18, 1980, was set into a circular orbit completing a revolution in 97 minutes. No further details were available in the newspapers. Determine the altitude and the launching velocity assuming that it was launched at zero angle of launching.

Solution Given that GM for the earth $= 3.9860 \times 10^{14} \text{ m}^3/\text{s}^2$ and r , the radius of the earth $= 6371$ km, the launching velocity V_0 can be determined from the equation

$$V_0 = \sqrt{\frac{GM}{r_0}} = \frac{20 \times 10^6}{\sqrt{r_0}} \text{ m/s} \quad (\text{i})$$

where r_0 is the radius of the circular orbit.

It is also given that time period for one revolution is 97 minutes.

Therefore,
$$97 \times 60 = \frac{2\pi r_0}{V_0},$$

whence
$$V_0 = 2\pi r_0 / (97 \times 60) \quad (\text{ii})$$

From Eqs. (i) and (ii), we get

$$\frac{2\pi r_0}{97 \times 60} = \frac{20 \times 10^6}{\sqrt{r_0}}$$

i.e.,
$$r_0^{3/2} = 18,493 \times 10^6$$

and
$$r_0 = 6,993,170 \text{ m}$$

or
$$r_0 = 6,993,17 \text{ km}$$

The satellite is launching at an altitude given by

$$h = r_0 - r = 6993.17 - 6371 = 622.17 \text{ km}$$

The launching velocity is given by

$$V_0 = \frac{20 \times 10^6}{\sqrt{6,993,170}} = 7563 \text{ m/s}$$

THE INDIAN SATELLITE STORY

| <i>Indian satellite</i> | <i>Launcher</i> | <i>Date of launch</i> |
|-------------------------|-------------------------------|-----------------------|
| APPLE | Ariane, France | 19 June 1981 |
| Aryabhata | SL-8, Russia | 19 April 1975 |
| Bhaskara-1 | SL-8, Russia | 07 June 1979 |
| Bhaskara-2 | SL-8, Russia | 20 November 1981 |
| Rohini-1B | SLV-3, India | 18 July 1980 |
| Rohini-2 | SLV-3, India | 31 May 1981 |
| Rohini-3 | SLV-3, India | 17 April 1983 |
| SROSS-C2 | PSLV-D2, India | 4 May 1994 |
| SROSS-C3 | ASLV-D3, India | 20 May 1992 |
| IRS-1A | SL-3, Russia | 17 March 1988 |
| | | |
| SRE-I | PSLV-C7, India | 10 January 2007 |
| CARTOSAT-2 | PSLV-C7, India | 10 January 2007 |
| INSAT-4B | Ariane-5ECA, French Guyana | 12 March 2007 |
| INSAT-4CR | GSLV-F04, India | 2 September 2007 |
| IMS-1 | PSLV-C9, India | 28 April 2008 |
| CARTOSAT-2A | PSLV-C9, India | 28 April 2008 |
| Chandrayaan-1 | PSLV-C11, India | 22 October 2008 |
| RISAT-2 | PSLV-C12, India | 20 April 2009 |
| ANUSAT | PSLV-C12, India | 20 April 2009 |
| Oceansat-2 | PSLV-C14, India | 23 September 2009 |

Concept Review Questions

Establish the correctness of the following statements:

- (a) The acceleration of a particle in a central-force motion must be directed towards or away from a fixed centre.
- (b) A central-force motion is a plane motion because the moment of momentum is conserved for the motion.

- (c) A particle projected to travel a short range over the surface of the earth traces a part of an ellipse which closely resembles a parabola.
- (d) The angle of launching has no influence on the magnitude of the escape velocity.

Show that the minimum velocity required to enable a body to escape the gravitational field of

the earth is $\sqrt{2}$ times the velocity to set it into a circular orbit at the same location.

Will a person travelling in a geo-stationary satellite feel weightlessness, heaviness, partial-weight relief or some other weight effect?

Show that the energy expended in enabling a body to escape the gravitational field of the earth

is twice the energy expended on it for a circular orbit at the same launching position.

Is the escape velocity greater at the surface of the earth or at the surface of the moon? Explain why? It is given that the moon's radius is about 0.27 that of the earth and its acceleration of gravity at the surface is 0.165 that of the earth at earth's surface.

Tutorial Problems

A satellite orbits the earth 900 km above the surface. Find the orbital speed and orbital time of the satellite if the radius of the earth is 6370 km.

(s 7410 m/s, 6165 s)

A satellite of mass 5000 kg is in an elliptic orbit around the earth. Its position varies between 300 km and 30,000 km above the surface of the earth of radius 6370 km. Determine the eccentricity of the orbit, its velocities at the nearest and farthest points from the earth and its angular momentum.

(s 0.690, 10,040 m/s, 18,420 m/s;
335 × 10¹³ kg m²/s)

A 5000 kg space vehicle is to be put into a circular orbit 10 000 km above the earth's surface. Find (a) the orbital velocity, (b) the orbital kinetic energy and (c) the total energy expended on it.

(s 4930 m/s, 6.08 × 10¹⁰ J)

With what speed must a 15,000 kg lunar module be launched from the moon's surface in order to escape into interplanetary space? What kinetic energy would be required?

A satellite is launched with a velocity of 7745 m/s parallel to the earth's surface at a distance of 6.624×10^6 m from the centre of the earth.

- What kind of trajectory will it have?
- What will be its greatest distance from the centre of the earth?
- Compute the time taken for it to travel from the minimum to the maximum distance from the earth.

- What would be the minimum escape velocity for this position of launching?

(s Elliptic orbit, $e = 0.0046$;
 6.564×10^6 m; 44.35 min; 10,970 m/s)

A particle is projected radially outwards, as shown in Fig. Prob. D2.6 from an altitude h . Compute the minimum velocity and minimum energy for it to escape the gravitational field.

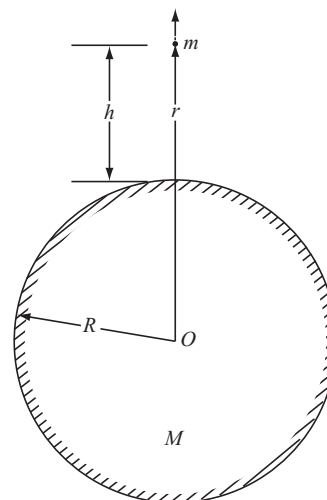


Fig. Prob. D2.6

Compute the radius of the moon's orbit in terms of the radius R of the earth, 6370 km, acceleration due to gravity g , 9.81 m/s² and the time period T , 27.3 days for the moon to revolve once around the earth.

- 8 A spacecraft describing a circular orbit at an altitude of 3000 km above the earth's surface must reduce its speed to V_0 in order to describe an elliptical trajectory for re-entry. It is desired that its splash-down occurs at a point S on the surface of the earth of radius 6370 km at a location where $\theta = 60^\circ$. Determine the speed V_0 at point A .

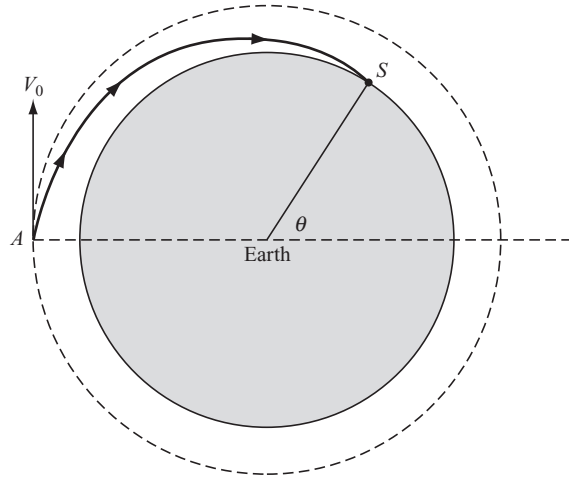


Fig. Prob. D2.8

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or the most appropriate response from among the available alternatives in the following multiple-choice questions:

- Constancy of areal velocity for a central-force motion of a particle *does not* imply that
- (a) equal areas are swept by the radius vector in equal times
 - (b) the tangential velocity varies inversely as the length of the radius vector
 - (c) the angular momentum is conserved
 - (d) the particle should stay in an orbit.

If satellite is launched with a velocity given by $V = \sqrt{2GM/r_0}$, with usual notation,

- (a) the trajectory would be parabolic only if the angle of launch is zero

- (b) the satellite will escape the gravitational field of the earth except if the launching is radially outwards
- (c) the energy expended on the satellite should be given by $GMm/2r_0$
- (d) the satellite will escape the gravitational field of the earth regardless of the angle of launch.

A geo-stationary satellite

- (a) stays at a fixed position in space
- (b) follows a circular orbit at any desired height
- (c) follows a circular orbit at any desired height but it has a time period of 24 h
- (d) follows a circular orbit at a unique altitude and has a time period of 24 h.

Answers to Multiple-Choice Questions

- 1 (d) 2 (d) 3 (d)

Virtual Work and Potential Energy Principles



10

10.1 INTRODUCTION

It is appropriate at this stage of presentation of the subject to look at the formulation of problems by methods other than Newton's laws of motion. We are here referring to the energy-based principles in mechanics. The advantages offered by the energy-based principles include freedom from drawing the free-body diagram, dealing with scalar quantities, namely energy and work instead of the vector quantities, such as force, velocity, acceleration, momentum, etc., ease of application to multibody systems and relative simplicity of analysis in the long run.

The principle of virtual work is introduced first for static equilibrium and then, together with the D'Alembert principle, it is extended to the analysis of dynamical systems. While we are on the subject of equilibrium, an alternative method known as the principle of potential energy, is also introduced. These principles are helpful in understanding the variational principles in mechanics.

10.2 VIRTUAL WORK

Consider a particle P subjected to a force \mathbf{F} . Work is said to be done by the force if the particle is actually displaced, say $d\mathbf{r}$, and the work dW is given by

$$dW = \mathbf{F} \cdot d\mathbf{r} \quad (10.1)$$

Although the force \mathbf{F} and displacement $d\mathbf{r}$ are vector quantities, work, the dot product of \mathbf{F} and $d\mathbf{r}$ is a scalar quantity. If the unit of force are N and units of displacement are m, the units of work must be Nm, i.e., joule.

If the particle is only imagined to be displaced, say $\delta\mathbf{r}$, hypothetically or virtually then we can imagine virtual work done by the force,

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} \quad (10.2)$$

It is not 'real' work because the particle does not actually get displaced. The concept of virtual work is important in analysing complex mechanical systems by imagining virtual displacements.

Whether a body is actually at rest or in motion, a virtual or possible displacement can be visualised in view of the constraints. *Such a displacement is hypothetical or imaginary it may or may not be along the actual displacement but it must be one of the physically possible displacements only.* Consider, for example, a body sliding up an incline under the action of some forces. The fact that the body can slide either up or down without violating the constraints, i.e., without being lifted from the incline or piercing into it or going sideways, the virtual displacement δ can be given either up or down the plane. Thus either of the modes of virtual displacements shown in Figs. 10.2(a) and (b) is acceptable. The fact that the body is actually moving up is of no consequence as far as the choice of virtual displacement is concerned. The actual mode of displacement is, however one of the ways a virtual displacement may be given. Virtual displacement δ is assumed to be very small or infinitesimal so that it is accompanied by an infinitesimally small amount of virtual work δW . *The term virtual work, therefore, implies the hypothetical work which would have been done to result in a virtual displacement under the application of the given system of forces and couple-moments.*

The virtual work δW due to a force resulting in virtual displacement δ at the point of application of the force is

$$\delta W = \mathbf{F} \cdot \delta$$

It follows that the virtual work for an infinitesimal virtual displacement of a particle subjected to a system of forces is

$$\delta W = \Sigma \mathbf{F} \cdot \delta \quad (10.3)$$

where $\Sigma \mathbf{F}$ is the resultant or equivalent force on the particle.

Alternatively, if a system of forces and couple-moments

$$F_1, F_2, \dots, F_n, \quad M_1, M_2, \dots, M_n$$

act on a rigid body in a specified manner and the linear possible virtual displacements at the points of application of the forces are

$$\delta_1, \delta_2, \dots \text{ respectively}$$

and the angular virtual displacements due to the moments are

$$\delta\theta_1, \delta\theta_2, \dots \text{ respectively}$$

as shown in Fig. 10.3, then the total virtual work done on the body must be

$$\delta W = F_1 \cdot \delta_1 + F_2 \cdot \delta_2 + \dots + F_n \cdot \delta_n + M_1 \cdot \delta\theta_1 + M_2 \cdot \delta\theta_2 + \dots + M_n \cdot \delta\theta_n \quad (10.4)$$

The summation of virtual work done by 'each force for its virtual displacement' may be written as the virtual work done by the resultant force $\Sigma \mathbf{F}$ for the virtual displacement of the centre of mass, i.e.,

$$\Sigma F_1 \delta_1 + \Sigma F_2 \delta_2 + \dots = \Sigma \mathbf{F} \cdot \delta \quad (10.5)$$

Likewise, the summation of virtual work done due to applied moments can be expressed as $\Sigma M \cdot \delta\theta$, which means that

$$\delta W = \Sigma \mathbf{F} \cdot \delta + \Sigma M \cdot \delta\theta \quad (10.6)$$

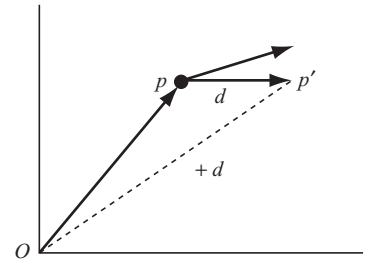


Fig. 10.1 Work of a force

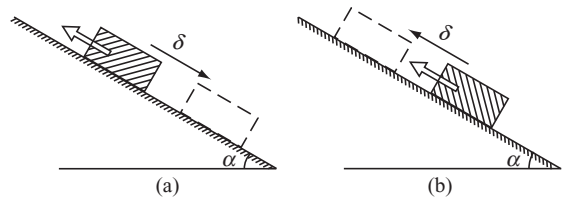


Fig. 10.2 A body on an incline

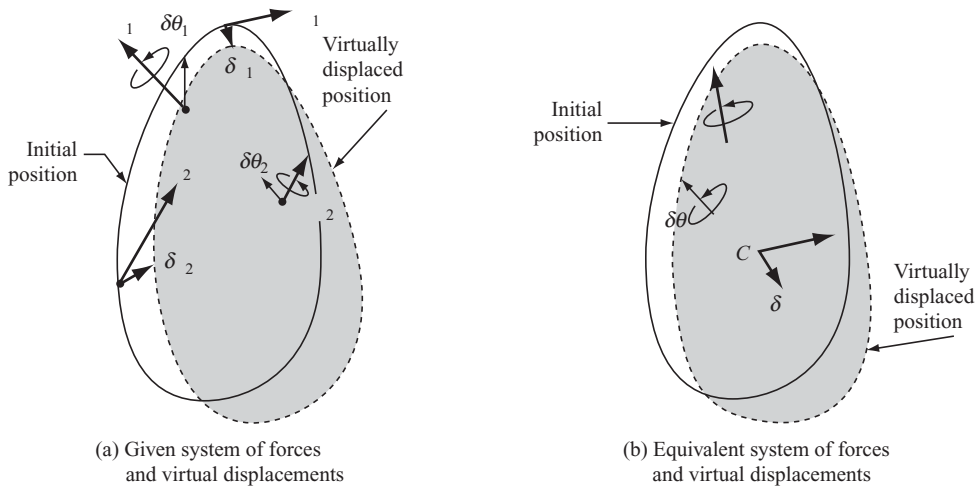


Fig. 10.3 System of forces acting on a rigid body

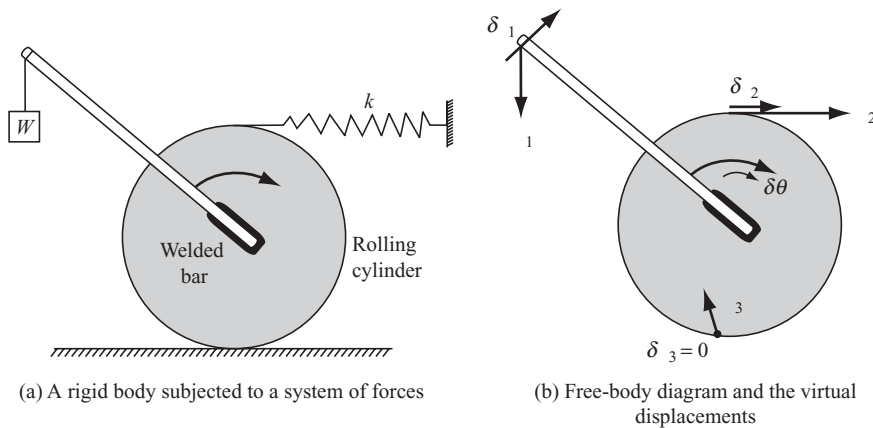


Fig. 10.4 Example of a rigid body subjected to a system of forces

For example, the rolling cylinder shown in Fig. 10.4 is subjected to a system of forces as shown in the free-body diagram. From the constraints on the motion, the virtual displacements at the points of action of the forces can be visualised readily and this procedure of finding the virtual work can be used with advantage.

10.3 PRINCIPLE OF VIRTUAL WORK

The condition of equilibrium of a body are

$$\Sigma F_x = 0 \quad \text{and} \quad \Sigma M = 0$$

The virtual work δW for infinitesimal virtual displacements is given by

$$\delta W = \Sigma F_x \delta x + \Sigma M \delta \theta$$

Since

$$\Sigma F_x = 0 \quad \text{and} \quad \Sigma M = 0,$$

δW must be zero!

This is the principle of virtual work. *It states that the virtual work should be zero for a body to be in equilibrium.*

The question that arises next is whether the virtual work principle offers any advantage over the conditions of equilibrium, which follow directly from Newton's law and Euler's equation. The answer is yes. There are definite advantages, some of which are explained as follows:

1. It may be easier and more convenient to determine the virtual work than to evaluate the resultant force and resultant moment. It is easier because the constraining forces which may be due to the normal reactions by the surfaces and the internal action-reaction forces do not contribute to the virtual work. For example, the reaction by a smooth surface can do no virtual work on a body when it is virtually displaced by sliding over the surface because the reaction force and displacement are at right angles to each other. The merit of the virtual work principle is, therefore, that the reaction forces and other constraining forces need not be determined at all. This step is unavoidable if we were to evaluate the resultant force for equating it to zero for equilibrium. Some other forces which do not contribute to the virtual work and need not be evaluated are:

- (i) Reaction at a smooth pin for rotation about the pin
- (ii) Reaction at a roller moving along a track
- (iii) Weight of a body when its centre of gravity moves horizontally
- (iv) Friction force acting on a wheel rolling without slip.

2. The principle of virtual work applies to a system of connected particles or of rigid bodies as good as to a single rigid body or to a single particle. Only the forces external to the system of connected bodies need be considered for virtual work; the internal forces cannot contribute to the virtual work. Care must be taken to select the virtual displacements in a manner which do not violate the constraints.

3. It may appear, in the first instance, that the virtual work principle applies only to a body in equilibrium and provides an alternative, single condition of equilibrium which is a little simpler. A little consideration of the D'Alembert's principle will show that if a body of mass m is accelerating at an acceleration a under the action of an external force F , then,

$$F + (-ma) = 0 \tag{10.8}$$

The form of the equation suggests that if a hypothetical force called inertia force

$$F_1 = -ma \tag{10.9}$$

is to act on the body in addition to the external force F , then the body would hypothetically come to a state of equilibrium. *It shows that the D'Alembert's principle used simultaneously with the virtual work principle should be able to extend the virtual work principle to bodies not in equilibrium.* As a closing comment, it should be mentioned here that this statement is so true that a complete formulation of energy method in mechanics has already overshadowed the Newtonian mechanics. Energy equations due to Langrange and Hamilton can be shown equivalent to the Newtonian formulation.

Example 10.1

A frictionless double-incline with angle α_1 and α_2 as shown in Fig. Ex. 10.1 carries a set of sliding masses m_1 and m_2 connected with an inextensible string and passing over a frictionless pulley at O . Obtain the relationship between α_1 and α_2 in terms of m_1 and m_2 for equilibrium and hence determine α_2 if

$$\alpha_1 = 30^\circ \quad \text{and} \quad m_1 = 2m_2$$

How would the angle change if the surfaces had a coefficient of friction μ instead of being smooth?

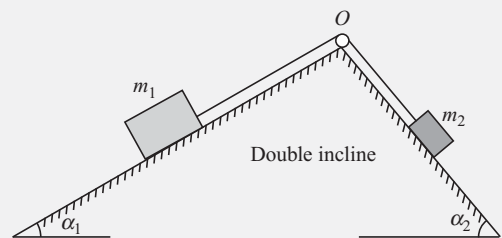


Fig. Ex. 10.1

Solution Let us draw and observe the free-body diagrams of the two masses in the absence of friction (Fig. Ex. 10.1 (Solution)).

Since the motion is possible only up and down the inclines, let us give a virtual displacement δr_1 up the α_1 incline to mass m_1 . The virtual displacement of mass m_2 must then be δr_1 down the α_2 incline because the string is inextensible and hence is of constant length.

The virtual work done by R_1 for the displacement δr_1 of mass m_1 must be zero because they are at right angles. Similarly, the virtual work by R_2 for the displacement δr_1 of mass m_2 must be zero. R_1 and R_2 need not be determined. Now, the virtual work done by T_1 for displacement δr_1 of mass m_1 along α_1 is $T_1 \delta r_1$ and the virtual work done by T_2 for displacement δr_1 of mass m_2 along α_2 is $T_2 \delta r_1$. The total work done by the tension forces in the inextensible string is

$$T_1 \delta r_1 + T_2 \delta r_1$$

which must be zero because T_1 is equal and opposite to T_2 , there being no friction in the pulley.

The only forces which indeed do virtual work are the external forces on the bodies. In this case $m_1 g$ and $m_2 g$ are the only external gravitational forces. The virtual work done by them adds up to

$$-m_1 g \sin \alpha_1 \delta r_1 + m_2 g \sin \alpha_2 \delta r_1$$

According to the virtual work principle, the virtual work must be zero for the system to be in equilibrium

$$-m_1 g \sin \alpha_1 \delta r_1 + m_2 g \sin \alpha_2 \delta r_1 = 0$$

Cancelling δr_1 and g from both terms,

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{m_2}{m_1}$$

It may appear that the virtual work method is also lengthy and requires careful elimination of the non-contributory forces. However, this is untrue. *It was unnecessary to draw the free-body diagram actually or even to eliminate the normal-reaction and internal equal tension forces.* Only a general statement is adequate and the required proof is between the two asterisks () above.

When $m_1 = 2m_2$ and $\alpha_1 = 30^\circ$

$$\sin \alpha_2 = 2 \sin 30^\circ = 1$$

whence $\alpha_2 = 90^\circ$

Let us now investigate the effect of friction at the inclined surfaces. The virtual displacement still being the same as before, the frictional forces would enter into the virtual work.

On mass m_1 , the frictional force would be

$$\mu m_1 g \cos \alpha_1$$

down the incline α_1 , and the virtual work contributed by it

$$-\mu m_1 g \cos \alpha_1 \delta r_1$$

Similarly, the virtual work done by the frictional force on mass m_2 would be

$$-\mu m_2 g \cos \alpha_2 \delta r_1$$

The virtual work principle would be applied to the total virtual work, i.e.,

$$-m_1 g \sin \alpha_1 \delta r_1 + m_2 g \sin \alpha_2 \delta r_1 - \mu m_1 g \cos \alpha_1 \delta r_1 - \mu m_2 g \cos \alpha_2 \delta r_1 = 0$$

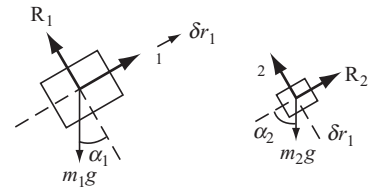


Fig. Ex. 10.1 (Solution)

Cancelling δr_1 and g ,

$$m_1(\sin \alpha_1 + \mu \cos \alpha_1) = m_2(\sin \alpha_2 - \mu \cos \alpha_2)$$

or

$$\frac{\sin \alpha_1 + \mu \cos \alpha_1}{\sin \alpha_2 - \mu \cos \alpha_2} = \frac{m_2}{m_1}$$

A noteworthy comment at this stage is that this relationship applies to the equilibrium against sliding in accordance with the chosen mode of virtual displacement. If δr_1 was reversed in direction, frictional forces would also reverse and the virtual work principle would yield

$$\frac{\sin \alpha_1 - \mu \cos \alpha_1}{\sin \alpha_2 + \mu \cos \alpha_2} = \frac{m_2}{m_1}$$

for equilibrium. The application of the virtual work principle for systems with dissipative forces such as friction, aero-dynamic drag, etc., requires as much attention as the application of Newtonian laws.

Example 10.2

Determine the relationship between the moment applied at the crank of radius R and the force applied at the crosshead in the slider crank mechanism shown in Fig. Ex. 10.2.

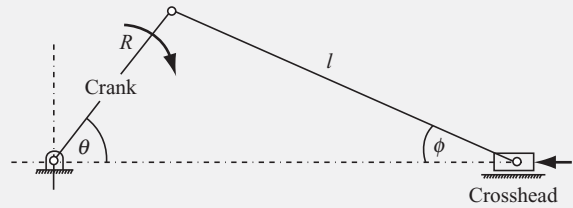


Fig. Ex. 10.2

Solution Let the system be virtually displaced as shown in Fig. Ex. 10.2 (Solution). The virtual work done by the moment and the force adds up to $- \delta\theta - \delta x$ (i) which must be equated to zero in the absence of any other external force or moment. Here, x is chosen positive to the right and θ positive anticlockwise as per our usual convention.

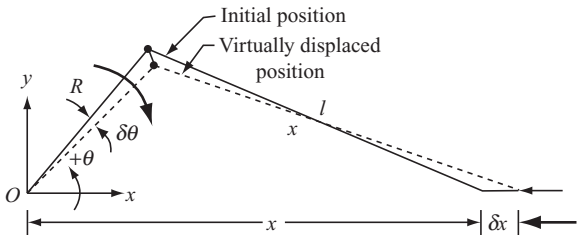


Fig. Ex. 10.2 (Solution)

From the geometry of the configuration,

$$x = R \cos \theta + l \cos \phi \tag{ii}$$

and

$$R \sin \theta = l \sin \phi$$

or

$$\cos^2 \phi = 1 - \sin^2 \phi = 1 - (R/l)^2 \sin^2 \theta \tag{iii}$$

and differentiating Eq. (ii) and using Eq. (iii)

$$\delta x = -R \sin \theta \cdot \delta\theta - R^2/l \frac{\sin \theta \cos \theta}{1 - (R/l)^2 \sin^2 \theta} \delta\theta$$

which when substituted in Eq. (i) and equated to zero results in the required relationship

$$= R \sin \theta \left(1 + \frac{R \cos \theta}{l \sqrt{1 - (R/l)^2 \sin^2 \theta}} \right)$$

Example 10.3 A ladder AB of weight mg and length l is held in equilibrium by a horizontal force P as shown in Fig. Ex. 10.3. Assuming the ladder to be idealised as a homogeneous rigid bar, relate P to mg . Assume the surfaces to be smooth.

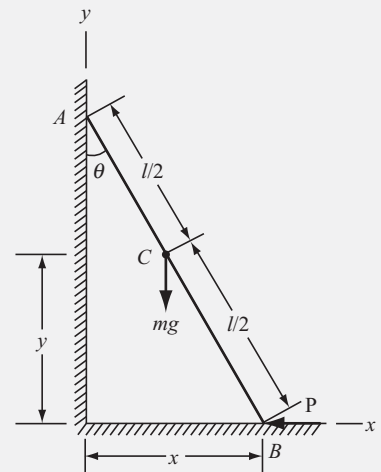


Fig. Ex. 10.3

Solution Let the virtual displacement of B be δx towards right, i.e., along the x -axis. The virtual work done by P is

$$W_p = -P \delta x$$

The virtual displacement of C will have x and y components, but the virtual work done by W must be

$$\delta W_w = -mg \delta y$$

where δy is the vertical component of the virtual displacement of C .

By the virtual work principle for equilibrium,

$$\delta W = \delta W_p + \delta W_w = 0$$

$$\text{or,} \quad -P \delta x - mg \delta y = 0 \tag{i}$$

From the geometry of placement

$$x = l \sin \theta \quad \text{and} \quad y = \frac{1}{2} l \cos \theta$$

$$\text{then,} \quad \delta x = l \cos \theta \delta \theta$$

$$\text{and} \quad \delta y = -\frac{1}{2} l \sin \theta \delta \theta$$

Substituting for δx and δy in terms of the single variable θ , for the one-degree of freedom system in Eq. (i),

$$-Pl \cos \theta \delta \theta + \frac{1}{2} mgl \sin \theta \delta \theta = 0$$

$$\text{whence} \quad P = \frac{1}{2} mg \tan \theta$$

It may be noted that the reactions by the smooth surfaces are normal to the respective surfaces and would, therefore, not contribute to the virtual work. *There is no need to draw a complete free-body diagram of the member it is only necessary to recognise the active or driving forces which contribute to the virtual work.*

If the surfaces were not smooth but rough with friction coefficients μ_A for the horizontal surface and μ_B for the vertical surface, the frictional forces would also be active forces at the points of contact. The

frictional forces would oppose the respective virtual displacements and contribute to the virtual work. For the positive δx_B and positive δy_A , the frictional virtual work is

$$-f_A \delta y_A - f_B \delta x_B$$

where f_A and f_B are the frictional forces at A and B respectively.

10.4 PRINCIPLE OF POTENTIAL ENERGY

If a system operates under the action of conservative forces, e.g., due to gravity, elastic elements, etc., the potential energy concept can be used to advantage. The potential energy PE for a conservative force $F(s)$ acting in a direction s as shown in Fig. 10.5(a) may be recollected to be given by

$$\frac{d(PE)}{ds} = -F(s) \tag{10.10}$$

For a gravitational force acting downward, as shown by Fig. 10.5(b)

$$F = -mg$$

$$\frac{d(PE)}{dz} = -(-mg) = mg$$

whence $PE = mg(z_2 - z_1) = mgh$

referred to zero potential energy at $h = 0$.

For an elastic element, the restoring force, as shown in Fig. 10.5(c), is

$$F = -kx$$

$$\frac{d(PE)}{dx} = -(-kx) = kx$$

and

$$PE = \frac{1}{2} kx^2$$

whence

referred to zero potential energy at the equilibrium position, $x = 0$.

The principle of potential energy for the equilibrium of a body or a system of bodies follows from the fact that the external force F should be zero. Then,

$$\frac{d(PE)}{ds} = 0 \tag{10.11}$$

In words, *the potential energy principle states that if the potential energy is a function of a single space coordinates, then the potential energy must have a stationary value at the position of equilibrium; the derivative of the potential energy with respect to that coordinate must vanish. The single coordinate in question may be x , y , z , θ or some other coordinate chosen for convenience. It can be appreciated from Calculus that a function may have a stationary value*

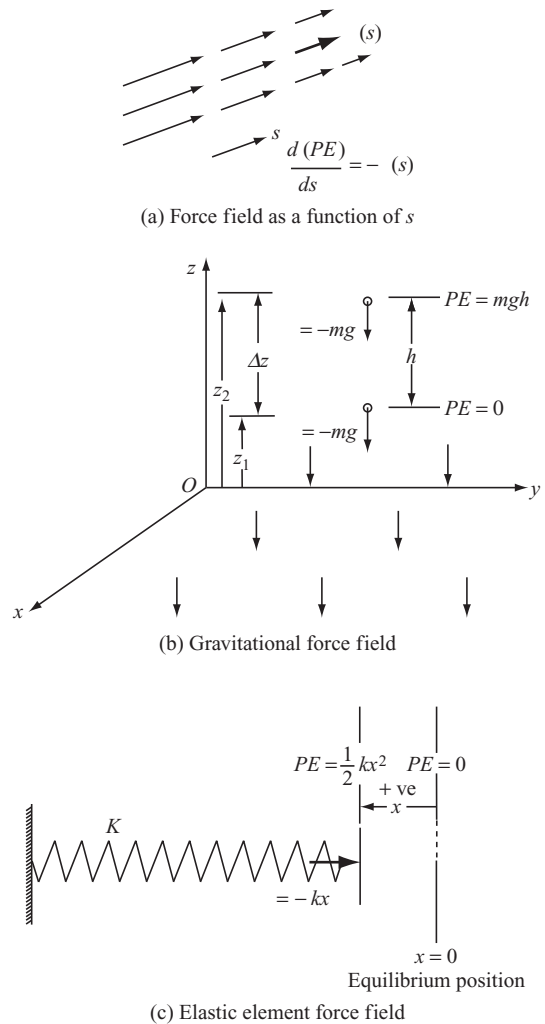


Fig. 10.5 Principle of potential energy: conservative fields

at a maxima, a minima, a point of inflexion or a flat region of the curve. For example, if a curve PE vs. s appears as that shown in Fig. 10.6, the body must be in equilibrium at points A, B, C, D, E, F , etc.

