Vector Integral Theorems

In this chapter we discuss three important vector integral theorems

1. Gauss Divergence Theorem
2. Green's Theorem
3. Stoke's Theorem

Applicaions: In solid mechanics, fluid mechanics, quantum mechanics, electrical engineering and various other fields these theorems will be of great use.

Gauss Divergence Theorem :-

➢ It is the relation between surface and volume integral.

Statement : If $\bar{F}$ is continuously differentiate vector point function defined in the region V bounded by the closed surface S then

$$\int_V Div\bar{F}\, dv = \int_s \bar{F} \cdot \bar{n}\, ds$$

Where n is the outward unit normal vector at any point on s.

Applications of Gauss Divergence Theorem :

1. In physics, Electrostatics, magnetism gravity can be expressed as Differential form and Integral form.
2. In fluid dynamics, quantum mechanics, electromagnetism, continuity equation.

Cartesian Form :

$$\text{Let } \bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\text{Div } \bar{F} = \nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\bar{n} = \bar{i}\cos\alpha + \bar{j}\cos\beta + \bar{k}\cos\gamma$$

Where α, β, γ are the angles with $\bar{n}$ makes the positive directions of x, y, z axis.

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\bar{n}$.

$$\bar{F} \cdot \bar{n} = f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma$$

$\therefore$ By Divergence theorem

$$\int_v \nabla \cdot \bar{F}\, \mathrm{d}v = \iint_s (f_1 \cos\alpha + f_2 \cos\beta + f_3 \cos\gamma)\, \mathrm{d}s$$

$$= \iint_s (f_1 dy\, \mathrm{d}z + f_2 dz\, \mathrm{d}x + f_3 dx\, \mathrm{d}y)$$

1. Verify the divergence theorem for $\bar{F} = 4xy\bar{i} - y^2\bar{j} + xz\bar{k}$ over the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0 and z = 1?

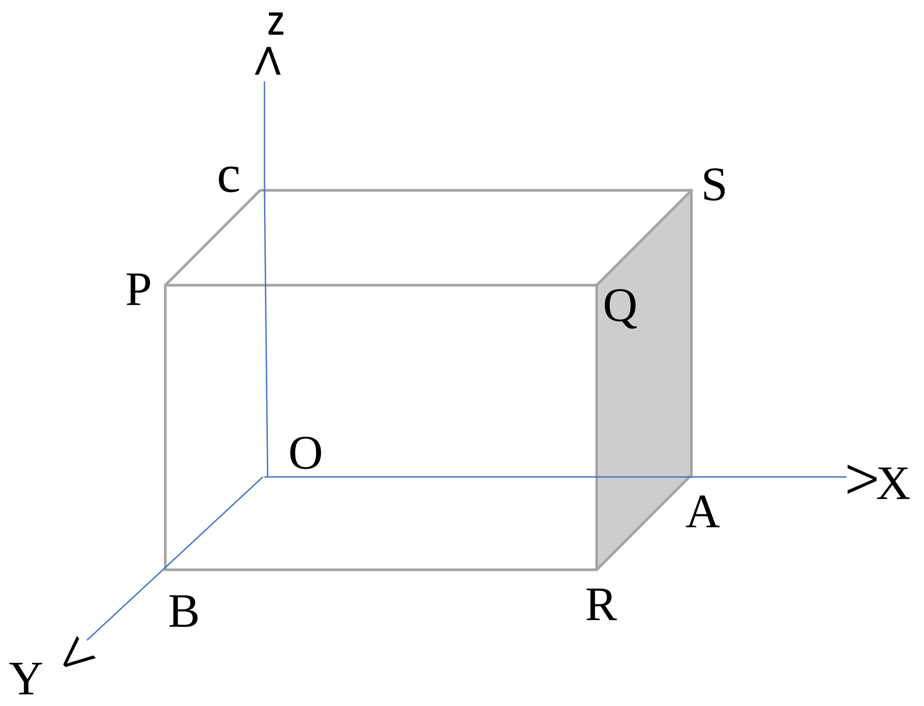
Sol : By Gauss Theorem $\int_V Div\bar{F}\, dv = \int_s \bar{F} \cdot \bar{n}\, ds$

