Mathematics (Subject Code)
3^{rd} Semester
Lecture Notes

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Mathematics (Subject Code)

3^{rd} Semester

Lecture Note # 1

Module - I

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Solution of non linear equation in one variable

In the science and engineering the solution or roots x of an equation $f(x) = 0$ occurs in many applications. If $f(x)$ is a linear or quadratic equation then we can find the roots of $f(x) = 0$ easily. However, many non linear equations, transdental equations can not be solved easily. Hence to find out the solution or root of such types of equations we need some numerical successive approximation methods or iterative methods.

Iterative method

These methods are based on the idea of successive approximations i.e starting with one or more initial approximations to the root or solution we obtain a sequence of approximations or iterates $\{x_k\}$, which is in the limit converges to the root. The methods may give only one root at a time.

Definition:

A sequence of iterates $\{x_k\}$ is said to be converges to the exact root or solution α if

$$\lim_{k\to\infty} |x_k - \alpha| = 0, \qquad k = 1,\ 2,\ 3,\ \ldots\ldots$$

Non linear / Polynomial / Algebric equations:

An expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \ \ldots\ldots\ + a_{n-1}x + a_n$$

Where a_1, a_2, ... a_n are constants provided $a_0 \neq 0$ and n is positive integer is called a polynomial in x of degree n. The polynomial equation $f(x) = 0$ is called algebric equation.

Example:

$$x^3 - 5x + 1 = 0$$
$$x^4 - 3x^3 + 4x - 3 = 0$$

Transdental equations:

If $f(x)$ is an expression involving trigonometric, logarithmic, exponential functions etc. then it is called transdental equation.

Example:

$$\cos x - xe^x = 0$$
$$\log_e x - e^x = 0$$

***Solutions / Roots*:**

A number α is said to be a root or solution of the equation $f(x) = 0$ if $f(\alpha) = 0$. One fundamental theorem is used to locate the interval in which the real root of the equation $f(x) = 0$ lies. The theorem is known as intermediate value theorem. We have to take that interval or any point on that interval as the initial approximation to the root of the equation in different iterative methods.

***Intermediate value theorem*:**

If $f(x)$ is a continuous function on some interval $[a, b]$ and $f(a)f(b) < 0$ i.e. $f(a)$ and $f(b)$ are of opposite sign then the equation $f(x) = 0$ has at least one real root or an odd number of roots in the interval (a, b).

***Example 1*:**

Find the interval in which the real root of the equation lies,

$$f(x) = x^3 - 5x + 1 = 0$$

***Solution*:**

We have

$$f(x) = x^3 - 5x + 1 = 0$$

Now

$$f(0) = 1 > 0$$
$$f(1) = -3 < 0$$

So, the root of the equation lies in the interval $(0, 1)$. '0' point is closer to the root of the equation than the point '1'. Because at 0 point the value of $f(x)$ is 1, where as at 1 point the value of $f(x)$ is -3. Since 1 is closer to 0 rather than -3, thus 0 point is closer to the root.

If we try to find out the closer interval in which the root of $f(x)$ lies, we have to check the points closer to 0.

$$f(0.1) = 0.501 > 0$$
$$f(0.2) = 0.008 > 0$$
$$f(0.3) = -0.473 < 0$$

Thus the closer interval in which the root of the equation lies $(0.2, 0.3)$. Among these two points 0.2 and 0.3, 0.2 is nearer to the root of the equation.

***Example 2*:**

Find out the interval in which the root the equation

$$f(x) = \cos x - xe^x = 0$$

lies.

***Solution*:**

First convert the calcular into radian mode because this equation contains trigonometric function.

We have

$$f(x) = \cos x - xe^x = 0$$
$$f(0) = 1 > 0$$
$$f(1) = -2.1779 < 0$$

So, the root of the equation lies in the interval $(0, 1)$.

Now we have to find out the closer interval in which the root of the equation lies

$$f(0.5) = 0.175 > 0$$
$$f(0.6) = -0.093 < 0$$

So, the root of the equation lies in the interval $(0.5, 0.6)$

There are several numerical methods for finding the root or solution of an equation $f(x) = 0$. Some methods are one point method because one initial approximation is required. Some of them are two point method because two initial approximations are required for finding out the root of the equation.

The methods are,

1. Bisection method (Two point method)
2. Secant method (Two point method)
3. Regular Falasi method (Two point method)
4. Newton Raphson method (One point method)
5. Fixed point iteration method (One point method)

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Lecture Note # 2
Module - I

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Bisection Method

This method is based on the repeated application of the intermediate value theorem. If we know that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$ we bisect I_0 at the point $m_1 = \frac{1}{2}(a_0+b_0)$. Denote by I_1, the interval (a_0, m_1) if $f(a_0 f(m_1)) < 0$ or the interval (m_1, b_0) if $f(m_1 f(b_0)) < 0$. Therefore the interval I_1 also contain the root. We bisect the interval I_1 and get the sub-interval I_2 at whose end point $f(x)$ takes the values of opposite signs and therefore contains the root. Containing this procedure we obtain a sequence of nested set of sub-intervals $I_0 \supset I_1 \supset I_2 \supset \ \ldots\ldots$. Such that each sub-interval contains the root. After repeating the bisection process n times we either find the root or find the interval I_n of length $\frac{b_0-a_0}{2^n}$.

We take the midpoint of the last sub interval as the desired approximation to the root.

This method is simple but slowly convergent method. The bisection method is convergent linearly. This is a two point formula because two interval approximation are required for finding out the root of the equation.

***Example 1*:**

Find the real root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

correct up to three decimal places by using bisection method.

***Solution*:**

We have,

$$f(x) = x^3 - 5x + 1 = 0$$

Now

$$f(0) = 1 > 0$$

$$f(0) = -2 < 0$$

So

$$f(0.1) = 0.501 > 0$$

$$f(0.2) = 0.008 > 0$$

$$f(0.3) = -0.473 < 0$$

Thus the root of the equation lies in the interval $(0.2, 0.3)$. Let $x_0 = 0.2$ and $x_1 = 0.3$ be the initial approximation to the root of the equation.

First approximation:

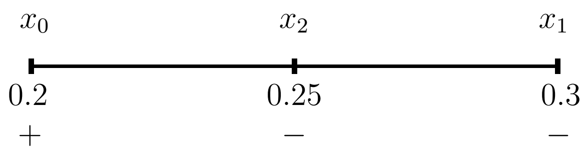
$$x_0 = 0.2$$

$$x_1 = 0.3$$

Let

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0.2 + 0.3}{2} = 0.25$$

$$f(x_2) = f(0.25) = -0.234 < 0$$



So the root of the equation lies in the interval $(x_0, x_2) = (0.2, 0.25)$

Second approximation:

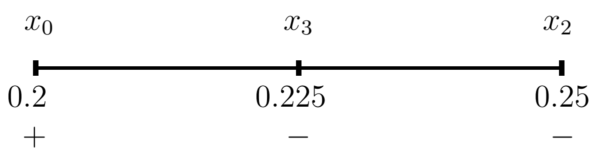
$$x_0 = 0.2$$

$$x_2 = 0.25$$

Let

$$x_3 = \frac{x_0 + x_2}{2} = \frac{0.2 + 0.25}{2} = 0.225$$

$$f(x_3) = f(0.225) = -0.113 < 0$$



So the root of the equation lies in the interval $(x_0, x_3) = (0.2, 0.225)$

Third approximation:

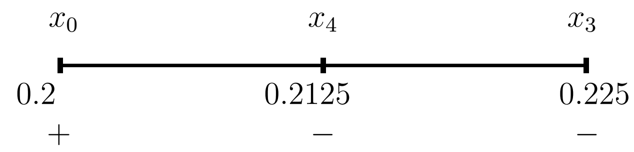
$$x_0 = 0.2$$

$$x_3 = 0.225$$

Let

$$x_4 = \frac{x_0 + x_3}{2} = \frac{0.2 + 0.225}{2} = 0.2125$$

$$f(x_4) = f(0.2125) = -0.052 < 0$$



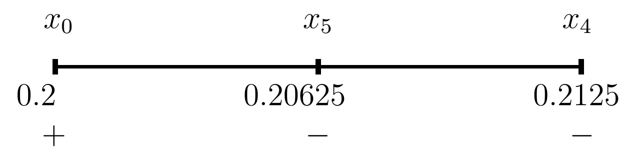
So the root of the equation lies in the interval $(x_0, x_4) = (0.2, 0.2125)$

Fourth approximation:

$$x_0 = 0.2$$
$$x_4 = 0.2125$$

Let

$$x_5 = \frac{x_0 + x_4}{2} = \frac{0.2 + 0.2125}{2} = 0.20625$$
$$f(x_5) = f(0.20625) = -0.0224 < 0$$



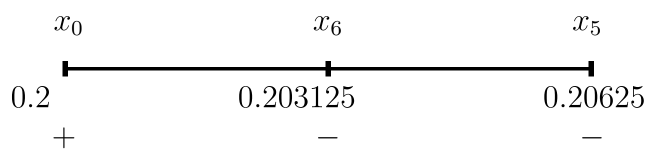
So the root of the equation lies in the interval $(x_0, x_5) = (0.2, 0.20625)$

Fifth approximation:

$$x_0 = 0.2$$
$$x_5 = 0.20625$$

Let

$$x_6 = \frac{x_0 + x_5}{2} = \frac{0.2 + 0.2125}{2} = 0.203125$$
$$f(x_6) = f(0.203125) = -0.0072 < 0$$



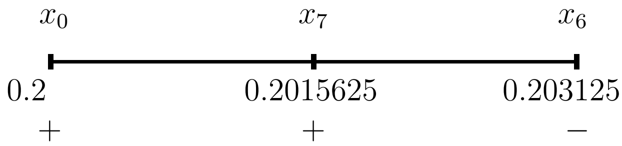
So the root of the equation lies in the interval $(x_0, x_6) = (0.2, 0.203125)$

Sixth approximation :

$$x_0 = 0.2$$
$$x_6 = 0.203125$$

Let

$$x_7 = \frac{x_0 + x_6}{2} = \frac{0.2 + 0.203125}{2} = 0.2015625$$
$$f(x_7) = f(0.2015625) = 0.00037 > 0$$



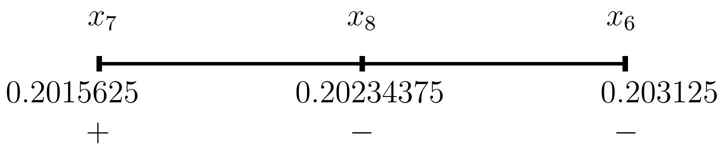
So the root of the equation lies in the interval $(x_7, x_6) = (0.2015625, 0.203125)$

Seventh approximation :

$$x_7 = 0.2015625$$
$$x_5 = 0.203125$$

Let

$$x_8 = \frac{x_7 + x_6}{2} = \frac{0.2015625 + 0.203125}{2} = 0.20234375$$
$$f(x_8) = f(0.20234375) = -0.003 < 0$$



So the root of the equation lies in the interval $(x_7, x_8) = (0.2015625, 0.20234375)$

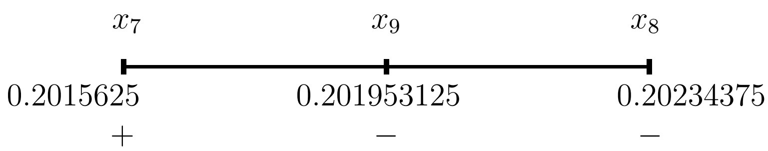
Eights approximation :

$$x_7 = 0.2015625$$
$$x_8 = 0.20234375$$

Let

$$x_9 = \frac{x_7 + x_8}{2} = \frac{0.2015625 + 0.20234375}{2} = 0.201953125$$

$$f(x_9) = f(0.201953125) = -0.001 < 0$$



So the root of the equation lies in the interval $(x_7, x_9) = (0.2015625, 0.201953125)$

Thus the required root of the equation correct up to three decimal places by bisection method is 0.201

***Example 2*:**

Find the real root of the equation

$$f(x) = x - e^{-x} = 0$$

by using bisection method.

OR

Perform five step to find the real root of the equation

$$f(x) = x - e^{-x} = 0$$

by using bisection method.

***Solution*:**

We have

$$f(x) = x - e^{-x} = 0$$

$$f(0) = -1 < 0$$

$$f(1) = 0.6321$$

$$f(0.5) = -0.1065 < 0$$

$$f(0.6) = 0.0511 > 0$$

Thus the root of the equation lies in the interval $(0.5, 0.6)$.

Let $x_0 = 0.5$, $x_1 = 0.6$ be the initial approximation to the root of the equation.