Following from the fact that the vanishing external force,

$$= 0$$

is a necessary condition and not a sufficient condition for equilibrium, it may be appreciated that Eq. (10.11) is not only a necessary but also an alternative condition for equilibrium.

Let us now turn to the three different states of equilibrium, viz. stable, unstable and neutral equilibrium. Consider a bar with its centre of gravity at C and suspended from a point O in three different ways.

Case (a): O above C

Case (b): O below C

Case (c): O coincident with C

as shown in Figs. 10.7(a), (b) and (c) respectively. In each case, the bar is in equilibrium when O and C are in the same vertical line, i.e., when the bar is vertical. Now, let us observe what happens to the bar when it is slightly displaced from its equilibrium position in each case.

In Case (a), the bar tends to return to its initial position. In Case (b), the bar continues to deviate from the initial position and in Case (c) the bar stays wherever it is left. This is so because the moment created by the shift of the centre of gravity from C to C' tends to restore equilibrium in the first case, deviate it further in the second case and there is no such moment in the third case. The state of equilibrium in which a slight disturbance from the equilibrium position is accompanied by a restoring moment so as to bring the body back into its initial state of equilibrium is called *stable equilibrium*. On the other hand, if the slight disturbance is accompanied by an adverse moment to displace the body further, the state is said to be *unstable* and if the slight disturbance fails to generate a restoring or a worsening moment, the body is said to be in a state of *neutral equilibrium*.

Since the potential energy of a suspended body reckoned to its point of suspension O is minimum when the centre of gravity C is below O , maximum when the centre of gravity C is above O and remains unchanged when C and O coincide it is clear that *the states of stable, unstable and neutral equilibrium correspond to the minimum, maximum and stationary potential energy respectively*. This fact can also be appreciated by the observation that a slight angular displacement of the bar is accompanied by a rise of C to C' in Case (a), lowering of C to C' in Case (b) and all-time coincidence of C and C' in Case (c) showing thereby that while in the vertical equilibrium position C must be the lowest possible in Case (a), highest in Case (b) and unchanged in Case (c).

From Differential Calculus, we can recollect that PE is a minima when

$$\frac{d^2 (PE)}{ds^2} > 0 \tag{10.12}$$

and a maxima when

$$\frac{d^2 (PE)}{ds^2} < 0 \tag{10.13}$$

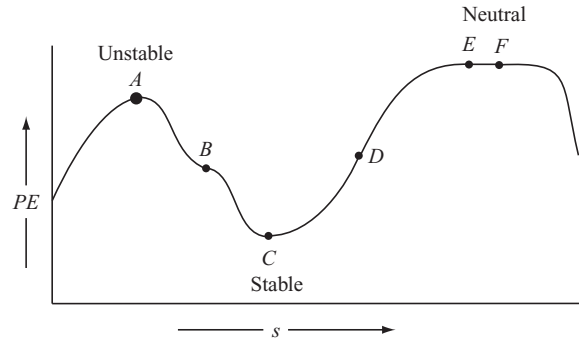


Fig. 10.6 Variation of potential energy and positions of equilibrium

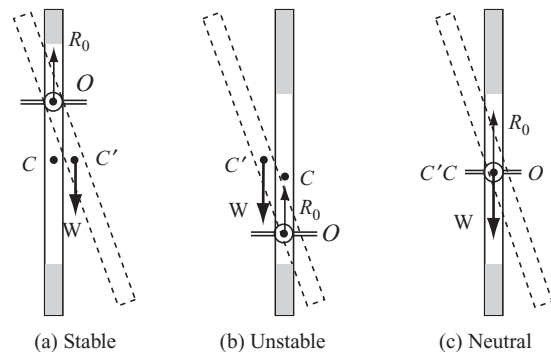


Fig. 10.7 State of equilibrium

and the point of inflexion and the state of 'stationary' P are characterised by

$$\frac{d^2(P)}{ds^2} = 0$$

It may be remarked that a point of inflexion of P vs. s is not classified as stable, unstable or neutral equilibrium. This is because slight displacement one way may imply stability but a slight displacement the other way may lead to instability. In fact, the body has little chance to restore to its initial position because it can become unstable as soon as it overshoots the equilibrium position. A summary of the results obtained so far is presented in Table. 10.1.

Table 10.1 States of equilibrium

| State | Stable | Unstable | Neutral |
|---|---------------------------------------|--------------------------------------|---------------------------------------|
| Common condition 'for all states': | E $\frac{d PE}{ds} = 0$ | | |
| Action of moment generated due to slight displacement Tendency of the body | Restoring Returns to Status Quo | Worsening Further tilting | Nil Stays as it is |
| Sign of $\frac{d^2 PE}{ds^2}$, the height of G above B | + ve | - ve | Zero |
| Potential energy Functional form of (P) | Minimum $\frac{d^2(P)}{ds^2} > 0$ | Maximum $\frac{d^2(P)}{ds^2} < 0$ | Constant $\frac{d^2(P)}{ds^2} = 0$ |

Referring again to Fig. 10.6, we can now qualify the states of equilibrium at the different points.

- At A : maxima, unstable equilibrium
- At B : inflexion, unstable equilibrium
- At C : minima, stable equilibrium
- At D : inflexion, unstable equilibrium
- At E and F : constant P , neutral equilibrium.

The concept of stability of equilibrium may be extended to a body capable of oscillations without any hinge or fulcrum. One such case of great importance is a floating body, e.g., a floating cylinder in a liquid or a floating ship in water. In the equilibrium position the weight of the body acting downward through the centre of gravity G is exactly balanced by the upthrust or buoyancy force acted by the surrounding liquid through the centre of buoyancy B .

$$W = B$$

A homogeneous cylinder floating upright in a liquid must be in a state of equilibrium as shown in Fig. 10.8(a).

If the body is tilted through a small angle $\delta\theta$, the centre of gravity is displaced in space but remains fixed with the body at G , whereas the centre of buoyancy shifts to new position B_1 because the volume of the body immersed in the liquid changes in shape. The buoyancy force B_1 , the same in magnitude as B , acts upward through B_1 . The weight acting downward through G and B_1 acting upward through B_1 constitute a

couple which may act to produce different effects: if it helps to restore the body to initial equilibrium as in Fig. 10.8(b), it is said to be in a stable state. If it acts to tilt the body further, as in Fig. 10.8(c), it is in unstable state and if it produces no effect, the body is said to enjoy neutral equilibrium as shown in Fig. 10.8(d).

A comparison of the mechanics of the states of equilibrium of a floating body with that for a body suspended from a point shows that there is some difference. The point of suspension for a suspended body is fixed and the potential energy is referred to this fixed point. The fixed point also provides a reaction force R_0 to constitute a couple which comes into play when the body is displaced. In a floating body there is no fixed point to provide the reaction; the reaction is provided by the surrounding liquid acting through the centre of buoyancy B or B_1 . The line of action of F_{B_1} through B_1 intersects the centreline of the body at M . The point M called the *metacentre* lying on the centreline of the body plays the same role as O , the point of suspension of the suspended body. This is quite clear from an observation of Fig. 10.8. The height of the metacentre M reckoned to the centre of gravity C , denoted by CM is called the metacentric height. The metacentric height should be positive for stability, zero for neutrality and negative for instability of equilibrium.

The states of equilibrium of a homogeneous cylinder placed on top of differently curved surfaces are shown in Table 10.2. In each case the initial equilibrium is assured by the fact that the tangent at the point of contact is a horizontal line showing that the potential energy would remain unaltered for an infinitesimal displacement $d\theta$ of the cylinder.

The principle of potential energy may be extended and it be applied to the problems concerning the state of equilibrium of a body or a system with more than one degree of freedom. The potential energy should have a stationary value with respect to each of the independent coordinates chosen to specify the orientation of the body or the system. For example, a system with n degrees of freedom requires that

$$\frac{\partial(PE)}{\partial x_1} = 0 = \frac{\partial(PE)}{\partial x_2} = \dots = \frac{\partial(PE)}{\partial x_n} \quad (10.14)$$

for it to be in equilibrium. The conditions for stability, neutrality and instability are, however, not so simple as to be stated or used conveniently.

For a two-degree-of-freedom system, in particular, where the independent coordinates are chosen as x and y , the condition for equilibrium is

$$\frac{\partial(PE)}{\partial x} = 0 = \frac{\partial(PE)}{\partial y} \quad (10.15)$$

and the stability of equilibrium demands that simultaneously

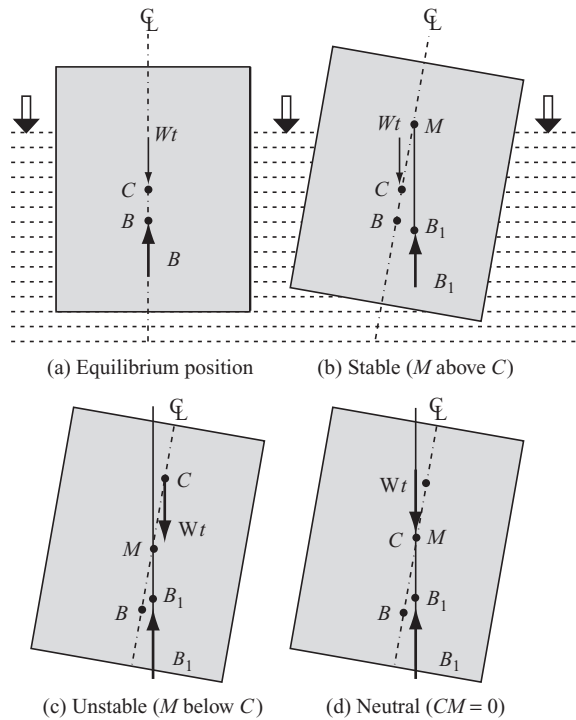


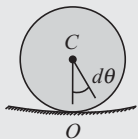
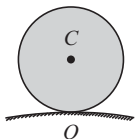
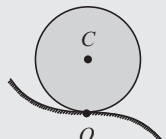
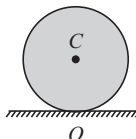
Fig. 10.8 Mechanics of equilibrium of a floating body

$$\left(\frac{\partial^2(PE)}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2(PE)}{\partial x^2}\right)\left(\frac{\partial^2(PE)}{\partial y^2}\right) < 0$$

and

$$\left(\frac{\partial^2(PE)}{\partial x^2} + \frac{\partial^2(PE)}{\partial y^2}\right) > 0 \tag{10.16}$$

Table 10.2 States of equilibrium of a homogeneous cylinder

| In each case, $\frac{d(PE)}{d\theta} = 0$ for equilibrium | | | | |
|---|---|---|---|---|
| |  |  |  |  |
| State | Stable | Unstable | Unstable | Neutral |
| PE | Minimum | Maxima | Inflexion | Constant |
| Function | $\frac{d^2(PE)}{d\theta^2} > 0$ | $\frac{d^2(PE)}{d\theta^2} < 0$ | $\frac{d^2(PE)}{d\theta^2} = 0$ | $\frac{d^2(PE)}{d\theta^2} = 0$ |

Example 10.4

The potential energy of a system is given in joules by the expression

$$PE = 10x^3 + 8x^2 - 9x$$

where x is in m.

Determine the equilibrium positions and the states of equilibrium.

Solution Given that $PE = 10x^3 + 8x^2 - 9x$

For equilibrium positions,

$$\frac{d(PE)}{dx} = 30x^2 + 16x - 9 = 0$$

whence

$$x_1 = -0.876 \text{ m}$$

$$x_2 = +0.3425 \text{ m}$$

Now, the second derivatives of the PE are

$$\frac{d^2(PE)}{dx^2} = 60x + 16$$

At

$$x_1 = -0.876 \text{ m}, \quad \frac{d^2(PE)}{dx^2} = -36.56; < 0$$

At

$$x_2 = 0.3425 \text{ m}, \quad \frac{d^2(PE)}{dx^2} = +36.55; > 0$$

The equilibrium position at $x = -0.876$ m corresponds to unstable equilibrium and that at $x = 0.3425$ m corresponds to stable equilibrium.

Example 10.5

A rectangular block and a semicylinder of the same length made of the same homogeneous material are secured together and placed on a flat rough surface in equilibrium as shown in Fig. Ex. 10.5. Determine the minimum radius R of the semicylinder in terms of the height H of the block if the width of the block is n times the radius of the semicylinder in order that the composite body be in stable equilibrium. Assume that the semicylinder may roll without slip on the flat surface.

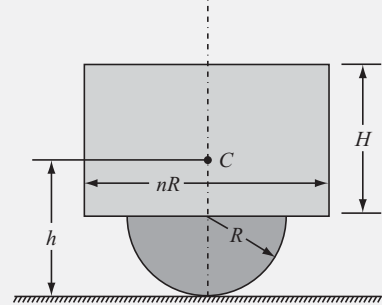


Fig. Ex. 10.5

Solution Let the centre of gravity of the composite body be h above the base in the equilibrium position. When it is tilted by a small angle θ as shown, the centre of gravity drops to a new height

$$R + (h - R) \cos \theta$$

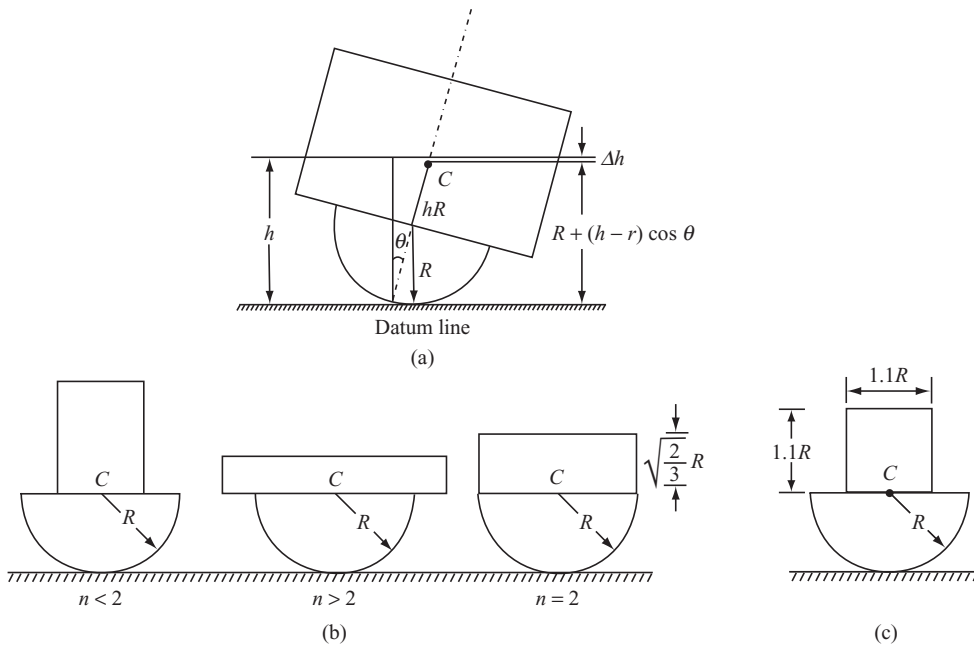


Fig. Ex. 10.5 (Solution)

Referring to the initial or equilibrium position, the potential energy of the body is

$$\begin{aligned} PE &= -mg \Delta h \\ &= mg (R - h)(1 - \cos \theta) \end{aligned}$$

where m is the mass of the composite body.

Now,

$$\frac{d(PE)}{d\theta} = mg(R - h) \sin \theta$$

This is zero when $\sin \theta = 0$ or $\theta = 0$ showing that the initial position is indeed the position of equilibrium,

$$\frac{d^2(PE)}{d\theta^2} = mg(R-h)\cos \theta$$

which, for $\theta = 0$, becomes

$$\frac{d^2(PE)}{d\theta^2} = mg(R-h)$$

This should be positive for stable equilibrium. The geometric condition for stability of the composite body is, therefore, that R should be greater than h . In other words, the centre of gravity should lie below the centre of the circle in the view shown in Fig. Ex. 10.5 (Solution). When $h = R$, the state is of neutral equilibrium: the body would stay where it is left. When h is greater than R , as initially assumed, the equilibrium is unstable.

In order that the centre of gravity be located at the centre of the circle for the limiting case,

$$\frac{H}{2} \times (nR \times H) = \frac{4R}{3\pi} \times \frac{\pi R^2}{2}$$

whence
$$H = \frac{2R}{\sqrt{3n}}$$

It is interesting to observe that the wider the block, more is n and less is H . The thinner the block, less is n and more is H .

In the limits,

$$n \rightarrow \infty, \quad H \rightarrow 0$$

and
$$n \rightarrow 0, \quad H \rightarrow \infty$$

In the special case when the width of the block equals the diameter of the semicylinder,

$$n = 2$$

and
$$H = \sqrt{\frac{2}{3}} R = 0.8165R$$

Three such cases of $n < 2$, $n > 2$ and $n = 2$ are shown in the neutral equilibrium condition. The bodies exhibit stable equilibrium of the height of the block is reduced, but are unstable if the height is slightly increased beyond the heights shown for neutral equilibrium.

Another interesting case is of securing a block of square cross-section on the semicylindrical block. The maximum side of the square cross-section for stable equilibrium would be

$$a = H = nR$$

which together with

$$H = \frac{2R}{\sqrt{3n}}$$

provides
$$n = (4/3)^{1/3} = 1.1$$

and
$$a = 1.1R$$

as shown in Fig. Ex. 10.5(c) (Solution).

10.5 GENERALISED COORDINATES

The position of a particle on a given line is described by one coordinate; on a plane by two coordinates and in space by three coordinates. The coordinates may refer to any set of axes; these may involve distances and angles in any combination. to describe the position of two particles in space, we use six coordinates. For a larger number of particles moving in a certain fashion, particularly when the particles refer to one or more of the rigid bodies, we

may have to use a larger number of coordinates. It is always possible to select the smallest number of independent variables to describe a system; The values
of these generalised variables, whatever they may be, completely define a system and can be varied to define other states in keeping with the constraints on the system. There is no unique choice of the generalised coordinates.

The number of such coordinates to describe the position is called the .
If a system can be defined by a minimum of 1, 2, ... generalised coordinates, it is said to have 1, 2, ..., degrees of freedom respectively. Such coordinates required to specify a system are denoted as q_1, q_2, \dots for degrees of freedom.

If time t is also independent coordinate in the description of a system, the number of degrees of freedom will remain the same as the number of coordinates excluding t .

A mechanical system consisting of n particles may have any number of degrees of freedom less than $3n$, depending upon the geometrical restrictions placed on their motions. Certain systems, such as a coil spring, vibrating string or beam which are assumed to consist of an infinite number of particles possess infinite degrees of freedom. In a variety of engineering problems, however, the connecting strings and masses of springs may be neglected and hence the system is taken to possess a finite number of degrees of freedom.

There is usually a wide range of the choice of coordinates as to which of the n coordinates are to be regarded as independent. For a simple pendulum the angular displacement of the string θ , or the x or y coordinate of the bob or any other one coordinate only is adequate to describe its configuration at any instant of time. The choice of such coordinates depends largely on insight and experience so as to be directly meaningful in the interpretation of the result in the light of the geometrical constraints imposed upon its motion.

If the number of independent coordinates employed is less than n , the problem is indeterminate; if a larger number of independent coordinates are employed, additional relationships interrelating the excess number of coordinates must be available, otherwise the entire analysis is null and void; such is the importance of the generalised coordinates for a system.

The transformation equations relating the rectangular coordinates of n particles to their n generalised coordinates are expressible as

$$\left. \begin{aligned} &= (q_1, q_2, \dots, q_n) \\ &= (x_1, x_2, \dots, x_n) \\ &= (y_1, y_2, \dots, y_n) \end{aligned} \right\} \begin{array}{l} \text{Continuous functions} \\ \text{with continuous derivatives} \end{array} \quad (10.17)$$

where n runs from 1 to n , the number of particles.

Concept Review Questions

State whether the following statements are true or false and justify your answer.

- (a) Virtual displacements are assumed to take place only instantaneously.
- (b) The virtual work concept applies only to systems in equilibrium.
- (c) Only the active or driving forces need be considered for the application of the virtual work principle.
- (d) The weight of a body is an active force for all modes of motion of the body.

(e) Frictionless constraints cannot provide any active force.

(f) The principle of virtual work applies to a system of bodies as well as to a single body.

If a body is actually moving under the application of a system of forces and there is actual work done by the forces, then will it be necessary, preferable or unnecessary to consider virtual work the same as actual work?

Under what circumstances will you use the principle of virtual work in preference to the other formulations for equilibrium?

Is the virtual work method restricted to single degree-of-freedom bodies and systems or is it valid for any system with any number of degrees of freedom? Support your comments by an example.

State the correctness of the following statements along with short comments.

- (a) The potential energy principle applies to a body in equilibrium, whether it is at rest or in uniform motion.
- (b) The potential energy principle applies as well to a system of interconnected bodies as to a single body.
- (c) A body may be in stable equilibrium although the potential energy is not a minima.
- (d) The point of inflexion on the curve showing the variation of potential energy with a space coordinate is potentially a point of unstable equilibrium.

- (e) The potential energy method is the only method which can decide the stability, neutrality and instability of equilibrium unambiguously.
- (f) The potential energy method may be extended to systems with any number of degree of freedom.

Relate the potential energy conditions for the states of equilibrium of a suspended body to the states of equilibrium of a floating body.

Imagine a right circular cone placed on a flat plate. Sketch the modes of placement for

- (a) stable equilibrium
- (b) unstable equilibrium
- (c) neutral equilibrium

and explain the states of equilibrium on the basis of the potential energy principle by sketching the variation of potential energy in each case.

- 8 Why are generalised coordinates so called? Must the number of generalised coordinates be equal to, less than or more than the degrees of freedom of a system?

Tutorial Problems

Using the method of virtual work, determine the value of F to hold the frame in equilibrium under the action of a force P as shown in Fig. Prob. 10.1. Each link is of length $2a$ and $\theta = 45^\circ$.

(Ans. $F = P/2 \tan \theta; P/2$)

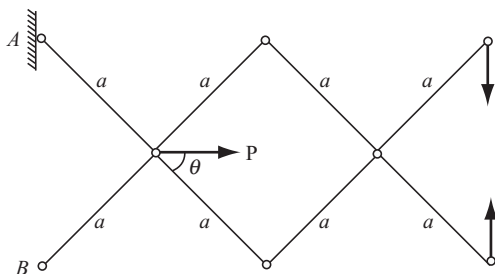


Fig. Prob. 10.1

Using the method of virtual work, determine the force in the top member of the simple truss consisting of equilateral triangles.

(Ans. $0.577 P$)

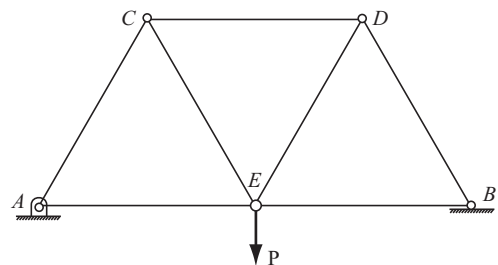


Fig. Prob. 10.2

A simple parallelogram linkage carries three forces as shown in Fig. Prob. 10.3 while the members AB and CD capable of oscillating

about A and D respectively are at 60° with the base line AD . Determine the moment M required to maintain equilibrium of the linkage.

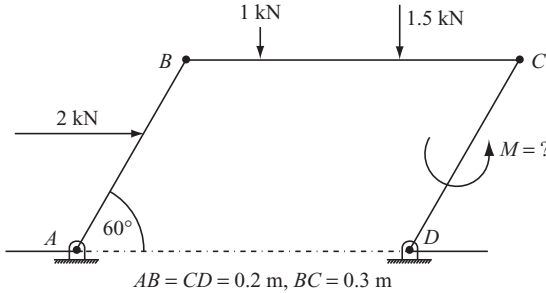


Fig. Prob. 10.3

A simple linkage consisting of two equal massless bars and a sliding block B is subjected to a set of two forces F_1 and F_2 as shown in Fig. Prob. 10.4. Express the angle θ as a function of F_1 and F_2 for equilibrium.

(s $\tan \theta = F_1/2F_2$)

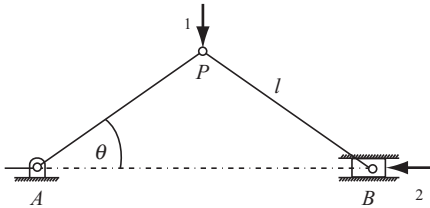


Fig. Prob. 10.4

Two inextensible strings of equal lengths and two equal concentrated masses comprise a double pendulum as shown in Fig. Prob. 10.5. A force F is applied horizontally at the lower end to keep it in equilibrium. Express the angles θ_1 and θ_2 with the vertical in terms of F and m .

(s $\tan \theta_1 = F/2 mg, \tan \theta_2 = F/mg$)

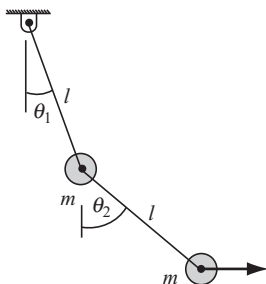


Fig. Prob. 10.5

A uniform rectangular plank of height h , base width b and mass m rests on top of a convex circular surface of radius R . Establish the relationship between the height of the plank and the radius of the convex surface for the stable and unstable equilibrium of the plank in rolling without slipping over the convex surface.

(s Equilibrium: OPQ vertical; Stable for $R > h/2$; Unstable for $R < h/2$)

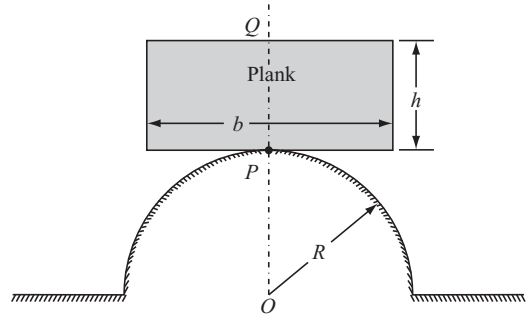


Fig. Prob. 10.6

Two weights W_1 and W_2 are placed at the ends of a massless bar of length l pivoted at a distance l_1 from the weight W_1 as shown in Fig. Prob. 10.7. If the bar is free to oscillate in the vertical plane discuss the criteria for the states of equilibrium.

(s Condition of equilibrium: Either $\theta = 0$ and any W_1 and W_2 at any l_1 and l_2 respectively and stable if $W_2 l_2 > W_1 l_1$ or $W_1 l_1 = W_2 l_2$ at any θ and neutral all the time)

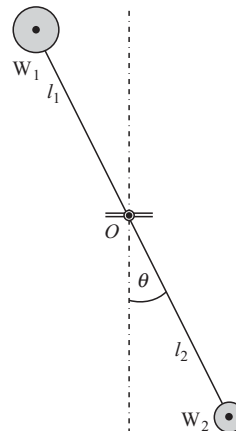


Fig. Prob. 10.7

- 8 A homogeneous body is composed of a hemisphere of radius R and a circular cylinder of radius R and height H as shown in Fig. Prob. 10.8. Discuss the relationship between R and H for equilibrium and hence find the maximum value of H consistent with the stability of the body. Assume the hemisphere to roll without slipping on the horizontal surface.

(s $H = R/\sqrt{2}$)

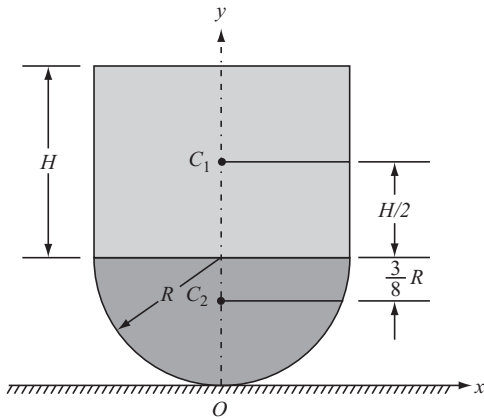


Fig. Prob. 10.8

A heavy uniform cube is balanced on the highest point of a fixed sphere of radius R . If the sphere is rough enough to prevent sliding and if the side of the square is $\pi R/2$, show that the cube can rock through a right angle without falling.

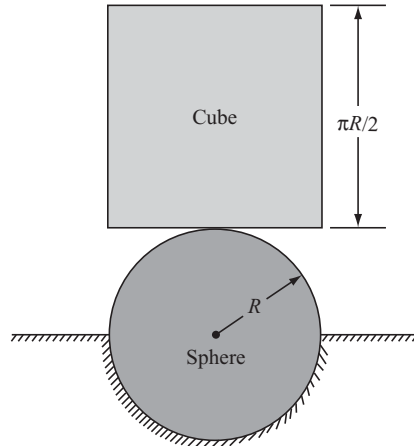


Fig. Prob. 10.9

A composite body made up of a cone and hemisphere of a homogeneous material rests on a rough flat surface. Show that the greatest height of the cone such that the equilibrium may be stable is $\sqrt{3}$ times the radius of hemisphere.

A solid circular cylinder of radius r and height h has a specific gravity of 0.6. Find the minimum ratio r/h for which the cylinder will float in water with its axis vertical and just be stable.

(s 0.693)

Look up Hints to Tutorial Problems at the end!

Multiple-Choice Questions

Select the correct or the most appropriate response from among the available alternatives in the following multiple-choice questions:

- Virtual work refers to work
- (a) by virtue of actual forces
 - (b) by virtue of actual displacements
 - (c) in overcoming the constraints
 - (d) associated with a possible displacement

The principle of virtual work states that the virtual work should be zero for a body

- (a) to be in equilibrium, in general
- (b) in equilibrium if it does not rotate
- (c) moving with a constant acceleration
- (d) rotating at a constant angular velocity

The existence of potential energy implies that

- (a) a general force field exists
- (b) a conservative force field exists

- (c) there must either be a spring or gravitational field
 (d) the body should be in equilibrium

Stability of equilibrium of a body requires that

(a) $\frac{d PE}{ds} = 0$

(b) $\frac{d PE}{ds} = 0$ and $\frac{d^2 PE}{ds^2} < 0$

(c) $\frac{d PE}{ds} = 0$ and $\frac{d^2 PE}{ds^2} > 0$

(d) $\frac{d^2 PE}{ds^2} = 0$

The time taken by a small frictionless bead to slide on a thin wire in the gravitational field is the minimum if the shape of the wire is

- (a) a straight line (c) a parabola
 (b) a cycloid (d) an involute

Answers to Multiple-Choice Questions

- 1 (d) 2 (a) 3 (b) 4 (c) 5 (b)

Vibrations of Simple Mechanical Systems

11



11.1 ELEMENTS OF MECHANICAL SYSTEMS

A system, by definition, is an assemblage of interacting elements constituted for a desired objective. The elements of a system may be simple mechanical, electrical, optical, thermo-mechanical, electro-dynamical or some other devices which are characterised by their behaviour. A system is termed *dynamical system* if the system response is a function of time. It is interesting, under the systems approach, to deal with different classes of systems in a unified fashion. We shall, however, confine ourselves to the mechanical systems. The study can, however, be extended to other systems.

A mechanical system comprises of mechanical elements which have the following properties or characteristics:

1. Inertia
2. Compliance
3. Damping.

These characteristics may either be translational or rotational. It may be noted that a suitable combination of the translational and the rotational characteristics can provide the desired characteristics in any motion, by simple superpositions, if the basic characteristics are linear. We shall restrict our discussion to the linear elements. Let us now focus our attention on each of the above characteristics.