Given $\bar{F} = 4xy\bar{i} - y^2\bar{j} + xz\bar{k}$

$$\nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(xz)$$

= 4y-2y+x = x+2y



Calculation of L.H.S

$$\int_V \nabla \cdot F \, dV = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x + 2y) \, dxd$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} (x + 2y) \, [z]_0^1 \text{ dxdy}$$

$$= \int_{x=0}^{1} \left[xy + \frac{2y^2}{2} \right]_0^1 dx$$

$$= \int_{x=0}^{1} (x + 1) dx$$

$$= \left[\frac{x^2}{2} + x \right]_0^1 = \tfrac{1}{2} + 1 = 3/2$$

R.H.S

Evaluation of surface integral $\iint_s \bar{F} \cdot \bar{n} \, ds$

$$\iint_s \bar{F} \cdot \bar{n} \, \mathrm{ds} = \iint_{s_1} \bar{F} \cdot \bar{n} \, \mathrm{ds} + \iint_{s_2} \bar{F} \cdot \bar{n} \, \mathrm{ds} + \iint_{s_3} \bar{F} \cdot \bar{n} \, \mathrm{ds} + \iint_{s_4} \bar{F} \cdot \bar{n} \, \mathrm{ds} + \iint_{s5} \bar{F} \cdot \bar{n} \, \mathrm{ds} + \iint_{s6} \bar{F} \cdot \bar{n} \, \mathrm{ds}$$

Since the surface of the cube divided into 6 faces.

1. <u>For s1(ASQR)</u> : $\bar{n} = \bar{i}, x = 1, ds = dydz$, y = 0 to 1 and z = 0 to 1

$$\iint_{s_1} \bar{F} \cdot \bar{n}\, ds = \int_{y=0}^{1}\int_{z=0}^{1} 4xy\, dydz = 4\int_{y=0}^{1}\int_{z=0}^{1} y\, dydz$$

$$= 4\int_{y=0}^{1} y[z]_0^1 dy = 4\int_{y=0}^{1} y\, dy = 4\left[\frac{y^2}{2}\right]_0^1 = 2$$

2. <u>For s2(OBPC)</u> : $\bar{n} = -\bar{i}, x = 0, ds = dxdz, y = 0\ to\ 1, z = 0\ to\ 1$

$$\iint_{S_2} \bar{F} \cdot \bar{n}\, dS = \int_{y=0}^{1}\int_{z=0}^{1} 4xy\, dydz = 0 \quad (\because x = 0)$$

3. <u>For s3(OBRA)</u> : $\bar{n} = \bar{j}, y = 1, ds = dxdz, x = 0\ to\ 1\ and\ z = 0\ to\ 1$

$$\iint_{s_3} \bar{F} \cdot \bar{n}\, ds = \int_{x=0}^{1}\int_{z=0}^{1} -y^2\, dxdz = \int_{x=0}^{1}\int_{z=0}^{1} -1\, dxdz$$

$$= \int_{x=0}^{1} -[z]_0^1 dx = -\int_{x=0}^{1} dx = -[x]_0^1 = -1$$

4. For s4(OACS) : $\bar{n} = -\bar{j}, y = 0, ds = dxdz, x = 0\ to\ 1\ and\ z = 0\ to\ 1$

$\iint_{s_4} \bar{F} \cdot \bar{n}\, ds = \int_{x=0}^{1} \int_{z=0}^{1} -y^2\, dxdz = o\ (\because y = 0)$

5. For s5(PQCS) : $\bar{n} = \bar{\kappa}, z = 1, ds = dxdy, x = 0\ to\ 1$ and y = 0 to 1

$\iint_{s5} \bar{F} \cdot \bar{n}\, dS = \int_{x=0}^{1} \int_{y=0}^{1} xz\, dxdy = \int_{x=0}^{1} \int_{y=0}^{1} x\, dxdy$

$= \left[\frac{x^2}{2}\right]_0^1 [y]_0^1 = ½$

6. For s6(BRPQ) : $\bar{n} = \bar{\kappa}, z = 0, ds = dxdy, x = 0\ to\ 1\ and\ y = 0\ to\ 1$

$\iint_{S_6} \bar{F} \cdot \bar{n}\, dS = \int_{x=0}^{1} \int_{y=0}^{1} xz\, dxdy = 0\ (\because z = 0)$

$\therefore \int_s \bar{F} \cdot \bar{n}\, ds = -2 - 1 + 0 + 0 + \frac{1}{2} + 0 = 3/2$

$\therefore$ Gauss Theorem verified

2.Evaluate $\iint_S x\, dydz + y\, dzdx + z\, dxdy$ over the sphere $x^2 + y^2 + z^2 = 1$

Sol : By Gauss Divergence then

$$\iint_s \bar{F} \cdot \bar{n}\, \mathrm{ds} = \int_v \nabla \cdot \bar{F}\, \mathrm{d}v$$

