HEAT TRANSFER

Introduction :-

Definition of heat transfer :-

→ Heat transfer may be defined as the transmission of energy from one region to another region as a result of temperature difference.

→ The study of heat transfer is carried out for the following purpose.

1. To estimate the rate of flow of energy as heat through the boundary of a system under study both study and transient conditions.
2. To determine the temperature field under steady and transient conditions.

→ The areas covered under the discipline of heat transfer are :-

- Design of thermal and nuclear power plants including heat engines, steam generators, condensers and other heat exchange equipments.
- Internal combustion engines.
- Refrigeration and air conditioning units.
- Design of cooling systems for electric motors, generators and transformers.
- Thermal control of space vehicles.

→ Differences between thermodynamics and Heat transfer :-

Thermodynamics	Heat transfer
1. It deals with the equilibrium states of matter, and precludes the existence of a temperature gradient.	1. It is a non equilibrium process. (since temperature gradient must exist for exchange of heat to take place.
2. when a system changes from one equilibrium state to another equilibrium state, thermodynamics helps to determine the quantity of work and heat interactions. It describes how much heat is to be exchanged during a process but does not hint how much the same could be achieved.	2. It helps to predict the distribution of temperature and to determine the rate at which energy is transfered across a surface of intrest due to temperature gradients at the surface, and difference of temperature between different surfaces.

→ Basic Laws of heat transfer :-

The following are the basic laws which govern heat transfer :-

1. First law of thermodynamics:-

when a system undergoes a thermodynamic cycle then the net heat supplied to the system from the sourroundings is equal to the net work done by the system on its sourroundings.

$$\oint dQ = \oint dW.$$

2. Second Law of thermodynamics:-

Heat will flow automatically from one reservoir to another at a lower temperature, but not in opposite direction.

3. Law of conservation of mass:-

This law is used to determine the parameter of flow.

4. Newton's law of motion:-

these laws are used to determine fluid flow parameters.

5. The rate equations:-

these equations are made applicable depending upon the mode of heat transfer being considered.

→ Modes of Heat Transfer:-

Heat transfer which is defined as the transmission of energy from one region to another as a result of temperature difference takes place by the following three modes.

i) conduction

ii) convection

iii) Radiation

1. conduction :-

The transfer of heat between two solid bodies is called conduction. It depends on the difference in temperature of the hot and cold body. Example of conduction heat transfer is two bodies at different temperature kept in contact with each other. Another example is heating one end of the metal like copper, due to conduction heat transfer the other end of the metal also gets heated. pure conduction is only found in solids.

→ Faurier's law of heat conduction :-

Faurier's law of heat conduction states as follows.

" the rate of flow of heat through a simple homogeneous solid is directly proportional to the area of the section at right angles to the direction of heat flow, and to change of temperature with respect to the length of the path of heat flow".

Mathematically it can be represented by

$$Q \propto A \frac{dt}{dx}.$$

$$Q = KA \frac{dt}{dx}.$$

where , Q = heat flow through a body per unit time.

A = surface area of heat flow, m^2.

dt = temperature difference in k or °c.

dx = thickness of body in the direction of flow, m.

K = constant of proportionality and is known as thermal conductivity of the body.

Thus $Q = -K \frac{dt}{dx}$.

The -ve sign of K is to take care of the decreasing temperature along with the direction of increasing thickness or the direction of heat flow.

Assumptions:-

1. conduction of heat takes place under steady state conditions.
2. temperature gradient is constant.
3. There is no internal heat generation.
4. the bounding surfaces are isothermal.
5. the material is homogeneous and isotropic.

Some essential features of fourier's law:-

1. It is applicable to all matter.
2. It is based on experimental evidence and cannot be derived from first principle.
3. It is a vector expression.
4. It helps to define thermal conductivity.

→ Thermal conductivity:-

$$Q = K A \frac{dt}{dx}$$

$$K = \frac{Q}{A} \frac{dx}{dt}$$

$$K = \frac{Q}{A} \frac{dx}{dt}$$

$$K = q \frac{dx}{dt} \qquad Q/A = q$$

where q = heat flux.

It is the amount of energy conducted through a body of unit area and unit thickness in unit time when the difference in temperature cause in the heat flow.

→ convection:-

" convection is the transfer of heat within a fluid by mixing of one portion of the fluid with another.

1. convection is possible only in a fluid medium and is directly linked with the transport of medium itself.

2. convection constitutes the macro-form of the heat transfer since macroscopic particles of a fluid moving in space cause the heat exchange.

3. the effectiveness of the convection depends upon the mixing motion of the fluid.

→ Newton's Law of cooling:-

The rate equation for the convective heat transfer between a surface and an adjacent fluid is prescribed by Newton's law of cooling.

$$Q = hA(t_s - t_f)$$

Q: Rate of conductive heat transfer

A = Area exposed to heat transfer

t_s = surface temperature.

t_f = fluid temperature,

h = co-efficient of convective heat transfer.

The units of h are,

$$h = \frac{Q}{A(t_s - t_f)} = \frac{W}{m^2\ ^oC} \text{ (or) } W/m^2\ ^oC \text{ or } W/m^2 K$$

→ Radiation:-

The heat transfer from one body to another without transmitting medium is known as radiation. It is an electro magnetic wave phenomenon.

The transfer of heat by radiation occurs because hot body emits more heat than its receives and a cold body receives more heat than it emits. Radiant energy requires no medium for propagation and will pass through vaccum.

Laws of Radiation:-

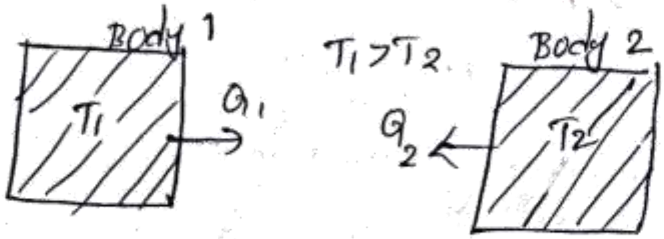
1. Wien's law:- It states that the wave length λ_m corresponding to the maximum energy is inversely proportional to the absolute temperature T of the hot body.

i.e $\lambda_m \propto \frac{1}{T}$ (or) $\lambda_m T = \text{constant}$.