1. Inertia (tendency to resist motion, e.g., by mass)

| Basic Concept | Translational Mass m | Rotational Moment of Inertia I |
|-----------------------|-------------------------|---|
| Acceleration | $= \frac{d^2x}{dt^2}$ | $\alpha = \frac{d^2\theta}{dt^2}$ |
| Action | Force | Moment |
| Constitutive Equation | $= m \frac{d^2x}{dt^2}$ | $= I\alpha = I \frac{d^2\theta}{dt^2}$ (11.1) |

Representation of inertial elements: Refer Fig. 11.1. Examples: Refer Fig. 11.2.

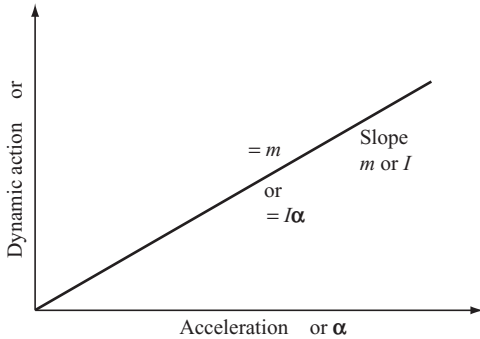


Fig. 11.1 Characteristic of an inertial element

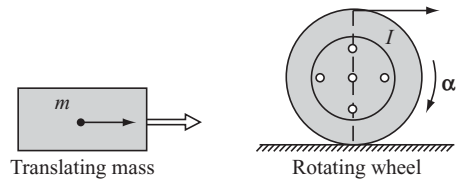


Fig. 11.2 Inertial elements

Comments Inertial elements, i.e., mass in translation and in rotation are pure translational and pure rotational elements respectively. By pure, it is implied that they have no other properties, such as deformability, etc. The motion of an inertial element is governed by the laws of Mechanics.

2. Compliance (with conservative elements, e.g., springs)

| | Translational | Rotational |
|---------------------------|--|---|
| Concept: Elastic Elements | Translational Spring (spring constant) k | Rotational Spring (spring constant) K |
| Displacement | x | θ |
| Action | Force | Moment M |
| Constitutive Equation | $= kx$ | $M = K\theta$ (11.2) |

Representation of elements producing compliance: Refer Fig. 11.3. Examples: Refer Fig. 11.4.

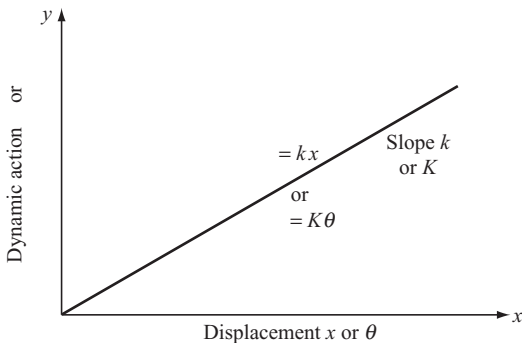


Fig. 11.3 Characteristic of a compliance element

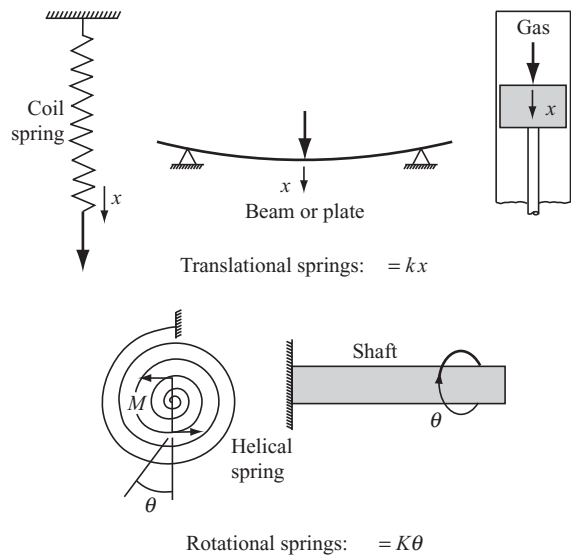


Fig. 11.4 Elements producing compliance

Comments A compliance element complies with a steady force to produce a corresponding steady deformation. Compliance elements such as springs are said to be pure if they are assumed to be devoid of mass and are capable of conserving energy under all modes of operation. A compliance element must return to its equilibrium position when the applied force is removed.

3. Damping (with dissipative elements, e.g., dashpots)

| | <i>Translational</i> | <i>Rotational</i> |
|--------------------------------------|---|---|
| <i>Concept: Dissipative Elements</i> | Translational Damper (damping constant) c | Rotational Damper (damping constant) C |
| <i>Velocity</i> | $V = \frac{dx}{dt}$ | $\omega = \frac{d\theta}{dt}$ |
| <i>Action</i> | Force | Moment |
| <i>Constitutive Equation</i> | $F = cV = c \frac{dx}{dt}$ | $M = C\omega = C \frac{d\theta}{dt}$ (11.3) |

Representation of damping elements: Refer Fig. 11.5. Examples: Refer Fig. 11.6.

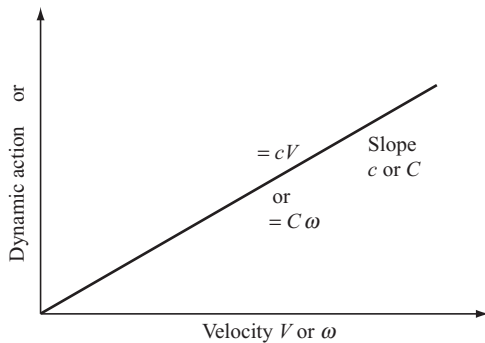


Fig. 11.5 Characteristic of a damping element

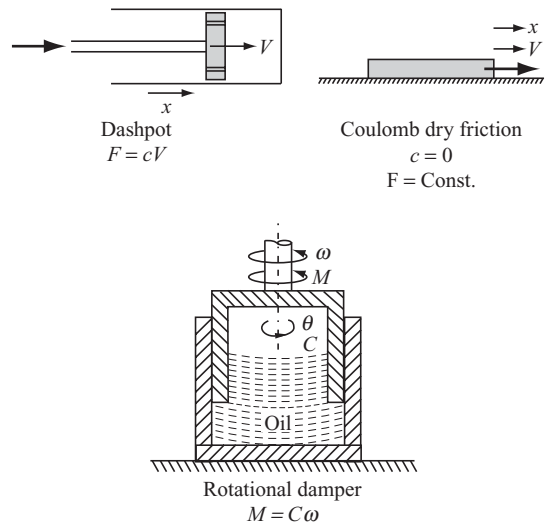


Fig. 11.6 Elements producing damping

Comments A damping element is associated with dissipative action. Dissipation of energy may be achieved through dry friction, viscous shearing or otherwise. Mass and elasticity of the components of the dampers are assumed to be zero for pure dampers. A damping element stays where it is, when the applied force is removed.

11.2 SIMPLE DYNAMICAL SYSTEMS

Simple mechanical systems are those which can be conceived as simple combinations of the system elements.

A spring-mass combination with the mass subjected to a force along the line of the spring and passing through the centre of mass of the spring is called a linear spring mass system. If the mass is also subjected to a damping force, it is called a damped spring-mass system. The system may be arranged to operate in translation or in rotation and the action of forces and moments may be adjusted to comprise a variety of systems, examples of which are shown in Figs. 11.7 and 11.8.

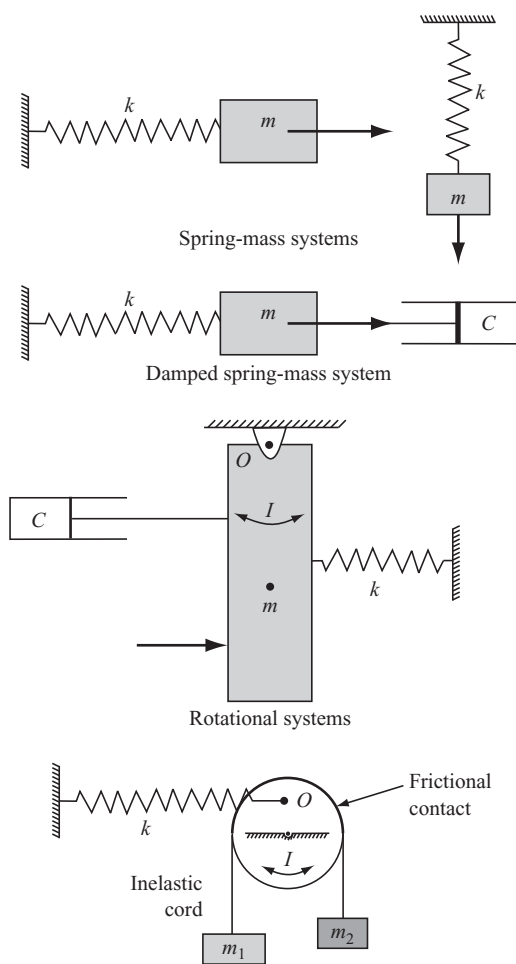


Fig. 11.7 Single-degree-of-freedom systems

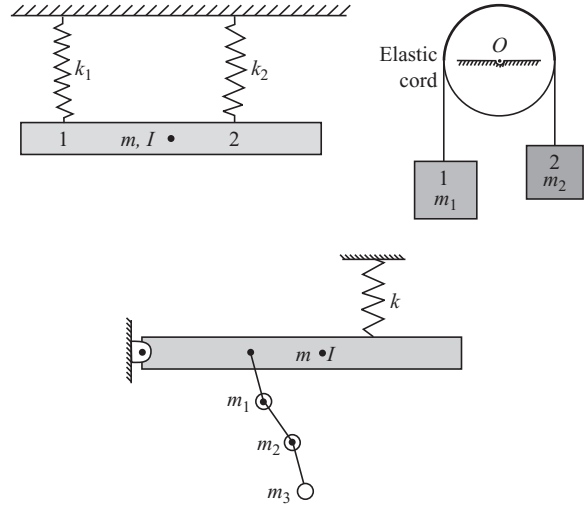


Fig. 11.8 Multi-degree-of-freedom systems

We next define the number of degrees of freedom of a mechanical system. The number of independent coordinates required to specify the position of a body has been defined as the degrees of freedom of the body. For constrained motion of a body, the number of constraints must be subtracted from six to obtain the degrees of freedom. It is, therefore, the possible modes of motion of a body which amount to the degrees of freedom. Now, for a mechanical system of n interconnected bodies there can be a maximum number of $6n$ degrees of freedom. The actual degrees of freedom which a system enjoys is the actual possible modes of motion. In other words, if there are c constraints on the system, then

$$\text{Degrees of freedom} = 6n - c \quad (11.4)$$

Mechanical systems may, therefore, possess one, two or more degrees of freedom up to a maximum of $6n$. It is not the number of masses or the springs or the complexity of appearance which increases the number of degrees it is the modes of motion to which a system can be subjected. For example, all the five simple systems

shown in Fig. 11.7 are one-degree-of-freedom systems. The mass in the first three spring-mass systems can only translate along the length of the spring. The bar in the rotational system can only oscillate about O in the plane of the figure. The disc carrying the masses can only oscillate about O , the cord being locked on its periphery due to friction; the up-and-down motion of the masses m_1 and m_2 are uniquely related to the angular motion of the disc. One-degree-of-freedom systems are also called simple systems. The systems shown in Fig. 11.8 possess more than one-degree-of-freedom each. The first two are two-degrees-of-freedom systems whereas the last one has four degrees of freedom.

Systems are sometimes classified as zero order, first order, second order, and so on. The order of a system is determined by the order of the differential equation which governs its dynamical behaviour.

Zero-order Systems These are those in which the input and output are related in direct proportion without the need for a differential equation. For example, a steadily operating spring balance shows the displacement of the pointer in direct proportion of the weight force applied

$$F = kx \quad \text{or} \quad x = \frac{F}{k}$$

Another example of a zero-order system is a potentiometer where the output voltage across the terminals varies directly with the displacement of the jockey.

First-order Systems These are systems which are described by first-order differential equations in respect of their dynamical behaviour. For example, a spring-dashpot system is governed by a first order equation

$$m \frac{dx}{dt} + kx = F$$

and hence it constitutes a first-order system. Another example of a first-order system is a liquid-in-glass thermometer where the displacement of the mercury level is related to the temperature surrounding the bulb by a first-order differential equation.

Second-order Systems These are likewise characterised by second-order differential equations governing their dynamical behaviour. For example, spring-mass systems, undamped and damped, constitute second-order system governed by the equations

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0$$

for undamped free and damped free vibrations respectively.

11.3 FEATURES OF VIBRATING SYSTEMS

A motion which repeats itself in equal intervals of time is called *periodic motion*. The representation of a system undergoing periodic motion is made in terms of its displacement-time plot as shown in Fig. 11.9.

One cycle of a periodic motion consists of the motion of the system from a given state or configuration to the same state or configuration after passing through the intermediate states only once. *The time period or period* τ of a periodic motion is the time interval required to execute one cycle. τ is usually expressed in seconds.

Cyclic frequency, or simply, frequency f is the number of cycles executed per unit time and is, therefore, inverse of the period τ , i.e.,

$$f = \frac{1}{\tau} \quad (11.5)$$

The cyclic frequency is usually expressed in cycles per second denoted as Hertz (Hz).

The *angular or circular frequency* ω is 2π times the cyclic frequency; usually expressed in rad/s units.

$$\omega = 2\pi f = \frac{2\pi}{\tau} \quad (11.6)$$

Amplitude of a periodic motion is the maximum displacement of the system from its equilibrium position.

The periodic motion or vibration of a system is caused by the application of disturbing forces which create displacements in the elastic elements of the system. By virtue of displacements, the elastic elements develop restoring forces which tend to bring the system to its initial state. The restoring motion may occur in such a way as to overshoot the initial state, i.e., to pass the initial state with some velocity. The overshooting action will be accompanied by displacements in the elastic elements which, by developing restoring forces, tend to restore the system to its initial state. The initial state may again be overshoot, and so on. In the absence of any resisting force, the vibrations thus set will continue to go on but in the presence of resisting forces, vibrations may be short-lived.

If a disturbing force is applied just to start the vibration and is then removed from the system leaving it to vibrate by itself, the system is said to undergo *free vibrations*. On the other hand, if the disturbing force continues to act at periodic intervals on the system, the system is said to undergo *forced vibrations*.

It is essential that a system possesses one or more elastic elements before it can vibrate. Resisting forces may act or may be developed by virtue of vibrations to oppose the elastic forces. The vibrations are then said to be damped. The presence of resisting forces in free vibrations makes them *damped-free vibrations* and in forced vibrations makes them *damped-forced vibrations*.

Free and forced vibrations, damped or undamped, constitute periodic motion. There are, however, some more terms generally used for vibrating systems.

The frequency of a system undergoing free vibrations damped or undamped, is called *natural frequency* and is denoted by ω_n . Free vibrations are also called *transient vibrations* because of their disappearance with time. On the other hand, *steady-state vibrations* are those which continue to repeat themselves with time. Forced vibrations are examples of steady-state vibrations.

A concept very frequently referred to in the study of periodic motion is *simple harmonic motion*. The simple harmonic motion of a particle is such that its acceleration is directed towards the equilibrium position and is proportional to the displacement from that position. It will be observed in the course of analysis that the periodic motion in the absence of external driving forces and resisting forces corresponds to simple harmonic motion. In other words, free undamped vibrations constitute simple harmonic motion.

A case of special interest is a *simple harmonic motion*. A particle P is said to execute a simple harmonic motion about a point O if its rectilinear displacement from O on either side of it is represented by the projection of a point p moving on the periphery of a circle with O as centre as shown in Fig. 11.10.

If the point p rotates about the centre O at a constant angular velocity ω and the time is referred from $t = 0$ at an extremity E_1 on the x -axis, the angular location of the point after time t is given by

$$\theta = \omega t$$

and the radial distance from O remains x_0 , the radius of the circle.

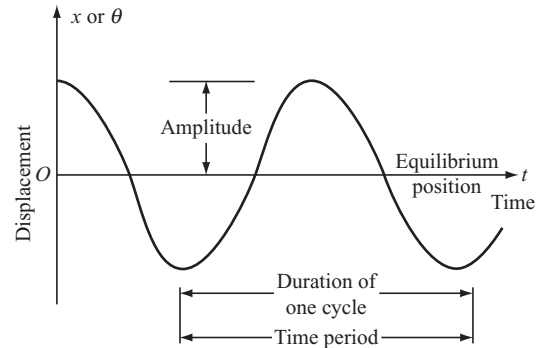


Fig. 11.9 Periodic motion—nomenclature

The projection of Op on the line of motion of the particle is the displacement of the particle at any instant

$$x = x_0 \cos \theta = x_0 \cos \omega t$$

Differentiating it with respect to time t ,

$$\begin{aligned} \frac{dx}{dt} = V &= -x_0 \omega \sin \omega t \\ &= -\omega \sqrt{x_0^2 - x_0^2 \cos^2 \omega t} = -\omega \sqrt{x_0^2 - x^2} \end{aligned}$$

and differentiating with respect to t again,

$$\frac{d^2x}{dt^2} = \frac{dV}{dt} = a = -x_0 \omega^2 \cos \omega t = -x \omega^2$$

Some conclusions from these relations for the simple harmonic motion are listed as follows:

1. A simple harmonic motion of a particle is a rectilinear motion; the displacement, velocity and acceleration all being along the same line.
2. The displacement from a central point O is represented by the projection of a point moving on the periphery of a circle with centre O .
3. The velocity of the particle varies sinusoidally, it being (i) a maximum of $x_0 \omega$ in magnitude at $\theta = \pi/2$ and $3\pi/2$ or $x = 0$, i.e., at the centre point O and (ii) a minimum of 0 in magnitude at $\theta = 0$ and π or $x = +x_0$ and $-x_0$, i.e., at the extremities.
4. The acceleration of the particle varies directly with its displacement x from the centre point O , it being (i) directed towards the centre point O , (ii) a maximum of $x_0 \omega^2$ in magnitude at $x = +x_0$, and $-x_0$ i.e., at the extremities, and (iii) a minimum 0 in magnitude at $x = 0$, i.e., at the centre point O .

In other words, the particle passes through the centre point O with a maximum velocity $x_0 \omega$ and zero acceleration. While passing through the extremities, the particle decelerates to zero velocity and accelerates from zero velocity with the maximum acceleration $x_0 \omega^2$.

5. The amplitude of motion of the particle is the maximum displacement, i.e., x_0 , from the centre point to either of the extremities.
6. The period of oscillation is the time the particle takes to complete one cycle. This is the time the point p takes to complete one revolution, i.e., angle 2π at an angular velocity of ω ,

$$\tau = \frac{2\pi}{\omega}$$

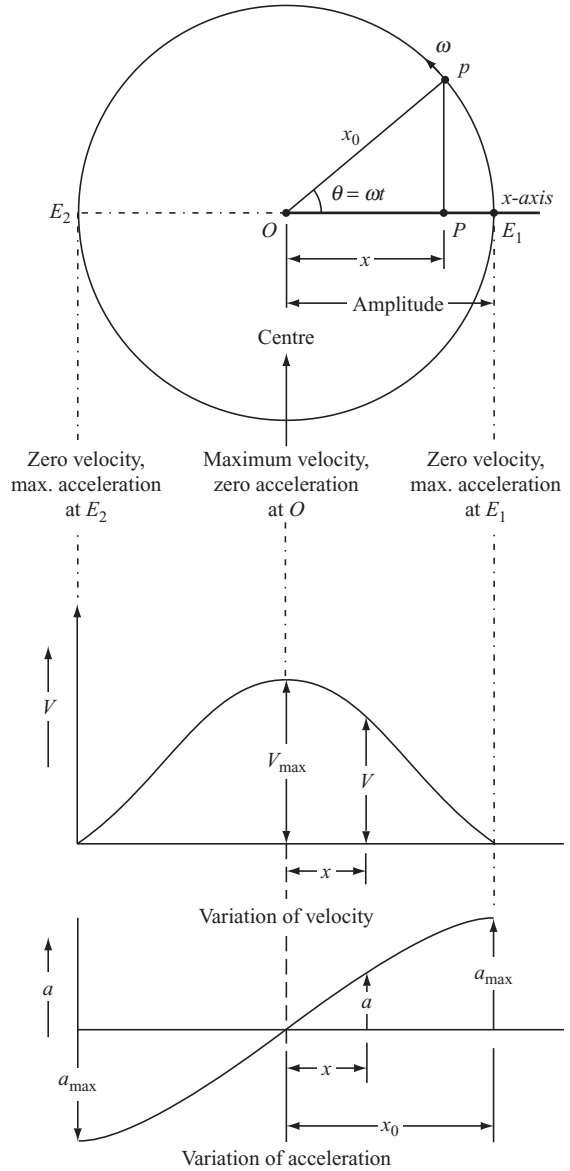


Fig. 11.10 Features of simple harmonic motion

7. The frequency of oscillation is the number of cycles per unit time described by the particle,

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

Example 11.1

A point moves with a simple harmonic motion such that it has a velocity of 9 m/s when it is at a distance of 2 m from the centre and a velocity of 4 m/s when it is at a distance of 3 m from the centre of the same side for the point moving in the same direction. Calculate

- (a) the amplitude of the motion
- (b) the time period of the motion
- (c) the time interval between the two positions
- (d) the acceleration of the point at these positions, and
- (e) the greatest acceleration.

Solution For the simple harmonic motion of a point,

$$x = x_0 \cos \omega t \quad \text{(i)}$$

$$V = -\omega \sqrt{x_0^2 - x^2} \quad \text{(ii)}$$

and $a = -x\omega^2 \quad \text{(iii)}$

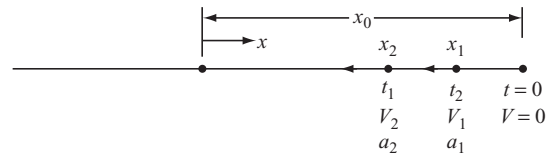


Fig. Ex.11.1 (Solution)

(a) For the data, from Eq. (ii),

$$9 = -\omega \sqrt{x_0^2 - 4} \quad \text{and} \quad 4 = -\omega \sqrt{x_0^2 - 9}$$

whence,

$$x_0 = 3.2 \text{ m} \quad \text{and} \quad \omega = 3.6 \text{ rad/s}$$

The amplitude of motion of the point is 3.2 m.

(b) The time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3.6} = 1.75 \text{ s}$$

(c) Assuming that $t = 0$ at $x = x_0$, times t_1 and t_2 for positions x_1 and x_2 are given by

$$3 = 3.2 \cos 3.6 t_1; \quad t_1 = 0.099 \text{ s}$$

$$2 = 3.2 \cos 3.6 t_2; \quad t_2 = 0.249 \text{ s}$$

and the time interval is

$$t_2 - t_1 = 0.249 - 0.099 = 0.15 \text{ s}$$

(d) From Eq. (iii), the accelerations are

$$a_1 = -3 \times 3.6^2 = -38.88 \text{ m/s}^2$$

$$a_2 = -2 \times 3.6^2 = -25.92 \text{ m/s}^2$$

(e) The maximum acceleration must be

$$\begin{aligned} a_{\max} &= -x_0 \omega^2 \\ &= -3.2 \times 3.6^2 = -41.47 \text{ m/s}^2 \end{aligned}$$

at the right extremity and the same in magnitude but opposite in direction at the left extremity.

11.4 LINEAR FREE VIBRATIONS

A system of a single mass constrained to move along the axis of a horizontal spring attached to it, the other end of which is fixed, undergoes free vibrations. The equilibrium position of the mass shown in Fig 11.11(a) corresponds to the unstretched length of the spring, which is also the reference position for the periodic motion.

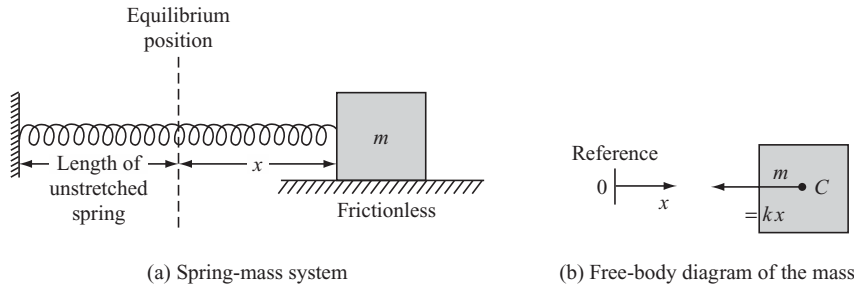


Fig. 11.11 Simple spring-mass system

At an instant when the mass is displaced by x , the restoring force acting on it is

$$F = -kx$$

The equation of motion in accordance with the Newton's law is

$$F = ma$$

or

$$-kx = m \frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

The most general solution of this second-order linear differential equation is

$$x = A \sin \omega_n t + B \cos \omega_n t$$

whence, by differentiating with respect to time t ,

$$\frac{dx}{dt} = \omega_n A \cos \omega_n t - \omega_n B \sin \omega_n t$$

$$\frac{d^2x}{dt^2} = -\omega_n^2 A \sin \omega_n t - \omega_n^2 B \cos \omega_n t = -\omega_n^2 x$$

Substituting these values in the equation of motion,

$$-\omega_n^2 x + \frac{k}{m} x = 0$$

whence,

Angular frequency,
$$\omega_n = \frac{\sqrt{k}}{m},$$

Cyclic frequency,
$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{and}$$

Time period,
$$\tau = \frac{1}{f} = 2\pi \sqrt{\frac{m}{k}}$$

It can be observed that the mass is set into vibrations by applying an initial force of kx_0 , say towards right, to bring the mass at rest at a distance x_0 from the equilibrium position. If the time there is reckoned at $t = 0$, then from the general solution

$$x = A \sin \omega_n t + B \cos \omega_n t$$

the initial condition

$$t = 0 \quad \text{at} \quad x = x_0$$

gives

$$A = 0 \quad \text{and} \quad B = x_0$$

rendering the solution as

$$x = x_0 \cos \omega_n t$$

The maximum displacement x_{\max} corresponds to $\cos \omega_n t = 1$, i.e., at the extremities of the periodic motion

$$x_{\max} = x_0$$

The maximum velocity v_{\max} corresponds to $\sin \omega_n t = 1$, i.e., at the instant when the system passes through the mean position.

$$v_{\max} = x_0 \omega_n$$

It follows that

$$\omega_n = \frac{|v_{\max}|}{|x_{\max}|}$$

The velocity of the mass at any instant is

$$v = \frac{dx}{dt} = -x_0 \omega_n \sin \omega_n t = -x_0 \sqrt{\frac{k}{m}} \sin \omega_n t$$

In particular, the velocity of the mass at the instant when it passes through the equilibrium position is obtained by substituting

$$t_E = \tau/4, \quad 3\tau/4, \quad 5\tau/4, \dots$$

Since

$$\tau = \frac{2\pi}{\omega_n},$$

$$t_E = \frac{\pi}{2\omega_n}, \frac{3\pi}{2\omega_n}, \frac{5\pi}{2\omega_n}, \dots$$

$$v_{E1} = -x_0 \sqrt{\frac{k}{m}} \sin \left(\omega_n \frac{\pi}{2\omega_n} \right) = -x_0 \sqrt{\frac{k}{m}}$$

$$v_{E2} = -x_0 \sqrt{\frac{k}{m}} \sin \left(\omega_n \cdot \frac{3\pi}{2\omega_n} \right) = x_0 \sqrt{\frac{k}{m}}$$

$$v_{E3} = -x_0 \sqrt{\frac{k}{m}},$$

and so on as the mass passes through the same position successively.

The value of the maximum velocity can as well be determined by using the principle of conservation of mechanical energy, i.e.,

$$PE + KE = \text{Const.}$$

The potential energy PE stored by the spring at $t = 0$ is

$$PE_1 = \frac{1}{2} kx_0^2$$

when the kinetic energy $KE_1 = 0$ for the mass, it being at rest. The kinetic energy of the mass when it passes through the equilibrium position is

$$KE_2 = \frac{1}{2} m v_E^2$$

when the potential energy $PE_2 = 0$ for the spring, it being the zero-stretch position.

By the conservation principle,

$$PE_1 + KE_1 = PE_2 + KE_2$$

$$\frac{1}{2} kx_0^2 = \frac{1}{2} m v_E^2$$

whence

$$v_E = x_0 \sqrt{\frac{k}{m}}$$

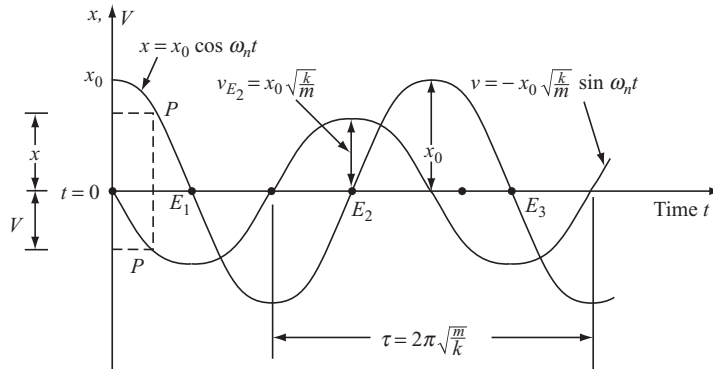
From the knowledge of the maximum velocity and maximum displacement of the system, the frequency and time period can also be estimated

$$\omega_n = \frac{|v_{\max}|}{|x_{\max}|} = \frac{x_0 \sqrt{k/m}}{x_0} = \sqrt{\frac{k}{m}}$$

$$\tau = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{k}}$$

$$f = \frac{1}{\tau} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

It is interesting to observe the periodic motion graphically. This is shown in Fig. 11.11(c). The mass is released at $t = 0$ from a position $x = x_0$ from a state of rest. The mass passes through the equilibrium positions E_1, E_2, E_3 , etc., when $x = 0$ and its velocity is maximum at these positions.



(c) Displacement and velocity vs time

Fig. 11.11 Simple spring-mass system

The case of free vibrations of a mass attached to a vertical spring fixed at the upper end is similar to the horizontal spring-mass system just considered. A reference to Fig. 11.12 shows that the equilibrium position of the spring-mass system in this case does not correspond to the unstretched state of the spring but to the spring stretched by the static deflection

$$\delta = \frac{W}{k}$$

At the equilibrium position, the weight W of the mass acting downward balances the tension T developed in the spring

$$W = mg = T = k\delta$$

For a displacement x from the equilibrium position, the restoring force acting on the mass equals

$$\begin{aligned} F &= -k(\delta + x) + W \\ &= -k(\delta + x) + k\delta \\ &= -kx \end{aligned}$$

which is precisely the same as that seen for a horizontal spring-mass system. The procedure to obtain and solve the equation of motion is therefore identical to that case. The reference position, i.e., the mean position of the oscillations is the equilibrium position corresponding to the static deflection δ .

It may also be seen that in this case

$$\delta = \frac{W}{k} = \frac{mg}{k}$$

or
$$\frac{k}{m} = \frac{g}{\delta}$$

The results are reproduced for ready reference:

| | | |
|--------------------|---|--------|
| Angular frequency, | $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta}}$ | (11.8) |
|--------------------|---|--------|

| | | |
|-------------------|--|--------|
| Cyclic frequency, | $f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}}$ | (11.9) |
|-------------------|--|--------|

| | | |
|--------------|---|---------|
| Time period, | $\tau = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{\delta}{g}}$ | (11.10) |
|--------------|---|---------|

| | | |
|-----------------------|---|---------|
| Velocity of the mass, | $v = -x_0 \sqrt{\frac{k}{m}} \sin \omega_n t = -x_0 \omega_n \sin \omega_n t$ | (11.11) |
|-----------------------|---|---------|

The maximum velocity is
$$v_E = x_0 \sqrt{\frac{k}{m}} = x_0 \omega_n$$

at the equilibrium position, directed alternatively upwards and downwards.

| | | |
|--------------------------|---------------------|---------|
| Acceleration of the mass | $a = -\omega_n^2 x$ | (11.12) |
|--------------------------|---------------------|---------|

| | |
|----------------------|------------------------------|
| Maximum acceleration | $a_{\max} = -\omega_n^2 x_0$ |
|----------------------|------------------------------|

at the extremity and directed towards the equilibrium position.