From cartesian form, we can write

$$\iint_S F_1\, \mathrm{d}y\, \mathrm{dz} + F_2 dx\, \mathrm{dz} + F_3 dx\, \mathrm{d}y = \int_V \nabla \cdot \bar{F}\, \mathrm{d}v$$

$$\mathrm{F} = F_1 \bar{\mathrm{i}} + F_2 \bar{j} + F_3 \bar{k}$$

$$= x\bar{\mathrm{i}} + y\bar{j} + z\bar{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

= 1+1+1 = 3

$\iint_S \bar{F} \cdot \bar{n}\, ds = \int_V 3\, dv = 3 \cdot \frac{4}{3}\pi r^3$ = $4\pi r^3$ (Sphere with unit radius i.e r =1)

= 4π($\because \int_V dv =$ Volume of the sphere $\frac{4}{3}\pi r^3$)

---

3. Evaluate $\iint_S x^3\, dy\, dz + x^2 y dx\, dz + x^2 z dx\, dy$ over the surface bounded by the planes z = 0, z = b and the cylinder $x^2 + y^2 = a^2$?

Sol : By Gauss Divergence Theorem

$$\iint_S \bar{F} \cdot \bar{n}\, ds = \int_V \nabla \cdot \bar{F}\, dv$$

$$\bar{F} = x^3 \bar{i} + x^2 y \bar{j} + x^2 z \bar{k}$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)$$

$= 3x^2 + x^2 + x^2 = 5x^2$

<u>Limits</u> : z = 0 to b , x = -a to a, y = $-\sqrt{a^2 - x^2}\ to\ \sqrt{a^2 - x^2}$

$$\iint_S \bar{F} \cdot \bar{n}\, \mathrm{d}s = \int_V \nabla \cdot \bar{F}\, \mathrm{d}v = \int_V 5\, x^2 \text{ dxdydz}$$

$$= \int_{x=-a}^{a} \int_{y=\ -\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} 5x^2\ dxdydz$$

$$= 5\int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 [z]_0^b dxdy$$

$$= 5\text{b}\int_{x=-a}^{a} x^2 [y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \text{dx}$$

Put x = asin θ , $dx = a\cos\theta\, \mathrm{d}\theta$ if x = -a then $\theta = -\frac{\pi}{2}\, and\ x = a\ then\ \theta = \frac{\pi}{2}$

$= 10b\int_{-\pi/2}^{\pi/2} a^2 \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\,.\,a\cos\theta\, d\theta$

$= 10b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4 \sin^2\theta \cos^2\theta \, d\theta$

$= a^4.10.2.b\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$

$= 20.\ a^4.b.1/2.1/2.\pi/2 = \frac{5\pi a^4 b}{4}$

Stoke's Theorem

➢It is the relation between line and surface integral.

Statement : If $\bar{F}$ is any continuous differentiable vector function and s is a surface enclosed by curve c, then $\int_c \bar{F} \cdot d\bar{r} = \int_c \text{curl}\,\bar{F} \cdot \bar{n}\, ds$ where $\bar{n}$ is the outward unit normal vector at any point of s and c is traversed in the positive direction.

Cartesian Form : Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ and $\bar{n} = \bar{i}\cos\alpha + \bar{j}\cos\beta + \bar{k}\cos\gamma$ where α, β, γ are angles with the positive direction of x, y, z axis $\cos\alpha$, $\cos\beta$, $\cos\gamma$

are directional cosines.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$F \cdot d\bar{r} = F_1\, dx + F_2\, dy + F_3\, dz$

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \bar{j}\left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right] + \bar{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

1.Using Stoke's theorem to evaluate $\oint \bar{A} \cdot d\bar{r}$ where $\bar{A} = 2y^2\bar{i} + 3x^2\bar{j} - (2x + z)\bar{k}$ and c is the boundary of the triangle whose vertices are (0,0,2), (2,0,0), (2,2,0)?

Sol : By Stoke's Theorem $\bar{A} = 2y^2\bar{i} + 3x^2\bar{j} - (2x + z)\bar{k}$

$$\text{curl } \bar{A} = \nabla \times \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -(2x + z) \end{vmatrix}$$

$$= \bar{i}(0 - 0) - \bar{j}(-2 - 0) + \bar{k}(6x - 4y)$$

$$= 2\bar{j} + \bar{k}(6x - 4y)$$

$$\int_C \bar{A} \cdot \mathrm{d}\bar{r} = \int_S (curl\ \bar{A} \cdot \bar{n})\,\mathrm{ds}$$

Since the surface is bounded by triangle with z-axis is 0. So surface lies in xy-plane.