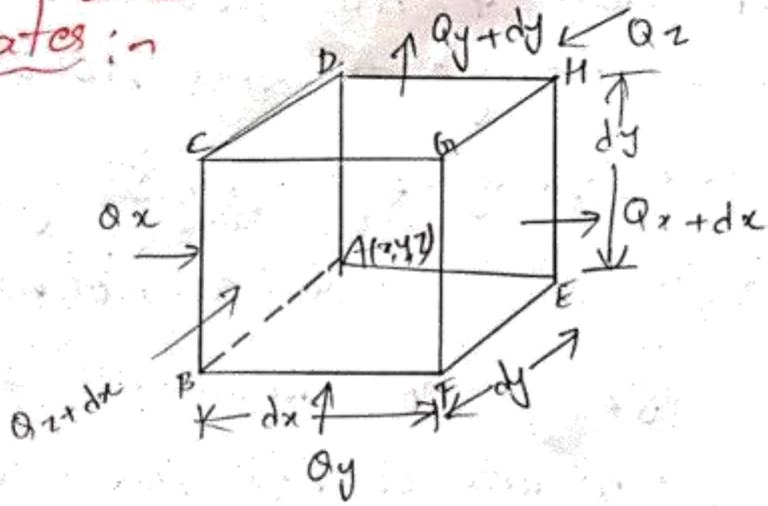
2. Kirchhoff's law:- It states that the emissivity of the body at a particular temperature is numerically equal to its absorptivity for radiant energy from body at the same temperature.

3. The stefan - Boltzmann law:- the law states that the emissive power of a black body is directly proportional to fourth power of its absolute temperature.

i.e. $Q \propto T^4$



→ General heat conduction equation in cartesian coordinates :-



Consider a small rectangular element of sides dx, dy, dz as shown in fig.

The energy balance of this rectangular element is obtained from first law of thermodynamics.

$$\Rightarrow \begin{bmatrix} \text{Net heat} \\ \text{conducted into} \\ \text{element from} \\ \text{all the coordinate} \\ \text{directions} \end{bmatrix} + \begin{bmatrix} \text{Heat generated} \\ \text{within the} \\ \text{element} \end{bmatrix} = \begin{bmatrix} \text{Heat} \\ \text{stored} \\ \text{in the} \\ \text{element} \end{bmatrix} \rightarrow (1)$$

→ net heat conducted into element from all the coordinate directions :-

Let q_x be the heat flux in a direction of face ABCD and q_{x+dx} be the heat flux in a direction of face EFGH.

The rate of heat flow into the element in x direction through the face ABCD is

$$Q_x = q_x \, dy\, dz = -k_x \frac{\partial T}{\partial x} dy\, dz. \longrightarrow ②$$

where k - thermal conductivity, W/mK.

$\frac{\partial T}{\partial x}$ - temp gradient.

The rate of heat flow into the element in x direction through the face ~~ABCD is~~ EFGH is

$$\cancel{Q_x = q_x \, dy\, dz = -k_x \frac{\partial T}{\partial x} dy\, dz}.$$

$$Q_{x+dx} = Q_x + \frac{\partial}{\partial x}(Q_x)\, dx$$

$$= -k_x \frac{\partial T}{\partial x} dy\, dz + \frac{\partial}{\partial x}\left[-k_x \frac{\partial T}{\partial x} dy\, dz\right] dx$$

$$Q_{x+dx} = -k_x \frac{\partial T}{\partial x} dy\, dz - \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] dx\, dy\, dz \longrightarrow ③$$

subtracting ② - ③,

$$Q_x - Q_{(x+dx)} = -k_x \frac{\partial T}{\partial x} dy\, dz - \left[-k_x \frac{\partial T}{\partial x} dy\, dz - \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] dx\, dy\, dz\right.$$

$$= \cancel{-k_x \frac{\partial T}{\partial x} dy}\, dz + k_x \cancel{\frac{\partial T}{\partial x}} dy\, dz + \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] dx\, dy\, dz$$

$$\Rightarrow Q_x - Q_{(x+dx)} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dx\, dy\, dz. \longrightarrow ④$$

similarly,

$$Q_y - Q_{(y+dy)} = \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] dx\, dy\, dz \quad \text{——} (5)$$

$$Q_z - Q_{(z+dz)} = \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] dx\, dy\, dz \quad \text{——} (6)$$

Adding (4) + (5) + (6)

net heat conducted $= \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] dx\, dy\, dz +$

$$\frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) dx\, dy\, dz + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] dx\, dy\, dz.$$

$$= \left\{\frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right]\right\} dx\, dy\, dz. \quad \text{——} (7)$$

Heat stored in the element :-

w.k.T.

Heat stored in the element = mass of the element × specific heat of the element × Rise in temp of element

$$= m \times c_p \times \frac{\partial T}{\partial t} \qquad \rho = \frac{m}{V}$$

$$= \rho \times V \times c_p \times \frac{\partial T}{\partial t}$$

$$= \rho \times (dx\, dy\, dz) \times c_p \times \frac{\partial T}{\partial t}. \quad \text{——} (8)$$

Heat generated within the element :-

Heat generated with in the element is given by

$$Q = \dot{q}\, dx\, dy\, dz. \quad \text{——} (9)$$

substituting 7, 8, 9 in Equ 1

$$\Rightarrow \left\{ \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] \right\} dx\, dy\, dz$$

$$+ \dot{q}\, dx\, dy\, dz = \rho\, c_p \frac{\partial T}{\partial t}\, dx\, dy\, dz.$$

$$\Rightarrow \frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] + \dot{q} = \rho\, c_p \frac{\partial T}{\partial t}.$$

considering the material is isotropic

so, $k_x = k_y = k_z = k =$ constant.

$$\left\{ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right\} k + \dot{q} = \rho\, c_p \frac{\partial T}{\partial t}$$

Divided by k.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{\rho\, c_p}{k} \frac{\partial T}{\partial t}.$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \quad \text{--- (10)}$$

It is a general three dimentional heat conduction equation in cartesian coordinates.

α = thermal diffusivity $= \dfrac{k}{\rho\, c_p}$ - m^2/s.

Thermal diffusivity is nothing but how fast heat is diffused through a material during changes of temperature with time.

case 1) :- No heat sources

In the absence of internal heat generation, the equ ⑩ reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad — ⑪$$

This equation is known as diffusion equation.

(or) Fourier's equation.

case 2) :- steady state conditions

In steady state conditions, the temperature does not change with time. so $\frac{\partial T}{\partial t} = 0$. The heat conduction equation reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = 0 \quad — ⑫$$

(or)

$$\nabla^2 T + \frac{\dot{q}}{k} = 0$$

This equation is known as poisson's equation.