First approximation:

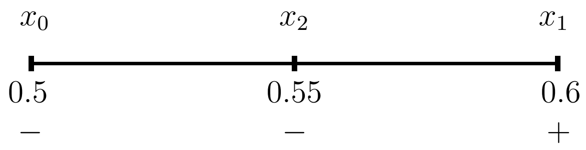
$$x_0 = 0.5$$

$$x_1 = 0.6$$

Let

$$x_2 = \frac{x_0 + x_1}{2} = \frac{0.5 + 0.6}{2} = 0.55$$

$$f(x_2) = f(0.55) = -0.0269 < 0$$



So the root of the equation lies in the interval $(x_2, x_1) = (0.55, 0.6)$

Second approximation:

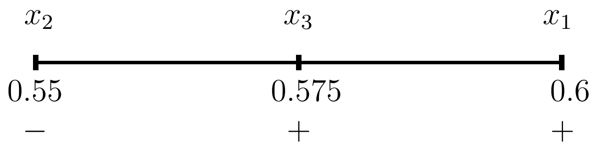
$$x_2 = 0.55$$

$$x_1 = 0.6$$

Let

$$x_3 = \frac{x_2 + x_1}{2} = \frac{0.55 + 0.6}{2} = 0.575$$

$$f(x_3) = f(0.575) = 0.012 > 0$$



So the root of the equation lies in the interval $(x_2, x_3) = (0.55, 0.575)$

Third approximation:

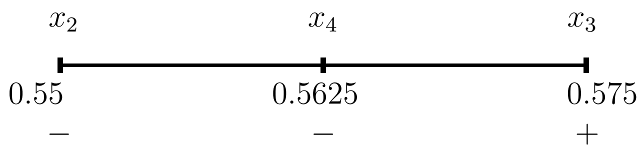
$$x_2 = 0.55$$

$$x_3 = 0.575$$

Let

$$x_4 = \frac{x_2 + x_3}{2} = \frac{0.55 + 0.575}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = -0.007 < 0$$



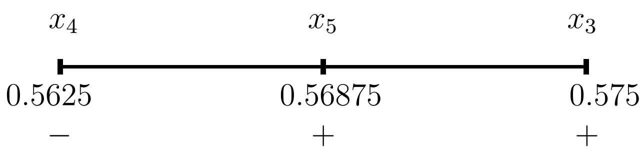
So the root of the equation lies in the interval $(x_4, x_3) = (0.5625, 0.575)$

Fourth approximation:

$$x_4 = 0.5625$$
$$x_3 = 0.575$$

Let

$$x_5 = \frac{x_4 + x_3}{2} = \frac{0.5625 + 0.575}{2} = 0.56875$$
$$f(x_5) = f(0.56875) = 0.002 > 0$$



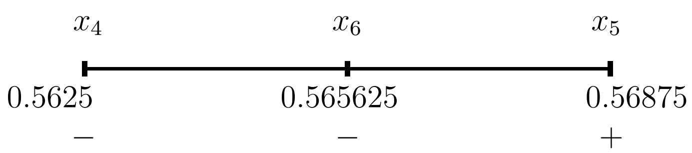
So the root of the equation lies in the interval $(x_4, x_5) = (0.5625, 0.56875)$

Fifth approximation:

$$x_4 = 0.5625$$
$$x_5 = 0.56875$$

Let

$$x_6 = \frac{x_4 + x_5}{2} = \frac{0.5625 + 0.56875}{2} = 0.565625$$
$$f(x_6) = f(0.565625) = -0.002 < 0$$



So the root of the equation lies in the interval $(x_6, x_5) = (0.565625, 0.56875)$

Sixth approximation :

$$x_6 = 0.565625$$
$$x_5 = 0.56875$$

Let

$$x_7 = \frac{x_6 + x_5}{2} = \frac{0.565625 + 0.56875}{2} = 0.5671875$$

So the root of the given equation by bisection method after fifth step is 5671875

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Lecture Note # 3
Module - I

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Secant method

The secant method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$

$$k = 1,\ 2,\ 3,\ \ldots .$$

First approximation (k=1)

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

Second approximation (k=2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

Third approximation (k=3)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} f(x_3)$$

and so on

Where x_0 and x_1 are called the initial approximation to the root of the equation.

Since two initial approximations are equal for finding out the root of the equation so it is called a two point formula.

NOTE: The rate of convergence of secant method is 1.618.

***Example 1*:**

Find the real root of the equation $f(x) = x^3 - 5x + 1 = 0$ correct upto three decimal places by using secant method.

***Solution*:**

We have $f(x) = x^3 - 5x + 1 = 0$

$$f(0) = 1 > 0, \qquad f(1) = -3 < 0$$

$$f(0.2) = 0.008 > 0, \qquad f(0.3) = -0.473 < 0$$

So the root of the equation lies in the interval $(0.2, 0.3)$

Let $x_0 = 0.2$ and $x_1 = 0.3$ be the initial approxiamtion to the root of the equation.

We have the secant method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$

$$k = 1,\ 2,\ 3,\ \ldots .$$

First approximation (k=1)

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$= 0.3 - \frac{(0.3 - 0.2)}{f(0.3) - f(0.2)} f(0.3) = 0.201663$$

Second approximation (k=2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.201663 - \frac{(0.201663 - 0.3)}{f(0.201663) - f(0.3)} f(0.201663) = 0.201639$$

So the root of the equation correct upto three decimal places by secant method is 0.201

***Example 2*:**

Find the real root of the equation $f(x) = \cos x - xe^x = 0$ by using secant method.

***Solution*:**

We have $f(x) = \cos x - xe^x = 0$

$$f(0) = 1 > 0, \qquad f(1) = -2.177979 < 0$$

$$f(0.5) = 0.053221 > 0, \qquad f(0.6) = -0.267935 < 0$$

So the root of the equation lies in the interval $(0.5, 0.6)$

Let $x_0 = 0.5$ and $x_1 = 0.6$ be the initial approximation to the root of the equation.

We have the secant method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$

$$k = 1,\ 2,\ 3,\ \ldots .$$

First approximation (k=1)

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$= 0.6 - \frac{(0.6 - 0.5)}{f(0.6) - f(0.5)} f(0.6) = 0.516571$$

Second approximation (k=2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.516571 - \frac{(0.516571 - 0.6)}{f(0.516571) - f(0.6)} f(0.516571) = 0.0.517678$$

Third approximation (k=3)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.517678 - \frac{(0.517678 - 0.516571)}{f(0.517678) - f(0.516571)} f(0.517678) = 0.0.517757$$

So the root of the equation by secant method after three steps is 0.517757

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Lecture Note # 4
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Newton Raphson method

The Newton Raphson method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0,\ 1,\ 2,\ 3,\ \ldots.$$

First approximation (k=0)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Second approximation (k=1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Third approximation (k=2)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

and so on

Where x_0 is known as the initial approximation to the root of the equation.

Since one initial approximations are equal for finding out the root of the equation so it is called a one point formula.

NOTE: The rate of convergence of Newton Raphson method is 2. So it has quadratic rate of convergence.

***Example 1*:**

Find the real root of the equation $f(x) = x^4 - x - 10 = 0$ by using Newton Raphson method.

***Solution*:**

We have $f(x) = x^4 - x - 10 = 0$, $f'(x) = 4x^3 - 1$

$$f(0) = -10 < 0, \qquad f(1) = -10 < 0, \qquad f(2) = 4 > 0$$

So the root of the equation lies in the interval $(1, 2)$,

Let $x_0 = 1.8$ be the initial approximation to the root of the equation.

We have the Newton Raphson method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0,\ 1,\ 2,\ 3,\ \ldots.$$

First approximation (k=0)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.8 - \frac{f(1.8)}{f'(1.8)} = 1.8583303$$

Second approximation (k=1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.8583303 - \frac{f(1.8583303)}{f'(1.8583303)} = 1.8555908$$

Third approximation (k=2)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.8555908 - \frac{f(1.8555908)}{f'(1.8555908)} = 1.8555845$$

So the root of the equation after three steps by Newton Raphson method is 1.8555845

***Example 2*:**

Find the real root of the equation $f(x) = \cos x - xe^x = 0$ by newton Raphson method correct up to three decimal places.

***Solution*:**

We have $f(x) = \cos x - xe^x = 0$, $\qquad f'(x) = -\sin x - xe^x - e^x$

$$f(0) = 1 > 0, \qquad f(1) = -2.177979 < 0$$

So the root of the equation lies in the interval $(0, 1)$

Let $x_0 = 0.5$ be the initial approximation to the root of the equation.

We have the Newton Raphson method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0,\ 1,\ 2,\ 3,\ \ldots.$$

First approximation (k=0)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5180260$$

Second approximation (k=1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.5180260 - \frac{f(0.5180260)}{f'(0.5180260)} = 0.517757$$

Third approximation (k=2)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.517757 - \frac{f(0.517757)}{f'(0.517757)} = 0.517757$$

So the root of the equation by Newton Raphson method correct up to three decimal places is 0.517.

***Example 3*:**

Find the cube root of 2 by newton Raphson method correct up to four decimal places.

***Solution*:**

$$\text{Let } x = \sqrt[3]{2}$$

$$\implies x^3 = 2$$
$$\implies x^3 - 2 = 0$$

Let $f(x) = x^3 - 2 = 0,\qquad f'(x) = 3x^2$

$$f(0) = -2 < 0, \qquad f(1) = -1 < 0, \qquad f(2) = 6 > 0$$

So the root of the equation lies in the interval $(1, 2)$

Let $x_0 = 1.2$ be the initial approximation to the root of the equation.

We have the Newton Raphson method for finding out the root of the equation $f(x) = 0$ is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0,\ 1,\ 2,\ 3,\ \ldots .$$

First approximation (k=0)

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1.2 - \frac{f(1.2)}{f'(1.2)} = 1.262962 \end{aligned}$$

Second approximation (k=1)

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.262962 - \frac{f(1.262962)}{f'(1.262962)} = 1.259928 \end{aligned}$$

Third approximation (k=2)

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 1.259928 - \frac{f(1.259928)}{f'(1.259928)} = 1.259921 \end{aligned}$$

Thus the cube root of 2 correct up to four decimal point by Newton Raphson method is 1.2599.

Mathematics (Subject Code)

3^{rd} Semester

Lecture Note # 5

Module - I

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Fixed point iteration method

Let the equation be $f(x) = 0$ and (a, b) be the interval in which the root of the equation $f(x)$ lies.

Write down the equation $f(x)$ in the form of $x = \varphi(x)$ such that $|\varphi'(x)| < 1$ for all $x \in (a, b)$ i.e the interval in which the root of the equation $f(x)$ lies.

Now the iteration method is

$$x_{k+1} = \varphi(x_k), \qquad k = 0,\ 1,\ 2,\ .\ .\ .\ .\ .$$

First approximation $(k = 0)$

$$x_1 = \varphi(x_0)$$

Second approximation $(k = 1)$

$$x_2 = \varphi(x_1)$$

Third approximation $(k = 2)$

$$x_3 = \varphi(x_2)$$

and so on.

Where x_0 is the initial approximation to the root of the equation. This is the one point formula. Since one initial approximation x_0 is required.

NOTE: Fixed point iteration method has a linear rate of convergence.

***Example 1*:**

Find the real root of the equation $f(x) = x^3 - 5x + 1 = 0$ by using fixed point iteration method.

***Solution*:**

We have $f(x) = x^3 - 5x + 1 = 0$

$$f(0) = 1 > 0, \qquad f(1) = -3 < 0$$

So the root of the equation lies in the interval $(0, 1)$.

Now $f(x) = x^3 - 5x + 1 = 0$ can be written as

$$x = -\frac{1}{(x^2 - 5)}$$

where

$$\varphi(x) = -\frac{1}{(x^2-5)}$$

Now

$$\varphi'(x) = \frac{2x}{(x^2-5)^2}$$

$$\left|\varphi'(x)\right| = \left|\frac{2x}{(x^2-5)^2}\right| < 0, \qquad \text{for all} \quad x \in (0,1)$$

i.e the interval in which the real root of $f(x)$ lies.

Thus the iteration scheme is

$$x_{k+1} = -\frac{1}{\left(x_k^2-5\right)} \qquad k = 0,\ 1,\ 2,\ \ldots\ldots$$

Let $x_0 = 0.2$ be the initial approximation to the root of the equation.

First approximation $(k = 0)$

$$\begin{aligned} x_1 &= -\frac{1}{\left(x_0^2-5\right)} \\ &= -\frac{1}{\left[(0.2)^2-5\right]} \\ &= 0.2016129 \end{aligned}$$

Second approximation $(k = 1)$

$$\begin{aligned} x_2 &= -\frac{1}{\left(x_1^2-5\right)} \\ &= -\frac{1}{\left[(0.2016129)^2-5\right]} \\ &= 0.2016392 \end{aligned}$$

Third approximation $(k = 2)$

$$\begin{aligned} x_3 &= -\frac{1}{\left(x_2^2-5\right)} \\ &= -\frac{1}{\left[(0.2016392)^2-5\right]} \\ &= 0.20016396 \end{aligned}$$

Thus the root of the equation by fixed point iteration after three steps is 0.2016396

***Example 2*:**

Find the real root of the equation $f(x) = 2x - \cos x - 3 = 0$ correct up to three decimal places by using fixed point iteration method.

***Solution*:**

We have $f(x) = 2x - \cos x - 3 = 0$

$$f(0) = -4 < 0, \qquad f(1) = -1.54 < 0, \qquad f(2) = 1.416 > 0$$

So the root of the equation lies in the interval $(1, 2)$.

Now $f(x) = 2x - \cos x - 3 = 0$ can be written as

$$x = \frac{\cos x + 3}{2}$$

where

$$\varphi(x) = \frac{\cos x + 3}{2}$$

Now

$$\varphi'(x) = -\frac{1}{2}\sin x$$

$$\left|\varphi'(x)\right| = \left|-\frac{1}{2}\sin x\right| = \left|\frac{1}{2}\sin x\right| < 0, \qquad \text{for all} \quad x \in (1, 2)$$

i.e the interval in which the real root of $f(x)$ lies. Thus the iteration scheme is applicable

So the iteration scheme is

$$\begin{aligned} x_{k+1} &= \varphi(x_k) \\ \Longrightarrow x_{k+1} &= \frac{\cos x_k + 3}{2} \qquad k = 0,\ 1,\ 2,\ \ldots\ldots \end{aligned}$$

Let $x_0 = 1.5$ be the initial approximation to the root of the equation.

