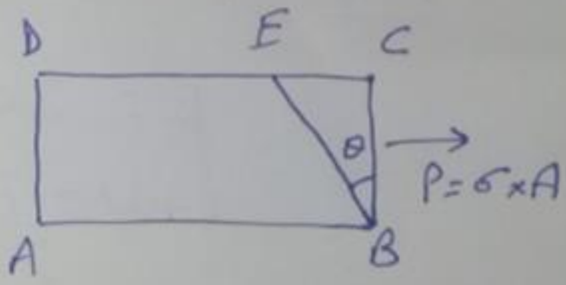
# SOLID MECHANICS

# Chapter 2
# Compound Stresses and Strains, Thin Pressure Vessels

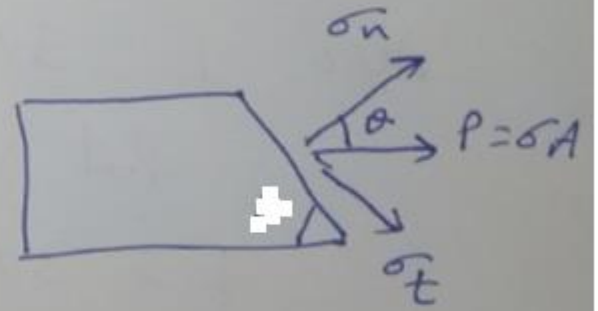
## Stress on Inclined plane




Area of plane BC = A

Let the force on plane BC, 'P' = $\sigma \times A$

On the inclined plane BE, the forces acting are $P_n$ and $P_t$




In Δ BEC,

$$\sec\theta = \frac{BE}{BC}$$

$$BE = BC \sec\theta = A \sec\theta$$

Now, the Normal component of force on plane BE

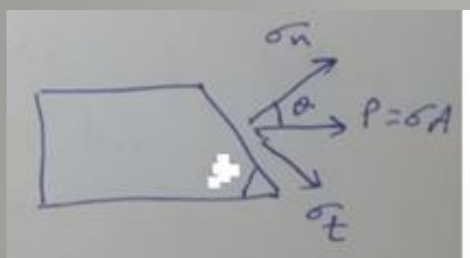
$$P_n = P\cos\theta = \sigma \times A\cos\theta$$

And Tangential component of force on plane BE

$$P_t = P\sin\theta = \sigma \times A\sin\theta$$

The Normal component of stress, $\sigma_n = \frac{P_n}{BE}$

$$= \frac{\sigma A\cos\theta}{A\sec\theta}$$




$$\boxed{\sigma_n = \sigma\cos^2\theta}$$

Tangential component of stress, $\sigma_t = \frac{P_t}{BE}$

$$= \frac{\sigma A\sin\theta}{A\sec\theta}$$

$$= \sigma\sin\theta\cos\theta$$

$$\boxed{\sigma_t = \frac{\sigma}{2}\sin 2\theta}$$

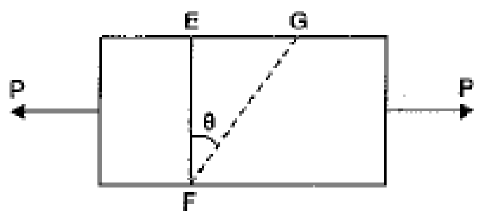
- Principal Planes:

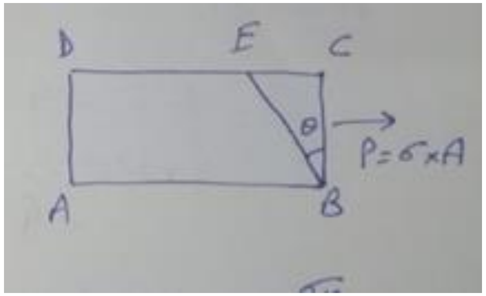
  - The planes which have no shear stress are known as principal planes. It mean planes of ‘zero shear stress’.
  - The normal stresses, acting on a principal plane, are principal stresses.

- Methods of determining stresses on oblique section:
  - Analytical Method
  - Graphical Method

# Analytical Method

- Member subjected to direct stress in one plane:




$\sigma_n = \sigma \cos^2\theta$

$\sigma_t = \frac{\sigma}{2} \sin 2\theta$

- A rectangular bar of area 10000mm$^2$, is subjected to an axial load of 20kN. Determine the normal and shear stresses on a section which is inclined at an angle of 30° with the normal cross-section of the bar.

a) 1.5N/mm2, 0.866N/mm2
b) 2.0N/mm2, 1.5N/mm2
c) 2.5N/mm2, 2.0N/mm2
d) 3.0N/mm2, 2.5N/mm2

Cross-sectional area of the rectangular bar,

$$A = 10000 \text{ mm}^2$$

Axial load, $P = 20 \text{ kN} = 20{,}000 \text{ N}$

Angle of oblique plane with the normal cross-section of the bar,

$$\theta = 30°$$

Now direct stress, $\sigma = \dfrac{P}{A} = \dfrac{20000}{10000} = 2 \text{ N/mm}^2$

Let $\sigma_n$ = Normal stress on the oblique plane

$\sigma_t$ = Shear stress on the oblique plane.