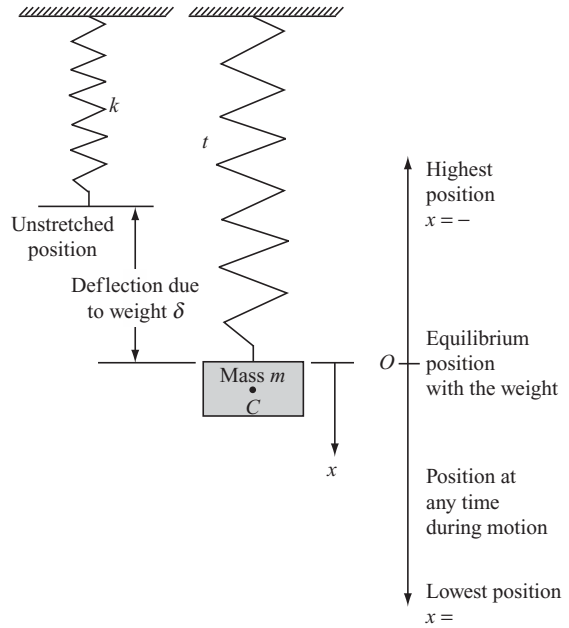


Fig. 11.12 Simple vertical spring-mass system

Example 11.2

A small motor of mass 20 kg is symmetrically mounted on four equal springs, each with a spring constant of 25 N/cm. Estimate the frequency and period of vibration of the motor.

Solution The four springs arranged ‘in parallel’ may be considered equivalent to a single spring located in line with the centre of mass of the motor with

$$k = 4 \times 25 = 100 \text{ N/cm} = 10,000 \text{ N/m}$$

Cyclic frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{10,000}{20}}$$

$$= 3.56 \text{ Hz}$$

Time period

$$\tau = \frac{1}{f} = 0.28 \text{ s}$$

Example 11.3

A vertical U-tube manometer contains a liquid of mass density ρ as shown in Fig. Ex. 11.3. A sudden increase of pressure on one column forces the level of the liquid down. When the pressure is released, the liquid columns start vibrating. Neglecting the frictional damping, determine the period of vibration. Comment if the period is affected by changing the liquid, diameter of the tube or length l of the liquid column.

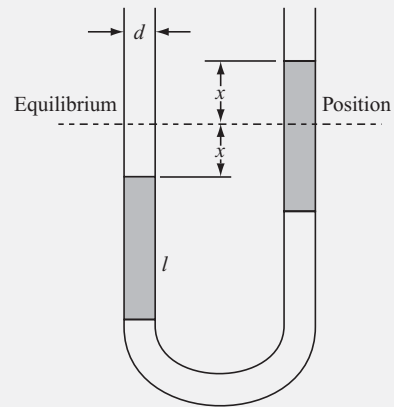


Fig. Ex. 11.3

Solution The equilibrium position of the liquid columns is when these are in level. At any instant during vibration, one column rises by a distance x while the other falls by the same distance. The restoring force, therefore, equals the weight of a liquid column $2x$ high.

$$F = -2x\pi(d^2/4)\rho g$$

This must equal the product of the mass of the total length of the liquid column in motion and the acceleration,

$$-2x \cdot \pi(d^2/4)\rho g = l\pi(d^2/4)\rho \frac{d^2x}{dt^2}$$

which, on simplification, yields the equation of motion

$$\frac{d^2x}{dt^2} + \frac{2g}{l}x = 0$$

The general solution of this equation is

$$x = A \cos \omega_n t + B \sin \omega_n t$$

whence

$$\frac{d^2 x}{dt^2} = -\omega_n^2 (A \cos \omega_n t + B \sin \omega_n t) = -\omega_n^2 x$$

and substituting it in the differential equation provides

$$\omega_n = \sqrt{\frac{2g}{l}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{2g}{l}}$$

$$\tau = 2\pi \sqrt{\frac{l}{2g}}$$

The period is unaffected by the density of the liquid and the diameter of the tube. The total length of the liquid column, however, does effect the period of vibration.

For a typical vertical manometer with a total column 0.5 m long,

$$\tau = 2\pi \times \sqrt{\frac{0.5}{2 \times 9.81}} = 1 \text{ s}$$

which implies that the manometer is capable of oscillating visibly at 1 cycle per second in the absence of damping effects which may arise from fluid friction between the liquid and the tube material.

Another interesting observation is that if the plane of the manometer is set at an angle θ to the horizontal, as is the case for low pressure-difference measurement, the restoring force becomes $\sin \theta$ times the above value and the period of vibration becomes

$$\tau = 2\pi \sqrt{\frac{l}{2g \sin \theta}}$$

which shows that the time period increases or the frequency decreases; again neglecting any damping forces.

EXPERIMENT E13 OSCILLATION OF A SIMPLE PENDULUM

OBJECTIVE TO DETERMINE THE TIME PERIOD OF A SIMPLE PENDULUM AND TO ESTIMATE THE VALUE OF G , THE ACCELERATION DUE TO GRAVITY.

Apparatus A simple pendulum consisting of a string and a bob, a metre rod and a stop watch.

Background Information A simple pendulum consists of a small but heavy bob with a smooth surface suspended by means of a long inextensible light string from a fixed support. Theoretically, it is a point mass m suspended by means of a massless string of constant length l . When displaced from its mean position by an angle θ , as shown in Fig. E13.1, the bob is subjected to a restoring force $mg \sin \theta$ which, for small displacement θ may be taken as $mg \theta$. The equation of motion for the bob is then,

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0 \quad (\text{E13.1})$$

whence, the time period τ may be obtained as

$$\tau = 2\pi \sqrt{\frac{l}{g}} \quad (\text{E13.2})$$

and

$$g = \frac{4\pi^2}{\tau^2} l \quad (\text{E13.3})$$

Observations and Calculations The time period may be obtained by observing the time for a number of oscillations, say 20. The observation may be repeated 5 to 10 times for the same length of string. If possible, the experiment may be repeated for a different length of string.

| S. No. | l (m) | Time for 20 oscillations | τ (s) | g (m/s^2) |
|--------|---------|--------------------------|------------|------------------------|
| | | | | |

The average value of g for the observations is the best estimate of g .

Result and Points for Discussion

Compare the value of g obtained by you with the standard value

$$g = 9.80665$$

and comment on the difference. Note that the difference is due to the latitude λ and height h above the sea level as well as the experimental errors. Count the sources of error.

Would an improved value of g be obtained if only

- (a) the length of the pendulum were increased?
- (b) the mass of the bob were increased?
- (c) the volume of the bob were decreased?
- (d) the surface finish of the bob were improved?
- (e) the simple pendulum were enclosed in a vacuum chamber?
- (f) the amplitude of oscillations is increased?

Is it necessary that the oscillations should be in one and the same vertical plane?

What would be the effect of the earth's spinning about its own axis on the motion of the simple pendulum over a prolonged period of time?

Compare the accuracy of this method of determining g with other methods known to you.

If a simple pendulum is taken to a location as far away from the surface of the earth as its radius, what would be the time period in relation to the time period at the surface of the earth?

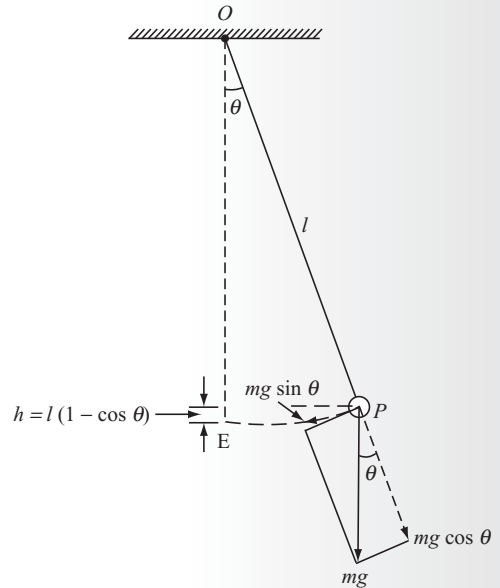


Fig. E13.1 Analysis of a simple pendulum

EXPERIMENT E14 MULTIPLE ELASTIC IMPACTS

OBJECTIVE TO UNDERSTAND THE IMPLICATIONS OF THE SIMULTANEOUS CONSERVATION OF MOMENTUM AND ENERGY IN MULTIPLE ELASTIC IMPACTS.

Apparatus A row of identical simple pendulums with steel bobs hanging in close contact with each other.

Background Information The momentum of a system of bodies is conserved if no external force acts upon them. The mechanical energy of a system of bodies is conserved if the motion or interaction of the bodies takes place in a conservative force field. Simultaneous conservation of momentum and mechanical energy may also take place. For example, if a number of balls, touching each other, and are struck by one or more balls directly and centrally, the motion is governed by the conservation of momentum and energy simultaneously. Assuming that the mass of each ball is m , let us analyse the effect of one or more balls striking the row of balls.

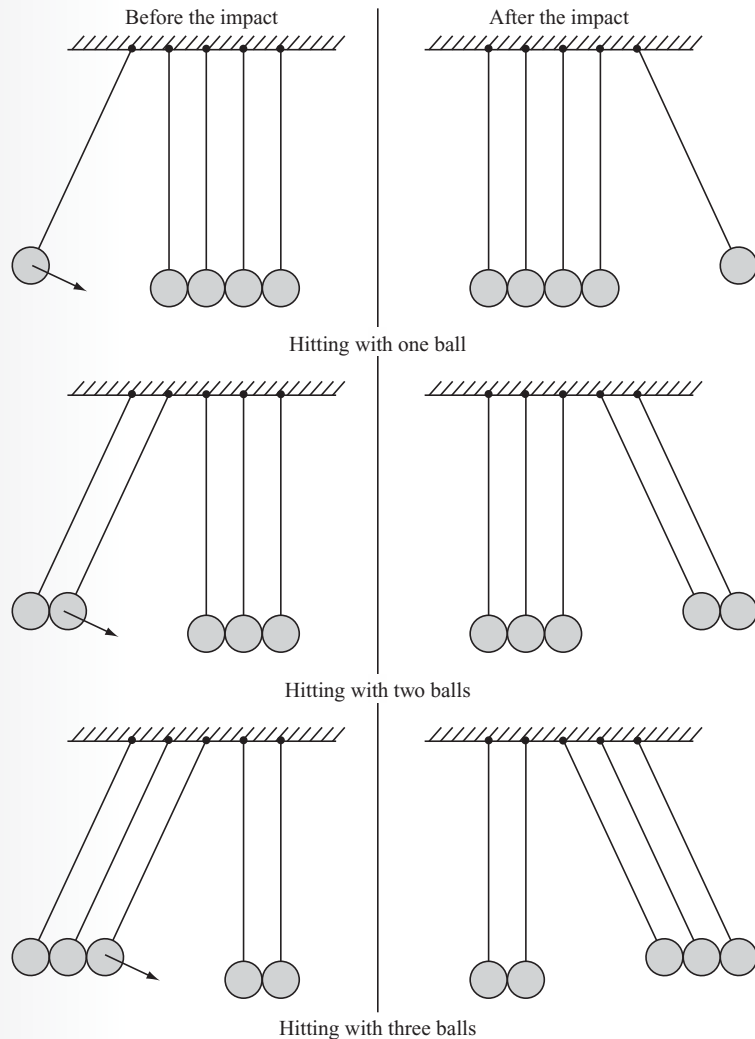


Fig. E14.1 *Playing with simple pendulums*

If the bob of a pendulum at one end is released from a small angle, its bob would move with a velocity v as it approaches the next bob. The momentum possessed by the first bob was mv and the mechanical energy was $\frac{1}{2}mv^2$. The number of balls that would be displaced, after multiple elastic impacts, would be such that the laws of conservation of momentum and energy are obeyed *simultaneously*.

If n balls moved out with velocities ($v_1, v_2, v_3, \dots, v_n$) respectively, then

$$v = v_1 + v_2 + v_3 + \dots + v_n$$

and

$$v^2 = v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2$$

These equations cannot be satisfied unless n equals 1, in which case $v = v_1$. The conclusion is that only the last one ball may move out with the same velocity with which the striking ball approached them. Only one pendulum will swing out. If two pendulums are released, causing two balls to approach with a velocity v each, the initial momentum of the system would be $2mv$ and the initial energy mv^2 . It will be discovered that, in this case, two end balls would move out, each with velocity v after the series of impacts. Supposing only one ball were to move out at velocity v_1 , then

$$2mv = mv_1; \quad v_1 = 2v$$

and

$$mv^2 = \frac{1}{2}mv_1^2; \quad v_1^2 = 2v^2$$

which is an inconsistent set of equations.

Similarly if three pendulums are released, then three pendulums would swing out!

Points for Discussion

When a ball rolls without slip on another surface, energy is conserved. Explain why and how?

Is the momentum conserved for every ball, taken one at a time, in the system? Recognise the balls for which momentum is conserved separately and for which it is not conserved separately. Is the energy conserved separately for each ball?

Would the pendulums keep on oscillating indefinitely?

Suggest some other experiment where you can gain visual experience on the conservation of momentum and energy, separately and simultaneously?

Example 11.4

A simple pendulum swings 5 oscillations in the same time as another 0.48 m longer swings 3 oscillations. Determine their lengths.

Solution Let the length of the first pendulum be l metres; the length of the second must be $(l + 0.48)$ metres.

In time T , the former swings 5 oscillations, its time period must be $T/5$.

Therefore,

$$T/5 = 2\pi \sqrt{\frac{l}{g}}$$

Similarly,

$$T/3 = 2\pi \sqrt{\frac{(l + 0.48)}{g}}$$

From these two equations,

$$\frac{9}{25} = \frac{l}{l + 0.48}$$

whence

$$l = 0.27 \text{ m}$$

The length of the second pendulum is $0.27 + 0.48 = 0.75 \text{ m}$.

Example 11.5

A spring of stiffness 7 N/m is cut into two halves and fixed to a block of mass m which is constrained to vibrate between two fixed end points. If it is observed to vibrate at 3 Hz, determine its mass. (Refer Fig. Ex. 11.5.)

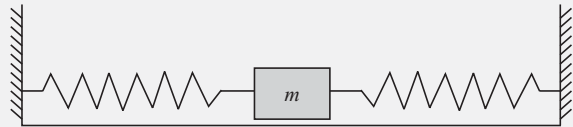


Fig. Ex. 11.5

Solution When the spring is cut into two halves, the stiffness of each half is doubled, i.e., it becomes 14 N/m. Now, two such springs are used to constrain the vibration of the mass, the effective spring stiffness is further doubled, i.e., 28 N/m.

Using the expression,

$$t = 2\pi \sqrt{\frac{m}{k}}$$

or

$$\frac{1}{3} = 2\pi \sqrt{\frac{m}{28}}$$

whence

$$m = 0.079 \text{ kg} = 79 \text{ grams}$$

Example 11.6

A spring-pan system consists of a spring of stiffness k and pan of mass M as shown in Fig. Ex. 11.6. If a mass of 1.5 kg is placed in the pan, it vibrates with a time period of 2.4 s; if a mass of 2 kg is placed, it vibrates with a time period of 2.6 s. Determine the values of k and M .

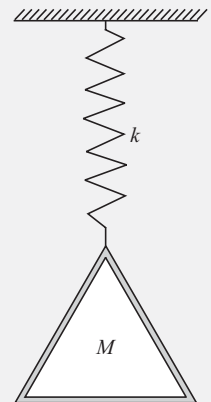


Fig. Ex. 11.6

Solution From the fact that

$$t = 2\pi\sqrt{\frac{m}{k}}$$

$$t_1 = 2\pi\sqrt{\frac{M + 1.5}{k}} = 2.4$$

and

$$t_2 = 2\pi\sqrt{\frac{M + 2}{k}} = 2.6$$

Solving the two equations,

$$M = 1.38 \text{ kg} \quad \text{and} \quad k = 19.7 \text{ N/m}$$

Example 11.7

A weightless bar is supported by two springs from above and carries a mass m suspended from a spring from its centre as shown in Fig. Ex. 11.7. Calculate the natural undamped frequency of the mass.

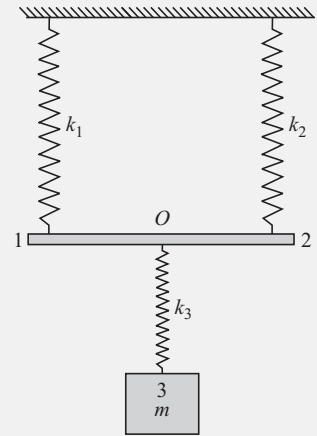


Fig. Ex. 11.7

Solution In the equilibrium position, the stretch of each spring is:

Spring 3: $\delta_3 = \frac{mg}{k_3}$

Spring 2: $\delta_2 = \frac{mg}{2k_2}$

Spring 1: $\delta_1 = \frac{mg}{2k_1}$

since the load is equally shared by springs 1 and 2 but acts wholly on spring 3.

The displacement δ of the mass in the equilibrium position is

$$\delta = \frac{\delta_1 + \delta_2}{2} + \delta_3$$

as can be visualised from the given configuration.

$$\begin{aligned}\delta &= \frac{1}{2} \left(\frac{mg}{2k_1} + \frac{mg}{2k_2} \right) + \frac{mg}{k_3} \\ &= mg \left(\frac{4k_1k_2 + k_1k_3 + k_2k_3}{4k_1k_2k_3} \right)\end{aligned}$$

Employing the expression

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}}$$

the frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{4k_1k_2k_3}{m(4k_1k_2 + k_1k_3 + k_2k_3)}} \quad (\text{i})$$

It has been intrinsically assumed that the mode of vibration in unequal stretching of the upper two supporting springs is so as to share the load equally by inclining the weightless bar.

If, instead, the bar was assumed to stay horizontal all the time, the stretch of the two supporting springs 1 and 2 would be equal irrespective of the proportion of sharing the load. The frequency would then be determined as follows:

The equivalent spring constant for the upper two supporting springs is

$$(k_1 + k_2)$$

and the equivalent k for all the three springs is given by

$$\frac{1}{k} = \frac{1}{(k_1 + k_2)} + \frac{1}{k_3}$$

whence

$$k = \frac{k_1k_3 + k_2k_3}{k_1 + k_2 + k_3}$$

and the natural frequency

$$\begin{aligned}f &= \frac{1}{2\pi} \sqrt{\frac{k}{m}} \\ &= \frac{1}{2\pi} \sqrt{\frac{k_1k_3 + k_2k_3}{m(k_1 + k_2 + k_3)}} \quad (\text{ii})\end{aligned}$$

Are the two answers (i) and (ii) very different? Not really, unless the load sharing is far from equal. One may like to try to find the result for assumed value of $k_1 = 20$, $k_2 = 25$ and $k_3 = 40$ N/cm. The answer is the same within working accuracy. But if k_1 and k_2 are widely different, the answers are incomparable; solution (i) is more realistic.

11.5 ANGULAR FREE VIBRATIONS

Angular periodic motion of a system can be analysed by considering an arbitrary angular displacement from its equilibrium position and by writing its equations of motion governed by (a) Newton's law, $F = ma$ and (b) Euler's equation $M = I\alpha$.

It may well be that in some cases only the Newton's law or the Euler's equation provides the equation of motion and can be solved to provide the frequency and time period. A summary of the salient results for rectilinear and rotational systems is given in Table 11.1.

Table 11.1 Analogy between rectilinear and rotational system

| Item | Rectilinear System | | Rotational System | |
|---------------------------|--|------------------|--|--------------------|
| | Symbol | unit | Symbol | unit |
| Time | t | s | t | s |
| Displacement | x | m | θ | rad |
| Velocity | $v \quad dx/dt$ | m/s | $\omega \quad d\theta/dt$ | rad/s |
| Acceleration | $a \quad d^2x/dt^2$ | m/s ² | $\alpha \quad d^2\theta/dt^2$ | rad/s ² |
| Inertia | m | kg | I | kg m ² |
| Effort | $m \quad d^2x/dt^2 \quad ma$ | N | $I\alpha$ | N m |
| Momentum | $m \quad v$ | N s | $I\omega$ | N m s |
| Impulse | $\int dt$ | N s | $\int T dt$ | N m s |
| Kinetic energy | $KE = \frac{1}{2}mv^2$ | J | $KE = \frac{1}{2}I\omega^2$ | J |
| Potential energy | $PE = \frac{1}{2}kx^2$ | J | $PE = \frac{1}{2}k\theta^2$ | J |
| Work | $\int dx$ | J | $\int d\theta$ | J |
| Spring constant | k | N/m | K | N m/rad |
| Damping coefficient | c | N s/m | C | N m s/rad |
| Damping factor | $\zeta = \frac{c}{C_c} = \frac{1}{2} \frac{c}{\sqrt{km}}$ | — | $\zeta = \frac{C}{C_c} = \frac{1}{2} \frac{C}{\sqrt{KI}}$ | — |
| Angular natural frequency | $\omega_n = \sqrt{k/m}$ | rad/s | $\omega_n = \sqrt{K/I}$ | rad/s |
| Natural frequency | $f_n = \omega_n/2\pi$ $= \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ | Hz | $f_n = \omega_n/2\pi$ $= \frac{1}{2\pi} \sqrt{\frac{K}{I}}$ | Hz |

Examples of angular periodic motions are a variety of pendulum motions which are discussed separately. Some examples in respect of spring-mass-inertia oscillatory systems are given as follows:

Example 11.8

A spring-mass system shown in Fig. Ex. 11.8 is capable of vibrating in the vertical plane of the figure. Determine its natural frequency.

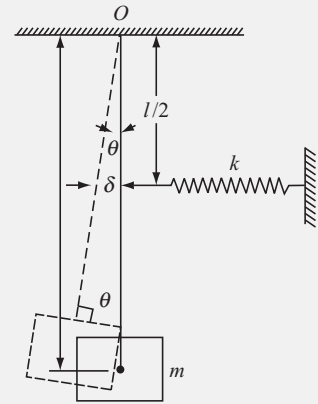


Fig. Ex. 11.8

Solution Consider a position of the vertical link displaced by an angle θ as shown. The restoring forces act at the location of the mass as well as at the junction of the spring. The restoring moment about O is

$$\begin{aligned} M &= -(mg \sin \theta l + k \delta l/2) \\ &= -(mgl\theta + k l/2\theta l/2) \\ &= -(mgl\theta + kl^2\theta/4) \end{aligned}$$

According to the Euler's law,

$$M = I\alpha$$

or
$$-(mgl\theta + kl^2\theta/4) = ml^2 \frac{d^2\theta}{dt^2}$$

whence,
$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l} + \frac{k}{4m}\right)\theta = 0$$

The general solution of the equation of motion is

$$\theta = A \sin \omega_n t + B \cos \omega_n t$$

Substituting for $\frac{d^2\theta}{dt^2} = -\omega_n^2\theta$ in the equation of motion

$$-\omega_n^2\theta + \left(\frac{g}{l} + \frac{k}{4m}\right)\theta = 0$$

whence,
$$\omega_n = \sqrt{\left(\frac{g}{l} + \frac{k}{4m}\right)}$$

and the cyclic frequency

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\left(\frac{g}{l} + \frac{k}{4m}\right)}$$

Example 11.9

A cylinder of radius r and mass m rests on a curved path of radius R as shown in Fig. Ex. 11.9. Show that the cylinder can oscillate about the bottom position when displaced and left to itself. Find the period of oscillation. Assume that the cylinder rolls without slipping.

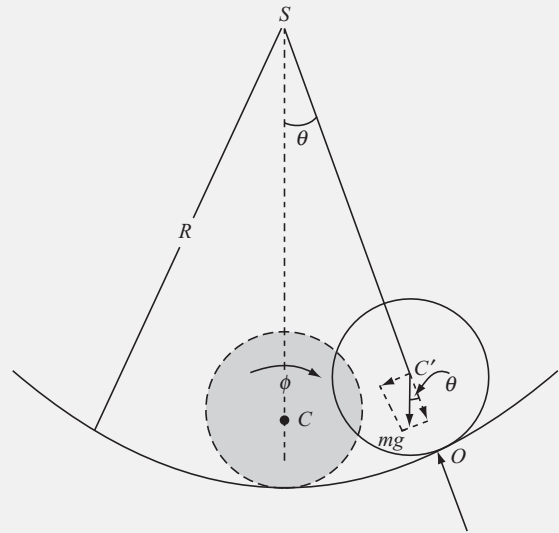


Fig. Ex. 11.9

Solution The cylinder, when displaced by a small angle θ , as shown, is acted upon by a restoring moment

$$M = -mg \sin \theta r = -mg r \theta$$

about an axis through the point of contact O with the curved path. This moment tends to roll the cylinder back to its equilibrium position at the bottom position. The presence of an equilibrium position and a restoring action to that position warrant oscillations.

The angular acceleration of the cylinder in roll is

$$\alpha = \frac{d^2 \phi}{dt^2}$$

where ϕ is the angle of rotation of the cylinder about its axis of symmetry.

It can be observed that

$$r\phi = (R - r) \theta$$

or

$$\phi = \frac{R - r}{r} \theta$$

Consequently

$$\alpha = \frac{d^2 \phi}{dt^2} = \frac{R - r}{r} \frac{d^2 \theta}{dt^2}$$

The moment of inertia I about an axis through the point of contact is

$$I = mr^2/2 + mr^2 = 3/2 mr^2$$

By the Euler's law,

$$M = I\alpha$$

$$-mgr\theta = 3/2 mr^2 \cdot \frac{R - r}{r} \frac{d^2 \theta}{dt^2}$$

or
$$\frac{d^2\theta}{dt^2} + \frac{g}{3/2(R-r)}\theta = 0$$

whence,
$$\omega_n = \sqrt{\frac{g}{3/2(R-r)}}$$

and the period of oscillation

$$\tau = 2\pi \sqrt{\frac{3/2(R-r)}{g}} = \pi \sqrt{\frac{6(R-r)}{g}}$$

Example 11.10 A system capable of undergoing angular periodic motion is shown in Fig. Ex. 11.10. Calculate its natural frequency.

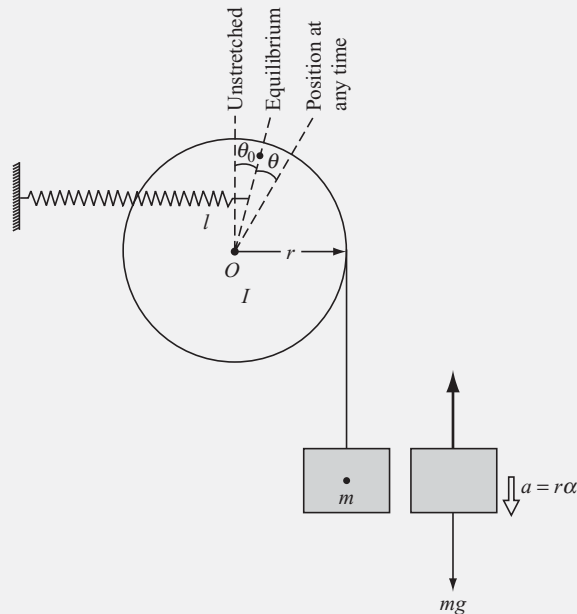


Fig. Ex. 11.10

Solution In the equilibrium position, the spring is stretched by

$$\delta = l\theta_0$$

such that

$$k \delta l = mgr$$

or

$$k l^2 \theta_0 = mgr \tag{i}$$

In a slightly displaced position at any time t , the equations of motion, due to Newton and Euler, for the mass and the cylinder respectively are

$$mg - T = ma = mr\alpha \tag{ii}$$

$$Tr - kl(\theta_0 + \theta)l = I\alpha \tag{iii}$$

Substituting the value of T from Eq. (ii) into Eq. (iii),

$$m(g - r\alpha) \cdot r - kl^2\theta_0 - kl^2\theta = I\alpha$$

Employing Eq. (i) and rearranging,

$$(I + mr^2) \alpha + kl^2\theta = 0$$

or

$$(I + mr^2) \frac{d^2\theta}{dt^2} + kl^2\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{kl^2}{I + mr^2} \theta = 0$$

from which the cyclic frequency f is obtained as

$$f = \frac{1}{2\pi} \sqrt{\frac{kl^2}{I + mr^2}}$$

Example 11.11

A uniform horizontal plank is resting symmetrically on two counter rotating drums, i.e., with equal and opposite angular velocities as shown in Fig. Ex. 11.11. Show that the plank performs simple harmonic motion, if displaced slightly from the equilibrium position. Find the distance between the drums for it to be equivalent to a seconds pendulum. Assume that the coefficient of friction μ between the plank and the drums is 0.8.

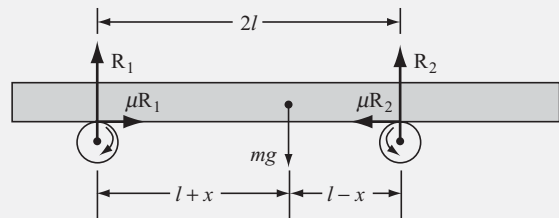


Fig. Ex. 11.11

Solution Consider the free-body diagram of the plank in a slightly displaced position, as shown.

$$R_1 + R_2 = mg \tag{i}$$

and

$$\mu R_1 - \mu R_2 = ma \tag{ii}$$

Taking moments about the left support

$$R_2 \times 2l = mg(l + x); R_2 = \frac{mg}{2l}(l + x)$$

Similarly,

$$R_1 = \frac{mg}{2l}(l - x)$$

Hence, from (ii),

$$\frac{\mu mg}{2l}(l - x - l - x) = ma$$

whence,

$$a = -\frac{\mu gx}{l} = -kx$$

which shows that the acceleration varies directly with the displacement and it is directed towards the mean position. The motion is, therefore, simple harmonic.

Time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{\mu g}}$

For a seconds pendulum $T = 2$ seconds,

Hence, $2\pi\sqrt{\frac{l}{0.8 \times 9.81}} = 2; \quad l = 0.8 \text{ m}$

Distance between the drums = $2l = 1.6 \text{ m}$.

11.6 PENDULUM MOTIONS

A pendulum is used to generate harmonic motion of a desired frequency. There are three different types of pendulums commonly used in engineering. These are illustrated in Fig. 11.12.

The *simple pendulum* consists of a concentrated mass m at end of an inextensible string, of negligible mass, the upper end of which is tied to a rigid support as shown in Fig. 11.13(a). The initial disturbance is given by displacing the bob through a small angle θ and then releasing it to perform a periodic motion.

The *compound pendulum* consists of a rigid body that oscillates about a horizontal axis through the body at some point other than the centre of mass as shown Fig. 11.13(b). The moment of inertia of the entire rigid body comes into play to establish the frequency of the periodic motion.

The *torsional pendulum* consists of a rigid body suspended by a vertical elastic shaft which when twisted develops a restoring moment as shown in Fig. 11.13(c). The rigid body is generally a horizontal circular disc or a spherical bob.

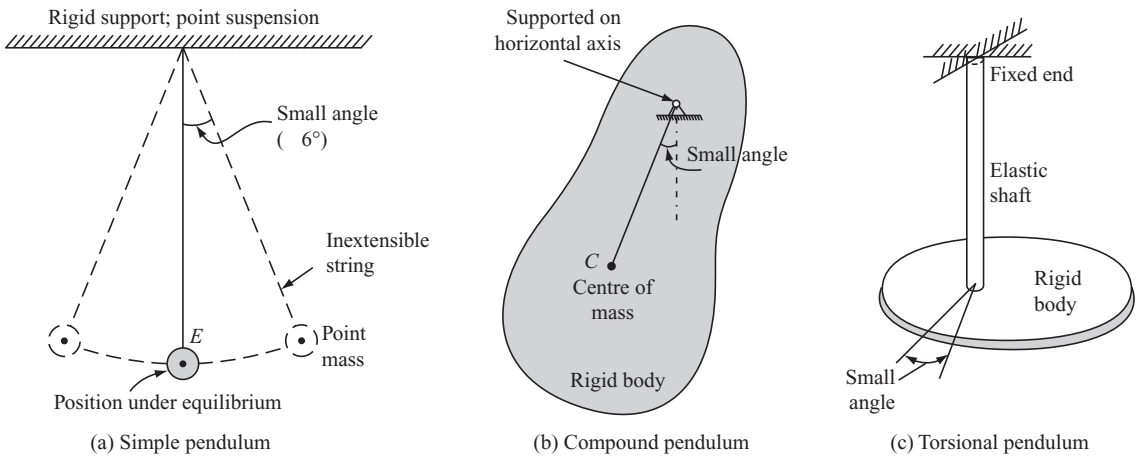


Fig. 11.13 Types of pendulums

There are other pendulums, such as the conical pendulum, ‘Foucault’s pendulum’, double and multiple-bob pendulums.