First approximation $(k = 0)$

$$\begin{aligned} x_1 &= \frac{\cos x_0 + 3}{2} \\ &= \frac{\cos(1.5) + 3}{2} \\ &= 1.5354 \end{aligned}$$

Second approximation $(k = 1)$

$$\begin{aligned} x_2 &= \frac{\cos x_1 + 3}{2} \\ &= \frac{\cos(1.5354) + 3}{2} \\ &= 1.5177 \end{aligned}$$

Third approximation $(k = 2)$

$$\begin{aligned} x_3 &= \frac{\cos x_2 + 3}{2} \\ &= \frac{\cos(1.5177) + 3}{2} \\ &= 1.5265 \end{aligned}$$

Fourth approximation $(k = 3)$

$$\begin{aligned} x_4 &= \frac{\cos x_3 + 3}{2} \\ &= \frac{\cos(1.5265) + 3}{2} \\ &= 1.5221 \end{aligned}$$

Fifth approximation $(k = 4)$

$$\begin{aligned} x_5 &= \frac{\cos x_4 + 3}{2} \\ &= \frac{\cos(1.5221) + 3}{2} \\ &= 1.5243 \end{aligned}$$

Sixth approximation $(k = 5)$

$$\begin{aligned} x_6 &= \frac{\cos x_5 + 3}{2} \\ &= \frac{\cos(1.5243) + 3}{2} \\ &= 1.5232 \end{aligned}$$

Seventh approximation $(k = 6)$

$$\begin{aligned} x_7 &= \frac{\cos x_6 + 3}{2} \\ &= \frac{\cos(1.5232) + 3}{2} \\ &= 1.5237 \end{aligned}$$

Thus the root of the equation $f(x) = 2x - \cos x - 3 = 0$ correct up to three decimal places by fixed point iteration method is 1.523

Mathematics (Subject Code)

3^{rd} Semester

Lecture Note # 6

Department of Mathematics

Ajay Binay Institute of Technology, Cuttack

Numerical Solutions of System of Linear Equations

Simultaneous linear equations have great importance in the field of engineering and science. In the field of science the analysis of electronic circuits having number of invariant elements, analysis of a network under sinusoidal steady stady state conditions and determination of output of a chemical plant are some of the problems which depends on the solution of system of linear algebraic equations. We shall discuss the solution of system of m linear equations with n unknowns where $m = n$ by different iterative and direct methods.

Some of the iterative methods are

1. Gauss seidel method
2. Successive over relaxation (SOR) method

Some of the direct methods are

1. Doolittle's method
2. Crout's method
3. Cholesky's method

Gauss seidel method

Let us explain the Gauss seidel method in the case of three linear equations with three unknowns. Similarly we can extend the method into n linear equations with n unknowns.

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$
$$a_2x + b_2y + c_2z = d_2 \tag{2}$$
$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

verify that,

$$|a_1| > |b_1| + |c_1|$$
$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

Then the Gauss Seidel iterative method can be used for the given system. Solving equation (1), (2) and (3) for x, y and z respectively we get

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Now we can start the solution process with initial values $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ for x, y and z respectively.

We calculate

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$
$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(0)}\right)$$
$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

Thus as soon as a new values for a variable is found, it is used immediately in the following equations.

If $x^{(k)}$, $y^{(k)}$, $z^{(k)}$ are the k^{th} iterates then the $(k+1)^{th}$ iterates will be

$$x^{(k+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(k+1)} - c_2 z^{(k)}\right)$$
$$z^{(k+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

The process is continued until the convergence is assured.

NOTE: In the absence of initial approximation $x^{(0)}$, $y^{(0)}$, $z^{(0)}$, they are taken as $(0, 0, 0)$.

***Example 1*:**

Solve the following system of linear equations by using Gauss seidel iterative method correct up to three decimal places.

$$10x - 5y - 2z = 3$$
$$x + 6y + 10z = -3$$
$$4x - 10y + 3z = -3$$

***Solution* :**

The above system of equations can be rewrite as

$$10x - 5y - 2z = 3 \tag{4}$$
$$4x - 10y + 3z = -3 \tag{5}$$
$$x + 6y + 10z = -3 \tag{6}$$

It can be easily verified that,

$$|10| > |-5| + |-2|$$
$$|-10| > |4| + |3|$$
$$|10| > |1| + |6|$$

We are from the equation (4), (5) and (6)

$$x = \frac{1}{10}(3 + 5y + 2z)$$
$$y = \frac{1}{10}(3 + 4x + 3z)$$
$$z = -\frac{1}{10}(3 + x + 6y)$$

Thus the iteration scheme for Gauss seidel method is

$$x^{(k+1)} = \frac{1}{10}\left(3 + 5y^{(k)} + 2z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{10}\left(3 + 4x^{(k+1)} + 3z^{(k)}\right)$$

$$z^{(k+1)} = -\frac{1}{10}\left(3 + x^{(k+1)} + 6y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

Let $x^0 = y^0 = z^0 = 0$ be the initial approximation to the solution.

First approximation ($k = 0$):

$$x^{(1)} = \frac{1}{10}\left(3 + 5y^{(0)} + 2z^{(0)}\right) = \frac{1}{10}(3 + 5 \times 0 + 2 \times 0) = 0.3$$
$$y^{(1)} = \frac{1}{10}\left(3 + 4x^{(1)} + 3z^{(0)}\right) = \frac{1}{10}(3 + 4 \times 0.3 + 3 \times 0) = 0.42$$
$$z^{(1)} = -\frac{1}{10}\left(3 + x^{(1)} + 6y^{(1)}\right) = -\frac{1}{10}(3 + 0.3 + 6 \times 0.42) = -0.582$$

Second approximation ($k = 1$):

$$x^{(2)} = \frac{1}{10}\left(3 + 5y^{(1)} + 2z^{(1)}\right) = \frac{1}{10}\left(3 + 5 \times 0.42 + 2 \times (-0.582)\right) = 0.3936$$
$$y^{(2)} = \frac{1}{10}\left(3 + 4x^{(2)} + 3z^{(1)}\right) = \frac{1}{10}\left(3 + 4 \times 0.3936 + 3 \times (-0.582)\right) = 0.28284$$
$$z^{(2)} = -\frac{1}{10}\left(3 + x^{(2)} + 6y^{(2)}\right) = -\frac{1}{10}(3 + 0.3936 + 6 \times 0.28284) = -0.509064$$

Third approximation ($k = 2$):

$$x^{(3)} = \frac{1}{10}\left(3 + 5y^{(2)} + 2z^{(2)}\right) = \frac{1}{10}\left(3 + 5 \times 0.28284 + 2 \times (-0.509064)\right) = 0.3396072$$
$$y^{(3)} = \frac{1}{10}\left(3 + 4x^{(3)} + 3z^{(2)}\right) = \frac{1}{10}\left(3 + 4 \times 0.3396072 + 3 \times (-0.509064)\right) = 0.283123$$
$$z^{(3)} = -\frac{1}{10}\left(3 + x^{(3)} + 6y^{(3)}\right) = -\frac{1}{10}(3 + 0.3396072 + 6 \times 0.283123) = -0.5038345$$

Fourth approximation ($k = 3$):

$$x^{(4)} = \frac{1}{10}\left(3 + 5y^{(3)} + 2z^{(3)}\right) = \frac{1}{10}\left(3 + 5 \times 0.283123 + 2 \times (-0.5038345)\right) = 0.340794$$
$$y^{(4)} = \frac{1}{10}\left(3 + 4x^{(4)} + 3z^{(3)}\right) = \frac{1}{10}\left(3 + 4 \times 0.340794 + 3 \times (-0.5038345)\right) = 0.285167$$

$$z^{(4)} = -\frac{1}{10}\left(3 + x^{(4)} + 6y^{(4)}\right) = -\frac{1}{10}(3 + 0.340794 + 6 \times 0.285167) = -0.505179$$

Fifth approximation $(k = 4)$:

$$x^{(5)} = \frac{1}{10}\left(3 + 5y^{(4)} + 2z^{(4)}\right) = \frac{1}{10}\big(3 + 5 \times 0.285167 + 2 \times (-0.505179)\big) = 0.341548$$

$$y^{(5)} = \frac{1}{10}\left(3 + 4x^{(5)} + 3z^{(4)}\right) = \frac{1}{10}\big(3 + 4 \times 0.341548 + 3 \times (-0.505179)\big) = 0.285065$$

$$z^{(5)} = -\frac{1}{10}\left(3 + x^{(5)} + 6y^{(5)}\right) = -\frac{1}{10}(3 + 0.341548 + 6 \times 0.285065) = -0.505193$$

Sixth approximation $(k = 5)$:

$$x^{(6)} = \frac{1}{10}\left(3 + 5y^{(5)} + 2z^{(5)}\right) = \frac{1}{10}\big(3 + 5 \times 0.285065 + 2 \times (-0.505193)\big) = 0.341493$$

$$y^{(6)} = \frac{1}{10}\left(3 + 4x^{(6)} + 3z^{(5)}\right) = \frac{1}{10}\big(3 + 4 \times 0.341493 + 3 \times (-0.505193)\big) = 0.285039$$

$$z^{(6)} = -\frac{1}{10}\left(3 + x^{(6)} + 6y^{(6)}\right) = -\frac{1}{10}(3 + 0.341493 + 6 \times 0.285039) = -0.505172$$

Thus the solution of the above system of linear equations by Gauss Seidel iteration method correct up to three decimal point each $x = 0.341$, $y = 0.285$, $z = -0.505$

***Example 2*:**

Solve the following system of linear equations by using Gauss seidel iterative method by taking the initial approximations as $x = 2$, $y = 1$ and $z = 1$.

$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$
$$8x - 3y + 2z = 10$$

***Solution* :**

The above system of equations can be rewrite as

$$8x - 3y + 2z = 10 \tag{7}$$
$$4x + 11y - z = 33 \tag{8}$$
$$6x + 3y + 12z = 35 \tag{9}$$

It can be easily verified that,

$$|8| > |-3| + |2|$$
$$|11| > |4| + |-1|$$
$$|12| > |6| + |3|$$

We are from the equation (7), (8) and (9)

$$x = \frac{1}{8}(10 + 3y - 2z)$$
$$y = \frac{1}{11}(33 - 4x + z)$$
$$z = \frac{1}{12}(35 - 6x - 3y)$$

Thus the iteration scheme for Gauss seidel method is

$$x^{(k+1)} = \frac{1}{8}\left(10 + 3y^{(k)} - 2z^{(k)}\right)$$
$$y^{(k+1)} = \frac{1}{11}\left(33 - 4x^{(k+1)} + z^{(k)}\right)$$

$$z^{(k+1)} = \frac{1}{12}\left(35 - 6x^{(k+1)} - 3y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots .$$

Given that $x^{(0)} = 2,\ y^{(0)} = 1$ and $z^{(0)} = 0$ is the initial approximation to the solution.

First approximation $(k = 0)$:

$$x^{(1)} = \frac{1}{8}\left(10 + 3y^{(0)} - 2z^{(0)}\right) = \frac{1}{8}(10 + 3 \times 1 - 2 \times 1) = 1.375$$

$$y^{(1)} = \frac{1}{11}\left(33 - 4x^{(1)} + z^{(0)}\right) = \frac{1}{11}(33 - 4 \times 1.375 + 1 \times 1) = 2.590909$$

$$z^{(1)} = \frac{1}{12}\left(35 - 6x^{(1)} - 3y^{(1)}\right) = \frac{1}{12}(35 - 6 \times 1.375 - 3 \times 2.590909) = 1.581439$$

Second approximation $(k = 1)$:

$$x^{(2)} = \frac{1}{8}\left(10 + 3y^{(1)} - 2z^{(1)}\right) = \frac{1}{8}\left(10 + 3 \times 2.590909 - 2 \times (1.581439)\right) = 1.826231$$

$$y^{(2)} = \frac{1}{11}\left(33 - 4x^{(2)} + z^{(1)}\right) = \frac{1}{11}\left(33 - 4 \times 1.826231 + 1 \times (1.581439)\right) = 2.479683$$

$$z^{(2)} = -\frac{1}{12}\left(35 - 6x^{(2)} - 3y^{(2)}\right) = \frac{1}{12}(33 - 6 \times 1.826231 - 3 \times 2.47968) = 1.383630$$

Third approximation $(k = 2)$:

$$x^{(3)} = \frac{1}{8}\left(10 + 3y^{(2)} - 2z^{(2)}\right) = \frac{1}{8}(10 + 3 \times 2.479683 - 2 \times 1.383630) = 1.833973$$

$$y^{(3)} = \frac{1}{11}\left(33 - 4x^{(3)} + z^{(2)}\right) = \frac{1}{11}(33 - 4 \times 1.833973 + 1 \times 1.383630) = 2.458885$$

$$z^{(3)} = \frac{1}{12}\left(35 - 6x^{(3)} - 3y^{(3)}\right) = -\frac{1}{12}(35 - 6 \times 1.833973 - 3 \times 2.458885) = 1.384958$$