Using equation (3.2) for normal stress, we get

$$\sigma_n = \sigma \cos^2 \theta$$

$$= 2 \times \cos^2 30° \qquad (\because \quad \sigma = 2 \text{ N/mm}^2)$$

$$= 2 \times 0.866^2 \qquad (\because \quad \cos 30° = 0.866)$$

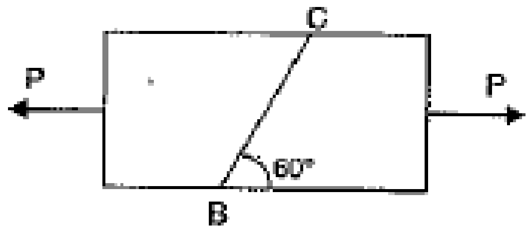
$$= \mathbf{1.5 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Using equation (3.3) for shear stress, we get

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta = \frac{2}{2} \times \sin (2 \times 30°)$$

$$= 1 \times \sin 60° = \mathbf{0.866 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

- A rectangular bar of cross-sectional area of 11000mm2 is subjected to a tensile load P as shown. The permissible normal and shear stresses on the oblique plane BC are given as 7N/mm$^2$ and 3.5N/mm$^2$, respectively. Determine the safe value of P.



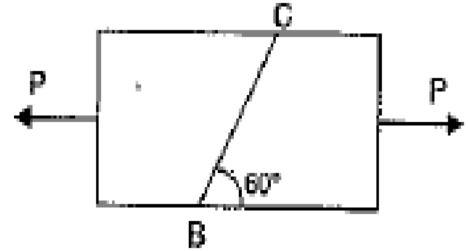
Area of cross-section, $A = 11000 \text{ mm}^2$

Normal stress, $\sigma_n = 7 \text{ N/mm}^2$

Shear stress, $\sigma_t = 3.5 \text{ N/mm}^2$

Angle of oblique plane with the axis of bar = 60°.

$\therefore$ Angle of oblique plane $BC$ with the normal cross-section of the bar,




$$\theta = 90° - 60° = 30°$$

Let $P$ = Safe value of axial pull

$\sigma$ = Safe stress in the member.

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30°$$

$$= \sigma (0.866)^2. \qquad (\because \cos 30° = 0.866)$$

$$\therefore \quad \sigma = \frac{7}{0.866 \times 0.866} = 9.334 \text{ N/mm}^2$$

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$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or

$$3.5 = \frac{\sigma}{2} \sin 2 \times 30^\circ = \frac{\sigma}{2} \sin 60^\circ = \frac{\sigma}{2} \times 0.866$$

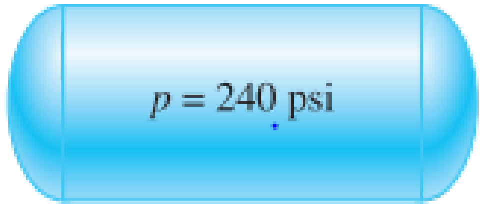
$$\therefore \quad \sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

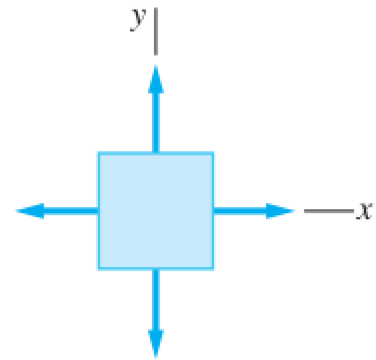
The safe stress is the least of the two, *i.e.*, 8.083 N/mm$^2$.