In the absence of Internal heat generation equation becomes.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad — ⑬$$

(or)

$$\nabla^2 T = 0.$$

This equation is known as Laplace equation.

case iii) :- one dimensional steady state heat conduction:-

If the temperature varies only in the x-direction the equation reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0. \quad \text{—— (14)}$$

(14) In the absence of internal heat generation, equation becomes

$$\frac{\partial^2 T}{\partial x^2} = 0. \quad \text{—— (15)}$$

case iv) :- Two dimensional steady state heat conduction:-

If the temperature varies only in the x and y directions, the equations (10) becomes.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{K} = 0. \quad \text{—— (16)}$$

In the absence of internal heat generation, equation (16) becomes.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

case v) :- unsteady state, one dimensional, without internal heat generation:-

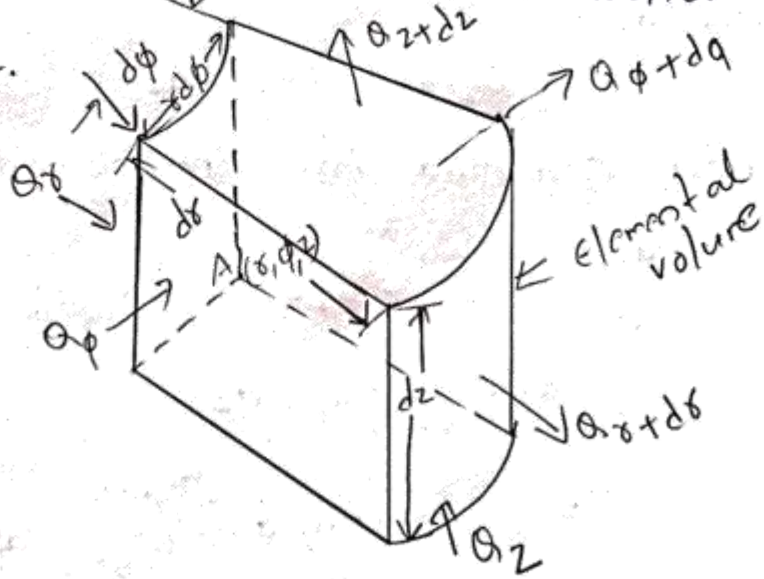
In unsteady state, the temperature changes with time, i.e $\frac{\partial T}{\partial t} \neq 0$. so the general conduction equation (10) reduces to

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}.$$

→ General heat conduction equation in cylindrical co-ordinates :-

The general heat conduction equation in cartesian coordinates derived in the previous section is used for solids with rectangular boundaries like squares, cubes, slabes etc. But the cartesian coordinate system is not applicable for solids like cylinders, cones, spheres etc. For cylindrical solids, a cylindrical coordinate system is used.

consider an elemental volume having the coordinates (r, ϕ, z) for three dimensional heat conduction analysis. as shown in fig.



The volume of the element $dv = r\,d\phi\,dr\,dz$.

Let us assume that thermal conductivity (k), specific heat (c_p), and density (ρ) are constant.

The energy balance of this cylindrical element is obtained from first law of thermodynamics.

$$\left\{\begin{array}{l}\text{Net heat}\\ \text{conducted into}\\ \text{element from}\\ \text{all the}\\ \text{coordinate}\\ \text{directions}\end{array}\right\} + \left\{\begin{array}{l}\text{Heat}\\ \text{generated}\\ \text{with in the}\\ \text{element}\end{array}\right\} = \left\{\begin{array}{l}\text{Heat}\\ \text{stored}\\ \text{in the}\\ \text{element}\end{array}\right\} \rightarrow (1)$$

Net heat conducted into element from all the co-ordinate directions :-

Heat entering in the element through (r, ϕ) plane in time $d\theta$.

$$Q_z = -k\,(r\,d\phi\,dr)\,\frac{\partial T}{\partial z}\,d\theta.$$

Heat leaving from the element through (r, ϕ) plane in time $d\theta$.

$$Q_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q_z)\,dz.$$

Net heat conducted into the element through (r,ϕ) plane in time $d\theta$.

$$= Q_z - Q_{z+dz}$$

$$= -\frac{\partial}{\partial z}(Q_z)\,dz.$$

$$= \frac{\partial}{\partial z}\left(k\,(r\,d\phi\,dr)\left[\frac{\partial T}{\partial z}\right]d\theta\right)dz.$$

$$= k\,\frac{\partial^2 T}{\partial z^2}\,(dr\cdot r\,d\phi\,dz)\,d\theta.$$

Net heat conducted through (r,ϕ) plane $= k\left[\frac{\partial^2 T}{\partial z^2}\right](dr\cdot r\,d\phi\,dz)\,d\theta$. $\rightarrow (2)$

Heat entering in the element through (ϕ, z) plane in time $d\theta$.

$$Q_r = -K(r\, d\phi\, dz)\,\frac{\partial T}{\partial r}\, d\theta$$

Heat leaving from the element through (ϕ, z) plane in time $d\theta$.

$$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\, dr.$$

Net heat conducted into the element through (ϕ, z) plane in time $d\theta$.

$$= Q_r - Q_{r+dr}.$$

$$= -\frac{\partial}{\partial r}(Q_r)\, dr.$$

$$= -\frac{\partial}{\partial r}\left[-k(r\, d\phi\, dz)\left[\frac{\partial T}{\partial r}\right] d\theta\right) dr.$$

$$= k(dr\, d\phi\, dz)\cdot\frac{\partial}{\partial r}\left[r\,\frac{\partial T}{\partial r}\right] d\theta.$$

$$= k(dr\, r\, d\phi.\, dz)\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] d\theta.$$

Net heat conducted through (ϕ, z) plane $= k(dr.\, r\, d\phi.\, dz)\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] d\theta.$ → (3).

Heat entering in the element through (z, r) plane in time $d\theta$.

$$Q_\phi = -K(dr.\, dz)\,\frac{\partial T}{r\,\partial\phi}\, d\theta.$$

Heat leaving from the element through (z, r) plane in time $d\theta$.

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\,\partial\phi}(Q_\phi)\, r\, d\phi.$$

Net heat conducted into the element through (z, r) plane in time $d\theta$.

$$Q_\phi - Q_{\phi+d\phi} = -\frac{\partial}{r\partial\phi}(Q_\phi)\, r\, d\phi.$$

$$= -\frac{\partial}{r\partial\phi}\left[-k(dr\,dz)\cdot\frac{\partial T}{r\partial\phi}\,d\theta\right] r\,d\phi.$$

$$= k\frac{\partial}{\partial\phi}\left[\frac{1}{r}\frac{\partial T}{\partial\phi}\right](dr\,d\phi\,dz)\,d\theta.$$

$$= k\left[\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right](dr\; r\,d\phi\,dz)\,d\theta.$$

Net heat conducted through (z, r) plane $= k\left[\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right](dr\; r\,d\phi\,dz)\,d\theta.$

→ Net heat conducted into element from all the coordinate directions

$$= k\frac{\partial^2 T}{\partial z^2}(dr\; r\,d\phi\,dz)\,d\theta + k(dr\; r\,d\phi\,dz)\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right]d\theta + k\left[\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right](dr\; r\,d\phi\,dz)\,d\theta.$$

$$= k(dr\; r\,d\phi\,dz)\,d\theta\left[\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right] \quad (4)$$

→ Heat generated within the element :-

Total heat generated within the element is given by

$$Q = q^{\cdot}(dr\; r\,d\phi\,dz)\,d\theta. \quad (5)$$

→ Heat stored in the element :-

The increase in internal energy of the element is equal to the net heat stored in the element.