Fourth approximation $(k = 3)$:

$$x^{(4)} = \frac{1}{8}\left(10 + 3y^{(3)} - 2z^{(3)}\right) = \frac{1}{8}(10 + 3 \times 2.45885 - 2 \times 1.384958) = 1.825842$$

$$y^{(4)} = \frac{1}{11}\left(33 - 4x^{(4)} + z^{(3)}\right) = \frac{1}{11}(33 - 4 \times 1.825842 + 1 \times 1.384958) = 2.459919$$

$$z^{(4)} = \frac{1}{12}\left(35 - 6x^{(4)} - 3y^{(4)}\right) = -\frac{1}{12}(35 - 6 \times 1.825842 - 3 \times 2.459919) = 1.389747$$

Fifth approximation $(k = 4)$:

$$x^{(5)} = \frac{1}{8}\left(10 + 3y^{(4)} - 2z^{(4)}\right) = \frac{1}{8}(10 + 3 \times 2.459919 - 2 \times 1.389747) = 1.825032$$

$$y^{(5)} = \frac{1}{11}\left(33 - 4x^{(5)} + z^{(4)}\right) = \frac{1}{11}(33 - 4 \times 1.8255032 + 1 \times 1.389747) = 2.462692$$

$$z^{(5)} = \frac{1}{12}\left(35 - 6x^{(5)} - 3y^{(5)}\right) = -\frac{1}{12}(35 - 6 \times 1.825032 - 3 \times 2.462692) = 1.388477$$

Sixth approximation $(k = 5)$:

$$x^{(6)} = \frac{1}{8}\left(10 + 3y^{(5)} - 2z^{(5)}\right) = \frac{1}{8}(10 + 3 \times 2.462692 - 2 \times 1.388477) = 1.826390$$

$$y^{(6)} = \frac{1}{11}\left(33 - 4x^{(6)} + z^{(5)}\right) = \frac{1}{11}(33 - 4 \times 1.826390 + 1 \times 1.388477) = 2.462083$$

$$z^{(5)} = \frac{1}{12}\left(35 - 6x^{(5)} - 3y^{(5)}\right) = -\frac{1}{12}(35 - 6 \times 1.826390 - 3 \times 2.462083) = 1.387950$$

Thus the solution of the above system of linear equations by Gauss Seidel iteration method after sixth step is $x = 1.826390$, $y = 2.462083$, $z = 1.387950$

Mathematics (Subject Code)
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Lecture Note # 7

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Successive over relaxation (SOR) method

Let us explain the SOR method in the case of three linear equations with three unknowns with relaxation parameter ω. Similarly we can extend the method into n linear equations with n unknowns.

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$

$$a_2x + b_2y + c_2z = d_2 \tag{2}$$

$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

In matrix form the above system of equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

STEP I : Verify the sufficient condition for the SOR method i.e the coefficient matrix $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is symmetric positive definite.

1. For symmetric verify $\mathbf{A^T} = \mathbf{A}$.
2. For positive definite show that

$$\begin{vmatrix} a_1 \end{vmatrix} > 0, \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} > 0, \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} > 0$$

STEP II : Verify that the relaxation parameter ω satisfies $0 < \omega < 2$.

STEP III : Write down the Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{a_1}\left(d_1 - b_1y^{(k)} - c_1z^{(k)}\right) \tag{4}$$

$$y^{(k+1)} = \frac{1}{b_2}\left(d_2 - a_2x^{(k+1)} - c_2z^{(k)}\right) \tag{5}$$

$$z^{(k+1)} = \frac{1}{c_3}\left(d_3 - a_3x^{(k+1)} - b_3y^{(k+1)}\right) \tag{6}$$

STEP IV : Multiply the RHS of the equation (4), (5) and (6) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1-\omega)$ respectively equation (4), (5) and (6) becomes

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega\frac{1}{a_1}\left(d_1 - b_1 y^{(k)} - c_1 z^{(k)}\right)$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega\frac{1}{b_2}\left(d_2 - a_2 x^{(k)} - c_2 z^{(k)}\right)$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \omega\frac{1}{c_3}\left(d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}\right)$$

$$k = 0,\ 1,\ 2,\ \ldots.$$

The process is continued until the convergence is assured.

NOTE: In the absence of initial approximation $x^{(0)}$, $y^{(0)}$, $z^{(0)}$, they are taken as $(0, 0, 0)$.

***Example 1*:**

Solve the following system of linear equations

$$3x - y + z = -1$$

$$-x + 3y - z = 7$$

$$x - y + 3z = -7$$

Check that the SOR method with the value $\omega = 1.25$ as the relaxation parameter can be used to solve the system of equations and then solve it.

***Solution* :**

STEP I : Here the coefficient matrix $\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Now $\mathbf{A^T} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Thus $\mathbf{A^T} = \mathbf{A}$, $\implies$ $\mathbf{A}$ is a symmetric matrix.

Now $\begin{vmatrix} 3 \end{vmatrix} = 3 > 0$, $\quad \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 > 0$, $\quad \begin{vmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 20 > 0$

So, the coefficient matrix $\mathbf{A}$ is symmetric positive definite.

STEP II :

The relaxation parameter $\omega = 1.25$ lies in the interval $0 < \omega < 2$. Thus SOR method is applicable for the given system of linear equations.

STEP III :

The iteration scheme by Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{3}\left(-1 + y^{(k)} - z^{(k)}\right) \tag{7}$$

$$y^{(k+1)} = \frac{1}{3}\left(7 + x^{(k+1)} + z^{(k)}\right) \tag{8}$$

$$z^{(k+1)} = \frac{1}{3}\left(-7 - x^{(k+1)} + y^{(k+1)}\right) \tag{9}$$

STEP IV :

Multiply the RHS of the equation (7), (8) and (9) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1 - \omega)$ respectively equation (7), (8) and (9) becomes

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega\frac{1}{3}\left(-1 + y^{(k)} - z^{(k)}\right) \tag{10}$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega\frac{1}{3}\left(7 + x^{(k+1)} + z^{(k)}\right) \tag{11}$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \omega\frac{1}{3}\left(-7 - x^{(k+1)} + y^{(k+1)}\right) \tag{12}$$

$$k = 0,\ 1,\ 2,\ \ldots.$$

Taking the initial approximation as $x^{(0)} = y^{(0)} = z^{(0)} = 0$

First approximation $(k = 0)$

$$\begin{aligned} x^{(1)} &= (1-\omega)x^{(0)} + \omega\frac{1}{3}\left(-1 + y^{(0)} - z^{(0)}\right) \\ &= (1 - 1.25) \times 0 + \frac{1.25}{3}(-1 + 0 - 0) \\ &= -0.416666 \end{aligned}$$

$$\begin{aligned} y^{(1)} &= (1-\omega)y^{(0)} + \omega\frac{1}{3}\left(7 + x^{(1)} + z^{(0)}\right) \\ &= (1 - 1.25) \times 0 + \frac{1.25}{3}(7 - 0.41666 + 0) \\ &= 2.743055 \end{aligned}$$

$$z^{(1)} = (1-\omega)z^{(0)} + \omega\frac{1}{3}\left(-7 - x^{(1)} + y^{(1)}\right)$$

$$= (1 - 1.25) \times 0 + \frac{1.25}{3}(-7 + 0.41666 + 2.743055)$$
$$= -1.600116$$

Second approximation $(k = 1)$

$$x^{(2)} = (1 - \omega)x^{(1)} + \omega\frac{1}{3}\left(-1 + y^{(1)} - z^{(1)}\right)$$
$$= (1 - 1.25)(-0.416666) + \frac{1.25}{3}(-1 + 2.743055 + 1.600116)$$
$$= 1.497154$$
$$y^{(2)} = (1 - \omega)y^{(1)} + \omega\frac{1}{3}\left(7 + x^{(2)} + z^{(1)}\right)$$
$$= (1 - 1.25)(2.743055) + \frac{1.25}{3}(7 + 1.497154 - 1.600116)$$
$$= 2.188002$$
$$z^{(2)} = (1 - \omega)z^{(1)} + \omega\frac{1}{3}\left(-7 - x^{(2)} + y^{(2)}\right)$$
$$= (1 - 1.25)(-1.600116) + \frac{1.25}{3}(-7 - 1.497154 + 2.188002)$$
$$= -2.228784$$

Third approximation $(k = 2)$

$$x^{(3)} = (1 - \omega)x^{(2)} + \omega\frac{1}{3}\left(-1 + y^{(2)} - z^{(2)}\right)$$
$$= (1 - 1.25)(1.497154) + \frac{1.25}{3}(-1 + 2.1880022 + 2.228784)$$
$$= 1.049372$$
$$y^{(3)} = (1 - \omega)y^{(2)} + \omega\frac{1}{3}\left(7 + x^{(3)} + z^{(2)}\right)$$
$$= (1 - 1.25)(2.188002) + \frac{1.25}{3}(7 + 1.049372 - 2.228784)$$
$$= 1.878244$$
$$z^{(3)} = (1 - \omega)z^{(2)} + \omega\frac{1}{3}\left(-7 - x^{(3)} + y^{(3)}\right)$$
$$= (1 - 1.25)(-2.228784) + \frac{1.25}{3}(-7 - 1.049372 + 1.878244)$$
$$= -2.014107$$

Thus the solution of above system of equations by SOR method with relaxation parameter $\omega = 1.25$ after three steps is given by $x = 1.049372, y = 1.878244, z = -2.014107$

***Example 2*:**

Solve the following system of linear equations

$$10x + y - z = 2$$

$$x + 10y - 2z = 5$$

$$-x - 2y + 10z = 3$$

Check that the SOR method with the value $\omega = 1.25$ as the relaxation parameter can be used to solve the system of equations and then solve it.

***Solution* :**

STEP I : Here the coefficient matrix $\mathbf{A} = \begin{bmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{bmatrix}$

Now $\mathbf{A^T} = \begin{bmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{bmatrix}$

Thus $\mathbf{A^T} = \mathbf{A}, \implies \mathbf{A}$ is a symmetric matrix.

Now $\begin{vmatrix} 10 \end{vmatrix} = 10 > 0,\qquad \begin{vmatrix} 10 & 1 \\ 1 & 10 \end{vmatrix} = 99 > 0,\qquad \begin{vmatrix} 10 & 1 & -1 \\ 1 & 10 & -2 \\ -1 & -2 & 10 \end{vmatrix} = 944 > 0$

So, the coefficient matrix $\mathbf{A}$ is symmetric positive definite.

STEP II :

The relaxation parameter $\omega = 1.25$ lies in the interval $0 < \omega < 2$. Thus SOR method is applicable for the given system of linear equations.

STEP III :

The iteration scheme by Gauss seidal iteration scheme for the above system of equations.

$$x^{(k+1)} = \frac{1}{10}\left(2 - y^{(k)} + z^{(k)}\right) \tag{13}$$

$$y^{(k+1)} = \frac{1}{10}\left(5 - x^{(k+1)} + 2z^{(k)}\right) \tag{14}$$

$$z^{(k+1)} = \frac{1}{10}\left(3 + x^{(k+1)} + 2y^{(k+1)}\right) \tag{15}$$

STEP IV :

Multiply the RHS of the equation (13), (14) and (15) by ω and adding to the vectors $x^{(k)}$, $y^{(k)}$ and $z^{(k)}$ by multiplying $(1-\omega)$ respectively equation (13), (14) and (15) becomes

$$x^{(k+1)} = (1-\omega)x^{(k)} + \omega\frac{1}{10}\left(2 - y^{(k)} + z^{(k)}\right) \tag{16}$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \omega\frac{1}{10}\left(5 - x^{(k+1)} + 2z^{(k)}\right) \tag{17}$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \omega\frac{1}{10}\left(3 + x^{(k+1)} + 2y^{(k+1)}\right) \tag{18}$$

$$k = 0,\ 1,\ 2,\ \ldots.$$

Taking the initial approximation as $x^{(0)} = y^{(0)} = z^{(0)} = 0$

First approximation $(k = 0)$

$$\begin{aligned} x^{(1)} &= (1-\omega)x^{(0)} + \omega\frac{1}{10}\left(2 - y^{(0)} + z^{(0)}\right) \\ &= (1-1.25)\times 0 + \frac{1.25}{10}(2 - 0 + 0) \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} y^{(1)} &= (1-\omega)y^{(0)} + \omega\frac{1}{10}\left(5 - x^{(1)} + 2z^{(0)}\right) \\ &= (1-1.25)\times 0 + \frac{1.25}{10}(5 - 0.25 + 0) \\ &= 0.59375 \end{aligned}$$

$$\begin{aligned} z^{(1)} &= (1-\omega)z^{(0)} + \omega\frac{1}{10}\left(3 + x^{(1)} + 2y^{(1)}\right) \\ &= (1-1.25)\times 0 + \frac{1.25}{10}(3 + 0.25 + 2\times 0.59375) \\ &= 0.55468 \end{aligned}$$

Second approximation $(k = 1)$

$$\begin{aligned} x^{(2)} &= (1-\omega)x^{(1)} + \omega\frac{1}{10}\left(2 - y^{(1)} + z^{(1)}\right) \\ &= (1-1.25)\times 0.25 + \frac{1.25}{10}(2 - 0.59375 + 0.55468) \\ &= 0.182616 \end{aligned}$$

$$y^{(2)} = (1-\omega)y^{(1)} + \omega\frac{1}{10}\left(5 - x^{(2)} + 2z^{(1)}\right)$$

$$= (1 - 1.25) \times 0.59375 + \frac{1.25}{10} (5 - 0.182616 + 2 \times 0.55468)$$
$$= 0.592404$$
$$z^{(2)} = (1 - \omega) z^{(1)} + \omega \frac{1}{10} \left(3 + x^{(2)} + 2y^{(2)}\right)$$
$$= (1 - 1.25) \times 0.555468 + \frac{1.25}{10} (3 + 0.182616 + 2 \times 0.592405)$$
$$= 0.407258$$

Third approximation $(k = 2)$

$$x^{(3)} = (1 - \omega) x^{(2)} + \omega \frac{1}{10} \left(2 - y^{(2)} + z^{(2)}\right)$$
$$= (1 - 1.25) \times 0.182616 + \frac{1.25}{10} (2 - 0.592405 + 0.407258)$$
$$= 0.181202$$
$$y^{(3)} = (1 - \omega) y^{(2)} + \omega \frac{1}{10} \left(5 - x^{(3)} + 2z^{(2)}\right)$$
$$= (1 - 1.25) \times 0.592405 + \frac{1.25}{10} (5 - 0.181202 + 2 \times 0.407258)$$
$$= 0.556063$$
$$z^{(3)} = (1 - \omega) z^{(2)} + \omega \frac{1}{10} \left(3 + x^{(2)} + 2y^{(3)}\right)$$
$$= (1 - 1.25) \times 0.407258 + \frac{1.25}{10} (3 + 0.181202 + 2 \times 0.556063)$$
$$= 0.434851$$

Thus the solution of above system of equations by SOR method with relaxation parameter $\omega = 1.25$ after three steps is given by $x = 0.181202, y = 0.556063, z = 0.434851$

Mathematics (Subject Code)

3^{rd} Semester

Lecture Note # 8

Department of Mathematics

Ajay Binay Institute of Technology, Cuttack

Introduction

Some basic Definitions

***Matrix*:** The rectangular array of real or complex numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ a_{31} & a_{32} & . & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{pmatrix}$$

is called a matrix. Where m is the number of rows and n is the number of columns of the matrix.