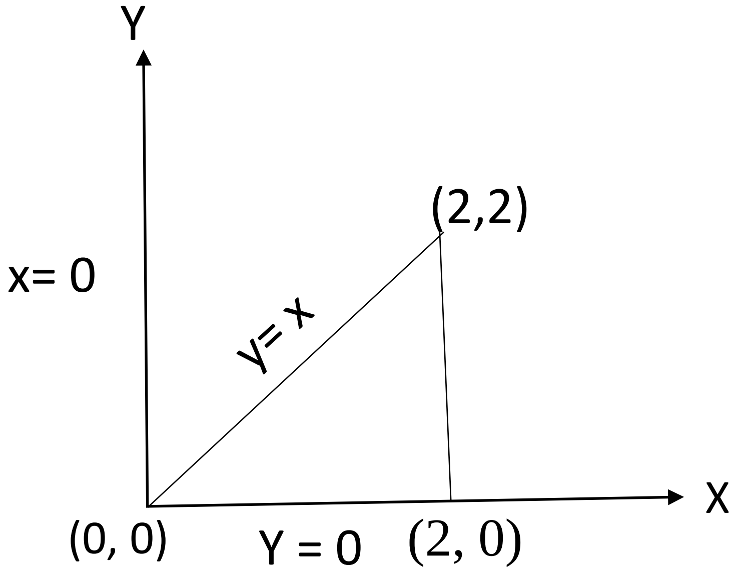
$$\therefore \bar{n} = \bar{\kappa}$$

$$\iint_S (curl\,\bar{A}\cdot\bar{n})ds = \iint_S (6x - 4y)dxdy$$

$$= \int_{x=0}^{2}\int_{y=0}^{x}(6x - 4y)dxdy$$

$$= \int_{x=0}^{2}[6xy - 2y^2]_0^x dx$$

$$= \int_{x=0}^{2} 4x^2 dx = \frac{4}{3}[x^3]_0^2 = 32/3$$



2.Evaluate $\oint(e^x dx + 2ydy - dz)$ $where\ c\ is\ the\ curve\ x^2 + y^2 = 9\ and\ z = 2?$

<u>Sol</u> By Stoke's theorem

$$\oint_S(\nabla \times \bar{F}).\bar{n}ds = \oint_C \bar{F}\cdot d\bar{r}$$

$\bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$= \bar{i}(0 - 0) - \bar{j}(0 - 0) + \bar{k}(0 - 0) = \bar{O}$

$\therefore \oint_c \bar{F} \cdot d\bar{r} = 0$

3. Verify stokes theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ where s is the circular disc $x^2 + y^2 \leq 1$, z = 0 ?

Sol Given $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ the boundary c of s is a circle in xy –plane. $x^2 + y^2 = 1$, z = 0 use the parametric coordinates x = $\cos\theta$, $y = \sin\theta$,

z = 0 ,0 $\leq \theta \leq 2\pi$, dx = -$\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\therefore \oint_C \bar{F} \cdot dr = \int_c F_1\, dx + F_2\, dy + F_3\, dz = \int_c -y^3\, dx + x^3\, dy$$

$$= \int_0^{2\pi} (-\sin^3\theta(-\sin\theta) + cos^3\theta cos\theta)\, d\theta$$

$$= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta) d\theta$$

$$= \int_0^{2\pi} d\theta - 1/2(\int_0^{2\pi} (2\sin\theta\cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi} d\theta - 1/2(\int_0^{2\pi} \sin^2 2\theta\, d\theta$$

$$= (2\pi - 0) - 1/4(\int_0^{2\pi} (1 - cos4\theta) d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{2\Pi}$$

$= 2\pi - 2\pi/4 = 6\frac{\pi}{4} = 3\frac{\pi}{2}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^3 & 0 \end{vmatrix}$$

$$= \bar{k}(3x^2 + 3y^2)$$

$$\therefore \int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3 \int_s (x^2 + y^2)\bar{k} \cdot \bar{n}\, \mathrm{ds}$$

$(\bar{k} \cdot \bar{n})ds = dxdy$ and R is the region on xy-plane.

$$\therefore \iint_s (\nabla \times \bar{F}) \cdot \bar{n}\, \mathrm{ds} = 3 \iint_R (x^2 + y^2) \mathrm{dxdy}$$

Put x = rcos ϕ , y = rsinϕ

$\therefore$ dxdy = rdrdϕ where r is varying from 0 to 1, $0 \le \theta \le 2\pi$.

$$\therefore \int (\nabla \times \bar{F}) \cdot \bar{n}\, \mathrm{ds} = 3\int_{\emptyset=0}^{2\pi} \int_{r=0}^{1} r^2 \cdot r dr\, d\phi = 3\frac{\pi}{2}$$

L.H.S = R.H.S

Hence theorem is verified.