Increase in internal energy

$= $ net heat stored in the element

$$= \rho\,(dr\, r\,d\phi\, dz)\, c_p \frac{\partial T}{\partial \theta} \times d\theta \longrightarrow (6)$$

substituting equations (4), (5), (6) in (1)

$$k(dr\, r\,d\phi\, dz)\, d\theta \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] +$$

$$\dot{q}\,(dr\, r\,d\phi\, dz)\, d\theta = \rho\,(dr\, r\,d\phi\, dz)\, c_p \frac{\partial T}{\partial \theta} \times d\theta.$$

Divided by $(dr\, r\,d\phi\, dz)\, d\theta$.

$$\Rightarrow k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \dot{q} = \rho\, c_p \frac{\partial T}{\partial \theta}.$$

$$\Rightarrow \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{\rho c_p}{k}\frac{\partial T}{\partial \theta}$$

It is a general three dimensional heat conduction equation in cylindrical coordinates.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}.$$

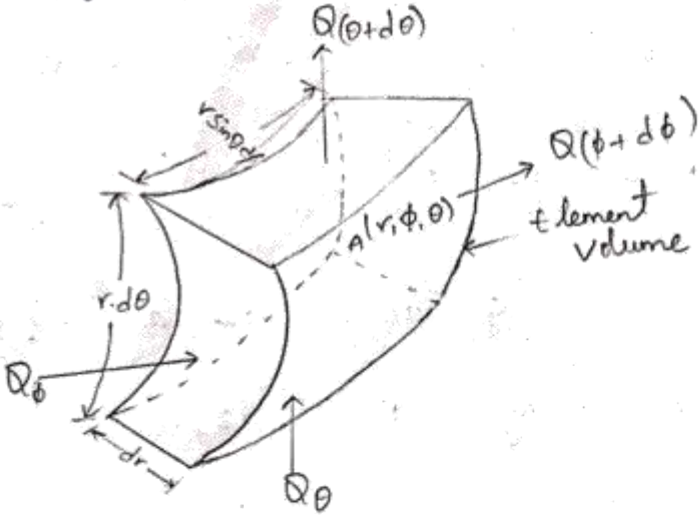
$$\left[\alpha = \frac{k}{\rho c_p}\right].$$

If the flow is steady, one dimensional and no heat generation, then the equation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0.$$

→ General heat conduction Equation in spherical coordinates :-

consider an elemental volume having the coordinates (r, ϕ, θ) for three dimensional heat conduction analysis as shown in fig.



The volume of the element $= dr\, r d\theta\, r\sin\theta\, d\phi$.

$$\left\{\begin{array}{c}\text{Net heat}\\ \text{conducted into}\\ \text{element from}\\ \text{all the coordinate}\\ \text{directions}\end{array}\right\} + \left\{\begin{array}{c}\text{Heat}\\ \text{generated}\\ \text{within the}\\ \text{element}\end{array}\right\} = \left\{\begin{array}{c}\text{Heat}\\ \text{stored in the}\\ \text{element}\end{array}\right\}$$

→ net heat conducted into element :-

Heat entering in the element through (r, θ) plane in ~~time~~ ~~$d\theta$~~ ϕ direction.

$$Q_\phi = -k(dr\; r d\theta)\, \frac{\partial T}{r\sin\theta\, \partial\phi}\, d\theta. \quad \text{---} (1)$$

Heat leaving from the element through (r, θ) plane in ϕ direction.

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\sin\theta\, \partial\phi}(Q_\phi)\, d\phi.\, r\sin\theta\, d\phi. \quad \text{---} (2)$$

Net heat conducted into the element through (r, θ) plane in time $d\theta$.

$$= Q_\phi - Q_{\phi+d\phi}$$

$$= -\frac{\partial}{\partial \phi}(Q_\phi)\, d\phi.$$

$$= -\frac{\partial}{\partial \phi}\left[\cancel{-}k\, (dr\ \cancel{r d\theta})\ \frac{\partial T}{\cancel{r\sin\theta\,\partial\phi}}\ d\theta\right] d\phi$$

$$= \cancel{\frac{1}{r\sin\theta}}\ \cancel{\frac{\partial}{\partial\phi}} = \cancel{k}\, (\cancel{dr}\ \cancel{r}\, d\theta.$$

$$= -\frac{1}{r\sin\theta}\ \frac{\partial}{\partial\phi}\ (Q_\phi)\ r\sin\theta\, d\phi.$$

$$= -\frac{1}{r\sin\theta}\ \frac{\partial}{\partial\phi}\left[-k\,(dr\ r d\theta)\cdot\frac{1}{r\sin\theta}\ \frac{\partial T}{\partial\phi}\ d\theta\right] r\sin\theta\ d\phi.$$

$$= k\,(dr\ r d\theta\ r\sin\theta\ d\phi)\ \frac{1}{r^2\sin^2\theta}\ \frac{\partial^2 T}{\partial\phi^2}\ d\theta. \quad \text{—— (3)}.$$

Heat flow in $r-\phi$ plane, θ - direction.

$$Q_\theta = -k\,(dr\ r\sin\theta\cdot d\phi)\,\frac{\partial t}{r\,\partial\theta}\ d\theta.$$

$$Q_{\theta+d\theta} = Q_\theta + \frac{\partial}{r\,\partial\theta}\ (Q_\theta)\, r\, d\theta$$

Net Heat conducted into the element through (r, ϕ) plane in time $d\theta$.

$$= Q_\theta - Q_{\theta+d\theta}$$

$$= -\frac{\partial}{r\,\partial\theta}\ (Q_\theta)\ r\, d\theta.$$

$$= -\frac{\partial}{r\,\partial\theta}\left[-k\,(dr\ r\sin\theta\ d\phi)\ \frac{\partial t}{r\,\partial\theta}\, d\theta\right] r\, d\theta.$$

$$= \frac{k}{r}\,\frac{dr\cdot r\,d\phi\cdot r\,d\theta}{r}\,\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial T}{\partial\theta}\right]d\theta.$$

$$= k(dr\; r\,d\theta\; r\sin\theta\; d\phi)\,\frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial T}{\partial\theta}\right]d\theta$$

Heat flow in θ-ϕ plane, r-direction.

$$Q_r = -k(r\,d\theta\; r\sin\theta\,.\,d\phi)\,\frac{\partial T}{\partial r}\,d\theta.$$

$$Q(r+dr) = Q_r + \frac{\partial}{\partial r}(Q_r)\,dr.$$

Net heat conducted in the element.