***Square Matrix*:** A matrix having equal number of rows and column is called a square matrix. Let us consider a square matrix of order n i.e.

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ a_{31} & a_{32} & . & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

a_{11}, a_{22}, a_{33}, ... and a_{nn} are called the diagonal elements of the square matrix. The line in which the diagonal elements lies is called principal diagonal.

***Upper Triangular Matrix*:** A square matrix whose elements below the principal diagonal are all zero is called an upper diagonal matrix.

Example:

$$\begin{pmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

***Unit Upper Triangular Matrix*:** An upper triangular matrix whose diagonal elements are $1's$ is called a unit upper triangular matrix.

Example:

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

***Lower Triangular Matrix*:** A square matrix whose elements above the principal diagonal are all zero is called an lower diagonal matrix.

Example:

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 9 & 2 \end{pmatrix}$$

***Unit Lower Triangular Matrix*:** An lower triangular matrix whose diagonal elements are $1's$ is called a unit lower triangular matrix.

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 1 \end{pmatrix}$$

Dolittle Method

Let us consider a system of three linear equations with three unknowns and is given by,

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

In matrix form the above set of equations can be represented as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e.

$$A\mathbf{x} = B \tag{1}$$

Where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let

$$A = LU \tag{2}$$

Where L is a unit lower triangular matrix and U is an upper triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ 0 & u_{22} & u_{32} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Using equation (2) in equation (1)

$$LU\mathbf{x} = B \tag{3}$$

Let

$$U\mathbf{x} = \mathbf{y} \qquad \text{where} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4}$$

Using equation (4) in equation (3)

$$L\mathbf{y} = B \tag{5}$$

Solving equation (5) by forward substitution method find out the value of $\mathbf{y}$. Putting the value of $\mathbf{y}$ in equation (4) find out the value of $\mathbf{x}$ by backward substitution method.

***Example 1*:**

Solve the following system of equations by using Dolittle method.

$$3x + y + z = 4$$
$$x + 2y + 2z = 3$$
$$2x + y + 3z = 4$$

***Solution* :**

In matrix form the above set of equations can be written as,

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$
$$\implies A\mathbf{x} = B \tag{6}$$

Let

$$A = LU \tag{7}$$

Where L is a unit lower triangular matrix and U is an upper triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Putting in equation (7)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Equating both sides, we have

$$u_{11} = 3, \qquad u_{21} = 1, \qquad u_{31} = 1$$

$$l_{21}u_{11} = 1, \qquad \Longrightarrow 3l_{21} = 1, \qquad \Longrightarrow l_{21} = \frac{1}{3}$$

$$l_{31}u_{11} = 2, \qquad \Longrightarrow 3l_{31} = 2, \qquad \Longrightarrow l_{31} = \frac{2}{3}$$

$$l_{21}u_{12} + u_{22} = 2, \qquad \Longrightarrow \frac{1}{3} \times 1 + u_{22} = 2, \qquad \Longrightarrow u_{22} = \frac{5}{3}$$

$$l_{21}u_{13} + u_{23} = 2, \qquad \Longrightarrow \frac{1}{3} \times 1 + u_{23} = 2, \qquad \Longrightarrow u_{23} = \frac{5}{3}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1, \qquad \Longrightarrow \frac{2}{3} \times 1 + l_{32}\frac{5}{3} = 1, \qquad \Longrightarrow l_{32} = \frac{1}{5}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3, \qquad \Longrightarrow \frac{2}{3} \times 1 + \frac{1}{5} \times \frac{5}{3} + u_{33} = 3, \qquad \Longrightarrow u_{33} = 2$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{5} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 0 & 2 \end{bmatrix}$$

Using equation (7) in equation (6)

$$LU\mathbf{x} = B \tag{8}$$

Put

$$U\mathbf{x} = \mathbf{y} \qquad \text{where} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

Using equation (9) in equation (8)

$$L\mathbf{y} = B \tag{10}$$

From equation (10) we have,

$$L\mathbf{y} = B$$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} y_1 \\ \frac{1}{3}y_1 + y_2 \\ \frac{2}{3}y_1 + \frac{1}{5}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Equating both sides and solving it by forward substitution method

$$y_1 = 4$$

$$\frac{1}{3}y_1 + y_2 = 3, \qquad \implies \frac{1}{3} \times 4 + y_2 = 3 \qquad \implies y_2 = \frac{5}{3}$$

$$\frac{2}{3}y_1 + \frac{1}{5}y_2 + y_3 = 4, \qquad \implies \frac{2}{3} \times 4 + \frac{1}{5} \times \frac{5}{3} + y_3 = 4 \qquad \implies y_3 = 1$$

Thus

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Putting the value of $\mathbf{y}$ in equation (9)

$$U\mathbf{x} = \mathbf{y}$$

$$\implies \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3x + y + z \\ \frac{5}{3}y + \frac{5}{3}z \\ 2z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Equating both sides and solving it by backward substitution method, we have

$$z = \frac{1}{2}$$

$$\frac{5}{3}y + \frac{5}{3}z = \frac{5}{3} \qquad \implies y + \frac{1}{2} = 1 \qquad \implies y = \frac{1}{2}$$

$$3x + y + z = 4 \qquad \implies 3x + \frac{1}{2} + \frac{1}{2} = 4 \implies x = 1$$

Thus $x = 1$, $y = \frac{1}{2}$ and $z = \frac{1}{2}$ is the solution of the above system of equations by dolittle method.

Mathematics (Subject Code)
3^{rd} Semester
Lecture Note # 9

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Crouts Method

Let us consider a system of three linear equations with three unknowns and is given by,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form the above set of equations can be represented as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e.

$$A\mathbf{x} = B \tag{1}$$

Where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let

$$A = LU \tag{2}$$

Where L is a lower triangular matrix and U is a unit upper triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Using equation (2) in equation (1)

$$LU\mathbf{x} = B \tag{3}$$

Put

$$U\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4}$$

Using equation (4) in equation (3)

$$L\mathbf{y} = B \tag{5}$$

Solving equation (5) by forward substitution method find out the value of $\mathbf{y}$. Putting the value of $\mathbf{y}$ in equation (4) find out the value of $\mathbf{x}$ in backward substitution method.

***Example 1*:**

Solve the following system of equations by using crouts method.

$$3x + y + z = 4$$
$$x + 2y + 2z = 3$$
$$2x + y + 3z = 4$$

***Solution* :**

In matrix form the above set of equations can be written as,

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$
$$\implies A\mathbf{x} = B \tag{6}$$

Let

$$A = LU \tag{7}$$

Where L is a lower triangular matrix and U is a unit upper triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Putting in equation (7)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Equating both sides, we have

$$l_{11} = 3, \qquad l_{21} = 1, \qquad l_{31} = 2$$

$$l_{11}u_{12} = 1, \qquad \implies 3u_{12} = 1, \qquad \implies u_{12} = \frac{1}{3}$$

$$l_{11}u_{13} = 1, \quad \implies 3u_{13} = 1, \quad \implies u_{13} = \frac{1}{3}$$

$$l_{21}u_{12} + l_{22} = 2, \quad \implies 1 \times \frac{1}{3} + l_{22} = 2, \quad \implies l_{22} = \frac{5}{3}$$

$$l_{31}u_{12} + l_{32} = 1, \quad \implies 2 \times \frac{1}{3} + l_{32} = 1, \quad \implies l_{32} = \frac{1}{3}$$

$$l_{21}u_{13} + l_{22}u_{23} = 2, \quad \implies 1 \times \frac{1}{3} + 1 \times \frac{1}{3}u_{23} = 2, \quad \implies u_{23} = 1$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3, \quad \implies 2 \times \frac{1}{3} + \frac{1}{3} \times 1 + l_{33} = 2, \quad \implies l_{33} = 2$$

Thus,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & \frac{5}{3} & 0 \\ 2 & \frac{1}{3} & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using equation (7) in equation (6)

$$LU\mathbf{x} = B \tag{8}$$

Put

$$U\mathbf{x} = \mathbf{y} \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

Using equation (9) in equation (8)

$$L\mathbf{y} = B \tag{10}$$

From equation (10) we have,

$$L\mathbf{y} = B$$

$$\implies \begin{bmatrix} 3 & 0 & 0 \\ 1 & \frac{5}{3} & 0 \\ 2 & \frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3y_1 \\ y_1 + \frac{5}{3}y_2 \\ 2y_1 + \frac{1}{3}y_2 + 2y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Equating both sides and solving it by forward substitution method

$$3y_1 = 4, \qquad \implies y_1 = \frac{4}{3}$$

$$y_1 + \frac{5}{3}y_2 = 3, \qquad \implies \frac{4}{3} + \frac{5}{3}y_2 = 3 \qquad \implies y_2 = 1$$

$$2y_1 + \frac{1}{3}y_2 + 2y_3 = 4, \qquad \implies 2 \times \frac{4}{3} + \frac{1}{3} \times 1 + 2y_3 = 4 \qquad \implies y_3 = \frac{1}{2}$$

Thus

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Putting the value of $\mathbf{y}$ in equation (9)

$$U\mathbf{x} = \mathbf{y}$$

$$\implies \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\implies \begin{bmatrix} x + \frac{1}{3}y + \frac{1}{3}z \\ y + z \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Equating both sides and solving it by backward substitution method, we have

$$z = \frac{1}{2}$$

$$y + z = 1 \qquad \implies y + \frac{1}{2} = 1 \qquad \implies y = \frac{1}{2}$$

$$x + \frac{1}{3}y + \frac{1}{3}z = \frac{4}{3} \qquad \implies x + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{4}{3} \implies x = 1$$

Thus $x = 1$, $y = \frac{1}{2}$ and $z = \frac{1}{2}$ is the solution of the above system of equations by crouts method.

Mathematics (Subject Code)
3^{rd} Semester
Lecture Note # 10

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Cholesky Method

Let us consider a system of three linear equations with three unknowns and is given by,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form the above set of equations can be represented as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e.

$$A\mathbf{x} = B \tag{1}$$

Where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let

$$A = LL^T \tag{2}$$

Where L is a lower triangular matrix and L^T is transpose of the matrix L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Using equation (2) in equation (1)

$$LL^T\mathbf{x} = B \tag{3}$$

Let

$$L^T\mathbf{x} = \mathbf{y} \qquad \text{where} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4}$$

Using equation (4) in equation (3)

$$L\mathbf{y} = B \tag{5}$$

Solving equation (5) by forward substitution method find out the value of $\mathbf{y}$. Putting the value of $\mathbf{y}$ in equation (4) find out the value of $\mathbf{x}$ by backward substitution method.