(a) Simple Pendulum

A simple pendulum, consisting of a massless and inextensible string of length l supporting a bob of concentrated mass m Fig. 11.14, generates a free harmonic motion under the action of the following factors:

1. The initial disturbing force is applied by hand to bring it to one of its extreme positions and then it is ‘let go’.

2. The restoring force is the component of the gravitational force of the bob, i.e., its weight along the motion of the bob.
3. When the initial disturbing force is removed, the restoring force is a function of the angular location of the bob; maximum at the extremities and zero at the equilibrium position.

The restoring force F is given by

$$mg \sin \theta \approx mg\theta$$

for small θ ($< 6^\circ$) directed towards the equilibrium position. Since the positive direction is denoted away from the equilibrium position,

$$F \approx -mg\theta$$

According to the Newton's law of motion,

$$F = ma$$

$$-mg\theta = m(l\alpha) = ml \frac{d^2\theta}{dt^2}$$

Cancelling the mass m from both sides and rearranging,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0 \tag{11.13}$$

This is the equation of motion of the simple pendulum.

The equation of motion for a simple pendulum may as well be set up with reference to the Euler's equation

$$M = I\alpha$$

The restoring moment about the origin O is

$$\begin{aligned} M &= -mg \sin \theta l \\ &= -mg\theta l \end{aligned}$$

for small θ .

Since the moment of inertia of the mass m about the same origin O is

$$I = ml^2$$

and the angular acceleration is

$$\alpha = \frac{d^2\theta}{dt^2}$$

the equation of motion becomes

$$-mg\theta l = ml^2 \frac{d^2\theta}{dt^2}$$

which, on rearranging gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0 \tag{11.13a}$$

The equation of motion can alternatively be obtained by employing the principle of conservation of energy:

At any instant, the height of the bob above the equilibrium position is given by

$$h = l(1 - \cos \theta)$$

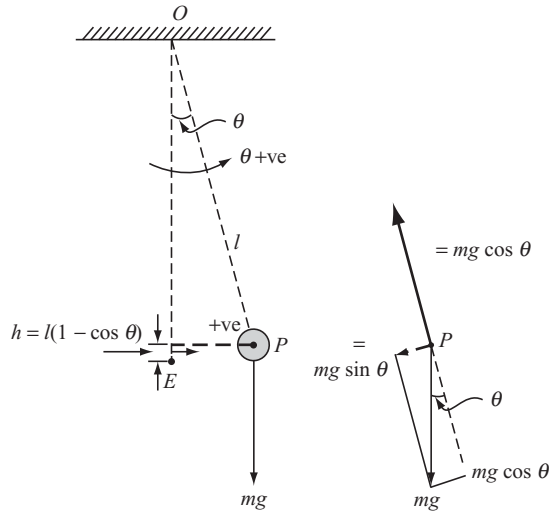


Fig. 11.14 Analysis of a simple pendulum

and the potential energy stored in the bob is

$$mgh = mgl(1 - \cos \theta)$$

The velocity of the bob is given by

$$v = l \frac{d\theta}{dt}$$

and the kinetic energy possessed by the bob is

$$\frac{1}{2}mv^2 = \frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2$$

By the principle of conservation of energy,

$$mgh + \frac{1}{2}mv^2 = \text{Constant}$$

or
$$mgl(1 - \cos \theta) + \frac{1}{2}ml^2 \left(\frac{d\theta}{dt} \right)^2 = \text{Constant}$$

Differentiating each term with respect to t and simplifying it,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad (11.13b)$$

which is the equation of motion.

This is a linear differential equation of the second order. Its solution is

$$\theta = A \sin \omega_n t + B \cos \omega_n t \quad (11.14)$$

whence,

$$\frac{d\theta}{dt} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t$$

and

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t \\ &= -\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) = -\omega_n^2 \theta \end{aligned}$$

Substituting the values of θ and $\frac{d^2\theta}{dt^2}$ in the differential equation,

$$-\omega_n^2 \theta + \frac{g}{l}\theta = 0;$$

whence,

$$\omega_n = \sqrt{\frac{g}{l}} \quad (11.15)$$

The frequency of the harmonic motion is

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \quad (11.16)$$

and the time period is

$$\tau = \frac{1}{f} = 2\pi \sqrt{\frac{l}{g}} \quad (11.17)$$

Some of the salient features of the periodic motion of a simple pendulum are:

1. The motion is along a plane arc of a circle as long as g is constant and the effects of spinning of the earth about its own axis as also of the rotation of the earth around the sun are negligible.
2. The velocity of the bob varies as

$$v = l \frac{d\theta}{dt} = l (A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t)$$

It is the maximum at the equilibrium position, $t = 0$

$$v_{\max} = lA\omega_n = lA \sqrt{\frac{g}{l}} = A \sqrt{lg}$$

and zero at the extremities.

3. The acceleration of the bob varies as

$$a = l \frac{d^2\theta}{dt^2} = -l\omega_n^2\theta$$

It is zero at the equilibrium position, $\theta = 0$ and increases with θ but it is always directed towards the equilibrium position. At the extreme positions, where the velocity is zero the acceleration is maximum and at the equilibrium positions, where the velocity is maximum, the acceleration is zero.

4. The restoring force F acting on the bob equals $mg \sin \theta$ which, for small θ , is

$$F \approx mg\theta$$

implying that F varies linearly with θ . This fact can be appreciated with reference to Fig. 11.15 where the variation of the restoring force is shown as a function of θ .

5. The frequency and time period of the motion are independent of the mass of the bob. The tension in the string is obviously proportional to the mass of the bob.
6. The frequency and time period are independent of the amplitude θ of motion. Their dependence upon the length of the string l and the acceleration due to gravity g is employed to determine the value of g .
7. If there were no resisting forces, the periodic motion would go on for ever but, in practice, there are resisting forces such as aerodynamic force on the bob which damp the motion to bring it to rest eventually.
8. The surface of the earth has been considered to provide an inertial frame. This is, however, untrue because the earth is spinning about its own axis. If the spinning of the earth is taken into account, the plane of oscillations of the pendulum precesses with time. Such precessional motion is ignored for a simple pendulum.

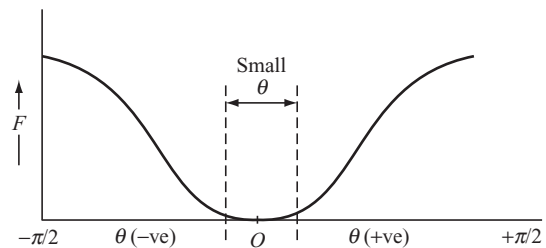


Fig. 11.15 Variation of the restoring force

Example 11.12

Determine the following parameters for a simple pendulum at the mean surface of the earth:

- (a) length of a 1-second pendulum
- (b) period of a 1-metre pendulum.

If these pendulums are taken to the mean surface of the moon where $g = 1.67 \text{ m/s}^2$, determine the corresponding periods.

Solution The value of g at the mean surface of the earth is 9.81 m/s^2 . Employing the relation,

$$\tau = 2\pi \sqrt{\frac{l}{g}}$$

(a) For a 1-second pendulum,

$$l = \frac{1}{4\pi^2} \times 9.81 = 0.284 \text{ m}$$

(b) For a 1-metre pendulum,

$$\tau = 2\pi \sqrt{\frac{1}{9.81}} = 2.01 \text{ s}$$

At the surface of the moon where $g = 1.67 \text{ m/s}^2$, the time periods of the two pendulums are

$$\tau_i = 2\pi \sqrt{\frac{0.248}{1.67}} = 2.42 \text{ s}$$

$$\tau_{ii} = 2\pi \sqrt{\frac{1}{1.67}} = 4.86 \text{ s}$$

The time period on the moon is indeed in the ratio of

$$\sqrt{\frac{g_{\text{earth}}}{g_{\text{moon}}}} = 2.42$$

(b) Compound Pendulum

A compound pendulum supported about a horizontal axis at a distance r above the centre of mass C , oscillating to produce a periodic motion, is shown in Fig. 11.16(a).

The restoring moment due to the weight of the pendulum about the pivot O is given by

$$M = (-mg \sin \theta) r$$

Considering the moment to be positive along θ increasing.

By Euler's law, the equation of motion is

$$M = I\alpha = I \frac{d^2\theta}{dt^2}$$

where I is the moment of inertia about the pivotal axis.

It follows that

$$-mgr \sin \theta = I \frac{d^2\theta}{dt^2}$$

which, for small θ , becomes

$$\frac{d^2\theta}{dt^2} + \frac{mgr}{I} \theta = 0 \tag{11.18}$$

The most general solution to the second order linear differential equation is given by

$$\theta = A \sin \omega_n t + B \cos \omega_n t$$

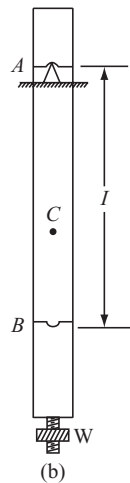
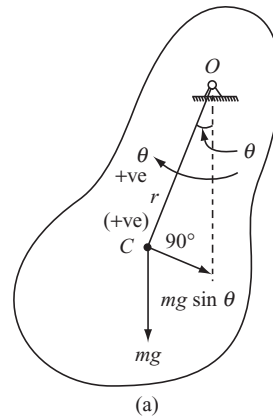


Fig. 11.16 (a) Analysis of compound pendulum
(b) Kater's pendulum

whence,

$$\frac{d\theta}{dt} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t$$

and

$$\frac{d^2\theta}{dt^2} = -\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) = -\omega_n^2 \theta$$

Substituting the value of θ and $\frac{d^2\theta}{dt^2}$ in the differential equation

$$-\omega_n^2 \theta + \frac{mgr}{I} \theta = 0$$

whence,

$$\omega_n = \sqrt{\frac{mgr}{I}} \quad (11.19)$$

The cyclic frequency is

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{mgr}{I}} \quad (11.20)$$

and the time period is

$$\tau = \frac{1}{f} = 2\pi \sqrt{\frac{I}{mgr}} \quad (11.21)$$

It is rather interesting to observe that the simple pendulum is a 'simple' case of the compound pendulum, i.e., when the mass is concentrated at the centre of mass and the distance r is large in comparison with the dimensions of the mass.

The moment of inertia about the pivotal axis is

$$I = m(k^2 + r^2)$$

where k is the radius of gyration.

For a simple pendulum $r \gg k$

hence,

$$I = m r^2$$

Substituting this value of I in the expression for the time period

$$\tau = 2\pi \sqrt{\frac{mr^2}{mgr}} = 2\pi \sqrt{\frac{r}{g}}$$

Recognising that r is essentially the length of the thread.

$$\tau = 2\pi \sqrt{\frac{l}{g}}$$

which is the expression desired for the simple pendulum.

An adaptation of the compound pendulum is the *Kater's pendulum*. It consists of a bar with two knife-edges A and B , as shown in Fig. 11.16(b), such that they are not obviously at the same distance from the mass centre C . The distance l between them can be measured with great precision. It is the accuracy and certainty with which this distance can be measured that makes it superior to the compound pendulum where the mass centre cannot be ascertained easily and the simple pendulum whose string cannot be weightless and inelastic. There

is a counterweight mg in the Kater's pendulum which can be adjusted so that the period of oscillation τ is the same when a either knife-edge is used. The relation

$$\tau = 2\pi \sqrt{\frac{I}{mgr}} = 2\pi \sqrt{\frac{l}{g}}$$

applies to the Kater's pendulum and the results obtained are more accurate than with either the simple or compound pendulum.

Example 11.13

A rod of length L and mass M is pivoted at one end to constitute a pendulum. Determine its period of oscillation and calculate its length if the period is desired to be 1 second.

If the rod was instead suspended from a point at one quarter of its length, what would be the expression for the period and what would be the length of the rod required for a period of 1 second?

Solution For the first case of suspension from the pivot O_1 ,

$$r = \frac{L}{2}$$

$$I = ML^2/12 + M(L/2)^2 = ML^2/3$$

and

$$\tau = 2\pi \sqrt{\frac{ML^2/3}{Mg L/2}} = 2\pi \sqrt{\frac{2L}{3g}}$$

For

$$\tau = 1 \text{ s, } L = 3g/8\pi^2 = 0.373 \text{ m}$$

For the second case of suspension from the pivot O_2 ,

$$r = L/4$$

$$I = ML^2/12 + M(L/4)^2 = 7/48 ML^2$$

and

$$\tau = 2\pi \sqrt{\frac{7/48 ML^2}{MgL/4}} = 2\pi \sqrt{\frac{7L}{12g}}$$

For

$$\tau = 1 \text{ s, } L = \frac{12g}{28\pi^2} = 0.426 \text{ m}$$

An extension of this example leads us to an interesting question: What will τ be if the rod is suspended at the centre of mass C ?

If the pivot is located at the centre of mass,

$$\tau = 2\pi \sqrt{\left(\frac{I}{mg \times 0}\right)} \rightarrow \infty$$

which shows that the periodic motion is not possible in this case; obviously because the restoring moment disappears as the weight of the body acts at the pivot itself.

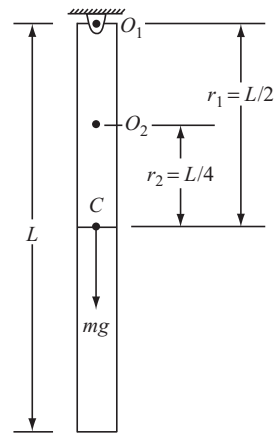


Fig. Ex. 11.13 (Solution)

Example 11.14 A disc of radius 10 cm is suspended from a point on its circumference. Determine its frequency of oscillation.

Solution Referring to Fig. Ex. 11.4 (solution). The moment of inertia about the pivotal axis is

$$I = M \frac{r^2}{2} + Mr^2 = \frac{3}{2} Mr^2$$

$$= \frac{3}{2} M \times 0.1^2 = 0.015M \text{ kg m}^2$$

where M is the mass of the disc.

The frequency is given by

$$f = \frac{1}{2\pi} \sqrt{\frac{M \times 9.81 \times 0.1}{0.015M}} = 1.287 \text{ s}^{-1}$$

and the time period

$$\tau = \frac{1}{f} = \frac{1}{1.287} = 0.777 \text{ s}$$

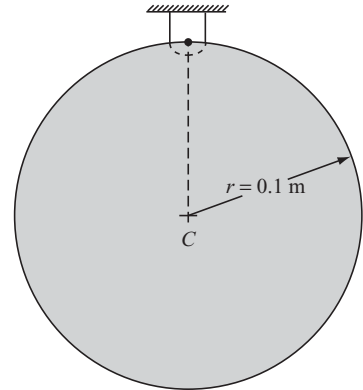


Fig. Ex. 11.14 (Solution)

EXPERIMENT E15 OSCILLATION OF A COMPOUND PENDULUM

OBJECTIVE TO DETERMINE THE TIME PERIOD OF OSCILLATION OF A COMPOUND PENDULUM AND TO ESTIMATE THE VALUE OF g , THE ACCELERATION DUE TO GRAVITY.

Apparatus A bar-shaped compound pendulum with provision to suspend it from over a knife edge at different points along its length as shown in Fig. E15.1. A beam compass, a stop watch and a metre rod.

Background Information From the equation of motion of a compound pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{mgr}{I} \theta = 0 \quad (\text{E15.1})$$

The time period is given by

$$\tau = 2\pi \sqrt{\frac{I}{mgr}} = 2\pi \sqrt{\frac{mk^2 + mr^2}{mgr}}$$

$$= 2\pi \sqrt{\frac{(k^2 + r^2)/r}{g}} \quad (\text{E15.2})$$

Comparing of this expression with the expression for the time period of a simple pendulum, i.e.,

$$\tau = 2\pi \sqrt{\frac{l}{g}}$$

shows that the length of an equivalent simple pendulum

$$l = \frac{k^2 + r^2}{r}$$

which, for a straight uniform bar becomes

$$l = \frac{L^2/12 + r^2}{r}$$

where L is the length of the bar.

From Eq. (E15.2), which is a quadratic equation in r ,

$$r^2 - \frac{2}{4\pi^2} gr + k^2 = 0$$

it is noticed that there must be two values of r for the same time period τ of the pendulum such that

$$\frac{k^2 + r_1^2}{r_1} = \frac{k^2 + r_2^2}{r_2}$$

whence

$$r_1 r_2 = k^2 \tag{E15.3}$$

Since k^2 must be a positive quantity, it follows that r_1 and r_2 must have the same sign, i.e., the two points of suspension at a distance r_1 and r_2 from the centre of mass must be on the same side of it. Similarly, there must be two points of suspension on the other side of the centre of mass which yield the same time period. In all, there are four points of suspension on a bar pendulum which provide the same time period.

The value of g , the acceleration due to gravity may be obtained from a single measurement of the time period for a known r and by applying Eq. (E15.2). Averaging on the basis of a number of isolated readings would be an improvement over the method. Better still is to draw a curve of the time period vs. the distance of the point of suspension from the centre of mass on one side or on both sides of it. Averaging for the value of g on the basis of pairs of values of the distances on either side of the centre of mass for the same time period has the added advantage of using the faired curves drawn on the basis of all the measurements.

Observations and Calculations The time period may be obtained by observing the time for a number of oscillations, say 30. The experiment is repeated for 10 different points of suspension of the rod on either or both sides of the centre of mass C . Distances may be measured accurately with the help of a beam compass.

| S. No. | r (m) | Time for 30 oscillations | τ (s) |
|--------|---------|--------------------------|------------|
| 1. | | | |
| 2. | | | |
| . | | | |
| . | | | |
| . | | | |

A curve or a set of curves is drawn, as shown in Fig. E15.2 and the acceleration due to gravity is obtained as follows:

$$r_1 = OA, r_1' = OA'; \text{ average } r_1 = (OA + OA')/2$$

$$r_2 = OB, r_2' = OB'; \text{ average } r_2 = (OB + OB')/2$$

and

$$k^2 = r_1 r_2$$

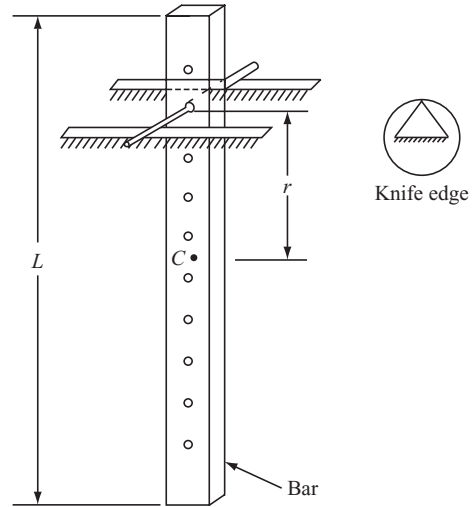


Fig. E15.1 Compound pendulum

From Eq. (E15.2),

$$g = \frac{4\pi^2}{\tau^2} \cdot \frac{k^2 + r^2}{r}$$

two estimates of g are obtained for the average r_1 and r_2 and k^2 as above and τ recorded from the plot as CO . Similarly, estimates of g are obtained from the set of points P, Q, P' and Q' , and so on and an average value of g is determined.

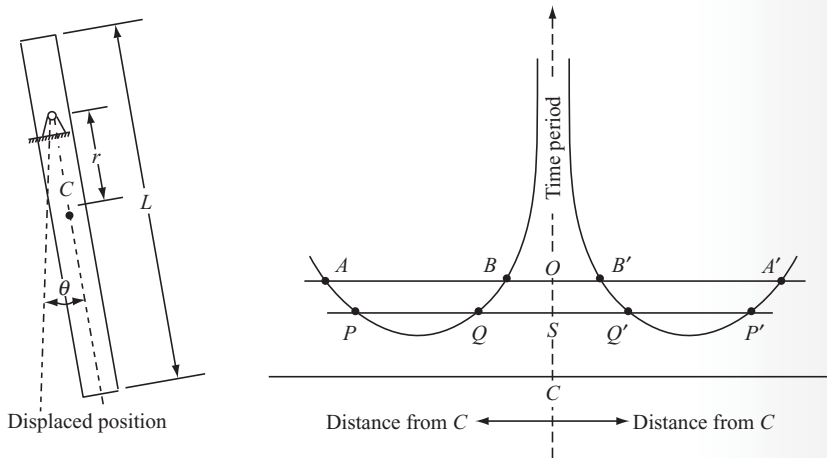


Fig. E15.2 Operation of a compound pendulum

Result and Points for Discussion

A value of g can as well be obtained from each observation of r and τ by employing Eq. (E15.2). What is the advantage of the method adopted by you over this procedure?

Theoretically, for a uniform bar,

$$k^2 = L^2/12$$

while it is actually,

$$k^2 = r_1 r_2$$

Which is closer to the true radius of gyration squared and why?

Determine the length of an equivalent simple pendulum for the compound pendulum and state why it must be less than the actual length of the pendulum below the point of suspension.

Which gives a more accurate value of the acceleration due to gravity, a simple pendulum or a compound pendulum?

What is the mean value of g over the surface of the earth and what is the value of g at the location of the experiment? Do the values compare with the estimate of g made by you?

Note that the moment of inertia of a rigid body about an axis may be determined by suspending the body about that axis and by noting the time period of oscillations by using Eq. (E16.2), if the acceleration due to gravity is known. Would you recommend this method of finding the moment of inertia of a rigid body?

What is the time period of oscillation of a bar when suspended from its centre of mass?

- 8 What is the minimum time period of the bar? Why can it not be less than this value?

(c) Torsional Pendulum

A torsional pendulum consisting of a rigid body attached to an elastic shaft operates by virtue of the restoring moment due to the elastic deformation of the shaft as shown in Fig. 11.17.

The restoring moment exerted by the shaft on the rigid body on a displacement θ is given by

$$M = -\frac{\pi r^4 G}{2L} \theta$$

By Euler's law, the equation of motion is

$$M = I\alpha$$

or
$$-\frac{\pi r^4 G}{2L} \theta = I \frac{d^2 \theta}{dt^2}$$

which on rearranging becomes

$$\frac{d^2 \theta}{dt^2} + \frac{\pi r^4 G}{2LI} \theta = 0 \tag{11.22}$$

The general solution of the second-order linear differential equation is expressed as

$$\theta = A \sin \omega_n t + B \cos \omega_n t$$

whence,
$$\frac{d^2 \theta}{dt^2} = -\omega_n^2 \theta$$

Substituting for θ and $\frac{d^2 \theta}{dt^2}$ in the equation of motion.

$$-\omega_n^2 \theta + \frac{\pi r^4 G}{2LI} \theta = 0$$

whence, the angular frequency

$$\omega_n = \sqrt{\frac{\pi r^4 G}{2LI}} \tag{11.23}$$

the cyclic frequency

$$f = \frac{1}{2\pi} \omega_n = \frac{1}{2\pi} \sqrt{\frac{\pi r^4 G}{2LI}} \tag{11.24}$$

and the time period

$$\tau = \frac{1}{f} = 2\pi \sqrt{\frac{2LI}{\pi r^4 G}} \tag{11.25}$$

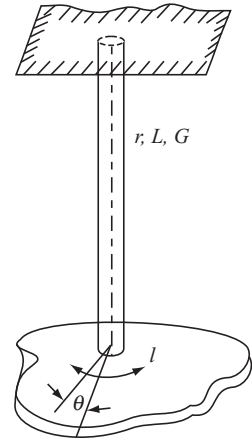


Fig. 11.17 Torsional pendulum

Example 11.15

A horizontal steel disc 30 cm in diameter and 3 cm thick is attached at its centre to a 1 m long vertical brass shaft of 1 cm diameter, the upper end of which is fixed. Assuming the density of steel as 7860 kg/m^3 and the modulus of rigidity for brass as $35 \times 10^9 \text{ N/m}^2$, calculate the frequency and time period of the assembly.

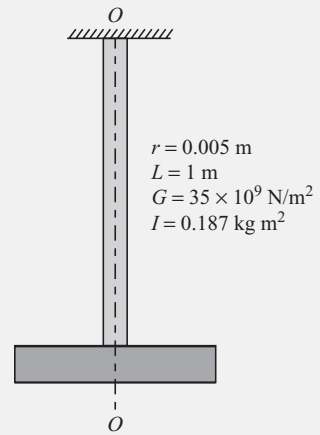


Fig. Ex. 11.15

Solution The mass of the disc is

$$m = \left(\frac{\pi \times (0.30)^2}{4} \times 0.03 \right) \times 7860 = 16.67 \text{ kg}$$

The moment of inertia about the axis of the shaft is

$$I = \frac{mr^2}{2} = \frac{16.67}{2} \times (0.15)^2 = 0.187 \text{ kg m}^2$$

The angular frequency is

$$\begin{aligned} \omega_n &= \sqrt{\frac{\pi r^4 G}{2LI}} = \sqrt{\pi \times \left(\frac{0.01}{2}\right)^4 \times 35 \times 10^9 / (2 \times 1 \times 0.187)} \\ &= 13.6 \text{ s}^{-1} \end{aligned}$$

The cyclic frequency f is given by

$$f = \frac{\omega_n}{2\pi} = \frac{13.6}{2\pi} = 2.16 \text{ s}^{-1}$$

The time period of oscillations is

$$\tau = \frac{1}{f} = \frac{1}{2.16} = 0.46 \text{ s}$$

(d) Conical Pendulum

A conical pendulum consists of a mass m suspended by an inextensible string of length l from a point O fixed on a vertical rotating rod as shown in Fig. 11.18. The equation of motion of the mass μ can be written as

$$= m$$

where , the net external force equals

$$-mg - T \sin \theta + T \cos \theta$$

and being unit vectors as shown in Fig. 11.18.

The acceleration of the mass in rotation at a constant angular velocity is purely centripetal

$$= \omega \times (\omega \times r) = \omega \times (\omega \times l \sin \theta) = -\omega^2 l \sin \theta$$

Hence

$$-mg - \sin \theta + \cos \theta = -m\omega^2 l \sin \theta$$

whence, the two scalar equations appear as

$$-mg + T \cos \theta = 0$$

or

$$T \cos \theta = mg$$

and

$$T \sin \theta = m\omega^2 l \sin \theta$$

It follows that

$$T = m\omega^2 l \tag{11.26}$$

and

$$\theta = \cos^{-1} (g/\omega^2 l) \tag{11.27}$$

The mass m describes a circle of radius r whereas the string and mass taken together describe the surface of a cone. The system has, therefore, acquired the name ‘conical pendulum’.

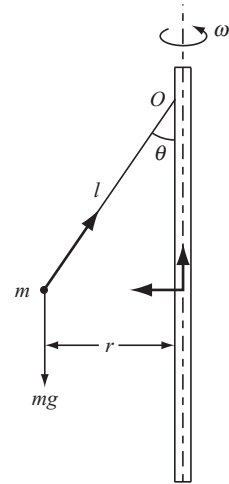


Fig 11.18 Conical pendulum

(e) Foucault’s Pendulum

The Foucault’s pendulum is essentially a simple pendulum with a very long string. A string of over 10 m length allows the bob to oscillate with an amplitude of 1 m with a small angular amplitude. The bob is capable of oscillation over a long period of time, say 24 hours or more. A fact which stands out is that the plane of oscillation of a simple pendulum should precess about the vertical axis due to the spinning of the earth as shown in Fig. 11.19. The period of precession would, of course, be different at different locations on the earth. The fact that the earth spins about its own axis is borne out by the experimental observations on the Foucault’s pendulum. One such pendulum is installed at the entrance of the Science Museum, London. The pendulum is ‘let go’ in a plane motion in the mornings and, as the time passes, the plane of oscillations changes continuously with time, thus demonstrating the spinning of the earth about its own axis, day in and day out.

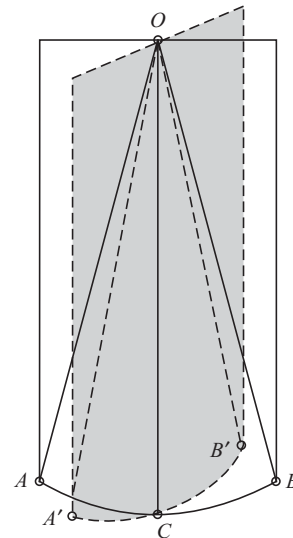


Fig 11.19 Shift of vertical place of oscillation from oacb to OA’CB’ with time of day due ot spinning of the earth

(f) Double Pendulum

A double pendulum consists of a mass m_1 suspended on a string of length l_1 from a fixed support and another mass m_2 suspended on a string of length l_2 from the first mass, as shown in Fig. 11.20. The mass m_2 can oscillate about the centre of mass m_1 whereas the mass m_1 can oscillate about the fixed point O . For a very special case when

$$m_1 = m_2 = m$$

$$l_1 = l_2 = l$$

and $\theta = \sin \theta, \cos \theta = 1$ for small angles θ_1 and θ_2 .

The equations of motion are

$$2 \frac{\ddot{\theta}_1}{2} + \frac{\ddot{\theta}_2}{2} = -2 \theta_1 \quad (11.28)$$

and

$$\frac{\ddot{\theta}_1}{2} + \frac{\ddot{\theta}_2}{2} = -\theta_2 \quad (11.29)$$

Substituting

$$\theta_1 = \theta_2 \cos \omega \quad \text{and} \quad \theta_2 = \theta_1 \cos \omega$$

$$2(-\omega^2) \theta_1 - \omega^2 \theta_2 = 0$$

and

$$-\omega^2 \theta_1 + (-\omega^2) \theta_2 = 0$$

whence, $\omega_1^2 = \frac{(2 + \sqrt{2})}{2}$ and $\omega_2^2 = \frac{(2 - \sqrt{2})}{2}$ (11.30)

and the normal frequencies are given by

$$\omega_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi} \sqrt{(2 + \sqrt{2})} \quad (11.31)$$

$$\omega_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi} \sqrt{(2 - \sqrt{2})} \quad (11.32)$$

It may be noted that ω_1 and ω_2 correspond to the normal modes in which the bobs move in the opposite and same directions respectively.

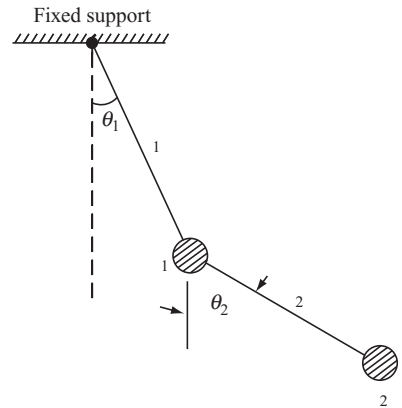


Fig 11.20 Double pendulum

Concept Review Questions

Elaborate the concept of ‘elements’ of a system. Is it necessary for an element to be pure and linear? Can every engineering system be simulated by an interconnection of mechanical elements? Are the results thus obtained fair approximations of the real situation?

The elements of a mechanical system are classified by the properties of inertia, compliance and damping. Some authors prefer to classify them by name, e.g., a rigid element, an elastic element and a damper. Show that the properties refer to the same elements.

Give one example each of the linear elements and the non-linear elements of a mechanical system.

Why are mechanical systems (for that matter, any system) prone to vibrations, so much so that the dynamics of a system implies its oscillatory characteristics?

Recognise the correct statements in the following:

- (a) The degrees of freedom of a system
 - (i) are the constraints applied on a system
 - (ii) refer to the order of a system
 - (iii) are the modes of motion of the system
 - (iv) equal the number of masses in a system.
- (b) A periodic motion of a system occurs
 - (i) if the initial state of the system is in equilibrium
 - (ii) if the restoring forces or moments come into play as soon as the system is disturbed.
 - (iii) if and only if the system undergoes simple harmonic motion
 - (iv) if the initial state as well as the disturbed states are in equilibrium.
- (c) The resisting force in an oscillatory system tends to
 - (i) reduce the time period
 - (ii) oppose the restoring force proportionately

- (iii) reduce the amplitude
- (iv) reduce the amplitude with time

Differentiate critically between

- (a) simple pendulum and compound pendulum
- (b) damping forces and driving forces
- (c) angular frequency and cyclic frequency.

Make simple sketches to illustrate three oscillating systems

- (a) capable of linear oscillations
- (b) capable of angular oscillations.

- 8 Can every oscillating system be reduced to an equivalent spring-mass-damper system? Establish the equivalence of any one linearly and one angularly oscillating system.