$$= Q_r - Q_{r+dr}.$$

$$= \frac{-\partial}{\partial r}\left[-(K\,r\,d\theta\; r\sin\theta\; d\phi)\,\frac{\partial T}{\partial r}\cdot d\theta\right]dr$$

$$= k\,d\theta\sin\theta\; d\phi\; dr\,\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right]d\theta$$

$$= k(dr\; r\,d\theta\cdot r\sin\theta\,.\,d\phi)\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]d\theta$$

Net heat accumulated in the element.

$$= k(dr\; r\,d\theta\cdot r\sin\theta\cdot d\phi)\left[\frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 T}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right]\right]d\theta.$$

→ Heat generated within the element :-

total heat generated within the element is given by.

$$Q = \dot{q}\,(dr\; r\,d\theta\cdot r\sin\theta\,.\,d\phi)\,d\theta.$$

→ Energy stored in the element:

The increase in internal energy of the element is given by is equal to the net heat stored in the element.

$$= \rho(dr \cdot r d\theta \cdot r\sin\theta \, d\phi)\, c_p \frac{\partial T}{\partial \theta} \cdot \partial\theta$$

Now,

$$K\, dr \cdot r d\theta \cdot r\sin\theta \cdot d\phi \left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial T}{\partial r}\right)\right] d\theta + q(dr\, r d\theta \cdot r\sin\theta \cdot d\phi)\, d\theta =$$

$$\rho(dr \cdot r d\theta \cdot r\sin\theta \cdot d\phi)\, c_p \cdot \frac{\partial T}{\partial\theta} \cdot d\theta.$$

dividing both sides by $k(dr \cdot r d\theta \cdot r\sin\theta\, d\phi)\, d\theta$.

we get

$$\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right]\right] +$$

$$\frac{q}{K} + \phi = \frac{\rho c_p}{K}\frac{\partial T}{\partial\theta} \cdot = \frac{1}{\alpha}\frac{\partial T}{\partial\theta}.$$

the above equation is the general heat conduction equation in spherical coordinates.

In case there are not heat sources present and the heat flow is steady and one dimensional, then the equation reduces to.

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dt}{dr}\right] = 0.$$

//.

→ one dimensional steady state conduction heat transfer :-

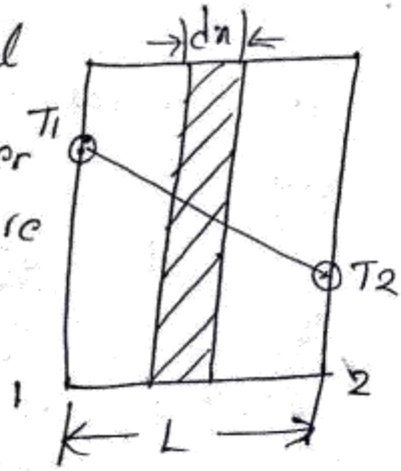
1. plane walls and composite walls.

2. Hallow cylinders and composite cylinders.

3. spheres.

1. conduction of heat through a slab (or) plane wall :-

consider a slab of uniform thermal conductivity k, thickness L, with inner temperature T_1 and outer temperature T_2.



Let us consider a small elemental area of thickness 'dx'.

from fourier law of conduction, w.k.T

$$Q = -kA \frac{dT}{dx}$$

$$Q\,dx = -kA\,dT.$$

Integrating the above equation between the limits of 0 to L, and T_1 and T_2.

$$\Rightarrow \int_0^L Q\,dx = -\int_{T_1}^{T_2} kA\,dT$$

$$\Rightarrow Q\left[x\right]_0^L = -kA\left[T\right]_{T_1}^{T_2}$$

$$\Rightarrow Q(L-0) = -kA(T_2-T_1)$$

$$\Rightarrow QL = kA(T_1-T_2)$$

$$\Rightarrow Q = \frac{kA}{L}\left[T_1-T_2\right].$$

$\Rightarrow$ ~~Q~~

$$Q = \frac{T_1 - T_2}{\frac{L}{KA}}$$

$$Q = \frac{\Delta T_{overall}}{R_{th}}$$

where $R_{th} = \frac{L}{KA}$ = thermal Resistance of slab.

$\Delta T = T_1 - T_2$. 1.

(or)

The general heat conduction equation in cartesian coordinates is given by.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}.$$

If the conduction takes place under the conditions steady state $\left(\frac{\partial T}{\partial \theta} = 0\right)$, one dimensional $\left[\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0\right]$ and with no internal heat generation $\left(\frac{q}{K} = 0\right)$ then the above equation reduces to.

$$\frac{\partial^2 T}{\partial x^2} = 0. \quad (or) \quad \frac{d^2 T}{dx^2} = 0 \quad — (1)$$

By integrating the above equation twice, we get.

$$\frac{dT}{dx} = C_1$$

$$T = C_1 x + C_2 \quad — (2)$$

C_1 & C_2 are arbitary constants.

at $x = 0$, $t = t_1$

at $x = L$ $t = t_2$.

substitute these values then.

$T_1 = 0 + C_2$. and $T_2 = C_1 L + C_2$

$C_2 = T_1$

$C_1 = \frac{T_2 - C_2}{L}$

$\therefore C_1 = \frac{T_2 - T_1}{L}$

$$T = \left(\frac{T_2 - T_1}{L}\right) x + C_2$$

$$\therefore T = \left(\frac{T_2 - T_1}{L}\right) x + T_1$$

$(T_2 - T_1) x + L T_1 = T L.$

$(T_2 - T_1) x = T L - L T_1$

$$\therefore x = \frac{(T - T_1) L}{(T_2 - T_1)} = \left(\frac{T_1 - T}{T_1 - T_2}\right) L.$$

→ conduction of heat through hollow cylinder:-

consider a hollow cylinder of inner radius r_1, outer radius r_2, inner tem T_1, outer temp T_2 and thermal conductivity k.

Let us consider a small elemental area of thickness "dr".

from fourier Law of conduction, w.k.T.

Area of the cylinder is $2\pi r L$.

$A = 2\pi r L$. , $Q = -KA \frac{dT}{dr}$

so, $Q = -K\, 2\pi r L \frac{dT}{dr}$.

$$Q \times \frac{dr}{r} = -K 2\pi L\, dT.$$

Integrating the above equations from r_1 to r_2 and T_1 to T_2.

$$Q \int_{r_1}^{r_2} \frac{dr}{r} = -k 2\pi L \int_{T_1}^{T_2} dT$$

$$Q (\ln r)_{r_1}^{r_2} = -k 2\pi L [T]_{T_1}^{T_2}$$

$$Q(\ln r_2 - \ln r_1) = -k 2\pi L (T_2 - T_1)$$

$$Q \ln\left(\frac{r_2}{r_1}\right) = 2\pi L k (T_1 - T_2)$$

$$Q = \frac{2\pi k L (T_1 - T_2)}{\ln(r_2/r_1)}$$

$$Q = \frac{T_1 - T_2}{\frac{1}{2\pi k L} \ln(r_2/r_1)}$$

$$Q = \frac{\Delta T_{overall}}{R}$$

$$\Delta T = T_1 - T_2, \quad R = \frac{1}{2\pi L k} \ln(r_2/r_1).$$

→ conduction of heat through Hollow sphere :-

consider a hollow sphere of inner radius r_1, outer radius r_2, inner tem T_1, outer temperature T_2 and thermal conductivity k.