***Example 1*:**

Solve the following system of equations by using cholesky method.

$$x + 2y + 3z = 5$$
$$2x + 8y + 22z = 6$$
$$3x + 22y + 82z = -10$$

***Solution* :**

In matrix form the above set of equations can be written as,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$
$$\implies A\mathbf{x} = B \tag{6}$$

Let
$$A = LL^T \tag{7}$$

Where L is a lower triangular matrix and L^T is a the transpose of the matrix L

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

From equation (7)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\implies \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{ll}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{ll}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Equating both sides, we have

$$l_{11}^2 = 1 \quad \implies l_{11} = 1, \quad l_{11}l_{21} = 2 \quad \implies l_{21} = 2, \quad l_{11}l_{31} = 3 \quad \implies l_{31} = 3$$
$$l_{21}^2 + l_{11}^2 = 8, \quad \implies 4 + l_{22}^2 = 8, \quad \implies l_{22} = 2$$
$$l_{31}l_{21} + l_{32}l_{22} = 22, \quad \implies 2 \times 3 + 2l_{32} = 22, \quad \implies l_{32} = 8$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82, \qquad \implies 9 + 64 + l_{33}^2 = 82, \qquad \implies l_{33}^2 = 9, \qquad \implies l_{33} = 3$$

Thus,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

$$L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

Using equation (7) in equation (6)

$$LL^T\mathbf{x} = B \tag{8}$$

Let

$$L^T\mathbf{x} = \mathbf{y} \qquad \text{where} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

Using equation (9) in equation (8)

$$L\mathbf{y} = B \tag{10}$$

From equation (10) we have,

$$L\mathbf{y} = B$$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$\implies \begin{bmatrix} y_1 \\ 2y_1 + 2y_2 \\ 3y_1 + 8y_2 + 3y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Equating both sides and solving it by forward substitution method

$$y_1 = 5$$

$$2y_1 + 2y_2 = 6, \qquad \implies 2 \times 5 + 2y_2 = 6 \qquad \implies y_2 = -2$$

$$3y_1 + 8y_2 + 3y_3 = -10, \qquad \implies 3 \times 5 + 8 \times (-2) + 3y_3 = -10 \qquad \implies y_3 = -3$$

Thus

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

Putting the value of $\mathbf{y}$ in equation (9)

$$L^T\mathbf{x} = \mathbf{y}$$

$$\implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$$\implies \begin{bmatrix} x + 2y + 3z \\ 2y + 8z \\ 3z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

Equating both sides and solving it by backward substitution method, we have

$$3z = -3, \qquad \implies z = -1$$

$$2y + 8z = -2 \qquad \implies 2y + 8 \times (-1) = -2 \qquad \implies y = 3$$

$$x + 2y + 3z = -3 \qquad \implies x + 2 \times 3 + 3 \times (-1) = -3 \implies x = 2$$

Thus $x = 2$, $y = 3$ and $z = -1$ is the solution of the above system of equations by choleskys method.

Mathematics (Subject Code)

3^{rd} Semester

Lecture Note # 11

Module - I

Department of Mathematics

Ajay Binay Institute of Technology, Cuttack

Interpolation and approximation

Let the explicity nature of a function $f(x)$ is not known. But the value of the function at $(n+1)$ distinct points $x_0,\ x_1,\ x_2,\ \ldots\ldots.\ x_n$ where $x_0 < x_1 < x_2,\ \ldots\ldots. < x_n$ is known. The process of interpolation is to find out another function $P(x)$ of degree n such that

$$P(x_i) = f(x_i) \qquad\qquad i = 0,\ 1,\ 2,\ \ldots\ldots\ldots\ n$$

The process by which $P(x)$ is determined is called interpolation. $P(x)$ is called the interpolating polynomial of function $f(x)$.

If $P(x)$ is a polynomial then it is called polynomial interpolation.

NOTE : The interpolating polynomial of a function $f(x)$ is unique.

Types of interpolation :

There are several types of interpolation. Some of them are listed below.

(i) Newton's forward interpolation

(ii) Newton's backward interpolation

(iii) Newton's divided difference interpolation

(iv) Lagrange's interpolation

NOTE :

(a) Newton's forward interpolation and Newton's backward interpolation are interpolation with equal intervals.

(b) Newton's divided difference interpolation and Lagrange's interpolation are interpolation with unequal intervals.

Finite differences

Suppose the function $y = f(x)$ has the values $y_0,\ y_1,\ y_2,\ \ldots\ldots.\ y_n$ for the values of $x = x_0 + h,\ x_0 + 2h,\ x_0 + 3h,\ \ldots\ldots.\ x_0 + nh$. To determine the values of $f(x)$ is based on the principle of finite differences.

Forward differences (Δ)

The differences $y_1 - y_0,\ y_2 - y_1,\ \ldots\ldots\ldots\ y_n - y_{n-1}$ are called first forward differences are denoted by $\Delta y_0,\ \Delta y_1,\ \Delta y_2,\ \ldots\ldots\ldots\ \Delta y_{n-1}$, where Δ is known as first forward difference operator.

Thus the first forward difference is given by

$$\Delta y_k = y_{k+1} - y_k$$

The second forward difference is given by

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$

In general the r^{th} forward difference is given by

$$\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k$$

Forward difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

The difference $\Delta^k y_0$ lies on a straight line sloping downward to the right.

Backward differences (∇)

The differences $y_1 - y_0,\ y_2 - y_1,\ \ldots\ldots\ldots\ y_n - y_{n-1}$ are called first backward differences are denoted by $\nabla y_1,\ \nabla y_2,\ \nabla y_3,\ \ldots\ldots\ldots\ \nabla y_n$, where ∇ is known as first backward difference operator.

Thus the first backward difference is given by

$$\nabla y_k = y_k - y_{k-1}$$

The second backward difference is given by

$$\nabla^2 y_k = \nabla y_k - \Delta y_{k-1}$$

In general the r^{th} backward difference is given by

$$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}$$

Backward difference table

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_4$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

The difference $\Delta^k y_5$ lies on a straight line sloping upward to the right.

Newton forward interpolation formula

This formula is applicable when the arguments are given with equal spaced.

Let $y = f(x)$ be a function which takes the values $f(a),\ f(a+h),\ f(a+2h),\ \ldots\ldots\ldots f(a+nh)$ for $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ (a+nh)$. Where h is the step size of the arguments.

Here $(n+1)$ arguments are given therefore the $(n+1)^th$ difference is zero. Thus $f(x)$ is a polynomial of degree n. So $f(x)$ can be written as

$$\begin{aligned} f(x) =& a_0 + a_1(x-a) + a_2(x-a)(x-a-h) + a_3(x-a)(x-a-h)(x-a-2h) \\ &+ \ \ldots\ldots\ldots + a_n(x-a)(x-a-h)\ \ldots\ldots\ (x-a-(n-1)h) \end{aligned} \tag{1}$$

Where $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ are constants.

Putting $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ a+nh$ in to equation (1) respectively, we get

$$\begin{aligned}
a_0 &= f(a) \\
a_0 + h &= f(a+h) \\
\text{or} \qquad a_1 h &= f(a+h) - f(a) \\
\implies a_1 &= \frac{f(a+h) - f(a)}{h} = \frac{\Delta f(a)}{h} \\
\text{and} \qquad a_0 + 2ha_1 + 2h^2 a_2 &= f(a+2h) \\
\implies 2h^2 a_2 &= f(a+2h) - a_0 - 2ha_1 \\
&= f(a+2h) - a_0 - 2\Delta f(a) \\
&= f(a+2h) - f(a+h) + f(a) \\
&= \Delta^2 f(a) \\
\implies a_2 &= \frac{\Delta^2 f(a)}{2!\ h^2}
\end{aligned}$$

Continuing in this way, we get

$$a_3 = \frac{\Delta^3 f(a)}{3!\ h^3} \ \ldots\ldots\ldots\ldots \ a_n = \frac{\Delta^n f(a)}{n!\ h^n}$$

Now substituting these values of $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ into equation (1), we get

$$\begin{aligned}
f(x) =& f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!\ h^2}(x-a)(x-a-h) \\
&+ \frac{\Delta^3 f(a)}{3!\ h^3}(x-a)(x-a-h)(x-a-2h) + \ \ldots\ldots\ldots \\
&+ \frac{\Delta^n f(a)}{n!\ h^n}(x-a)(x-a-h)\ \ldots\ldots\ (x-a-(n-1)h)
\end{aligned} \tag{2}$$

Further put $x = a + Uh$, then

$x-a = Uh,\ x-a-h = (U-1)h,\ x-a-2h = (u-2)h,\ \ldots\ldots\ x-a-(n-1)h = U-(n-1)h$

Using these values in equation (2)

$$\begin{aligned}
f(a+hU) =& f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \ \ldots\ldots\ldots \\
&+ \frac{U(U-1)\ \ldots\ldots\ (U-n+1)}{n!}\Delta^n f(a)
\end{aligned}$$

This formula is known as Newton's froward interpolation with equal intervals.

***Example 1*:**

From the following table find the number of students who obtain less than 45 marks

Range of Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No of Students	31	42	51	35	31

Using Newton's forward interpolation formula

***Solution*:**

The difference table for the given data is as follows

Marks x	No of students $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Less 40	31				
		42			
Less 50	73		9		
		51		-25	
Less 60	124		-16		37
		35		12	
Less 70	159		-4		
		31			
Less 80	190				

Here $h = 10$, $a = 40$, and $x = 45$

$$U = \frac{x-a}{h} = \frac{45-40}{10} = \frac{1}{2}$$

By Newton forward interpolation formula

$$\begin{aligned} f(a+hU) =& f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \frac{U(U-1)(U-2)}{3!}\Delta^3 f(a) \\ &+ \frac{U(U-1)(U-2)(U-3)}{4!}\Delta^4 f(a) \\ \implies f(45) =& f(40) + \frac{1}{2}\Delta f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\Delta^2 f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\Delta^3 f(40) \\ &+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}\Delta^4 f(40) \\ =& 31 + \frac{1}{2} \times 42 - \frac{1}{8} \times 9 - \frac{1}{16} \times 25 - \frac{5}{128} \times 37 \\ =& 47.867 \end{aligned}$$

Thus the number of students who obtain less than 45 marks are 48.

***Example 2*:**

For the following data calculate the differences and obtain Newton forward interpolating polynomial

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

***Solution*:**

The difference table is given by

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

Here $a = 0, \quad h = 1$

$$U = \frac{x-a}{h} = \frac{x-0}{1} = x$$

The Newton forward interpolation formula is

$$\begin{aligned} f(a+hU) =& f(a) + U\Delta f(a) + \frac{U(U-1)}{2!}\Delta^2 f(a) + \frac{U(U-1)(U-2)}{3!}\Delta^3 f(a) \\ &+ \frac{U(U-1)(U-2)(U-3)}{4!}\Delta^4 f(a) \\ \implies f(x) =& f(0) + x\Delta f(0) + \frac{x\,(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) \\ &+ \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(0) \\ =& 3 + 3x + \frac{x(x-1)}{2}(2) + 0 + 0 \\ =& 3 + 3x + x^2 - x \\ =& x^2 + 2x + 3 \end{aligned}$$

Mathematics (Subject Code)
3^{rd} Semester
Lecture Note # 12
Module - I

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Newton backward interpolation formula

Let $y = f(x)$ be a function which takes the values $f(a),\ f(a{+}h),\ f(a{+}2h),\ \ldots\ldots\ldots f(a{+}nh)$ for $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ (a+nh)$. Where h is the step size of the arguments.

Let us consider the function $f(x)$

$$\begin{aligned} f(x) =& a_0 + a_1(x-a-nh) + a_2(x-a-nh)(x-a-(n-1)h) \\ &+ a_3(x-a-nh)(x-a-(n-1)h)(x-a-(n-1)h) \\ &+ \ \ldots\ldots\ldots\ + a_n(x-a-nh)(x-a-(n-1)h)\ \ldots\ldots\ (x-a-h) \end{aligned} \tag{1}$$

Where $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ are constants.

Putting $x = a,\ a+h,\ a+2h,\ \ldots\ldots\ldots\ a+nh$ in to equation (1) respectively, we get

$$\begin{aligned} a_0 =& f(a+nh) \\ a_0 - a_1 h =& f(a+(n-1)h) \\ \implies a_1 =& \frac{f(a+nh) - f(a+(n-1)h)}{h} \\ \implies a_1 =& \frac{\nabla f(a+nh)}{h} \\ \text{and} \qquad a_0 - 2ha_1 + 2h^2 a_2 =& f(a-(n-2)h) \\ \implies a_2 =& \frac{\nabla^2 f(a+nh)}{2!\ h^2} \end{aligned}$$

Continuing in this way, we get

$$a_3 = \frac{\nabla^3 f(a+nh)}{3!\ h^3}\ \ldots\ldots\ldots\ a_n = \frac{\nabla^n f(a+nh)}{n!\ h^n}$$

Now substituting these values of $a_0,\ a_1,\ a_2,\ \ldots\ldots\ldots\ a_n$ into equation (1), we get

$$\begin{aligned} f(x) =& f(a+nh) + \frac{\nabla f(a+nh)}{h}(x-a-nh) \\ &+ \frac{\nabla^2 f(a+nh)}{2!\ h^2}(x-a-nh)(x-a-(n-1)h) \\ &+ \frac{\nabla^3 f(a+nh)}{3!\ h^3}(x-a-nh)(x-a-(n-1)h)(x-a-(n-2)h) + \ \ldots\ldots \\ &+ \frac{\nabla^n f(a)}{n!\ h^n}(x-a-nh)(x-a-(n-1)h)\ \ldots\ldots\ (x-a-h) \end{aligned} \tag{2}$$

Further put $x = a+nh+Uh,\ x-a-(n-1)h = (U+1)h,$
$x-a-(n-2)h = (U+2)h,\ \ldots\ldots\ldots\ x-a-h = (U+n-1)h$

Using these values in equation (2)

$$f(a+nh+Uh) = f(a+nh) + U\nabla f(a+nh) + \frac{U(U+1)}{2!}\nabla^2 f(a+nh) + \ \dots\dots\dots \\ + \frac{U(U+1)(U+2)\ \dots\dots\ (U+n-1)}{n!}\nabla^n f(a+nh)$$

This formula is known as Newton's backward interpolation with equal intervals.