$\therefore$ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$

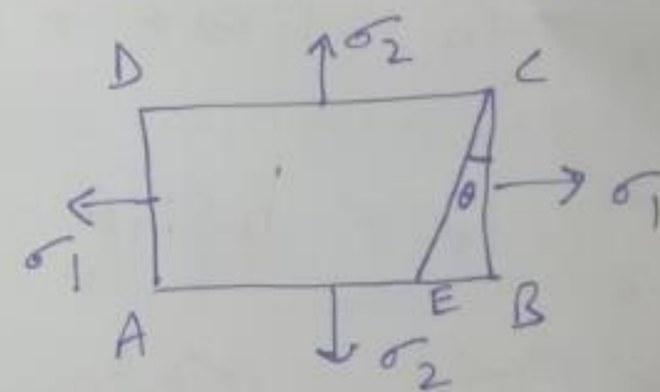
$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913 \text{ kN}}. \quad \textbf{Ans.}$$




## Stresses on inclined plane in 2-D stress system

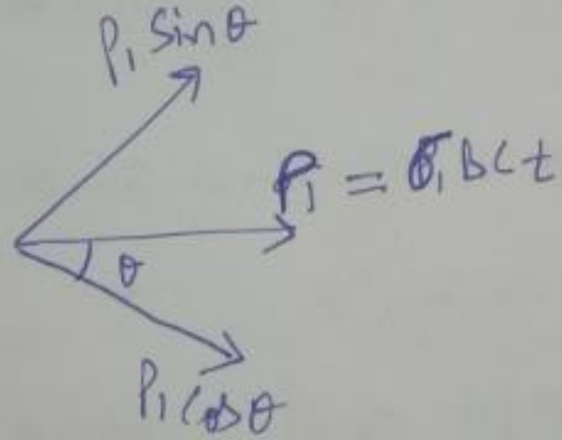
Consider a rectangular body ABCD of thickness 't'.

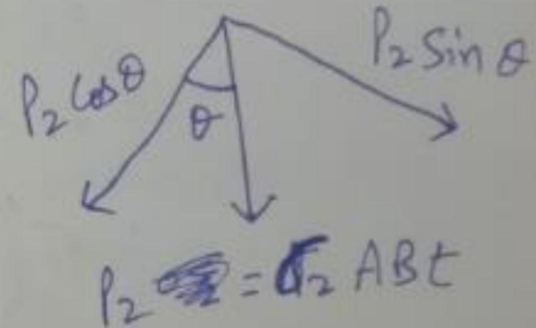



Plane CE is inclined at an angle $\theta$ with vertical plane BC.

$\sigma_1$ is the major stress and $\sigma_2$ is minor stress on planes BC and AB respectively.

For evaluating Normal and shear stresses on the inclined plane CE, Resolve the stresses $\sigma_1$ & $\sigma_2$ in horizontal and vertical components i.e. Normal and Tangential components.



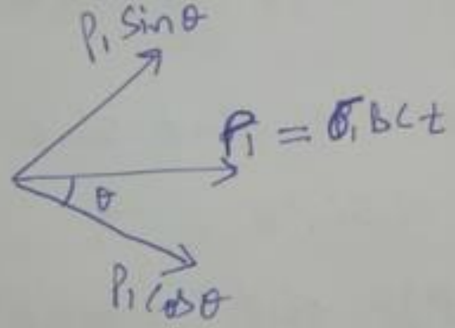



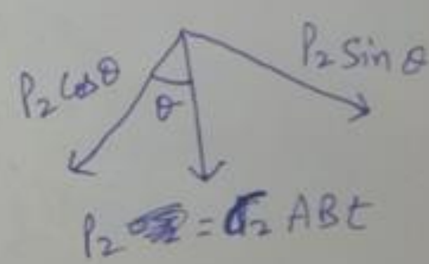

Normal Force on plane BC

$$P_1 = \sigma_1 BC\, t$$

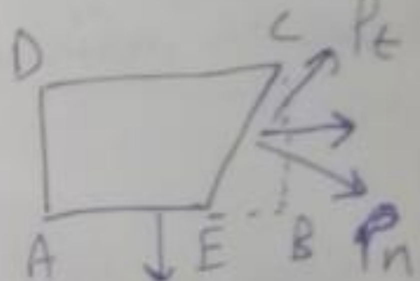
Normal Force on plane AB

$$P_2 = \sigma_2 AB\, t$$



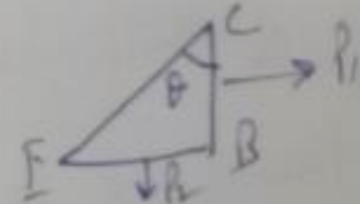




Now Finding the Normal and Tangential Resisting forces on the inclined plane :-




Normal force $P_n = P_1 \cos\theta + P_2 \sin\theta$ — (1)

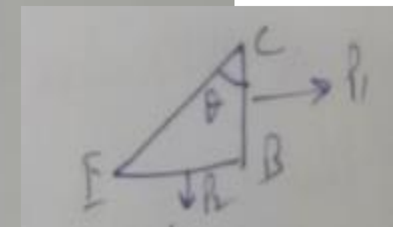
Shear force $P_t = P_1 \sin\theta - P_2 \cos\theta$ — (2)