Ill-effects of the resonance of a system can be avoided

- (a) by keeping the resonating frequency as low as possible

- (b) by making the frequency of the driving force as low as possible
- (c) by passing the system through the state of resonance as quickly as possible.

Why is it that the equation of motion of a simple pendulum can be obtained either by Newton's law or by Euler's law or by the principle of conservation of mechanical energy, whereas the equation of motion for a compound pendulum cannot be obtained by the application of Newton's law alone?

Assuming that $g = \frac{GM}{(R+h)^2}$

where G is the gravitational constant, M is the mass of the earth, h is the vertical distance from the surface of the earth and R is the radius of the earth, establish a relationship between h and the length of a simple pendulum of period 1 second at all altitudes.

Tutorial Problems

A small ball of mass m is fixed at the mid-length of a taut wire of length l with tension P in it as shown in Fig. Prob. 11.1. Show that the ball executes a simple harmonic motion for small displacements. Also calculate the time period of the ball.

$$\left(\quad \right) \quad s \quad \tau = 2\pi \sqrt{\frac{ml}{4P}}$$

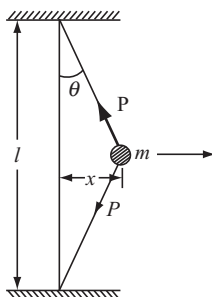


Fig. Prob. 11.1

A solid aluminium sphere with 50 cm diameter is attached to the lower end of 10 m long aluminium rod of 5 cm diameter, the upper end of which is fixed. Find the period of this pendulum. Take $G = 24 \times 10^9 \text{ N/m}^2$ and $\rho = 27,000 \text{ kg/m}^3$ for aluminium.

(s 0.12 s)

A block of steel of mass 50 kg is supported by an alternative spring arrangements as shown in Figs. Prob. 11.3(a) and (b) and (c). Determine the natural frequency of the block in vertical motion. Estimate also the maximum velocity and acceleration of the block if the amplitude is 5 cm. Take $k_1 = 40 \text{ N/cm}$ and $k_2 = 50 \text{ N/cm}$.

- (s (a) 0.214 s^{-1} , 0.067 m/s , 0.09 m/s^2
- (b) 0.106 s^{-1} , 0.0334 m/s , 0.0222 m/s^2
- (c) 0.257 s^{-1} , 0.08 m/s , 0.13 m/s^2)

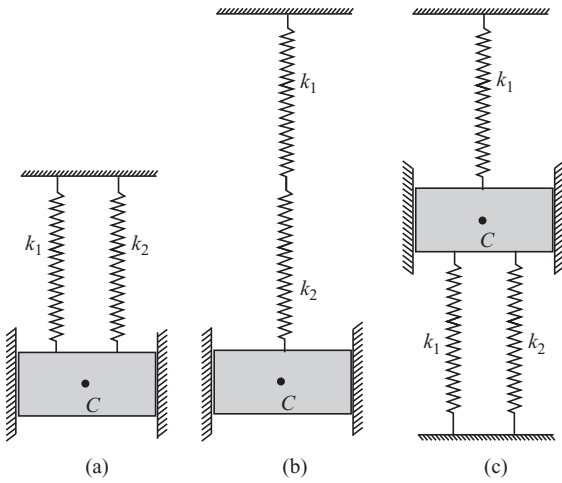


Fig. Prob. 11.3

A vertical *U*-tube manometer containing mercury for a total length of 0.3 m is subjected to a sudden pressure differential which is removed and the columns are set to vibrate. Determine the frequency of vibration and comment on its value if

- (a) the size of the manometer is changed to a tube of half its diameter and a length half as much and if
 - (b) the mercury is replaced by water
 - (c) the *U*-tube is inclined with its plane making an angle of 30° with the horizontal.
- (s $f = 1.29$ Hz; (a) no change with diameter but $f = 1.82$ Hz for half length
 (b) no change, (c) 0.912 Hz)

A semicircular cylinder of radius r and mass density m rests in the equilibrium position on its curved surface on a flat plate. Establish that the semi-cylinder is capable of oscillating and determine its period of oscillation.

$$\left(\text{ s } \tau = 2\pi \sqrt{\frac{(9\pi - 16)r}{2g}} \right)$$

A mass μ attached to a rigid rod which is held in position by two horizontal springs is made to vibrate in the plane of the springs (Fig. Prob. 11.6). What condition must be satisfied for equilibrium in the vertical position?

Obtain the equation of motion and express the natural frequency of the system in terms of the given parameters.

When will the period of oscillations be infinite?

$$\left(\text{ s } l_1^2 = \frac{mgl_2}{2k}; \tau = 2\pi \sqrt{\frac{2kl^2}{ml_2^2} - \frac{g}{l_2}} \right)$$

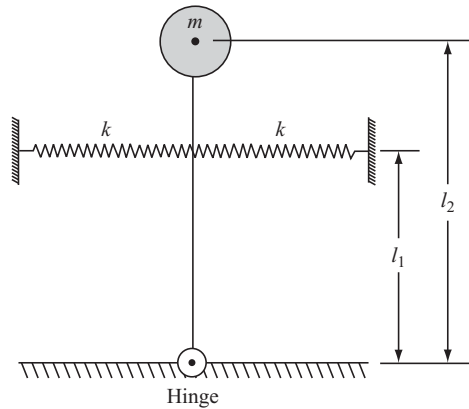


Fig. Prob. 11.6

Calculate the frequency of oscillation of a system consisting of a plate and cylinder attached to a spring as shown in Fig. Prob. 11.7.

Evaluate the frequency for

$$l = 1 \text{ m}, r = 0.2 \text{ m}, m_1 = 50 \text{ kg},$$

$$m_2 = 20 \text{ kg and } k = 50 \text{ N/cm}$$

$$\left(\text{ s } f = 0.98 \text{ Hz} \right)$$

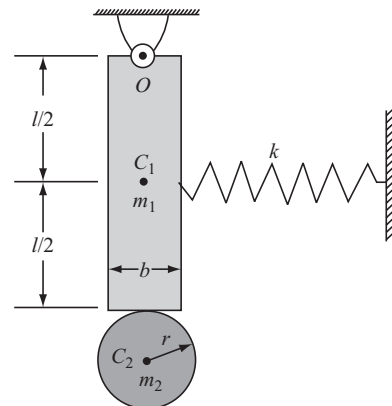


Fig. Prob. 11.7

- 8 A lever of length L is placed on a fulcrum, O off-centre by a distance l , and is held by two springs on the other end. Obtain an expression for the frequency of the system. Assuming the mass of a 2 m lever as 20 kg and the spring constant of each spring as 100 N/cm, calculate the distance l for the largest frequency of vibration.

$$\left(s f = \frac{1}{\pi} \sqrt{\frac{6k(Ll + l^2)}{m(L^2 + 12l^2)}} \right)$$

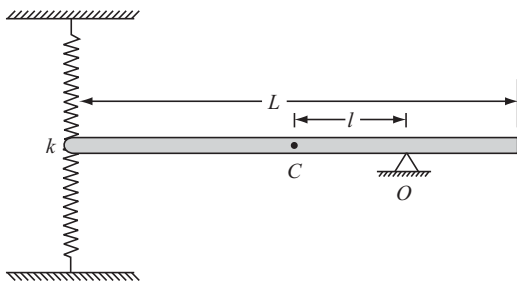


Fig. Prob. 11.8

A particle P of mass m oscillates along a vertical line in a system of springs shown in Fig. Prob. 11.9. Determine the frequency of oscillation assuming all other members as massless elements.

$$\left(s \omega^2 = \frac{4}{m \left(\frac{1}{4k_1} + \frac{1}{4k_2} + \frac{1}{k_3} + \frac{1}{k_4} \right)} \right)$$

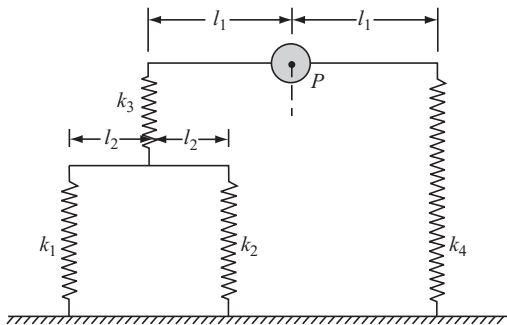


Fig. prob. 11.9

A circular disc of diameter d and mass m is free to roll without sliding on a horizontal plane. Two identical springs of stiffness k are attached to as shown in Fig. Prob. 11.10. Determine the period of small oscillations of the disc.

$$\left(s f = \frac{1}{\pi} \sqrt{\frac{6k(L/2 + l)^2}{m(L^2 + 12l^2)}} \right)$$

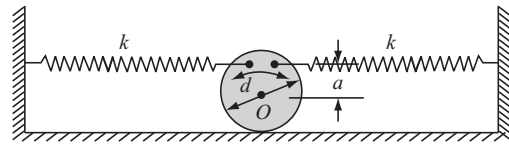


Fig. Prob. 11.10

Find the period for small oscillations if a rod of length L and mass M is suspended from a point $L/4$ from one end.

$$\left(s 2 \sqrt{\frac{7L}{24g}} \right)$$

A pulley having a moment of inertia I about its axis of rotation supports a rope which carries a mass m at one end, while the other end is connected to a spring of spring constant k as shown in Fig. Prob. 11.12. Find the period of oscillation of the system. Assume that the rope does not slip on the pulley.

$$\left(s 2\pi \left(\frac{I + mR^2}{kR^2} \right) \right)$$

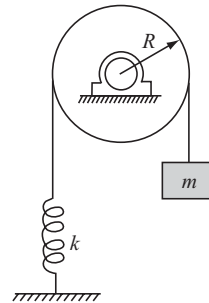


Fig. Prob. 11.12

Two parallel cylindrical rollers rotate in opposite directions as shown in Fig. Prob. 11.13. The distance between the centres of the rollers is a . A straight, uniform horizontal rod of length l and weight W rests on top of the rollers. The coefficient of kinetic friction between the rod and the roller is μ . Taking x as the distance from the centre of the rod to the midpoint between the rolls, write the equation of motion of the rod, assuming that it has been initially displaced from the central

position. Find the frequency of the resulting vibratory motion.

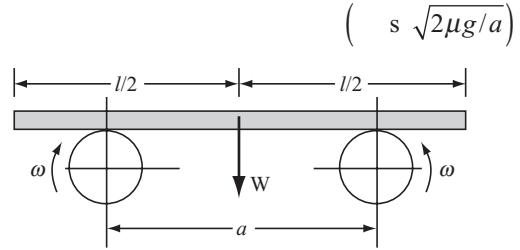


Fig. Prob. 11.13

Look up Hints to Tutorial Problems at the End!

Multiple-Choice Questions

Select the correct or the most appropriate response from among the available alternatives in the following multiple-choice questions:

- A linear response implies that
- (a) the elements are in one line
 - (b) the response is along the given line
 - (c) the sensitivity is constant
 - (d) the response is not exponential

- A vibrating system
- (a) does not pass through an equilibrium position
 - (b) passes through the equilibrium position once every cycle

- (c) passes through the equilibrium position twice every cycle
- (d) starts from an equilibrium position and does not return to it

Resonance occurs when

- (a) a freely vibrating system is made to vibrate at increasingly higher frequencies
- (b) the forcing frequency equals the natural frequency of the system
- (c) the system vibrates at its natural frequency
- (d) the amplitude of vibrations exceeds twice the amplitude of free vibrations

Answers to Multiple-Choice Questions

- 1 (c) 2 (a) 3 (b)

Appendices A1 and A2

This section consists of appendices with information useful for the study of mechanics:

-
- **APPENDIX 1 PROPERTIES OF PLANE FIGURES (LINE AND AREA ELEMENTS)**
 - **APPENDIX 2 PROPERTIES OF SOLID BODIES (THIN, SOLID AND HOLLOW BODIES)**
-

Appendix 1

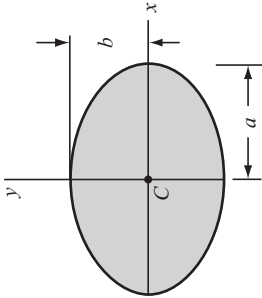
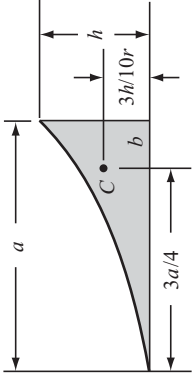
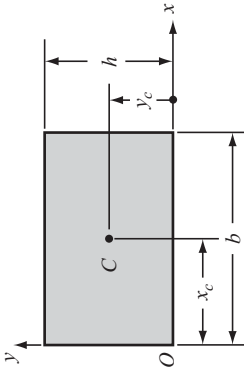
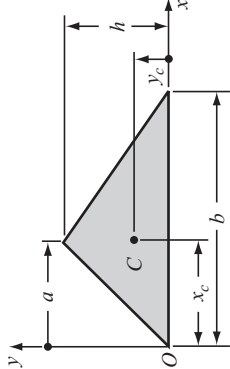
Properties of Plane Figures (Line and Area Elements)

| Figure | Description | Area | Centroid | Area Moment of Inertia |
|--------|--|--------------|--|----------------------------------|
| | 1. General circular arc $L = 2r\theta$ | Circular Arc | $x_c = \frac{r \sin \theta}{\theta}$ $y_c = 0$ | |
| | 2. Quarter circular arc, $\theta = \frac{\pi}{4}$ $L = \frac{\pi}{2} r$ | | $x_c = \frac{2r\sqrt{2}}{\pi}$ $y_c = 0$ | |
| | 3. Semicircular arc, $\theta = \frac{\pi}{2}$ $L = \pi r$ | | $x_c = \frac{2r}{\pi}$ $y_c = 0$ | |
| | Circular Area Sector $A = r^2 \theta$ | | 1. General circular sector $x_c = \frac{2r \sin \theta}{3\theta}$ $y_c = 0$ | $I_{xx} = I_{yy} = \pi r^4 / 16$ |
| | 2. Quarter circle, $\theta = \frac{\pi}{4}$ $x_c = \frac{4r\sqrt{2}}{3\pi}$ $y_c = 0$ | | $I_{zz} = \pi r^4 / 8$ | |
| | 3. Semicircle, $\theta = \frac{\pi}{2}$ $x_c = \frac{4r}{3\pi}$ $y_c = 0$ | | $I_{xx} = I_{yy} = \pi r^4 / 8$ $I_{zz} = \pi r^4 / 4$ | |

(Continued)

Appendix 1 (Continued)

| Figure | Description | Area | Centroid | Area Moment of Inertia |
|--------|----------------|-------------------------|--------------------------------------|--|
| | Quarter circle | $A = \frac{\pi r^2}{4}$ | $x_c = \frac{4r}{3\pi}$ $y_c = 0$ | $I_{xx} = I_{yy} = \frac{\pi r^2}{16}$ $I_{zz} = \frac{\pi r^4}{8}$ |
| | Semicircle | $A = \frac{\pi r^2}{2}$ | $x_c = 0$ $y_c = \frac{4r}{3\pi}$ | $I_{xx} = I_{yy} = \frac{\pi r^4}{8}$ $I_{zz} = \frac{\pi r^4}{4}$ |
| | Circle | $A = \pi r^2$ | $x_c = 0$ $y_c = 0$ | $I_{xx} = I_{yy} = \frac{\pi r^4}{4}$ $I_{zz} = \frac{\pi r^4}{2}$ |

| | a | a | a | a |
|---|--------------|--|--|-----|
|  <p>Ellipse</p> | $A = \pi ab$ | $x_c = 0$ $y_c = 0$ | $I_{xx} = \pi ab^3/4$ $I_{yy} = \pi ba^3/4$ $I_{zz} = \frac{\pi ab}{4}(a^2 + b^2)$ | |
|  <p>Parabolic spandrel</p> | $A = ah/3$ | $x_c = \frac{3a}{4}$ $y_c = \frac{3h}{10r}$ | — | |
|  <p>Rectangle</p> | $A = bh$ | $x_c = b/2$ $y_c = h/2$ | $I_{xx} = bh^3/3$ $I_{yy} = hb^3/3$ $I_{xy} = b^2h^2/4$ $I_{xc} = bh^3/12$ $I_{yc} = hb^3/12$ | |
|  <p>Triangle</p> | $A = bh/2$ | $x_c = (a + b)/3$ $y_c = h/3$ | $I_{xx} = bh^3/12$ $I_{yy} = bh(a^2 + ab + b^2)/12$ $I_{xy} = bh^2(2a + b)/24$ $I_{xc} = bh^3/36$ | |

(Continued)

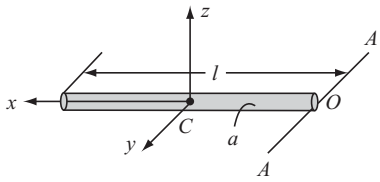
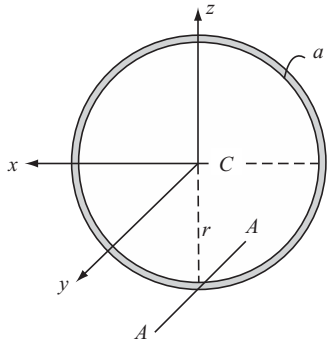
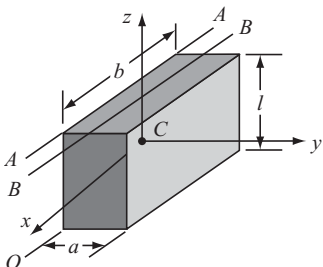
Appendix 1 (Continued)

| Figure | Description | Area | Centroid | Area Moment of Inertia |
|--------|-----------------|----------------------|--|---|
| | Trapezoid | $A = h(b_1 + b_2)/2$ | $x_c = \frac{h(b_1 + 2b_2)}{3(b_1 + b_2)}$ $y_c = \frac{(b_1^2 + b_1b_2 + b_2^2)}{3(b_1 + b_2)}$ | $I_{xx} = h(b_1^4 - b_2^4)/12(b_1 - b_2)$ $I_{yy} = h^2(b_1 + 3b_2)/12$ $I_{xy} = h^2(b_1^2 + 2b_1b_2 + 3b_2^2)/24$ |
| | Parallelogram | $A = ab \sin \theta$ | $x_c = (a \cos \theta + b)/2$ $y_c = a \sin \theta/2$ | $I_{xx} = a^2b \sin^3 \theta/3$ $I_{yy} = ab(2a^2 \cos^2 \theta + 3ab \cos \theta + 2b^2)/6 \sin \theta$ $I_{xy} = a^2b \sin^2 \theta (4a \cos \theta + 3b)/12$ |
| | Parabola | $A = 2ab/3$ | $x_c = 2a/5$ $y_c = 3b/8$ | $I_{xx} = 2ab^3/15$ $I_{yy} = 16a^2b/15$ $I_{xy} = 7a^2b^2/60$ |
| | Quarter-ellipse | $A = \pi ab/4$ | $x_c = 4a/3\pi$ $y_c = 4b/3\pi$ | $I_{xx} = \pi ab^3/16$ $I_{yy} = \pi a^3b/16$ $I_{xy} = a^2b^2/8$ |

Appendix 2

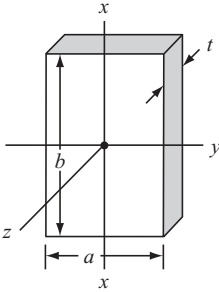
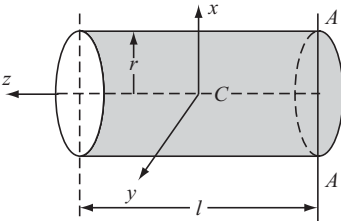
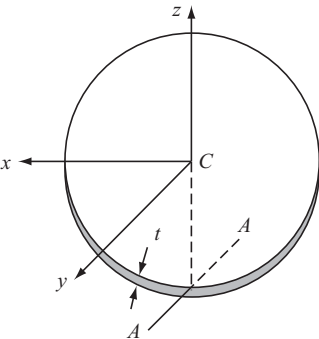
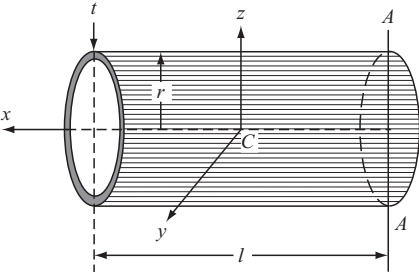
Properties of Solid Bodies

(Thin, Solid and Hollow Bodies)

| Figure | Description | Mass | Moment of Inertia |
|---|----------------------------------|----------------------------|---|
|  | Slender (Thin) Rod | $m = \rho l \cdot a$ | $I_{yy} = I_{zz} = m l^2 / 12$ $I_{xx} = 0$ $I_{AA} = m l^2 / 3$ |
|  | Thin Hoop (Ring) | $m = 2 \rho \pi r \cdot a$ | $I_{xx} = I_{zz} = m r^2 / 2$ $I_{yy} = m r^2$ $I_{AA} = 2 m r^2$ |
|  | Solid Rectangular Parallelepiped | $m = \rho a b l$ | $I_{xx} = m (a^2 + l^2) / 12$ $I_{yy} = m (b^2 + l^2) / 12$ $I_{zz} = m (a^2 + b^2) / 12$ $I_{xy} = I_{yz} = I_{xz} = 0$ $I_{AA} = m (a^2 + l^2) / 3$ $I_{BB} = m (a^2 + 4l^2) / 12$ |

(Continued)

Appendix 2 (Continued)

| Figure | Description | Mass | Moment of Inertia |
|---|-------------------------------|----------------------------------|--|
|  | Thin Rectangular Plate | $m = \int abt$ | $I_{xx} = \frac{1}{12} ma^2$ $I_{yy} = \frac{1}{12} mb^2$ $I_{zz} = \frac{1}{12} m(a^2 + b^2)$ |
|  | Solid Right Circular Cylinder | $m = \rho \pi r^2 l$ | $I_{zz} = m r^2 / 2$ $I_{yy} = I_{xx} = m (3r^2 + l^2) / 12$ $I_{AA} = m (3r^2 + 4l^2) / 12$ |
|  | Thin Circular Plate | $m = \rho \cdot \pi r^2 \cdot t$ | $I_{xx} = I_{zz} = m r^2 / 4$ $I_{yy} = m r^2 / 2$ $I_{AA} = 3m r^2 / 2$ |
|  | Thin Cylindrical Shell | $m = 2\pi r t l$ | $I_{xx} = m r^2$ $I_{yy} = I_{zz} = m (6r^2 + l^2) / 12$ $I_{AA} = m (3r^2 + 2l^2) / 6$ |

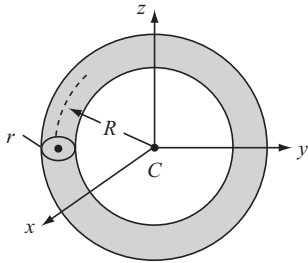
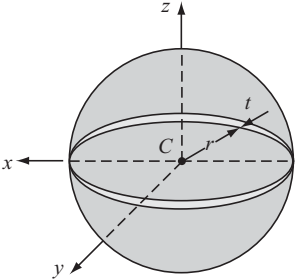
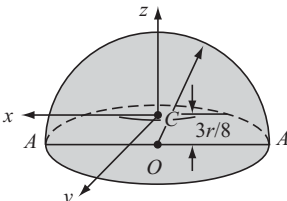
(Continued)

Appendix 2 (Continued)

| Figure | Description | Mass | Moment of Inertia |
|--------|---------------------------------|--|---|
| | Right Rectangular Solid Pyramid | $m = \rho ab l / 3$ $OC = l / 4$ | $I_{xx} = m (4b^2 + 3l^2) / 80$ $I_{yy} = m (4a^2 + 3l^2) / 80$ $I_{zz} = m (a^2 + b^2) / 20$ $I_{AA} = m (b^2 + 2l^2) / 20$ $I_{BB} = m (a^2 + 2l^2) / 20$ |
| | Right Circular Solid Cone | $m = \rho \pi r^2 l / 3$ $OC = l / 4$ | $I_{yy} = I_{xx} = 3 m (4r^2 + l^2) / 80$ $I_{zz} = 3 m r^2 / 10$ $I_{AA} = m (3r^2 + 2l^2) / 20$ $I_{BB} = 3 m (r^2 + 4l^2) / 20$ |
| | Solid Sphere | $m = 4 \rho \pi r^3 / 3$ | $I_{xx} = I_{yy} = I_{zz} = 2 / 5 m r^2$ |
| | Solid Ellipsoid | $m = 4 \rho \pi abc / 3$ | $I_{xx} = m (b^2 + c^2) / 5$ $I_{yy} = m (a^2 + b^2) / 5$ $I_{zz} = m (a^2 + c^2) / 5$ |

(Continued)

Appendix 2 (Continued)

| Figure | Description | Mass | Moment of Inertia |
|--|------------------------------|---|---|
|  | Solid Torus (like a tyre) | $m = 2\rho\pi^2 r^2 R$ | $I_{xx} = m(4R^2 + 3r^2)/4$ $I_{yy} = I_{zz} = m(4R^2 + 5r^2)/8$ |
|  | Thin Spherical Shell | $m = \rho \cdot 4\pi r^2 \cdot t$ | $I_{xx} = I_{yy} = I_{zz} = 2/3 m r^2$ |
|  | Solid Hemisphere | $m = \rho \cdot 2/3 \pi r^3$ $OC = 3r/8$ | $I_{xx} = I_{yy} = 83mr^2/320$ $\approx 1/4 m r^2$ $I_{zz} = 2/5 m r^2$ $I_{AA} = 2/5 m r^2$ |

Hints to Tutorial Problems

CHAPTER R2

- R Vector $\mathbf{P} = (10 - 3)\mathbf{i} + (5 + 1)\mathbf{j} + (8 - 2)\mathbf{k}$ $\sqrt{7^2 + 6^2 + 6^2}$.
- R The feasible path is from $P(0, 0)$ to x, y passing through $(3, 4)$ and then to the left horizontally to reach $(0, 10)$. Unit vector along the incline is $\bar{\mathbf{u}} = (3\mathbf{i} + 4\mathbf{j})/5$. One must go until $y=10$; hence $x = 7.5$.
Distance along the incline is $\sqrt{10^2 + 7.5^2} = 12.5$ units.
Total distance travelled = $12.5 + 7.5 = 20$ units.
- R $\bar{\mathbf{r}} = (-5 - 0)\mathbf{i} + (2 - 0)\mathbf{j} + (14 - 0)\mathbf{k}$
Magnitude of $\bar{\mathbf{r}}$ is $\sqrt{5^2 + 2^2 + 14^2} = 15$ units.
 $\bar{\mathbf{r}} = 15(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$; By comparison with above,
 $l = -5/15, m = 2/15, n = 14/15$, etc.
- R For an arbitrary triangle ABC , drop a perpendicular from B to side AC and use Pythagorus theorem to prove the two laws.
- R The direction of projection is along the cross product of A and B vectors which is given by
- $$\begin{vmatrix} 2 & -6 & -3 \\ 4 & 3 & 1 \end{vmatrix}, \text{ i.e., } 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}.$$
- Find the unit vector along it and multiply by 70 m/s.
- R By definition, $\bar{\mathbf{r}} \cdot \mathbf{B} = AB \cos 60$
Find $A = 101$ units, $B = \sqrt{18 + x^2}$ units and hence, by comparing,
 $x = -35.15$ units as the feasible answer.
- R Use the rules as stated in the text.
- R 8 (a) You need to show that
 $e_1 \cdot e_2 = 0,$
 $e_2 \cdot e_3 = 0$ and
 $e_3 \cdot e_1 = 0$
(b) Find $\bar{\mathbf{r}} \times \mathbf{B}$ and $\mathbf{B} \times \bar{\mathbf{r}}$.
Now show that the dot product of the two results is not zero; hence non-coplanar.
- R For (a), differentiate term-by-term. For (b), differentiate the scalar after dot product, $16t^3$ and get the answer $48t^2$. Likewise for (c) and (d).

R Velocity $\mathbf{v} = d \mathbf{r} / dt$ and substitute $t = 2$ seconds.
 Acceleration $\mathbf{a} = d \mathbf{v} / dt$ and substitute $t = 2$ seconds.

R Unit vector along $PA = (4-0) \mathbf{i} + 0 \mathbf{j} + (0-8) \mathbf{k} / \sqrt{4^2 + 8^2}$

Force along $PA = 20(4\mathbf{i} - 8\mathbf{k}) / \sqrt{80}$

Forces in other cables are $T_1(-\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}) / \sqrt{81}$

and $T_2(-2 \mathbf{i} - 3 \mathbf{j} - 8 \mathbf{k}) / \sqrt{77}$.

Equating the sum of all three \mathbf{i} and \mathbf{k} components and zero, $T_1 = 22$ kN, $T_2 = 28.6$ kN.

Forces exerted on the pole = sum of \mathbf{j} components which is 63 kN.

R Get $\mathbf{r}_{PQ} = -2 \mathbf{i} + 2 \mathbf{j}$ with $\mathbf{u} = (-2 \mathbf{i} + 2 \mathbf{j}) / 3$. The force is 100 N. The distance PO is $6 \mathbf{i} - 4 \mathbf{j}$ and the moment $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ can be found.

R Use the definitions and apply the rules.

- R (a) that \mathbf{u} is a constant vector or zero.
- (b) that the two are collinear or parallel or one is zero.
- (c) irrotationality!

R $\mathbf{r} \times \mathbf{V} = \begin{vmatrix} 2 & 3 & 4 \\ 2x & -2y & xy \end{vmatrix}$, etc.

R $\times \mathbf{B} = \begin{vmatrix} 100 & 25 & 0 \\ 0.01 & -0.001 & 0 \end{vmatrix}$, etc.

R $\omega = 2\pi \times 50/60 = 5.236$ rad/s
 For C_1 , $\mathbf{r} = 0.5 \mathbf{i}$, $\mathbf{v} = \omega \times \mathbf{r}$, etc.
 For C_2 , $\mathbf{r} = 0.2 \mathbf{i} + 0.5 \mathbf{j}$, $\mathbf{v} = \omega \times \mathbf{r}$, etc.

R 8 Construct the force triangle with the desired conditions and show that the components are 200 and 282.6 N. For equally inclined components,

$$F_1 \cos \theta + F_2 \cos \theta = R$$

$$F_1 \sin \theta = F_2 \sin \theta; F_1 = F_2 = R / (2 \cos \theta)$$

As θ increases, $\cos \theta$ decreases until $\theta = 90^\circ$ where $\cos \theta = 0$ and F_1 and F_2 tend to infinity.

R For equilibrium,
 $-T_1 \cos 30 - T_2 \cos 25 - T_3 \cos 30 = -1000$
 and $T_1 \sin 30 \cos 30 - T_3 \sin 30 \cos 30 = 0$
 and $T_1 \sin 30 \sin 30 + T_2 \sin 25 - T_3 \sin 30 \sin 30 = 0$
 which provide, $T_1 = T_3 = 356.6$ N, $T_2 = 421.9$ N.

R The resultant force $100(0.6 \mathbf{i} + 0.8 \mathbf{j})$ must be the vectorial sum of $F_1(0.5 \mathbf{i} + 0.5 \mathbf{j} + 0.707 \mathbf{k})$, etc. Find the values of the magnitudes F_1 , F_2 and F_3 .

CHAPTER 2

While at $(2, 1)$, it exerts a moment $(2 + 1) \times (-50)$, i.e., $100 - 50$ about O . Now, while at $(1, 2)$, it exerts $50 - 100$. The difference is required to maintain equivalence.

By the principle of parallel transfer of force, it is necessary to accompany with a moment 100×0.25 for O and 100×0.5 for B .

Resolve them into components and add, e.g., $150 \cos 30 + 150 \sin 30 + 180 \cos 45 - 180 \sin 45 \dots$

Resolve them into x , y and z components and add.

They result in a couple of moment 0.3×10 Nm clockwise normal to the plane. If one force was 11 N, the resultant force would be 1 N in that direction and additional moment of 0.15×1 Nm.

Resultant force is zero but the resultant moment about any arbitrary point is -30 Nm.

Let the fourth force be F located at a distance x from A . Write the equations for resultant action and solve for F and x .

- 8 Proceed by considering the resultant as the summation of forces and summation of moments about any arbitrary chosen points.

Summation of forces and moments about any chosen point provides the answer. Addition of a force along the bar is made vectorially.