***Example 1*:** For the following data

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

***Solution*:**

The difference table for the given data is as follows

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$	$\nabla^6 f(x)$	$\nabla^7 f(x)$
1	1							
		7						
2	8		12					
		19		6				
3	27		18		0			
		37		6		0		
4	64		24		0		0	
		61		6		0		0
5	125		30		0		0	
		91		6		0		
6	216		36		0			
		127		6				
7	343		42					
		169						
8	512							

Here $a + nh = 8$, $h = 1$, and $x = 7.5$ then

$$U = \frac{x - (a + nh)}{h} = \frac{7.5 - 8}{1} = -0.5$$

By Newtons backward interpolation formula

$$\begin{aligned} f(a + nh + Uh) =& f(a + nh) + U\nabla f(a + nh) + \frac{U(U+1)}{2!}\nabla^2 f(a + nh) \\ &+ \frac{U(U+1)(U+2)}{3!}\nabla^3 f(a + nh) \\ \implies f(7.5) =& f(8) + (-0.5)\nabla f(8) + \frac{(-0.5)(-0.5+1)}{2!}\nabla^2 f(8) \\ &+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}\nabla^3 f(8) \\ =& 512 - 84.5(169) - \frac{(0.5)(0.5)}{2}(42) - \frac{(0.5)(0.5)(1.5)}{6}(6) \\ =& 512 - 84.5 - 5.25 - 0.375 \\ =& 421.875 \end{aligned}$$

***Example 2*:**

Given the following table

x	0.1	0.2	0.3	0.4	0.5
e^x	1.10517	1.2140	1.34986	1.49182	1.64872

Find $e^{0.411}$ by using Newtons backward interpolation.

***Solution*:**

The difference table is given by

x	$y = e^x$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.1	1.10517				
		0.10883			
0.2	1.2140		0.02703		
		0.13586		-0.02093	
0.3	1.34986		0.0061		0.02997
		0.14196		0.00884	
0.4	1.49182		0.01494		
		0.1569			
0.5	1.64872				

Here $a + nh = 0.5, \quad h = 1, \quad x = 0.411$

$$U = \frac{x - (a + nh)}{h} = \frac{0.411 - 0.5}{0.1} = -0.89$$

By Newtons backward interpolation formula

$$\begin{aligned} y(x) =& y_n + U\nabla y_n + \frac{U(U+1)}{2!}\nabla^2 y_n + \frac{U(U+1)(U+2)}{3!}\nabla^3 y_n \\ &+ \frac{U(U+1)(U+2)(U+3)}{4!}\nabla^4 y_n \\ \Longrightarrow \quad y(0.411) =& 1.64872 + (-0.89)(0.1569) + \frac{(-0.89)(-0.89+1)}{2!}0.01494 \\ &+ \frac{(-0.89)(-0.89+1)(-0.89+2)}{3!}(0.00884) \\ &+ \frac{(-0.89)(-0.89+1)(-0.89+2)(-0.89+3)}{4!}(0.02977) \\ =& 1.507903164 \end{aligned}$$

Hence, $e^{0.411} = 1.507903164$

Mathematics (Subject Code)
3^{rd} Semester
Lecture Note # 13
Module - I

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Let $f(x_0),\ f(x_{1)},\ f(x_2),\ .\ .\ .\ .\ .\ .\ .\ f(x_n)$ be the values of the function $f(x)$ corresponding the points $x_0,\ x_1,\ .\ .\ .\ .\ .\ .\ .\ .\ x_n$ not equally spaced.

Then we define the first divided difference of $f(x)$ between x_0 and x_1 as follows.

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The second divided difference of $f(x)$ between $x_0,\ x_1$ and x_2 as follows.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Where

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

In similar way, the n^{th} divided difference is given by

$$f[x_0, x_1,\ .\ .\ .\ .\ .\ .\ x_{n-1},\ x_n] = \frac{f[x_1, x_2,\ .\ .\ .\ .\ x_n] - f[x_0, x_1,\ .\ .\ .\ .\ x_{n-1}]}{x_n - x_0}$$

Newton's divided difference formula

Let $f(x_0),\ f(x_{1)},\ f(x_2),\ .\ .\ .\ .\ .\ .\ .\ f(x_n)$ be the values of the function $f(x)$ at the points $x_0,\ x_1,\ .\ .\ .\ .\ .\ .\ .\ .\ x_n$ respectively which are not equally spaced.

By the definition of divided difference, the first divided difference of $f(x)$ is given by

$$f[x, x_0] = \frac{f(x_0) - f(x)}{x_0 - x}$$

$$\implies f(x) = f(x_0) + (x - x_0)f[x, x_0] \tag{1}$$

Again the second divided difference is given by

$$f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$$\implies f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]$$

$$\implies \frac{f(x) - f(x_0)}{x - x_0} = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]$$

$$\implies f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x, x_0, x_1] \tag{2}$$

Similarly,

$$\begin{aligned} f(x) =& f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)f[x, x_0, x_1, x_2] \end{aligned} \tag{3}$$

Proceeding in the same way

$$\begin{aligned}f(x) =& f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \;\ldots\ldots \\ &+(x - x_0)(x - x_1)(x - x_2)\;\ldots\ldots\;(x - x_{n-1})f[x_0, x_1,\;\ldots\ldots\;x_n] \\ &+(x - x_0)(x - x_1)(x - x_2)\;\ldots\ldots\;(x - x_n)f[x, x_0, x_1,\;\ldots\ldots\;x_n]\end{aligned} \tag{4}$$

Since, $f(x)$ is a polynomial of degree n so

$$f[x, x_0, x_1,\;\ldots\ldots\;x_n] = 0$$

Thus the equation (4) becomes

$$\begin{aligned}f(x) =& f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \;\ldots\ldots \\ &+(x - x_0)(x - x_1)(x - x_2)\;\ldots\ldots\;(x - x_{n-1})f[x_0, x_1,\;\ldots\ldots\;x_n]\end{aligned}$$

This formula is known as Newtons divided difference interpolation formula.

***Example 1*:**

Find the unique polynomial of fegree two or less such that $f(0) = 1,\; f(1) = 3,\; f(3) = 55$ by using Newtons divided difference interpolation.

***Solution*:**

Here, $x_0 = 0,\; x_1 = 1,\; x_2 = 3$
and $f(x_0) = 1,\; f(x_1) = 3,\; f(x_2) = 55$

By Newtons divided difference interpolation

$$f(x) = P(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$\begin{aligned}f[x_0, x_1] =& \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{3 - 1}{1 - 0} = 2 \\ f[x_1, x_2] =& \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{55 - 3}{3 - 1} = 26 \\ f[x_0, x_1, x_1] =& \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{26 - 2}{3 - 0} = 8\end{aligned}$$

x	$f(x)$				
0	1				
1	3	2	$f[x_0, x_1]$		
3	55	26		8	$f[x_0, x_1, x_2]$

Putting these values in equation (1) we get,

$$\begin{aligned} f(x) =& P(x) = 1 + (x-0)2 + (x-0)(x-1)8 \\ =& 8x^2 - 6x + 1 \end{aligned}$$

***Example 2*:**

Use Newton divided difference formula to calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

***Solution*:**

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 5,\ x_5 = 6$
and $f(x_0) = 1,\ f(x_1) = 14,\ f(x_2) = 15,\ f(x_3) = 5,\ f(x_4) = 6,\ f(x_5) = 19$

We have from the Newtons divided difference interpolation formula

$$\begin{aligned} f(x) = P(x) =& f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] \\ &+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] \\ &+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0,x_1,x_2,x_3,x_4] \\ &+(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)f[x_0,x_1,x_2,x_3,x_4,x_5] \end{aligned} \tag{5}$$

Divided difference table

x	$f(x)$					
0	1					
1	14	13	$f[x_0,x_1]$			
2	15	1	-6	$f[x_0,x_1,x_2]$		
4	5	-5	-2	1	$f[x_0,x_1,x_2,x_3]$	
5	6	1	2	1	0	$f[x_0,x_1,x_2,x_3,x_4]$
6	19	13	6	1	0	0
						$f[x_0,x_1,x_2,x_3,x_4,x_5]$

Putting the values in equation (5)

$$\begin{aligned} f(x) = P(x) =& 1 + (x-0)13 + (x-0)(x-1)(6) + (x-0)(x-1)(x-2)(1) \\ &+(x-0)(x-1)(x-2)(x-4)(0) + (x-0)(x-1)(x-2)(x-4)(x-5)(0) \\ =& 1 + 13x - 6x(x-1) + x(x-1(x-2)) \end{aligned}$$

Now

$$
\begin{aligned}
f(3) = P(3) =& 1 + 39 - 18 \times 2 + 3(3-1)(3-2) \\
=& 1 + 39 - 36 + 6 \\
=& 10
\end{aligned}
$$

Mathematics (Subject Code)
3^{rd} Semester
Lecture Note # 14
Module - I

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Lagrange's interpolation formula

Let $y = f(x)$ be a polynomial in x which takes the values $y_0 = f(x_0)$, $y_1 = f(x_1)$, $\ldots\ldots\ldots$ $y_n = f(x_n)$ corresponding to $x_0,\ x_1,\ x_2,\ \ldots\ldots\ldots\ldots\ x_n$. There are $(n+1)$ values of $f(x)$. So $(n+1)^{th}$ difference is zero. Thus $f(x)$ is a polynomial in x of degree n. Let this polynomial be

$$\begin{aligned} y = f(x) \ =\ & a_0(x-x_1)(x-x_2)\ \ldots\ldots\ (x-x_n) \\ & + a_1(x-x_0)(x-x_2)\ \ldots\ldots\ (x-x_n) \\ & + a_2(x-x_0)(x-x_1)(x-x_3)\ \ldots\ldots\ (x-x_n) \\ & +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_n(x-x_0)(x-x_1)\ \ldots\ldots\ (x-x_{n-1}) \end{aligned} \tag{1}$$

where $a_0, a_1, a_2,\ \ldots\ldots\ a_n$ are the constants can be determined by putting $y = y_0$ at $x = x_0$, $y = y_1$ at $x = x_1$, $\ldots\ldots\ldots\ldots\ldots$ etc.

Now putting $y = y_0$ and $x = x_0$ in equation (1), we get

$$\begin{aligned} y_0 =& a_0(x_0-x_1)(x_0-x_2)\ \ldots\ldots\ (x_0-x_n) \\ \implies a_0 =& \frac{y_0}{(x_0-x_1)(x_0-x_2)\ \ldots\ldots\ (x_0-x_n)} \end{aligned}$$

Similarly putting $y = y_1$ and $x = x_1$ in equation (1), we get

$$\begin{aligned} y_1 =& a_1(x_1-x_0)(x_1-x_2)\ \ldots\ldots\ (x_1-x_n) \\ \implies a_0 =& \frac{y_1}{(x_1-x_0)(x_1-x_2)\ \ldots\ldots\ (x_1-x_n)} \end{aligned}$$

Proceeding in this way, we get $a_2, a_3, a_4,\ \ldots\ldots\ a_n$. Now putting all the values of $a_0, a_1, a_2,\ \ldots\ldots\ a_n$ into equation (1) we get,

$$\begin{aligned} f(x) =& \frac{(x-x_1)(x-x_2)\ \ldots\ldots\ (x-x_n)}{(x_0-x_1)(x_0-x_2)\ \ldots\ldots\ (x_0-x_n)} y_0 \\ & + \frac{(x-x_0)(x-x_2)\ \ldots\ldots\ (x-x_n)}{(x_1-x_0)(x_1-x_2)\ \ldots\ldots\ (x_1-x_n)} y_1 \\ & +\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \frac{(x-x_0)(x-x_1)\ \ldots\ldots\ (x-x_{n-1})}{(x_{n-1}-x_0)(x_{n-1}-x_2)\ \ldots\ldots\ (x_{n-1}-x_n)} y_n \end{aligned}$$

is known as Lagranges interpolating polynomial of degree n.

***Example 1*:**

Find a unique polynomial of degree two or less by using Lagranges interpolation given that, $f(0) = 1,\ f(1) = 3,\ f(3) = 55$.

Solution:

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 3$ and $y_0 = f(x_0) = 1,\ y_1 = f(x_1) = 3,\ y_2 = f(x_2) = 55$

By Lagranges interpolation

$$\begin{aligned}
f(x) =& \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \\
=& \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3) + \frac{(x-0)(x-1)}{(3-0)(3-1)}(55) \\
=& \frac{1}{3}(x-1)(x-3) - \frac{3}{2}x(x-3) + \frac{55}{6}x(x-1) \\
=& 8x^2 - 6x + 1
\end{aligned}$$

Example 2:

Find the $f(4)$ from the following table by using Lagranges interpolation

x :	0	1	2	5
$f(x)$:	2	5	7	8

Solution:

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$

and $y_0 = f(x_0) = 2,\ y_1 = f(x_1) = 5,\ y_2 = f(x_2) = 7,\ y_3 = f(x_3) = 8$

$$\begin{aligned}
f(x) =& \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \\
& \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \\
=& \frac{(x-1)(x-2)(x-5)}{(0-1)(0-3)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(5) + \\
& \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(7) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(8) \\
\implies f(4) =& \frac{(4-1)(4-2)(4-5)}{(0-1)(0-3)(0-5)}(2) + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)}(5) + \\
& \frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)}(7) + \frac{(4-0)(4-1)(4-2)}{(5-0)(5-1)(5-2)}(8) \\
=& 1.2 - 10 + 14 + 3.2 \\
=& 8.4
\end{aligned}$$

Mathematics (Subject Code)
3^{rd} Semester
Question Bank & Assignment
Module - I

Department of Mathematics
Ajay Binay Institute of Technology, Cuttack

Multiple Choice Questions

1. Which theorem locate an interval in which the real root of an equation $f(x)$ lies.

   (a) Mean value theorem
   (b) Rolls theorem
   (c) Lagranges theorem
   (d) Intermediate value theorem

2. In which interval the real root of the equation $x^2 - \log_e x - 12 = 0$ lies.

   (a) $(0, 1)$
   (b) $(2, 3)$
   (c) $(3, 4)$
   (d) $(1, 2)$

3. The rate of convergence of bisection method is

   (a) Linear
   (b) Quadratic
   (c) Cubic
   (d) None of the above

4. The rate of convergence of secant method is

   (a) 1.618
   (b) 1.168
   (c) 1.816
   (d) 1.268

5. The rate of convergence of Newton Raphson method is

   (a) 1
   (b) 2
   (c) 3
   (d) 4

6. Which method has a faster rate of convergence to the root of the equation

   (a) Bisection method
   (b) Secant method
   (c) Newton Raphson Method
   (d) Iteration method

7. In fixed point iteration method to find out the root of the equation $f(x) = 0$ we write $f(x)$ in the form $x = \varphi(x)$. The function $\varphi(x)$ must satisfies which condition in (a, b) the interval in which the root of the equation $f(x)$ lies.