Let us consider a small elemental area of thickness 'dr'. from fourier law of heat conduction, w.k.T.

$$Q = -KA\frac{dT}{dr}.$$

Area of sphere is $4\pi r^2$

$$A = 4\pi r^2.$$

$$Q = -K\,4\pi r^2\frac{dT}{dr}.$$

$$Q\frac{dr}{r^2} = -K\,4\pi\,dT$$

Integrating on both sides

$$Q\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_1}^{T_2}4\pi k\,dT.$$

$$Q\int_{r_1}^{r_2}\frac{dr}{r^2} = -4\pi k\int_{T_1}^{T_2}dT$$

$$Q\left[\frac{-1}{r}\right]_{r_1}^{r_2} = -4\pi k\,[T]_{T_1}^{T_2}$$

$$Q\left[\frac{1}{r_1}-\frac{1}{r_2}\right] = -4\pi k(T_2-T_1)$$

$$Q\left[\frac{r_2-r_1}{r_1 r_2}\right] = 4\pi k(T_1-T_2)$$

$$Q = \frac{4\pi k(T_1-T_2)}{\frac{r_2-r_1}{r_1 r_2}}$$

$$Q = \frac{T_1-T_2}{\left(\frac{r_2-r_1}{4\pi k(r_1 r_2)}\right)}$$

$$\therefore Q = \frac{\Delta T\ \text{overall}}{R}.$$

where $\Delta T = T_1 - T_2$

$$R = \frac{r_2-r_1}{4\pi k(r_1 r_2)}$$

→ Heat conduction through a composite wall:-

consider the transmission of heat through a composite wall consisting of a number of slabs.

Let L_A, L_B, L_C = Thickness of slabs A, B, C

k_A, k_B, k_C = thermal conductivities of the slabs A, B, C.

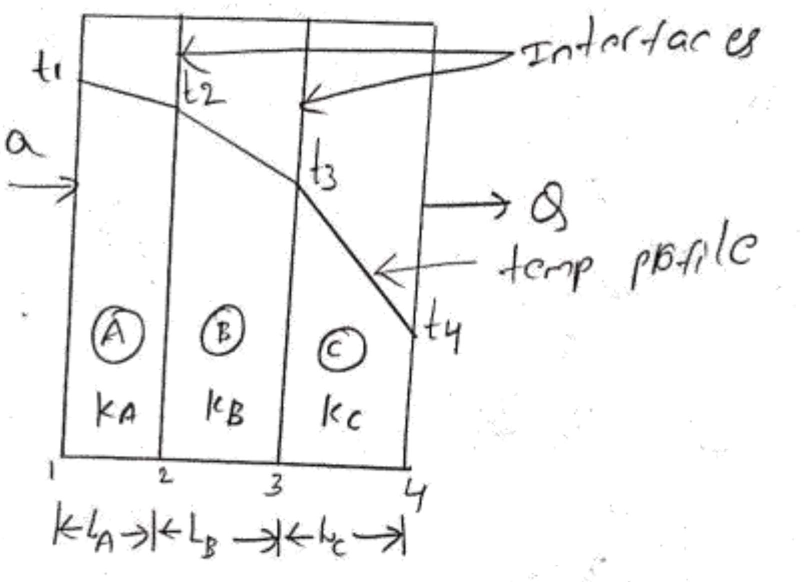
t_1, t_4 $(t_1 > t_4)$ = temp at the wall surfaces 1 and 4.

t_2, t_3 = temp at the interfaces 2 and 3.

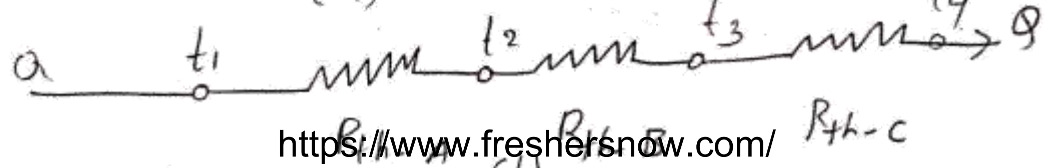
since the quantity of heat transmitted per unit time through each slab/layer is same,

$$Q = \frac{k_A A \cdot (t_1 - t_2)}{L_A} = \frac{k_B \cdot A (t_2 - t_3)}{L_B} = \frac{k_C A (t_3 - t_4)}{L_C}.$$

Assuming that there is a perfect contact between the layers and no temp drop occurs at the interface between the materials.



(a)



(b)

Rearranging the above equation we get,

$$t_1 - t_2 = \frac{Q L_A}{k_A A} \quad \text{--- (1)}$$

$$t_2 - t_3 = \frac{Q L_B}{k_B A} \quad \text{--- (2)}$$

$$t_3 - t_4 = \frac{Q L_C}{k_C A} \quad \text{--- (3)}$$

Adding 1, 2, 3 equations.

$$t_1 - t_4 = Q\left[\frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A}\right]$$

$$Q = \frac{A(t_1 - t_4)}{\left[\frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C}\right]}$$

$$Q = \frac{(t_1 - t_4)}{\left[\frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A}\right]} = \frac{(t_1 - t_4)}{\left[R_{th\text{-}A} + R_{th\text{-}B} + R_{th\text{-}C}\right]}$$

If the composite wall consists of n slabs/layers, then,

$$Q = \frac{(t_1 - t_{(n+1)})}{\sum_1^n \frac{L}{kA}} \Rightarrow Q = \frac{\Delta T_{overall}}{\sum R_{th}}$$

In order to solve more complex problems involving both series and parallel thermal resistances, the electrical analogy may be used.

→ Thermal contact resistance :-

In composite wall, the calculations of heat-flow are made on the assumptions.