The resultant force is $(3 - 0.5)$ kN. Find the magnitude and the angle.

The pivot provides force-reactions to balance the forces. Sum of moments is zero. If the 10 N force moves to P , there is a net moment!

Proceed by summing the forces and moments and locating the single equivalent force.

The equivalent action at O should consist of F and M which are summations of forces and their moments about O .

Resolve the loads into x and y components and add. Find the magnitude of the load and the unit vector only along which it acts.

Consider the sum of forces and moments about the fulcrum.

Single resultant = $-400 \cos 60 - (-300 - 200 + 400 \sin 60)$ with the moment $400 \sin 60 \times 0.3 \times 50 - 200 \times 0.5 = F_y \times x$.

at A and $B = 3/0.4 = 7.5$ N. The force R referred to A is 100 acting at A together with a moment 100×0.2 in the z -direction. Equivalent couple force are 50 N, 0.4 m apart. Hence the replacement.

- 8 $= -50 + 80 - 50$ passing through O and $M_0 = 1250 - 500 + 750$. Find the unit vector along and the component of the moment along it is $M_0 = 351.1$ units. This could be eliminated by transferring the force parallel to itself such that $\times = n$.

Find the angle θ_1 of the 50 N force with the base diagonal as 20.4 degrees. The angle θ_2 between the x -axis and the base diagonal is 68.2 degrees. Write for the force and then the moments as \times . Find the resultant force and moment.

Consider two parallel elementary discs of radii r_1 and r_2 of the spheres at a distance $(r-x-y)$ between them. Write the expression for the elementary force of attraction and integrate over the radii of the spheres.

Consider two rod elements equidistant from the point mass. The horizontal components of the forces cancel and the vertical components add up. Mass per unit length of the rod is $M/2a$. Integrate the force from 0 to a .

Limiting friction = $0.3 \times 20 \times 10^3 = 6$ kN. With relative motion the frictional force decreases.

It all depends upon the masses and coefficient of friction between different pairs of surfaces. Draw a figure with, say, five plates and apply a horizontal force at the mid plate 3 with plates 2 and 1 above it and 4 and 5 below it. If F is greater than $\mu_3 (m_1 g + m_2 g + m_3 g)$ and $\mu_2 (m_1 g + m_2 g)$ is greater than F_1 then the plate number 3 and those above it will slide. Like this, discuss different possibilities.

Resultant force = $k + k + k$ but $+ = -$ by the parallelogram law of forces. Therefore, the resultant is zero. Moment about an arbitrary point equals 2 times area of the triangle. A polygon can be considered to be composed of constituent triangles; hence the fact!

Resultant = $200 - 400 - 500$ units. If it is passed through the point O , the accompanying moment would be $3 \times (-500) + 6 \times (-400)$.

$M_A = 4 \times 4 \sin 40 - T \times 6 \cos (90 - (180 - (90 + 30 + 40)))$. Equate it to zero for equilibrium and find T .

Consider a particle of mass m at a distance d from the earth. Then,

$$GM_e \cdot m/d^2 = GM_m \cdot m/(3.8 \times 10^5 - d)^2$$

Find d .

- 8 The upward force at B must equal the weight of the punch, 2×9.81 N. Force in $AB = 2 \times 9.81/\cos 30$ and the moment = $0.2 \times$ force in AB .

Express the force in components and express the position vector $\vec{r}_{E \text{ or } B}$ to find the moment about C , similarly, about H . Moment along the axis CF is given by the dot product of the moment about C with the unit vector along CF .

Proceed by taking summation of forces and take moments about the point D . Next, employ the 'parallel force transfer' principle to get the answer for the second part.

CHAPTER 3

Consider the free-body diagram of the boat including force exerted by the flow and tension T in OC to find their magnitudes. Now draw the f.b.d without T and determine the tensions in OA and OB . Both being positive, the boat will remain in equilibrium.

Consider the forces at O for equilibrium and apply the Lami's theorem.

For a concurrent force system, $\Sigma M = 0$. Check if it is true in this case.

Find ΣM and $\Sigma F = 0$. Let $(x + y) \times 8 = -12 + 15$.

For (d), yes by choosing three forces which add to 8 and their $\Sigma M = 12 - 15$.

First, consider the equilibrium at O . Force in OC is 29 kN.

Then, $P = 29 \cos 15 = 28$ kN.

First, consider the f.b.d of the cylinder, and then for the link taking into account the 570 N force exerted by the cylinder normal to the link.

For (a), apply the Lami's theorem. For (b), substitute for T and W ; $d = 10$ m.

- 8 Find the tensions from the f.b.d of the rod. Determine x from A so that $\Sigma M_A = 0$.

First, use $\Sigma M_0 = 0$ for the arm. Then, at a time t , let the load be at x and at $t + \Delta t$, it be at $x + \Delta x$. By subtraction and using $x = vt$, $dx = v dt$ get the answers! Substitute $\theta = 30^\circ$ at the end.

From the f.b.d of cylinder 1, get the force in the bar as 1200 N. Then, from the f.b.d of the cylinder 2, get P .

For the tetrahedron joining the four centres, locate the centre of the top ball; $\sin \theta = 1/\sqrt{3}$. From the f.b.d of the top ball, $3N \cos \phi = mg$. For a lower ball, $N \sin \theta = 2T \cos 30^\circ$.

For equilibrium at B apply the Lami's theorem.

Use $\Sigma M = 0$ and take moments about the axis 2 – 3 and then about axis CD for $\Sigma M = 0$.

Taking the origin at O , the unit vector along OA is

$(b\mathbf{i} - a\mathbf{j} - h\mathbf{k})/\sqrt{b^2 + a^2 + h^2}$. Also, find for OB and consider the equilibrium at O .

Tension in BC is $T_1(-8\mathbf{k} + 4\mathbf{j} - 2\mathbf{i})/\sqrt{84}$ and in DE is

$T_2(-5\mathbf{k} - 4\mathbf{j})/\sqrt{41}$. Consider $\Sigma M_A = 0$ for equilibrium. $R_x = 0$, $R_y = 840$ and $R_A = 5100$ N.

Locate B , C , G and H . Find the unit vector along BH and BG and consider the f.b.d of the boom.

CHAPTER 51

Reaction at D consists of x and y components, D_x and D_y . Reaction at roller E is only horizontal, E_x . Take moments about E and then at D and use $\Sigma F_y = 0$.

Consider equilibrium at A ; then at C and B .

Consider equilibrium at A to get forces in AB and AE . Then consider equilibrium at B and then at E . Find reaction at D and force in DC by equilibrium at D .

Determine the inclinations of AB and AC by geometry. Reactions are found by considering the equilibrium of whole truss. By equilibrium at A , find forces in AB and AC . Force in BC can be found by considering equilibrium at B or C . Same way, attempt the next problem.

Reaction at R has both components A_x and A_y ; reaction at D is only vertical, D_y . Find them by equilibrium of the total truss. Then consider equilibrium at D , C , F and A in turn.

Determine F_x , F_y and E_y . Cut a section through AH , AC and BC and consider the equilibrium of the upper part. Either cut another section through HG , CG and CD or proceed by taking joints B , H , C and D , in turn.

Determine F_x , F_y and D_y by equilibrium of the whole truss. Then use the method of joints by taking D , C , H , etc., reaching F at the end only.

Find the reactions A_x , A_y and C_y first. Then take up joints C , D and A or B , in turn. One of the last two can be used for verification.

- 8 Find the reactions A_x , A_y and B_y by equilibrium of the whole frame. Consider the f.b.d. separately for BEC and ADC and DE . Each of them is in equilibrium. BE must be a 2-force member (only in compression) because reaction at B must be vertical only.

Reactions can be found by considering that Σ and Σ are both zero at all hinges. All forces have 2 or 3 components depending upon the geometry of the members.

There being only horizontal force P at B and only one member to take horizontal force, force in BC must be zero and AB must be in tension. Then consider the joint B .

CHAPTER S2

- (a) Find $R_B = 48.3$ kN, $R_D = 11.7$ kN. SF from A to B is 20 kN; it becomes 28.3 beyond B and rises to 11.67 beyond C , until D . BM is zero at A , changes and drops linearly to 50 kNm at B and rises linearly to 85 kNm at C and drops linearly to zero again at D .
- (b) Reaction at A is a force 6 kN upward and a 17 kNm moment anticlockwise. The SF is 6 kN from A to B , reduces linearly to zero at C and remains zero C to D . BM is 17 kNm at A , rises linearly to +1 at B , rises parabolically to +10 kNm at C and remains constant until D .
- (c) The 10 kN force implies that there is a 10 kN force plus a 10 kNm moment at C . Find R_A as 14.4 kN and $R_D = 9.6$ kN. SF changes from 14.4 to 5.6 at B ; to 15.6 at C , to 6 kN at D , stays 6 kN until E and then drops linearly to zero at F . Likewise BM varies linearly between AB , BC , CD and DE . At C , it changes from 17.6 to 27.6 kNm. From E to F , it drops to zero parabolically.

SF remains constant at M/L from A to B . BM rises linearly from zero at A to Ma/L at P ; drops to $M(L - a)/L$ at P and decreases linearly to zero at B .

Find $R_A = 5.44$ kN, $R_c = 7.56$ kN.

SF at A is 5.44 kN, rises parabolically to 3.56 at B and linearly to 5.76 at C , where it drops to 2 kN and then rises to zero at D . The BM diagram consists of parabolic and cubic segments with 3.52 kNm at B and 1 kNm at C .

Reactions are 2.5 kN each.

$$SF = W/L x + 2.5 W - 2W - 2W + 2.5W.$$

Likewise,

$$BM = W/L x^2/2 - 2.5 (x - L/3) + 2W (x - 4L/9) \text{ etc.}$$

For mid-load,

$$BM = W/2 \cdot L/2 + W/2 \cdot L/8 = 0; BM = 3/16 WL$$

For the train reaching the pier B ,

$$BM = 3/4 \cdot L/2 + W \cdot L/4 = 0; BM = 1/8 WL \text{ (This is less).}$$

Reaction at the fixed end is 4 kN upward force and a 12 kNm moment. SF from free end to mid point is 2 kN, changes to 4 kN at mid point and remains constant until the fixed end. BM is zero at the free end, changes linearly to 4 at mid point or the linearly to 12 kNm at the fixed end.

Loading; A to C should be 2.5 kN/m ud; C to E 5 kN/m ud. No load from E to B . BM diagram must be parabolic where SF is linear; or linear where SF is constant. BM is 5 kNm at C , +17.5 kNm at D , +15 kNm at E .

- 8 SF should be linearly varying from zero at A to 80 kN at mid point, where it rises to +80 kN; it again drops to zero at B . The loading must be 'uniformly distributed' from A to B , with a mid-load (or reaction) R for equilibrium.

CHAPTER S3

Reaction at front wheels is $4/7 Mg$; hind wheel $3/7 Mg$. For rear-wheel drive, the maximum frictional force equals $0.7 \times 4/7 Mg = 0.4 Mg$, which is also the maximum possible force to accelerate the car. For just front wheel drive, $F = 0.7 \times 3/7 Mg = 0.3 Mg$. For the 4-wheel drive

$$F = 0.7 Mg$$

which is the sum of the above two types of drives!

From the f.b.d. of the wedge, at the verge of impending motion, frictional forces $\mu_1 N_1$ and $\mu_2 N_2$ act upon it where N_1 is the normal reaction above and N_2 is the normal reaction below it.

$$F = \mu_1 N_1 + \mu_2 N_2 \cos \theta + N_2 \sin \theta.$$

$$N_1 = W; N_2 \cos \theta - \mu_2 N_2 \sin \theta = N_1, \text{ etc.}$$

At the position of impending slip, the horizontal reaction at the upper end is N_1 and the vertical reaction at the lower end is N_2 , together with frictional force μN_2 opposing motion. Take moments about the lower end or equate to zero. From equilibrium, $N_1 = 296$, $N_2 = 1185$ N. Determine the inclination as 70° and $h = 4.41$ m.

For the f.b.d. of the block,

$$W = N \cos \theta - \mu N \sin \theta$$

$$F = N \sin \theta + \mu N \cos \theta.$$

Solve the equations to get F in terms of W .

For the f.b.d. of the block,

$$W = \mu N \sin \theta + N \cos \theta \text{ and}$$

$$\mu N \cos \theta - N \sin \theta = Wa/g, \text{ etc.}$$

Let the pivot be at ' hL ' from the left as the force P is applied at ' KL ' from the left end. Consider an element dx , frictional force on that element equals $\mu (m/L) g \cdot dx$. Integrate over the left of the O and right of O or using P as an external force, the net force must be zero. Likewise, the net moment about O must be zero.

From the f.b.d. of the peg for equilibrium, $W \sin \beta$ equals or is less than $(\mu_1 N_1 + \mu_2 N_2)$ and $W \cos \beta = N_1 + N_2$. Taking moments about the pegs, proceed to determine $\tan \beta$.

- 8 Draw the f.b.d. of the ball for impending sliding motion. Application of F at height h causes a turning moment Fh about the point of contact but $F(h - r)$ about the centre of mass.

Draw the f.b.d. at the instant of impending sliding at both surfaces. Use summation of forces and moments to zero to solve for equilibrium and get $P = 3/8 W$.

Draw the f.b.d. of the block and solve for equilibrium at the instant of impending motion.

The reaction at the corner point must pass through the centre of the cylinder. $0.275 P \cos 30 + 0.08(P \sin 30 - 25g) = 0$

whence $P = 70.5$ N

Draw the f.b.d. at that instant and solve for equilibrium, assuming μ as the coefficient of friction. Get $\mu = 0.21$.

Employ $T_1/T_2 = e^{\mu\theta}$ and θ is 3.5 times 2π radians.

Employ $T_1/T_2 = e^{\mu\theta}$ and notice that the friction coefficients are μ_1, μ_2, μ_3 , etc. and θ s are different for all cylinders.

For a screw jack,

$$F = W \tan (\alpha + \phi) \times r/L$$

Use $\tan \phi = 0.1$

Here, $\theta = 4$ times 2π radians.

$$T_1 = 450 \text{ g and } T_2 = 45 \text{ N}$$

Find μ by using $T_1/T_2 = e^{\mu\theta}$.

Employ $T_1/T_2 = e^{\mu\theta}$ to find T_2 .

8 Use the relations $\mu = FL$ and $F = W \tan (\alpha + \phi) \times r/L$ for the jack.

Let the pressure be expressed as $p = Ar^2 + C$. The total force P is given by

$$P = \int_0^R p \cdot 2\pi r dr \text{ and } T = \int_0^R 2\pi r^2 \cdot p \cdot \mu \cdot dr.$$

Notice that $\theta = 2\pi$ radians. If P is less than 658.6 N, there is no lifting; it just stays there. Minimum P required to hold it in position is 15.2 N, which is $100/(658.6/100)$ N.

Proceed by assuming that the pressure p acts uniformly at the area of contact. Integrate it over the limits to get the total force P .

CHAPTER 4

Consider an element of arc of length $R d\theta$. Its moment about the diameter is $x \cdot R d\theta$. Substitute $x = R \cos \theta$ and integrate 0 to π to get $2R^2$. Now, $y_c = 2R^2 / \int R d\theta$.

Consider a strip of area $dA = x dy$. Use $y = \sin x$ and $dy = \cos x dx$. Now, $y_c = \int y dA / \int dA$.

Take an elementary area contained by radii r and $r + dr$ and angle $d\phi$ at ϕ from one end. $dA = r d\phi dr$ and $y = r \sin \phi$. Apply $y_c = \int y dA / \int dA$.

Divide the trapezium into a right angled triangle and a rectangle. Find the x_c and y_c for each and hence for the composite area, e.g.

$$x_c = (x_{c_1} A_1 + x_{c_2} A_2) / (A_1 + A_2)$$

For y_c , consider a strip parallel to x -axis; $y_c = \int y(xdy) / \int xdy$. Substitute the given relation and integrate zero to 1.

Divide the L-section into two rectangles, locate x_c and y_c for each and proceed.

Volume = $\pi(R_2^2 - R_1^2)/h$ in principle. Here,

$V = \pi(2.5^2 - 2.3^2) \times 0.5 + (2.3^2 - 2.2^2) \times 0.25$. Alternatively, locate the radius of the centre of arc r_c and also the area of the given cross section.

Volume = $2 \pi r_c \times A$.

- 8 Consider the centroids of the composite areas $(-\pi d^2/4)$ and (πd^2) and proceed.

Using Pappus and Guldinus theorem, $A = (2\pi r) \times (2\pi R)$.

Draw the intersecting curves tentatively, intersecting in the first quadrant. Locate the points of intersection $(0, 0)$ and $(6.87, 7.86)$. Now,

$x_c = \int x dA / \int dA = \int x(y_2 - y_1) dx / (y_2 - y_1) dx$ and simplify. Similarly proceed for y_c .

Use $y_c = (y_{c_1} m_1 - y_{c_2} m_2 + y_{c_3} m_3) / (m_1 - m_2 + m_3)$ where suffix 1 stands for the complete perspex cylinder, 2 for the hole and 3 for the lead filling.

$y_c = 0$ by symmetry about the x -axis. Side of the square is $R/\sqrt{2}$ and area is $R^2/2$, etc.

$x_0 = 0$ by symmetry about the y -axis. Volume = $\pi/3 (r_1^2 (h_1 + h_2) - r_2^2 h_2)$. Use the fact that y_c for a complete cone is at $h/4$ from the base.

For the hemisphere, $y_c = (R - 3/8R) = 5/8R$ and $V = 2/3\pi R^3$. For the cylinder $y_c = R + h/2$, etc.

For equilibrium, the centre of gravity should be vertically below the point of suspension and hence at the lowest position.

For the circular arc, $l_s = \pi R$ and $Y_{cs} = 2R/\pi + A$. Find a such that

$L_s \cdot y_{cs} + 2a \cdot a \cdot 2 = (L_s + 2a) \cdot a$. Get $a = \sqrt{2}R$ and total length = $(\pi + 2\sqrt{2})R$.

The centroid of the semicircular wire, which is at $2R/\pi$ from the diameter should be vertically below the point of suspension, for equilibrium. If W is suspended from B , then take moments about the hinge and equate the sum to zero.

- 8 Employ the Pappus–Guldinus theorem. For a hemisphere, $4\pi r^2/2 = \pi r \times$ distance covered by the generating arc πy_c . Hence $y_c = 2r/\pi$ and $x_c = 0$. Similarly y_c for a semicircular disc = $2/3\pi r^3 / (\pi r^2/2) = 4r/3\pi$.

For the arc, $y_c = 0.2 + 2 \times 0.15/\pi = 0.295$ m; $l = \pi \times 0.15$ and proceed.

CHAPTER 5

Integrate $a = \frac{dv}{dt}$ w.r.t. t in order to get $v = t^4/4 - t^3 + 5t + c_1$ and evaluate the constant c_1 by substituting $t = 1$ s, $v = 6.25$ m/s. Integrate again to get s .

From $a = \frac{dv}{dt}$, find v and then find s using $v = 0$ at $t = 0$ and $s = 0$ at $t = 0$. It will stop again when $v = 0$. Find t at that instant.

Final velocity is $(2 - 3) \times 10$ more than the initial velocity! Determine the x , y and z components by employing $s = ut + 1/2at^2$ along each direction.

At $t = 0$, $x = 0$, $y = 3$, $z = 0$; at $t = 25$, $x = 20$, $y = 3$, $z = -4$. Determine the constants A , B and C to come out as 5, 0 and -2 respectively. Then find the velocity at $t = 5$ s.

Let $x = x_0 \cos \omega t$, $v = -\omega x_0 \sin \omega t$ and $a = -\omega^2 x_0 \cos \omega t$ referred to an extremity. Use $x_0 = 1$ m, $r = 2\pi/\omega = 2$ s.

Let $x = x_0 \sin \omega t$, referred to the mid-point. Here $\omega = \pi/3$. Use the boundary conditions.

Here $\tau = 0.6 = 2\pi/\omega$. At the mean position, $v_{\max} = \omega x_0 = 1.5$. At half way position, $0.5 = \sin \omega t$. So, $\omega t = 0.523$ and $v = 1.5 \cos 0.523 = 1.3$ m/s.

- 8 For the projectile path, $x_{\max} = v \cos \theta \cdot t$ and $z_{\max} = v \sin \theta \cdot t/2 - \frac{1}{2} g (t/2)^2$. Equate x_{\max} to z_{\max} and use the fact that $z = 0$ at x_{\max} ; $t = 2v \sin \theta/g$.

The horizontal distance between them, $30 \cos 30^\circ$ is the sum of the distances traversed by the bullets, i.e. $350 \cos 30^\circ \cdot t + 300 \cos (-30^\circ) \cdot t$ whence $t = 0.0462$ s, etc.

For simplicity consider the target at x_m horizontally $= v \cos \theta \cdot t$ and $t = 2v \sin \theta/g$. $x_m = v^2 \sin 2\theta/g$. Since $\sin \psi = \sin (\pi - \psi)$, $\theta_1 = 1/2 \sin^{-1} (gx/v^2)$; $\theta_2 = \pi/2 - \theta_1$.

$z = x \tan \alpha - gx^2/(2v^2 \cos^2 \alpha)$. For $x = 5z \geq 2$ and $x = 0, z = 0$. From the equation, $\sin 2\alpha = 0.2725$, $2\alpha = 15.81^\circ$ and 164.2° . Hence find the appropriate α .

$z = -120 = x \tan \alpha - gx^2/(2r^2 \cos^2 \theta)$. Find x for $\theta = 30^\circ$. Use $x = v \cos \theta \cdot t$.

At any time t , separation between them is given by $x = h - (2v \cos \alpha t)$ and $z = z_2 - z_1$. Distance d between them is $\sqrt{(x^2 + z^2)}$. For it to be minimum, differentiate it w.r.t. time and equate to zero!

For the first half distance $s/2 = \frac{1}{2} g(t-1)^2$ and for the next half distance, $s/2 = g(t-1) + 1/2 g \cdot 1^2$. Equate the two expressions and proceed.

The path of the ball is given by $z = -gx^2/(2 \times 1.5^2 \times 1)$, i.e., $z = -2.18x^2$. Find the point of its intersection with $z = -x$, the slope of the tips of the stairs; $x = 0.45$ m which means the third step.

For the path, $z = \tan \theta - gx^2/(2V^2 \cos^2 \theta)$, substitute $x = 12$ m, $z = -2$ m, $\theta = 30^\circ$, and with $g = 9.81$ m/s², determine the speed V as 37 km/hour.

Assuming that the helicopter is descending with a vertical u , initial height h , use $h = ut + 1/2 gt^2$. For the second packet, the height is $(h - 5u)$, etc.

- 8 Obtain $dx/dt = 8t$ and also dy/dt by using $xy = 16$. Substitute $t = 1$ second to get the velocity components. For $x = 2$ m, get $t = 0.707$ second and proceed the same way.

Given that, for $\alpha = 90^\circ$, $h = z_{\max}$

Now, $x_{\max} = V_0^2 \sin 2\alpha/g = 2h$ at $\alpha = 45^\circ$.

Proceed by taking $s/2 = 1/2 a t_1^2$, $s/2 = 1/2 r t_2^2$, $t = t_1 + t_2$, $a t_1 = r t_2$.

Given that $\alpha = kt = d\omega/dt$. Intergrate w.r.t. time and substitute the data.

Draw a line joining G to E and write vector expressions for position, velocity and acceleration.

First differentiate the given relations w.r.t. time for velocity and acceleration and then substitute the data each time.

The time taken for the bullet to hit it is expressed as $h/(V \cos \alpha - v) \tan \alpha$ from $\tan \alpha = h/x'$ and $x = x' + vt$ where x' is the horizontal distance at the instant of firing and x is at the instant of striking the shell. Then, find h by using the expression $V \sin \alpha t - 1/2 gt^2$.

Use $z = 1/2 gt^2$ to find the time and $x = V_x \cdot t$. $x/h = \tan \theta$; $\theta = 48.9^\circ$.

Differentiate r and θ w.r.t. time and use $\dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r\dot{\theta} \mathbf{e}_\theta$ and substitute $t = 3$ seconds. Differentiate again w.r.t. time to get the acceleration.

Notice that $\omega = 10$ rad/s,

$\alpha = 2$ rad/s, $\dot{r} = 3$ m/s, $\ddot{r} = 2$ m/s², $r = 0.5$ m. Substitute in the terms for velocity and acceleration in cylindrical co-ordinates.

- 8 From the given expression for r and θ , determine, r , θ , at $t = 20$ seconds and differentiate twice to get velocity and acceleration respectively in cylindrical co-ordinates.

From $y = 2x^3$, $\dot{y} = 6x^2 \cdot \dot{x}$. Now, $\mathbf{v} = \dot{x} \mathbf{i} + \dot{y} \mathbf{j}$ and $|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} = 3$ whence $\dot{x} = 0.493$ and $\dot{y} = 2.96$ m/s. Similarly, from $y = 2x^3$, get \ddot{y} and proceed.

By energy conservation until A and B , $V_A = V_B = V = \sqrt{2gh}$; $h = 0.5 + l \cos 30^\circ$. The acceleration at A is $g \cos 30^\circ$ along the slope and hence $-g \cos 60^\circ$ tangential to the arc; the radial component being V^2/r towards the centre of the arc. Using $v = 5.18$ m/s from the above, obtain the total acceleration.

Use $\mathbf{a}_{p/f} = \mathbf{a}_p + \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{p/f} + 2\boldsymbol{\omega} \times \mathbf{v}_{p/f} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{p/f})$. Place the moving frame at B on the link AB .

Then $\alpha = 4$, $\omega = 3$, $\mathbf{r}_{p/f} = 2 \times 5 = 10$, $\mathbf{v}_{p/f} = 4 + 50$, $\mathbf{a}_0 = 0$. Substitute these values above to get the result.

$\mathbf{a}_p = 5$ m/s radial plus 5×10 , i.e. 50 m/s tangential velocity. It can never reach B .

From $x = l \tan \theta$, get \dot{x} by using $\omega = 2\pi N/60$ rad/s. Again, differentiate and substitute the data and get the acceleration.

Consider the chain at an instant t when a length x has dropped and $(l - c - x)$ is flat on the table. Consider the tension T in the chain at the edge; then, $T + (c + x)w/g \ddot{x} = (c + x)w$ or $T = (c + x)w(1 - \ddot{x}/g)$ for the vertical part. Also, $T = (l - c - x)w/g \ddot{x}$ for the horizontal part. The two to get the differential equation describing the motion. Solve the equation as well.

$V_r = V \sin \alpha (= \dot{r})$, $L_r = l \sin \alpha (= r)$. Now, $\mathbf{a} = -r\omega^2 \mathbf{e}_r + 2\dot{r}\omega \mathbf{e}_\theta$. The magnitude of acceleration is given by $\sqrt{l \sin \alpha \omega^2 + (2V \sin \alpha \omega)^2}$ which simplifies to yield the answer.

CHAPTER 6

The splash of water is heard after the stone drops a distance and the sound travels up the same distance. Hence $2.5 = \sqrt{2d/g} + d/330$ which gives $d = 28.6$ m.

Here $9\sqrt{x} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}$ whence $v dv = 9x^{1/2} dx$, $v = \sqrt{12}x^{3/4}$

Now $v = \frac{dx}{dt}$. Substitute and integrate to get $x = 67.5$ m, $v = 81.6$ m/s and $a = 73.9$ m/s².

For the free-body diagram of the block, write the equations of motion along the horizontal and vertical axes. Also, write for the free-body diagram of the wedge. Solve the equations.

Find the acceleration by using $v = u + at$ and write the equation of motion along the incline for the verticle. Then

$$F = 2000 \times 0.33 + 200 + 2000 \times 9.81 \times 1/20 = 1848 \text{ N.}$$

For the total system of masses accelerating at a , the net external force along the motion is $m_3g - \mu_1 m_1 g - \mu_2 m_2 g$; equate it to $(m_1 + m_2 + m_3)$ times acceleration. Then, consider the free-body diagram of the blocks m_3 and m_1 and solve the equations of motion.

Consider the free-body diagram of block B . $ma = \mu m g$; $a = \mu g$. The force F on block A causes resistance $\mu m g$ and inertia force ma on B .

Net force on the object is $mg - kv^2 = ma = m \frac{dv}{dt}$. Separating the variables, $dt = dv / (g - k/m \cdot v^2)$.

Intergrate it logarithmically and obtain the expression.

- 8 For the upward motion of the particle, $-mg - kv - ma = 0$. Expressing $a = \frac{dv}{dt}$ and separating the variables and integrating both sides logarithmically, get the answers. For the second case $-mg - kv^2 - ma = 0$ and proceed the same way.

From the free-body diagram of the block A , $F = \mu_A N_A - T \cos \theta - Ma = 0$ and $N_A - Mg - T \sin \theta = 0$. Eliminate N_A . Now consider block B and get $T = m(a + \mu_B g) / (\cos \theta + \mu_B \sin \theta)$. Then use the given values and solve for T and a .

For the position at an angle θ , $mv^2/r - mg \cos \theta - N = 0$ and $N = 0$ at the instant of leaving the sphere. Referred to the initial position, mechanical energy is conserved. Solve for θ .

For the conical pendulum, at the instant shown, $T \cos \theta - mg = 0$ and $T \sin \theta - m v^2/r = 0$. Use $v = r \omega$ and $\sin \theta = r/l$ to get the answers.

For the free-body diagram of the rotating block,

$$T \cos 30 = m \omega^2 r + N \cos 60 \quad \text{and}$$

$$T \sin 30 = mg - N \sin 60 .$$

Solve the equations. At the instant it loses contact, $N = 0$. Find T and ω .

Kinetic energy of the wagon just before it touches the spring is $1/2 m v^2$ and the potential energy stored in the spring after compression is $1/2 (k_1 x_1^2 + k_2 x_2^2)$. Equate the two, using $x_2 = 2/3 x_1$. Next, for the free

body diagram of the wagon, $ma + kx = 0$. Using $a = \frac{dv}{dt} \cdot \frac{dx}{dt}$, separating the variables and integrating we get $-v^2/2 = k/m \cdot x^2/2 - 50$, etc.

For (a), $t = 22/V \cos 45$, $V \sin 45 - t - 1/2g t^2 = 0$.

For (b), $t = 22/V \cos 45$, $V \sin 45 - t - 1/2g t^2 = -2$.

Use conservation of mechanical energy principle,

$$v = \sqrt{2 \times 9.81 \times 3 (\cos 45^\circ - \cos 30^\circ)} = 3.06 \text{ m/s}$$

For the free-body diagram at the top position, $m v^2/r - m g - N = 0$. For no contact, $N = 0$, $V = \sqrt{gr}$.

Equating the mechanical energies at the top and bottom positions, $1/2 m V_t^2 + mgr = 1/2 m V_b^2 - mgr$. Solve for V_b .

- 8 At the top position, $T_t = m \omega^2 r - m g$ and at the bottom position,

$$T_b = m \omega^2 r + m g .$$

If mass is halved, both tensions are also halved. If ω is halved, substitute $\omega = \pi$ and solve.

At the highest position, $m V^2/r = m g + N$ and $N = 0$ for loss of contact.

The centripetal force, mV^2/r must be less than or equal to the frictional force μmg .

At the top position of the loop, $v \geq \sqrt{gr}$ and at the bottom position, $1/2 m v^2 + mg \times 2r = 1/2 m v_b^2$; $v_b \geq \sqrt{5gr}$. If it starts from a height h , $mgh = 1/2 m v_b^2$. Then, $h = 2.5r$ for it to complete the loop. For $h = 0.5gr$, it will loop until the top point and separate thereafter!

Use conservation of angular momentum in the absence of external moment, i.e., $d/dt (I\omega) = 0$ and hence $\omega r^2 = \text{constant}$. Differentiate it to get $\alpha r^2 + 2r \dot{r} \omega = 0$ and proceed.

Find the work done in lifting the elementary length dx from the lower end and then integrate it from 0 to 20 m.

As such, the bullet speed reduces by 360 m/s. What should be the thickness, with the same properties of the plank, to reduce the speed to zero, i.e., by 600 m/s? It is the energy lost by the bullet which equals the work done in overcoming resistance by the plank material!

Find the kinetic energy of the bob when it just passes the vertical position. Then use the simple pendulum of 1 m length to determine where the potential energy becomes zero! That's the angle required.

Apply the 'conservation of angular momentum' principle because there is no external moment acting upon it!