   (a) $|\varphi(x)| < 0$
   (b) $|\varphi'(x)| < 0$
   (c) $|\varphi(x)| > 0$
   (d) $|\varphi'(x)| > 0$

8. In SOR method the coefficient matrix **A** must be

(a) Symmetric and positive definite
(b) Skew symmetric and positive definite
(c) Only symmetric
(d) Only skew symmetric

9. In Cholesky's method the co-efficient matrix **A** is
   (a) Skew symmetric
   (b) Symmetric
   (c) Orthogonal
   (d) None of the above

10. In SOR method the relaxation parameter ω lies in the interval
    (a) $0 < \omega < 3$
    (b) $\omega > 2$
    (c) $0 < \omega < 2$
    (d) $-1 < \omega < 1$

11. In crouts method the coefficient matrix **A** can be written as the product of a
    (a) Lower and upper triangular matrix
    (b) Unit lower and upper triangular matrix
    (c) Unit lower and unit upper triangular matrix
    (d) Lower and unit upper triangular matrix

12. In Dolitte method the coefficient matrix **A** can be written as the product of
    (a) Lower and upper triangular matrix
    (b) Unit lower and upper triangular matrix
    (c) Lower and unit upper triangular matrix
    (d) Unit lower and unit upper triangular matrix

13. Which method has a greater rate of convergence for solving a system of linear equations?
    (a) Gauss Jacobi method
    (b) Gauss Seidel method
    (c) SOR method
    (d) None of these

14. Which interpolation method is used for equal intervals?
    (a) Lagranges method
    (b) Newton divided difference
    (c) Newton forward
    (d) None of these

15. Which interpolation method is used for unequal intervals?

(a) Lagranges method

(b) Newton backward

(c) Newton forward

(d) None of these

16. Let $f(0) = 1, \quad f(1) = 3, \quad f(3) = 55$, then the value of $l_0(x)$ is

(a) $\frac{1}{3}(x-1)(x-3)$

(b) $-\frac{1}{3}x(x-3)$

(c) $\frac{55}{6}x(x-1)$

(d) None of these

17. Let $f(0) = 1, \quad f(1) = 3, \quad f(3) = 55$, then the value of $f[0,1,3]$ is

(a) 2

(b) 26

(c) 8

(d) 10

18. One function has how many number of interpolating polynomials

(a) More than 1

(b) Exactly 1

(c) ∞

(d) None of these

19. For three points the degree of the interpolating polynomial is less tan or equals to

(a) 1

(b) 2

(c) 3

(d) 4

20. Let $f(x) = \frac{1}{x}$ the $f[x_0, x_1]$ is

(a) $\frac{1}{x_0 x_1}$

(b) $\frac{1}{x_0}$

(c) $\frac{1}{x_1}$

(d) $-\frac{1}{x_0 x_1}$

Short Answer Type Questions

1. State intermediate value theorem.
2. Write down two one point method and two twopoint method for finding out the root of the equation $f(x) = 0$.
3. State secant method for finding out the root of the equation $f(x) = 0$.
4. State secant method for finding out the root of the equation $f(x) = 0$.
5. Write down the Newton Raphson scheme of iteration for finding out the value of $\sqrt{3}$.
6. Describe fixed point iteration method.
7. What is the geometrical interpolation of Secant method?
8. What is the geometrical interpolation of Newton Raphson method?
9. What is interpolation?
10. Give two methods for interpolation in equal intervals and two methods for interpolation in unequal intervals.
11. Write Newton forward and Newton backward interpolation formula of degree n.
12. Write Lagranges and Newtons divided difference interpolation formula of degree n.
13. Let $f(x) = \frac{1}{x}$ then find $f[x_0, x_1, \;.\;.\;.\;.\;.\; x_{n-1}, x_n]$.
14. Write down the convergence condition of Gauss seidel method.
15. Verify that SOR method is applicable for the given system of equation with relaxation parameter 1.25 or not.

    $3x_1 - x_2 + x_3 = -1, \quad -x_1 + 3x_2 - x_3 = 7, \quad x_1 - x_2 + 3x_3 = -7$
16. Let x_0, x_1, x_2 be equi spaced point with step length h then express $f[x_0, x_1, x_2]$ in terms of forward difference.
17. Let x_0, x_1, x_2 be equi spaced point with step length h then express $f[x_0, x_1, x_2]$ in terms of backward difference.
18. Verify that choleskys method is applicable for the given system of equations or not.

    $x + 2y + 3z = 8, \quad 2x + 8y + 22z = 6, \quad 3x + 22y + 82z = -10$
19. Let $f(0) = 1$ and $f(2) = 5$ write down the Lagranges linear interpolation polynomial at $x = 1$.
20. Let $f(0) = 1$ and $f(2) = 5$ write down the Newton divided difference linear interpolation polynomial at $x = 1$.

Long Questions

1. Find the real root of the equation by bisection method (perform five iterations)

   (a) $x^3 - x - 1 = 0$

   (b) $3x - \sqrt{1 + \sin x} = 0$

   (c) $x \log_{10} x = 1.2$

2. Find the real root of the equation by bisection method correct up to three decimal places

   (a) $\cos x - xe^x = 0$

   (b) $x^3 - 4x - 9 = 0$

   (c) $x^4 - x - 10 = 0$

3. Find the real root of the equation by secant method (perform three iterations)

   (a) $x^3 - 9x - 1 = 0$

   (b) $xe^x - 3 = 0$

   (c) $x^2 - \log_e x - 12 = 0$

4. Find the real root of the equation by secant method correct up to three decimal places

   (a) $x^4 - x - 10 = 0$

   (b) $x^3 - x^2 - 2 = 0$

   (c) $x - e^{-x} = 0$

5. Find the cube root of 2 by using Newton Raphson method correct upto three decimal places.

6. Find the real root of the equation by Newton Raphson method (perform four iterations)

   (a) $x^3 - 5x + 1 = 0$

   (b) $x - e^{-x} = 0$

   (c) $3x = \cos x + 1$

7. Find the real root of the equation vy Newton Raphson method correct upto three decimal places.

   (a) $3x - \cos x - 1 = 0$

   (b) $x \log_{10} x - 1.2 = 0$

   (c) $\log_e x - \cos x = 0$

8. Find the real root of the equation by fixed point iteration (perform four iterations)
   (a) $x^3 - x^2 - 1 = 0$
   (b) $\cos x - xe^x = 0$
   (c) $x = 0.21\sin(0.5 + x)$
9. Find the real root of the equation by fixed point iteration method correct up to three decimal places
   (a) $e^x - x = 0$
   (b) $x^3 + x^2 = 100$
   (c) $5x^3 - 20x + 3 = 0$
10. Solve by using Gauss seidal iteration method (perform five steps)
   (a) $27x + 6y - z = 85$
   $6x + 15y + 2z = 72$
   $x + y + 54z = 110$
   (b) $7x + 52y + 13z = 104$
   $83x + 11y - 4z = 95$
   $3x + 8y + 29z = 71$
11. Apply Gauss seidal iteration method to solve the following system of equations correct upto three deimal places.
   (a) $x + 2y + 5z = 20$
   $5x + 2y + 2z = 12$
   $x + 4y + 2z = 15$
   (b) $3x + 20y - z = -18$
   $2x - 3y + 20z = 25$
   $20x + y - 2z = 17$
12. Solve by using successive over relaxation (SOR) method with relaxation parameter $\omega = 1.25$.
   (a) $2x + z = 6$
   $2y + z = 3$
   $y + 2z = 4.5$
   (b) $4x + 3y = 2y$
   $3x + 4y - z = 30$
   $-y + 4z = -24$
13. Solve by using crouts method
   (a) $x + y + z = 1$
   $3x + y - 3z = 5$
   $x - 2y - 5z = 10$
   (b) $5x + 2y + z = -12$
   $-x + 4y + 2z = 20$
   $2x - 3y + 10z = 3$

14. Solve by using dolittle method

(a) $2x - 6y + 8z = 24$
$5x + 4y - 3z = 2$
$3x + y + 2z = 16$

(b) $x + 2y + 3z = 6$
$2x + 3y + z = 9$
$3x + y + 2z = 8$

15. Solve by using cholesky's method

(a) $2x + y + 3z = 1$
$x + 4y - z = 3$
$3x - y + z = 4$

(b) $4x + 3y - z = 3$
$3x + y + 5z = 2$
$-x + 5y + 2z = 5$

16. From the following data find y at $x = 43$ and $x = 84$ by using Newtons forward and backward interpolation formula

x :	40	50	60	70	80	90
y :	184	204	226	250	276	304

17. From the following data

Temp (°C) :	140	150	160	170	180
Pressure Kgf/cm^2 :	3685	4854	6302	8076	10225

using Newtons interpolation formula find the pressure of the stream for a temperature of 142°C.

18. From the following data given I, the indicated HP and V the speed in knots developed by a ship

V :	8	10	12	14	16
I :	1000	1500	3250	5400	8950

find I when V=9 using Newtons interpolation formula.

19. Estimate the value of y at $x = 28$ from the following data by using Newtons backward interpolation formula.

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

20. Estimate the value of y at $x = 42$ from the following data by using Newtons backward interpolation formula.

x :	20	25	30	35	40	45
y :	354	332	291	260	231	204

21. By means of Newtons divided difference formula find the value of $f(15)$ from the following table

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

22. Use Newtons divided difference formula to find $f(x)$ from the following table

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

23. Find $f'(10)$ from the following table by using Newtons divided difference interpolation formula

x :	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606

24. Using Lagranges interpolation formula calculate $f(3)$ from the following table

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	9

25. Using Lagranges interpolation formula find $f(x)$ at $x = 2.5$ from the following table

x :	1	2	3	4
$f(x)$:	1	8	27	64

26. Find the cubic Lagranges interpolating polynomial from the following table

x :	0	1	2	5
$f(x)$:	2	3	12	147

Assignment I

Module - I
Answer All Questions

(Group - A)

1. (a) Write the Newton Raphson scheme of iteration for finding out the cube root of 2.
   (b) State secant method for finding out the root of $f(x) = 0$.
   (c) State intermediate value theorem and find out the interval in which the root of the equation $f(x) = x - e^{-x} = 0$ lies.
   (d) Describe fixed point iteration method.
   (e) Write down the convergence condition for Gauss seidel method.
   (f) Write down the sufficient condition of SOR method for solving a system of linear equation.
   (g) What is interpolation?
   (h) Write down Lagranges interpolation formula of degree n.

(Group - B)

2. Perform five iterations for finding out the real root of the equation $f(x) = x^3 - x - 11 = 0$ by bisection method.
3. Perform five iterations for finding out the real root of the equation $xe^x = 1$ by iteration method.
4. Solve by using Crout's method

$$2x + y + 4z = 12, \qquad 8x - 3y + 2z = 20, \qquad 4x + 11y - z = 33$$

5. Solve by using Dolittle method

$$2x + y + 3z = 13, \qquad x + 5y + z = 14, \qquad 3x + y + 4z = 17$$

6. Find the value if y at $x = 5$ byusing Lagranges interpolation formula.

x :	1	3	4	8	10
y :	8	15	19	32	40

7. Given values

x :	5	7	11	13	17
$f(x)$:	150	392	1452	1451	5202

Evaluate $f(9)$ by using Newtons divided difference interpolation.

(Group - C)

8. (a) Perform five steps to solve the given system of equation by Gauss seidel method.
$$5x + 2y + z = 12, \qquad x + 4y + 2z = 15, \qquad x + 2y + 5z = 20$$
(b) Find the real root of the equation $f(x) = xe^x - 3 = 0$ by secant method correct upto three decimal places.

9. (a) Solve by using SOR method by taking the relaxation parameter as $\omega = 1.25$.
$$2x + z = 6, \qquad 2y + z = 3, \qquad y + 2z = 4.5$$
(b) Solve the given system of equation by Cholesky method.
$$x + 2y + 3z = 1, \qquad 2x + 5y + 4z = 2, \qquad 3x + 4y + z = 3$$

10. (a) Find the value of y when $x = 47$ and $x = 63$ from the following table by using Newton forward and backward interpolation.

x :	45	50	55	60	65
y :	114.84	96.16	83.32	74.48	68.48

(b) Find the real root correct upto three decimal places by using Newton Raphson method.
$$f(x) = x^3 + 3x^2 - 3 = 0.$$