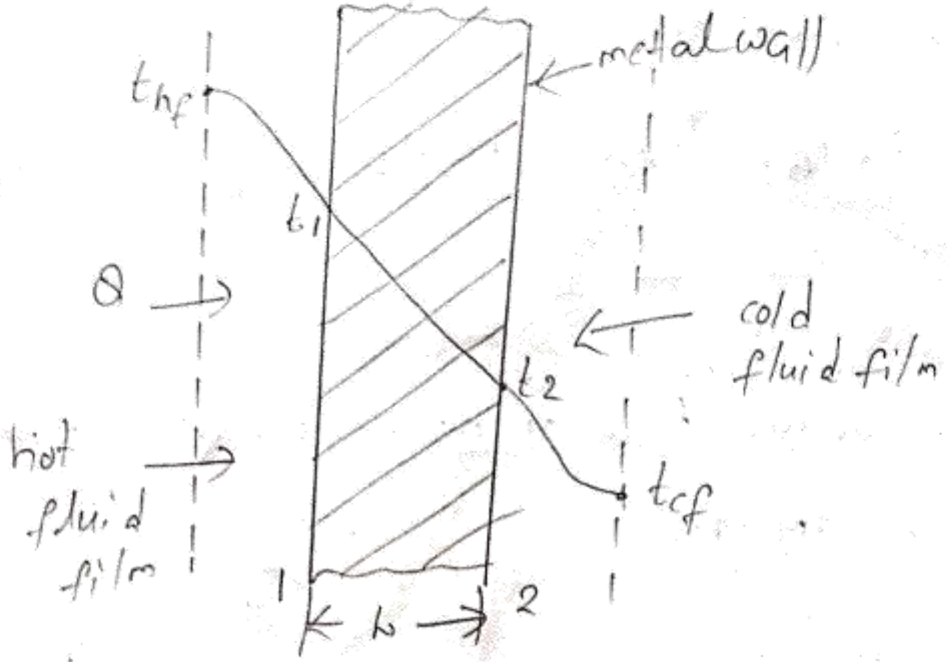
i) The contact between the adjacent layers is perfect.

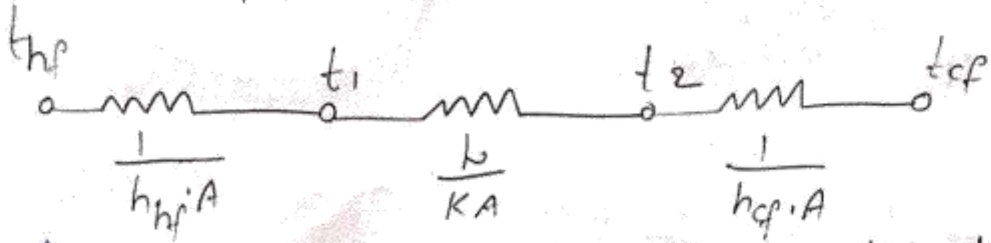
ii) At the interface there is no-fall of temperature.

iii) At the interface the temp is continuous,

In real systems, however due to surface roughness and spaces (filled with air) the contact surfaces touch only at discrete locations. That the area available for the flow of heat at the interface will be small compared to geometric face area. Due to this reduced area and presence of air, a large resistance to heat flow at the interface occurs. This resistance is known as thermal contact resistance.

→ The overall heat transfer coefficient :-





h_{hf} = heat transfer coefficient from hot fluid to metal metal surface.

h_{cf} = heat transfer coefficient from metal surface to cold film.

The equations of heat flow through fluid and metal surfaces are given by.

$$Q = h_{hf} A (t_{hf} - t_1) \quad \text{— ①}$$

$$Q = \frac{KA(t_1 - t_2)}{b} \quad \text{— ②}$$

$$Q = h_{cf} A (t_2 - t_{cf}). \quad \text{— ③}$$

Rearranging 1, 2, 3 we get.

$$t_{hf} - t_1 = \frac{Q}{h_{hf} \cdot A}$$

$$t_1 - t_2 = \frac{Qb}{KA}$$

$$t_2 - t_{cf} = \frac{Q}{h_{cf} \cdot A}$$

adding those equations.

$$t_{hf} - t_{cf} = Q\left[\frac{1}{h_{hf} A} + \frac{L}{KA} + \frac{1}{h_{cf} A}\right]$$

$$Q = \frac{A(t_{hf} - t_{cf})}{\frac{1}{h_{hf} A} + \frac{L}{K} + \frac{1}{h_{cf}}}$$

If U is the overall heat transfer coefficient, then

$$Q = UA(t_{hf} - t_{cf})$$

$$U = \frac{1}{\frac{1}{h_{hf}} + \frac{L}{K} + \frac{1}{h_{cf}}}$$

→ critical radius of insulation:-

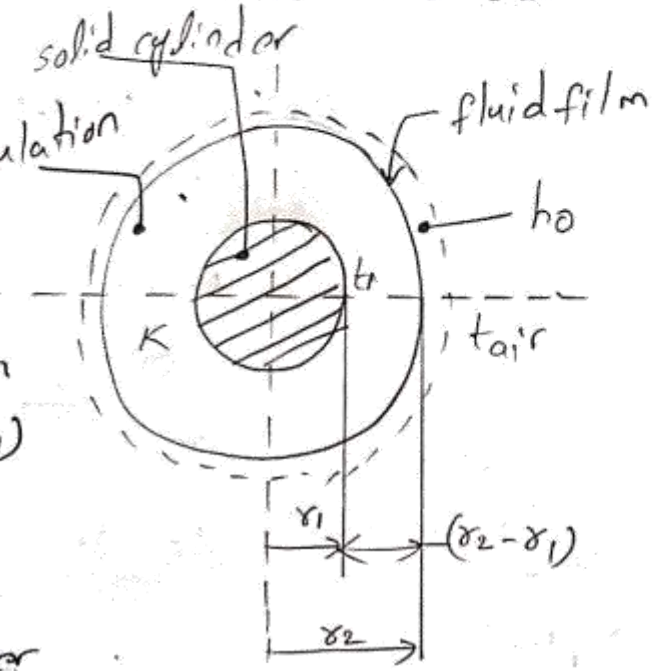
Insulation definition:- A material which retards the flow of heat with reasonable effectiveness is known as Insulation. Insulation serves the following two purposes:

i) It prevents the heat flow from the system to surroundings.

ii) It prevents the heat from the surronding to the system.

The addition of insulation material on a surface does not reduce the amount of heat transfer rate always. Infact under certain circumstances it actually increases the heat loss up-to certain thickness of insulation. the radius of insulation for which the heat transfer is maximum is called critical radius of insulation and the corresponding thickness is called critical thickness. If the thickness is further increased, the heat loss will be reduced.

→ critical radius of insulation for a cylinder



consider a solid cylinder of radius r_1 insulated with an insulation of thickness $(r_2 - r_1)$ as shown in fig.

Let L = length of cylinder

t_1 = surface tem of cylinder

t_{air} = tem of air.

h_0 = heat transfer coefficient at the outer surface of the insulation.