Apply the 'conservation of linear momentum' principle because there is no external force acting on the system. Water is assumed to float the boat frictionlessly and the force by the goat is internal to the goat-plus-boat system.

- 8 Use geometry to determine the change in length of the spring. The energy released by the spring should equal the work done!

CHAPTER 7

$v_A = \omega_{0A} \times r_{0A}$ and $v_B = \omega_B \times r_B = v_A + v_{BA}$. Find ω_B and v_{BA} . For maximum v_B , differentiate w.r.t. θ and equate to zero. Use the fact, $0.2 \sin \theta = 1 \sin \phi$. When the angle at A is 90° , find θ . The position has maximum acceleration at $\theta = 0 = \pi$. Then, $V_B = 0$.

$$v_C = -10 \text{ m/s}, \omega = +100 \text{ rad/s}, \alpha_A = \omega \times (\omega \times r_{AC}) + \alpha \times r_{AC} = 100 \text{ m/s}^2 \text{ along } A \text{ to } C.$$

$$v_C = 10 (\cos 30^\circ + \sin 30^\circ), \omega = -25 \text{ rad/s}, v_A = v_C = v_{AC} = 18.66 + 5 \text{ m/s}.$$

$d = s \sin 60^\circ = (10 + 3 \cos 60^\circ) \tan \phi$, $\phi = 12.74^\circ$. Write for ω , v_A , V_C and find v_{CA} , etc. Then, $\omega = -V_C/15 = -0.87 \text{ rad/s}$ and determine v_C and solve for α .

For a plane problem of kinematics, one may proceed graphically by drawing a line oa perpendicular to OA , equal to 0.524 m/s. Also, from a , draw a line perpendicular to $O_2 B_1$, etc. Finally, from o draw a line ox perpendicular to O and measure it.

Let θ be up the slope and ϕ normal to it. $\omega = 5 \text{ rad/s}$, $\alpha = -2 \text{ rad/s}^2$, etc. and substitute terms in the expression for a_c .

$v_B = 10 \times 5.25$, $\alpha_B = -10 \times 23$. One may draw the velocity diagram and the acceleration diagram or proceed vectorially.

- 8 $V_C (\cos 45^\circ - \sin 45^\circ) = 2 + V_{CA} (-\cos 45^\circ - \sin 45^\circ)$.

Obtain $V_C = V_{CA} = 1.41$ m/s and ω of CA is $1.414/0.2 = 7.07$ rad/s.

$20 = 0 + \omega \times 0.25 = -0.25 \omega$ whence, $\omega = -80$ rad/s. Then,
 $v_C = 20 + (-80 \times 0.25) = 40$ m/s.

Let a moving frame of reference be placed on pin P fixed with CD and let Q be on the link AB . Use cosine law and sine law. $Q = 12.5 - 6.4$ and $P = Q + PQ$. Finally, ω of $CD = V_P/12 = -10.7/12 = -0.89$ rad/s.

From geometry, $AO = 11.33$ cm, $OP = 10$ cm, etc. Take A' on CD , coincident with A . Then $v_{A'} = v_A + v_{A'A}$. Also, from geometry, $\theta = 27.9^\circ$. Finally, ω of $OP = 99.7/10 = 9.97$ rad/s.

In this case, angular momentum must be conserved. $\Sigma m v r = \text{constant}$, $r^2 m \dot{\omega} = \text{constant} = R^2 m \omega_0^2$. From geometry of moment,

$R = r + 2\pi a$. $\theta/2\pi = r/a + \theta$. Express r and \dot{r} and integrate w.r.t. t to get the desired expressions.

Let the inner cylinder rotate at ω_i . The velocity at the bottom is $r_3 \omega_i$. For 'no-slip' condition, this is also the velocity of the inside of the hollow cylinder. The tangential velocity of the hollow cylinder is $r_3 \omega_i / r_2 = v / r_1$, whence $\omega_i = r_2 / (r_1 r_3) \cdot v$. Similarly, proceed for the angular acceleration.

The velocity of C must be the average of the two. It can also be determined by equating the values of the rotational velocity ω of the cylinder as seen from C , both above and below; $(\dot{x}_1 - v) / R = (v - \dot{x}_2) / R$.

Write the expressions for v_A and v_B and use $v_B = v_A + v_{BA}$. Solve the equation.

From the geometry, $\angle OCQ = 25.7^\circ$, $\angle OCQ = 94.3^\circ$, etc. Find V_C and V_{CP} and hence compute V_B and V_A , taking into account the fact that A can move only horizontally.

From the geometry, find the angle ACB and proceed either graphically or vectorially.

CHAPTER 8

- 8 Take a strip of width dy and length $2x$ at a distance y from the x axis. Use the equation $x^2/a^2 + y^2/b^2 = 1$ and substitute $x = a \cos \theta$ and $y = b \sin \theta$. Integrate, $\theta = -\pi/2$ to $+\pi/2$.
- 8 Equate the I_{xx} to I_{yy} , i.e. $bh^3/12 = hb^3/48$. Hence $b = 2h$.
- 8 Consider an elemental strip of length a , cross section $dx \times dy$ and find I_{cc} . Integrate between the limits $-a/2$ to $+a/2$.
- 8 Consider an elemental disc of radius r_0 at a distance z from the apex. Then, $r_0/z = R/h$ and $m = \rho \cdot 1/3 \pi R^2 h$. Find I_{zz} by double integration.
- 8 The proof is given in Ex. 8.14 for a solid sphere, set R_1 to zero and $R_2 = R$.
- 8 Consider a circular disc of radius y at a distance x from the y axis. First, find the volume and the mass of the ellipsoid. Then, use $I_{xx} = my^2/2$ for the disc and integrate.
- 8 $I_{xx} = I_{yy} = m R^2/4 = 100 R^2$. Further, $I_{AA} = 100 R^2 + 400 (0.5)^2 = 109$ etc.
- 8 8 Consider an element dx at a distance x from the centre. Find I_{yy} about the desired points and substitute $l = 30$ cm.
- 8 Polar moment of inertia $I_0 = \pi D^4/32 = \pi R^4/2$ for the circular section and $\pi r^2/4$ for the semicircle removed therefrom.

- 8 First find the height of the centroid above the base, 4.48 cm and then find I_{cc} by using components.
- 8 Find I_{xxc} by the method of components. Locate $x_c = 2.14$ cm and find I_{yye}, I_{zzc} .
- 8 The given section is equivalent to the circumscribing rectangle minus two semicircles. Area of the net section is 101.46 cm^2 . Then locate $x_p = 2.12$ cm and find the I_{xx} and I_{yy} .
- 8 It is equivalent to a composite hollow cylinder of diameters 20 cm and 3 cm minus a hollow cylinder of diameters 18 cm and 6 cm with widths 5 cm and 4 cm respectively. Use $I_0 = m(R_1^2 + R_2^2)/2$.
- 8 For the composite body,

$$I_0 = m_{\text{rod}} l^2/12 + m_{\text{rod}} (l/2)^2 + 2/5 m_{\text{bob}} R^2 + m_{\text{bob}} (l + R/2)^2.$$
- 8 For the flywheel considered as a solid cylinder of radius 0.5 m width 4 cylinders of radii 0.1 m each removed,

$$I_0 = M \times 0.5^2/2 - 4 (m \times 0.1^2/2 + m \times 0.2^2), \text{ where}$$

$$M = \pi \times 0.5^2 \times 0.1 \times 7500 = 580 \text{ kg and } m = 23.5 \text{ kg.}$$

CHAPTER 9

Applying the equation of motion along the horizontal direction to the free-body diagram of the body, $F - \mu mg = ma$, whence, $a = 1.06 \text{ m/s}^2$. The cabinet would tip about the corner B if the moment Fh of the applied force exceeds the resisting moment $50 \times 9.81 \times 0.5 \times 1.06 \times 0.7$ where 50×1.06 is the inertia force.

The ball can leave the frame only if the system is accelerated to the right with an acceleration a . Then, $mg \sin 30 - ma \sin 30 - N_1 \cos 30 = 0$. At the verge of loss of contact, $N_1 = 0$; $a = g \tan 30 = 5.66 \text{ m/s}^2$.

$I = mk^2$; $T \cdot r = mk^2 \cdot \alpha$ and $T + W/g \cdot a - W = 0$. From kinematics, $\alpha = a/r$. Find T and a .

For the free-body diagrams, $N - m_1g - T = 0$ and $T + m_2a_2 - m_2g = 0$. Also, $T \cdot r - fR = I\alpha$ and $a_1 = R\alpha$ by kinematics. Find a_1 and α . Use $s = ut + 1/2 a t^2$ to find t and $\omega_f = \omega_i + \alpha t$ to find ω_f .

For the free-body diagrams of the pulley and the two weights,
 $T_1 - W_1 - W_1a/g = 0$, $T_2 - W_2 - W_2a/g = 0$ and $(T_2 - T_1) \cdot r = mk^2 \cdot \alpha$. Also, from kinematics, $a = r\alpha$. Solve the equations.

For the two discs of polar moments of inertia I_1 and I_2 respectively,
 $T_1 - f r_1 = I_1 \alpha_1$, $f r_2 = I_2 \alpha_2$ and by kinematics, $a = \alpha_1 r_1 = \alpha_2 r_2$. Eliminate f and α_1 to get the answer.

For the free-body diagram of the sphere (with just a force F at the centre and a frictional force f at the base) $F - f = ma$ and $f \cdot r = I\alpha$. Using $a = r\alpha$, and $I = 2/5 mr^2$ for the sphere, $a = 5F/7 m$. For the cylinder, $I = mr^2/2$ and $a = 2F/3 m$.

- 8 $F \cdot p = I_0 \cdot \alpha$ and $a = \alpha \cdot l/2$. For no horizontal component at the pivot,

$$F = \mu a = m\alpha l/2 \text{ and } p = 2I_0/m l.$$
 For a bar, $I_0 = ml^2/3$; $p = 2/3 l$.
 For a cylinder, $I_0 = mr^2/2 + mr^2$; $p = 3/4 l$.
 For a sphere, $I_0 = 2/5 mr^3 + mr^2$; $p = 7/10 l$.

Given that $v = 138.9 \text{ m/s}$ and $r = 5000 \text{ m}$ Resolving forces along the wing of the plane, $mv^2/r \cos \theta = mg \sin \theta$; $\tan \theta = v^2/gr$, $\theta = 21.4$.

Kinetic energy = $1/2 mV_c^2 + 1/2 I_c \omega^2$; $\omega = V_c/r$.

For a cylinder, $I_c = mr^2/2$, $KE = 3/4 m V_c^2$

For a sphere, $I_c = 2/5 mr^2$, $KE = 7/10 mV_c^2$ which is less than that for a cylinder.

For the bar, $F = R = M \cdot a_b$. Taking moments about A , $F \cdot L/2 = R \cdot L = I_A \cdot \alpha_b$. Using $a_b = L/2 \cdot \alpha_b$ and $I_A = mL^2/3$ for the bar and $R = m a_d$ and $I_0 = m r^2/2$, $a_d = r \cdot \alpha_d$ for the disc, solve the equations.

$I_0 = 1/3 m (L/2)^2 \times 2 = mL^2/6$. It will have the maximum angular velocity at the mean, i.e. the lowest hanging position when $y = L/(4\sqrt{2})$. Use conservation of mechanical energy, i.e. $(PE + KE) = \text{constant}$.

In the inclined position of the bar under horizontal acceleration, $mg \sin \theta = m a \cos \theta$; $\tan \theta = a/g$. For an acceleration along an incline at an angle ϕ with the horizontal, $\tan \psi = (a \cos \phi)/(a \sin \phi + g)$.

For the free-body diagram of the bar, $N_1 + f_2 - mg = m a_y$ and $N_2 - f_1 = m a_x$. Taking moments about the centroid, using $I_c = mL^2/12$ and $a_x = L/2 \cos \theta$. α , also using $f_1 = \mu_1 N_1$ and $f_2 = \mu_2 N_2$. Substitute the data!

When the lever accelerates, the suspended masses need be considered to result in forces T_1 and T_2 respectively, $T_1 = m_1(g + a)$, $T_2 = m_2(g - a)$.

Also, $(T_2 - T_1) \times 0.25 = I_c \alpha$ and $\alpha = 4a$. If the masses were not suspended but attached at the ends, one could simply apply $\alpha = MI_c$ where M is the net moment due to weights and I_c is for the composite system.

For the rod, $I_0 = mL^2/3$. The roller reaction must be vertical only;

$R - mg = m a_y$ and $a_x = 0$. Use $a_c = L/2 \cdot \alpha$. Find R .

At what instant of slip, draw the free-body diagram of the bar making an angle θ with the horizontal. Now, $I_0 = mL^2/12 + m a^2$, $h = a \sin \theta$ and $m g h = 1/2 I \omega^2$. Also, $-mg a \cos \theta = I \cdot \alpha$. Solve the equations.

8 By angular momentum conservation, $0.25 \times 500 \times 1 = I\omega = 7.5$ units.

$$I_0 = \int_{0.5}^{1.5} (m/L)x^2 dx = 21.67. \text{ Find } \omega \text{ and then } V_E = r \cdot \omega.$$

Initial kinetic energy = $1/2 I\omega^2 = 1.3 \text{ J} = mg(1 - \cos \theta)$.

From $1/2 mv^2 = mgh$, $v = 4.43 \text{ m/s}$.

Now, $I\omega + m V r = m v r = 132.9 \text{ kg m}^2/\text{s}$ where V is after striking.

Using $\omega = V/r$, $I = 1/2 Mr^2$, solve the equations.

$KE = 1/2 I\omega^2 + 1/2 mV^2 = 110.3 \text{ J}$, $PE = 147.2 \text{ J}$, then find

$$V_f \text{ from } 1/2 (I\omega_\phi^2 + mV_f^2) = 257.5 \text{ J}. V_f = 2.54 \text{ m/s}$$

From the free body diagrams of the cylinder and the sphere, taken separately, $F + f_1 - mg \sin \theta = -m a$, $f_2 - F - m g \sin \theta = -m a$ $f_1 r = I_1 \alpha$, $f_2 r = I_2 \alpha$ and $I_1 = m r^2/2$, $I_2 = 2/5 m r^2$, solve the equations.

From the free-body diagram of the shell $W = R$, $f = \mu R$, $F - f = W/g \cdot A$ where A is the horizontal acceleration. Also, $F \cdot a = W/g \cdot A \cdot r/2$ and solve the equations.

$I_0 = 1/12 mL^2 + m (L/2 - a)^2$. Equate the loss in potential energy the gain in its kinetic energy and get the answer.

Loss in potential energy of the small disc ($5 \times 9.81 \times 0.1 \times 2$) equals the gain in the kinetic energy for the same movement,

i.e., $1/2 I_{c\text{-big}} \cdot \omega^2 + 1/2 mV_c^2 + 1/2 I_{0\text{-small}} \cdot \omega^2$. Hence, the answer.

Loss in kinetic energy from the given initial velocity and coming to rest equals the gain in the potential energy up the incline.

For the free-body diagrams of the cylinder and the falling weight, write the equations of motion and use the kinematics equation, $a = r\alpha$.

First find the force in the bar BC by employing the triangle law of forces. For the free-body diagram of the cylinder with no force along AB , as the string is cut, write and solve the equations of motion. The cylinder has rotational kinetic energy for rotation about C .

Draw the free-body diagram with no force in the string, as the string is cut, write and solve the equations of motion.

- 8 Loss in potential energy equals gain in kinetic energy of the body. Rolling down the same distance along the same incline, they lose equal potential energies. The cylinder, with higher moment of inertia, would achieve less velocity to acquire the same kinetic energy. Hence the sphere reaches first. For the sphere, $mg L \sin \theta = 1/2 mV_s^2 + 1/2 \cdot 2/5 mR^2 \omega_s^2$. Similarly for the cylinder, $mg(L - 0.12) \theta = \text{change in kinetic energy}$. Dividing one by the other and solving, $L = 3.48$ m.

Draw the free-body diagram at the instant of removal of pin B , with zero reaction at B . Write and solve the equations of motion together with the kinematic condition, $a_c = r \times \alpha$ about the pivot A .

The kinetic energy of the block is due to translation of the mass horizontally and its rotation about the given axis. The two, being scalar, are additive!

CHAPTER D1

As it reaches the floor first time, $V_1 = \sqrt{2gh}$. After the first impact, $V_1' / V_1 = e \cdot \sqrt{2gh} = \sqrt{2ge^2h}$. Similarly, after the second impact $V_2' = \sqrt{2ge^4h}$ since it rises to $h/2$, $e^4h = h/2$; $e = 0.814$.

Along the line of impact $m_1V_{1n} + m_2V_{2n} = m_1V_{1n}' + m_2V_{1n}'$ whence

$V_{1n}' = 1.155, V_{2n}' = 2.02$ m/s. In the absence of relative slippage, tangential components of velocities remain unaltered.

Hence, $V_1' = \sqrt{1.155^2 + 0.5^2} = 1.26$ m/s, $V_2' = 2.52$ m/s.

Along the line of impact, $V_1 \cos(90 - \theta) = V_1' \sin(\theta - 30) + V_{2n}'$ and $(V_{2n}' - V_{1n}') / (V_{1n} - V_{2n}) = 0.8$. Given that V_1' is along the line towards the pocket, find θ !

$(V_{2n}' - V_{1n}') / (0 - 30) = 0.7$ and $30 \times 0.15 = 0.2V_1' + 0.15V_{2n}'$, which provide $V_1' = 21.86$, $V_1' = 0.86$ m/s. Retarding force is μmg . Use the work energy principle to find the distance s .

Now, $V_1 = \sqrt{2(1 - \cos 30)g}$ and $(V_1' - V_{2n}') / V_1 = 0.8$. By momentum conservation, $mV_1 = m(V_1' + V_{1n}')$. Next, equate the loss in kinetic energy to gain in potential energy to find the rise and hence the angle.

For the elastic impact, $20 \cos 30 - (-8 \cos 60) / (V_{2n}' - V_{1n}') = 1$ and employ the momentum conservation to get the answers. For the plastic impact, $V_{2n}' = V_{1n}'$ the two being stuck together and $0.5 V_{1n}' + 0.2 V_{2n}' = 7.86$. Solve the equations to get the answers.

Initial angular momentum about the pivot = $0.02 \times 1 \times 400 / \sqrt{2} = 5.66 \text{ kg m}^2/\text{s}$ Finally, $0.02 v \times 1 + I\omega = 5.66$. For perfectly elastic collision, there being no energy loss, equate the kinetic energies that are required to reach the vertically upward position, the bar will continue to rotate at 1154 radius/second. For perfectly plastic collision, the object is embedded on the bar. Find the swing of the bar by finding the potential energy gained.

When the object pierced through the bar,

$$0.02 (400 - 200) / \sqrt{2} = 0.96 \omega \text{ and } 1/2 I \omega^2 = mg \times 0.6 (1 - \cos \theta) \theta = 49.5^\circ$$

8 Here, $-V_n' / 25 \cos 70 = 0.75$ and the tangential velocity

$$V_t' = V_n' = 25 \sin 70.$$

Initial kinetic energy = $1/2 \times 0.018 \times 600^2$ and $m_1 V_1 = (m_1 + m_2) V'$, whence $V_n' = 2.155^2 \text{ m/s}$. The final kinetic energy being

$$1/2. (5 + 0.018) \times 2.155^2 = 11.6 \text{ J only, the remaining, i.e., 99.65\% is lost.}$$

By momentum conservation, $V' = (1 \times 60 - 15 \times 20) / (1 + 15)$. The difference between the total initial and final kinetic energies is manifested in noise, deformation, friction and heating!

Now $m v = (M + m) V$. Retarding frictional force = $(M + m) g \cdot \mu$. By the work-energy principle, $1/2 (M + m) (m / (M + m))^2 v^2 = (M + m) g \cdot \mu \cdot s$. Solve for the distances.

For the first collision, both momentum and kinetic energy being conserved,

$$MV = mV_1' + mV_2'; V = V_1' + V_2' \tag{i}$$

$$1/2 mV^2 = 1/2 mV_1'^2 + 1/2 mV_2'^2; V^2 = V_1'^2 + V_2'^2 \tag{ii}$$

Squaring (i) and comparing with (ii) shows that $V_1' V_2' = 0$.

Since V_2' cannot be zero for a body on a smooth surface, $V_1' = 0$.

Hence one body moves at a time and the motion propagates to the last body.

If d approaches zero, the four bodies are taken together as a single entity of mass 4 m. Then, $mV = mV_1' - 4mV_2'$, etc. would show that

$$V_1' = -3/5 V \text{ and } V_2' = 2/5 V.$$

$$V_1 = \sqrt{2g \cdot 9} \text{ and } V_1' \sqrt{2g \cdot 6 \cdot e} = V_1' / V_1 = \sqrt{6/9} = 0.82.$$

Draw a tangent at C. Then $V_t = V \sin \theta = V_t' = V' \sin (90 - \theta) = V' \cos \theta$.

$$\text{Now, } e = V' \cos (90 - \theta) / V \cos \theta = V' / V \tan \theta \quad \tan^2 \theta.$$

$$V_1 = \sqrt{2g \times 2} = 6.26 \text{ m/s. By momentum conservation,}$$

$$30 \times 6.26 = 30 V_1' + 10 V_2' \text{ and } (V_2' - V_1') / 6.26 = 1 \text{ whence } V_2' = 9.4 \text{ m/s}$$

$$\text{Now, } 1/2 \times 10 \times 9.4^2 + 10 \times g \delta = 1/2 \times 20 \times 1000 \times \delta^2 \text{ whence, } \delta = 0.23 \text{ m.}$$

$$V_1 = \sqrt{2g \times 2.4} = 6.86 \text{ m/s and } (0 - V_n^1) / 6.86 \cos 15 = 0.8; V_{1n}' = 5.30 \text{ m/s, etc.}$$

Determine V_1 for the small bob before impact and V_2' for the bigger bob after the impact. Conservation of mechanical energy for the bigger bob provides the angular displacement. For equal masses and elastic impact, θ must be the same as the initial angle of the striking bob.

CHAPTER D2

$r_0 = 6370 + 900 = 7270$ km. For circular orbit,

$$V_0 = \sqrt{GM/r_0} = \frac{\sqrt{3.986 \times 10^{14}}}{7270 \times 10^3} = 7410 \text{ m/s}$$

Orbital time $= 2\pi \times 7270/7410 \times 1000 = 6165$ seconds $= 1.712$ hours.

At perigee, $r_{\min} = 300 + 6370 = 6670$ km, at apogee $r_{\max} = 36370$ km.

$a = (6670 + 36370)/2 = 21520$ km and

$$b = \sqrt{6670 \times 36370} = 15575 \text{ km} \quad 21520\sqrt{1-e^2}; \quad e = 0.69.$$

Determine $D = 1/p$ from trigonometry and apply central force formulae.

$$V_0 = 20 \times 10^6 / \sqrt{(10,000 + 6370 \times 1000)} = 4940 \text{ m/s}; \quad KE = 1/2 \times 5000 \times 4940^2 = 6.1 \times 10^{10} \text{ J}$$

Total energy expended on it is

$$GMm/(2r_0) = 3.986 \times 10^{14} \times 5000 / (2 \times 16370 \times 1000)$$

Escape velocity equals

$$\sqrt{2GM/r_0} = (2 \times 3.986 \times 10^{14} \times 0.0123 / (6370 \times 1000 \times 0.273))^{1/2}$$

$$KE \text{ required} = 1/2 \times 15,000 \times 2375^2 = 4.2 \times 10^{10} \text{ J.}$$

$$C = 7745 \times 6.624 \times 10^6 = 5.13 \times 10^{10} \text{ m}^2/\text{s},$$

$V_c = \sqrt{GM/r_0} = \sqrt{3.986 \times 10^{14} / 6.624 \times 10^6} = 7757$ m/s. Since the velocity is less than V_0 , it must be an elliptic orbit. Its escape velocity would have been $7757 \times \sqrt{2} = 10,970$ m/s. The satellite must have been launched at the perigee, $r_{\min} = 6.624 \times 10^6$ m. Find d , then $e = 0.0046$. Use ellipse geometry and find the time as half of the time period, $2\pi a b/C$.

$$V_e = \sqrt{\frac{2GM}{r_0}} = \sqrt{2 \times GM/(r+R)} \text{ irrespective of the direction of escape.}$$

$$E = \sqrt{GMm/r_0} = \sqrt{GMm/(r+R)}$$

Assuming the moon to orbit on a circular path, i.e., ellipse with low eccentricity, use the expression for time period and for velocity along a circular orbit.

- 8 The problem reduces to finding V_0 for an elliptic orbit such that the apogee is at 9370 km. Calculate the virtual perigee from the geometrical data.

CHAPTER 10

Let the virtual displacement of C the point of application of force be δx . Then the virtual displacement δy of B is given by

$$(a \cos \theta + \delta x)^2 + (a \sin \theta + \delta y)^2 = a^2 \text{ whence, } \delta y = -\delta x \tan \theta.$$

Virtual work $P \cdot \delta x + 2\alpha F \cdot \delta y = 0$ whence, $F = P/2 \tan \theta$. At $\theta = 45^\circ$, $F = P/2$.

Let ED be virtually displaced through an angle $\delta \theta$. $\sin \theta$.

Virtual work $F_{CD} \cdot a \delta \theta \sin \theta - P/2 a \delta \theta = 0$ whence $F_{CD} = 0.577P$.

Let the virtual displacements of AB and BC be through an angle $\delta\theta$. $\delta x_m = 0.1 \delta\theta \sin \theta$, $\delta y_N = 0.2\delta\theta \cos \theta$.

$2x \delta x_M + (1 + 1.5) \delta y_N \quad M\delta\theta = 0$ whence, $M = 0.533 \text{ kNm}$

For virtual displacement at P , $\delta x = l\delta\theta$ and $\delta y = -l\delta\theta \cos \theta$.

$-F_1 \times \delta y + (-F_2) \times \delta x \times 2 = 0$, whence $F_1/2 F_2 = \tan \theta$.

In a 2-degree of freedom system, one can vary θ_1 and θ_2 independent of each other. Let us keep θ_1 constant and change θ_2 by $\delta\theta_2$.

Then, $\delta x_2 = l \delta\theta_2 \cos \theta_2$, $\delta y_2 = -l \delta\theta_2 \sin \theta_2$ and $F \cdot \delta x_2 = mg \delta y_2 = 0$;

$\tan \theta_2 = F/mg$. Now change only θ_1 by $\delta\theta_1$ and proceed to get

$\tan \theta_1 = F/2 mg \delta\theta$

Initial height of centroid is $R + h/2$. After virtual displacement, it is $(R + h/2) \cos \theta + R\theta \sin \theta$. Find the drop δh and $PE = -mg \delta h$. For equilibrium, $\partial PE/\partial \theta = 0$ whence, $R/(h/2) = \tan \theta/\theta$.

For a virtual displacement of $\delta\theta$, $\delta y_A = l_1 \delta\theta \sin \theta$ and $\delta y_B = -l_2 \delta\theta \sin \theta$. Find PE and $\partial PE/\partial \theta = 0$ for equilibrium. Also the second derivative should be greater than zero for stability.

- 8 The centroid should be at the centre of the circle, as in a solved example, $1/2 \cdot 4/3 \cdot \pi R^3 \times 3/8 R - \pi R^2 H \cdot H/2$ whence, $H = R/\sqrt{2}$ for stable equilibrium.

For equilibrium, $R/(h/2) = \tan \theta/\theta$; $\theta = h/2R \cdot \tan \theta$.

For stability, $R \cos \theta - h/2 (\sin^2 \theta/\cos \theta + \cos \theta) > 0$. Using the fact, $h = \pi R/2$ $\theta = \pi/4 = 45$ degrees on either side of the top position.

The centroid is located by $1/2 \cdot 4/3 \cdot \pi R^3 \cdot 3/8 R - 1/3 \pi R^2 \cdot HH/4$; $H = \sqrt{3}R$.

The buoyancy force acts at the centre of buoyancy, B but the body oscillates about a metacentre M .

From fluid mechanics, $BM = I/V = \pi R^4/4$.

$1/(\pi r^2 d) = r^2/(2 \cdot 4h)$. $BC = h/2 - d/2 = 0.2h$. $CM = r^2/(2 \cdot 4h) - 0.2h$. For stable equilibrium, M should be above C or at most coincide with C , i.e., $r/h = 0.693$.

CHAPTER 11

For the initially displaced position, the restoring force on the mass is $2 \cdot P \sin \theta = 4Px/l$. The equation of motion is $4P/L + md^2x/dt^2 = 0$, whence $\tau = 2\pi\sqrt{ml/4p}$.

Mass of the bob = 1413.7 kg, $I = 2/5 \times 1413.7 \times 0.5^2$, $\tau = 0.124$ second.

Equivalent spring constants, k are: for (a), $40 + 50$, for (b),

$40 \times 50/(40 + 50)$ and for (c), $40 + 50 + 40$. For each case,

$50 d^2x/dt^2 + kx = 0$ and $\omega_n = \sqrt{k/50}$, $f = \omega_n/2 \pi$ and $v_{\max} = x_0 \omega_n$, $a = x_0 \omega_n^2$

Restoring force in the displaced position is $-2x \cdot \pi d^2/4 \cdot \rho g$ which equals mass times acceleration, $l \cdot \pi d^2/4 \cdot \rho \cdot D^2 x/dt^2$. Solve to get $f = 1.29$ Hertz. No change with d . If l is halved, f becomes $\sqrt{2}$ times. If the U-tube is inclined at 30 degrees with the horizontal, restoring force is $\sin 30$ times and g is replaced by $g \sin 30$. Then, f becomes $l/1/\sqrt{2}$ times.

$I_0 = mr^2/2$ where m is the mass of the semicylinder. Also,

$I_0 = I_c + m (4r/3\pi)^2$ and $I_p = I_c + m (r - 4r/3\pi)^2$, etc. Restoring moment = $mg \cdot 4r \cdot 3\pi \cdot \theta$. It can, therefore, oscillate. For the equation of motion, find τ .

Displacements $x_1 = l_1 \theta$ and $x_2 = l_2 \theta$

and $-2kx_1l_1 + mgx_2 = m l_2^2 \cdot d^2\theta/dt^2$. Solve and get τ .

$PE = 2 \cdot 1/2, k x_1^2 - m g l_2 (1 - \cos \theta)$, $\partial P/\partial E = 0$ and get the answer.

Restoring force in the spring $= -k x = -k \cdot l/2 \cdot \theta$,

Restoring moment $= -k l/2 \theta \cdot l/2 - m_1 g l/2 - m_2 g (L + r)$

I_0 for the system $= m_1 l^3/3 + m_2 r^2/2 + m^2 (r + l)^2$

Apply Euler's equation $M = I_0 \cdot \alpha$, and solve for f . Substitute the data now!

- 8 Initial deflection $\delta = (L/2 + l) \cdot \alpha$. Write the equation of motion and get f and f_{\max} by differentiating f and equating to zero.

Deflections in the springs are $\delta_4 = mg/2k_4$, $\delta_3 = mg/2k_3$,

$\delta_1 = mg/4k_1$, $\delta_2 = mg/4k_2$. Deflection of P is the average of the two springs. Then find ω .

For displacement θ subtended at the bottom of the disc, horizontal displacement x in each spring is $(d/2 + a) \theta$. Now, write the equation of motion and find τ .

$I_0 = m l^2/12 + m (l/4)^2 = 7/48 m l^2$ about the pivot. Restoring moment $= m g l/4$. Hence $m g l/4 \theta + 7/48 m l^2 \cdot d^2\theta/dt^2 = 0$; $\tau = 2\pi\sqrt{7l/12g}$.

Initial extension due to hanging mass $= R\theta$. Restoring moment $= k R \theta \cdot R$ about the axis of rotation of the pulley. $(I + mR^2) d^2\theta/dt^2 + kR^2 \theta = 0$.

Consider the slightly displaced position towards right of the equilibrium by x . Find the reactions, $R_1 = W(a + 2x)/2a$, $R_2 = W(a - 2x)/2a$. The frictional force f_1 tends to restore it and frictional force f_2 tends to take it away. Hence $-f_1 + f_2 = W/g \cdot d^2x/dt^2$. Use $f_1 = \mu R_1$ and $f_2 = \mu R_2$, etc. and find ω_n and f as required.

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