K = thermal conductivity of insulating material.

the rate of heat transfer from the surface of the solid cylinder to the surroundings is given by.

$$Q = \frac{2\pi L (t_1 - t_{air})}{\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_0 r_2}} \quad \text{—— (1)}$$

from the above equ that as r_2 increases, the factor $\frac{\ln(r_2/r_1)}{K}$ increases but the factor $\frac{1}{h_0 r_2}$ decreases.

thus Q becomes maximum when the denominator become minimum. the required condition is

$$\frac{d}{dr_2}\left[\frac{\ln(r_2/r_1)}{K} + \frac{1}{h_o r_2}\right] = 0.$$

$$\frac{1}{K}\frac{1}{r_2} + \frac{1}{h_o}\left(\frac{-1}{r_2^2}\right) = 0.$$

$$\frac{1}{K}\frac{1}{\cancel{r_2}} = \frac{1}{h_o}\frac{1}{r_2^{\cancel{2}}}$$

$$\frac{1}{K} = \frac{1}{h_o}\frac{1}{r_2}$$

$$r_2 h_o = K$$

$$r_2 = r_c = \frac{K}{h_o}$$

for sphere:- $r_2 = r_c = \frac{2K}{h_o}$

pro:-

1. calculate the critical radius of insulation for asbestos ($k = 0.172$ W/m K) surrounding a pipe and exposed to room air at 300K with $h = 2.8$ W/m K. calculate the heat loss from a 475K, 60mm diameter pipe when covered with the critical radius of insulation, and without insulation.

sol:- $K = 0.172$ W/m K, $T_1 = 475K$, $T_2 = 300K$.

$h_o = 2.8$ W/m^{2}k, $r_1 = \frac{60}{2} = 30mm = 0.03$ m.

i) the critical radius of insulation,

$$r_c = \frac{K}{h_o} = \frac{0.172}{2.8} = 0.06143\text{ m (or) } 61.43\text{ mm}$$

ii) Q (with insulation) $= \dfrac{2\pi (T_1 - T_2)}{\dfrac{\ln(r_c/r_1)}{K} + \dfrac{1}{h_o r_c}}$

$= \dfrac{2\pi (475-300)}{\dfrac{\ln(0.06143/0.03)}{0.172} + \dfrac{1}{2.8 \times 0.06143}}$

$= \dfrac{1099.56}{4.167 + 5.814} = 110.16\,W/m.$

iii) Q (without insulation) $= h_o\, 2\pi r_1 (T_1 - T_2)$

$= 2.8 \times 2\pi \times 0.03 (475-300)$

$= 92.36\,W/m.$

→ **Heat conduction with internal heat generation:-**

In many cases, there is a heat generation within the system. Typical examples are.

1. Electrical coils.
2. Resistance heater
3. Nuclear reactor
4. Combustion of fuel in the fuel bed of boiler furnace.

→ **plane wall with uniform heat generation:-**

Consider a plane wall of thickness L of uniform thermal conductivity K. Let the wall surfaces are maintained at temperatures t_1 and t_2.

Let us assume that the heat flow is

* one dimensional
* steady state condition

P1

Heat conducted in at a distance x,

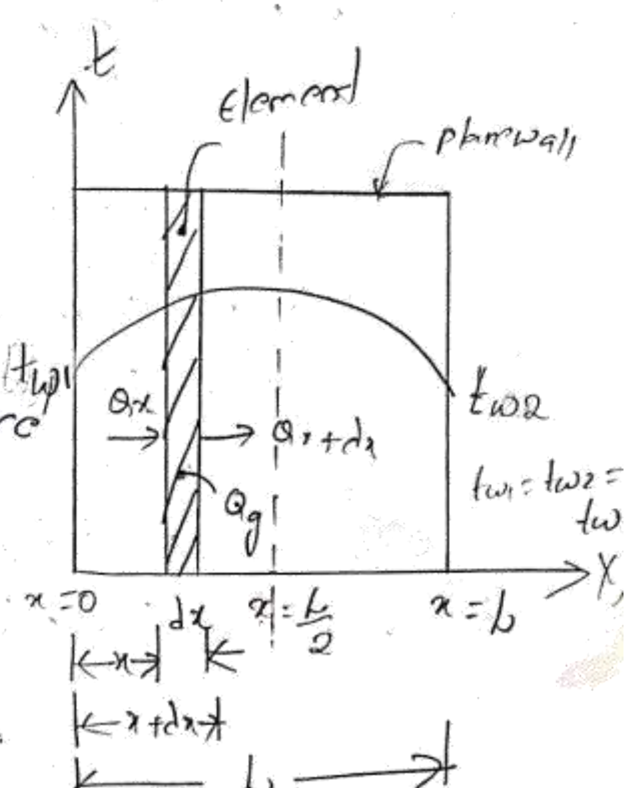
$$Q_x = -kA\frac{dt}{dx}$$

Heat generated in the element

$$Q_g = A\,dx\,q_g.$$

Heat conducted out at distance $(x+dx)$,

$$Q_{x+dx} = Q_x + \frac{d}{dx}(Q_x)\,dx$$



An energy balance on the element of thick dx is given by.

$$Q_x + Q_g = Q_{x+dx}$$

$$\cancel{Q_x} + Q_g = \cancel{Q_x} + \frac{d}{dx}(Q_x)\,dx$$

$$\cancel{= Q_x + \frac{d}{d}}$$

$$Q_g = \frac{d}{dx}(Q_x)\,dx$$

$$A\,dx\,q_g = \frac{d}{dx}\left[-kA\frac{dt}{dx}\right]dx$$

$$= -kA\frac{d^2t}{dx^2}\,dx$$

$$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0 \longrightarrow (1)$$

The first and second integration of equ (1) gives respectively.

$$\frac{dt}{dx} = \frac{-q}{k}x + c_1$$

$$\therefore t = \frac{-q}{k}\frac{x^2}{2} + c_1 x + c_2.$$

Both the surfaces have the same temperature:

At $x=0$ $\quad t = t_1 = t_w$

At $x=b$ $\quad t = t_2 = t_w$

using these boundary conditions in the above equation

$$t = -\frac{q_g}{2k}(0)^2 + c_1(0) + c_2$$

$$c_2 = t = t_w$$

$$\therefore \underline{c_2 = t_w}$$

$$\cancel{t} = -\frac{q_g}{2k}(b)^2 + c_1(b) + \cancel{t_w}$$

$$\frac{q_g}{2k} b^{\cancel{2}} = c_1 \cancel{b}$$

$$\therefore c_1 = \frac{q_g}{2k} \cdot b$$

substituting these values of c_1 and c_2 in the above equ.

$$t = -\frac{q_g}{2k}x^2 + \frac{q_g}{2k} b\,x + t_w.$$

$$t = \frac{q_g}{2k}(b-x)x + t_w$$

Inorder to determine the location of the maximum temp, differentiating the equ. w.r.t. 'x' and equate to zero.

$$\frac{dt}{dx} = \frac{q_g}{2k}(b - 2x) = 0$$

$$b - 2x = 0.$$

$$x = \frac{b}{2}.$$

$$t_{max} = \left[\frac{q_g}{2k}(b-x)x\right]_{x=\frac{b}{2}} + t_w.$$