If $\begin{pmatrix}\sigma_n \\ \sigma_\theta\end{pmatrix}$ and $\begin{pmatrix}\sigma_t \\ \tau_\theta\end{pmatrix}$ are normal and shear stresses developed on the inclined plane CE, then

Eq$^n$ (1) becomes,

$$\sigma_\theta \cdot CE \cdot t = \sigma_1 \cdot BC \cdot t \cos\theta + \sigma_2 \cdot EB \cdot t \sin\theta$$




$$\sigma_\theta = \sigma_1 \frac{BC}{CE} \cos\theta + \sigma_2 \frac{EB}{CE} \sin\theta$$

$$= \sigma_1 \cos\theta \cdot \cos\theta + \sigma_2 \sin\theta \cdot \sin\theta$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

$$= \sigma_1 \left( \frac{1 + \cos 2\theta}{2} \right) + \sigma_2 \left( \frac{1 - \cos 2\theta}{2} \right)$$

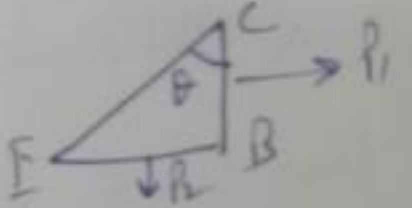
on simplifying,

$$\boxed{\begin{matrix}\sigma_\theta \\ \sigma_n\end{matrix} = \left( \frac{\sigma_1 + \sigma_2}{2} \right) + \left( \frac{\sigma_1 - \sigma_2}{2} \right) \cos 2\theta}$$

Similarly,

Eq$^n$ (2) $P_t = P_1 \sin\theta - P_2 \cos\theta$ becomes

$$\tau_\theta \cdot CE \cdot t = \sigma_1 \, CB \cdot t \sin\theta - \sigma_2 \, EB \cdot t \cos\theta$$




$$\tau_\theta = \sigma_1 \frac{CB}{CE} \cdot \sin\theta - \sigma_2 \frac{EB}{CE} \cdot \cos\theta$$

$$= \sigma_1 \cos\theta \sin\theta - \sigma_2 \sin\theta \cos\theta$$

$$= (\sigma_1 - \sigma_2) \sin\theta \cos\theta$$

OR $$\tau_\theta \; (\sigma_t) = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta$$

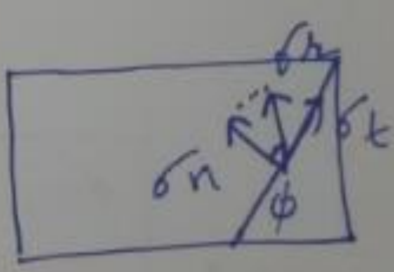
The resultant stress on the inclined plane is given by:

$$\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

<u>obliquity</u>:

Angle made by the resultant stress with the normal of the oblique plane is called obliquity.

$$\tan\phi = \frac{\sigma_t}{\sigma_n}$$



## Max. Shear stress :

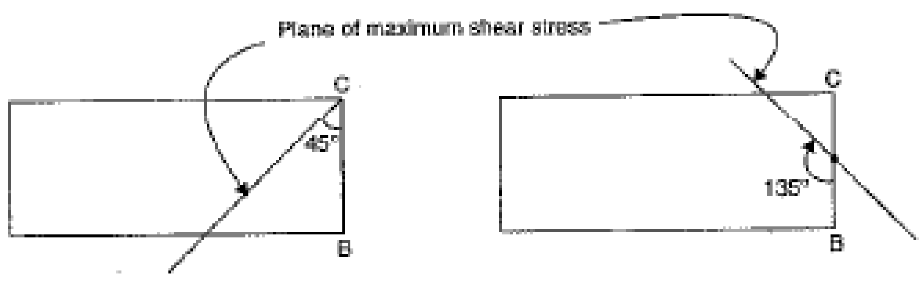
$$\tau_\theta = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta$$

Shear stress is max. when $\sin 2\theta = 1$

or $2\theta = 90^\circ$ or $270^\circ$

i.e. $\theta = 45^\circ$ or $135^\circ$

And $$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}}$$



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## Principal Planes:

For principal planes, $\tau_\theta = 0$

i.e.

$$\left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta = 0$$

$$\sin 2\theta = 0$$

$$\Rightarrow \quad 2\theta = 0 \text{ or } 180^\circ$$

$$\boxed{\theta = 0 \text{ or } 90^\circ}$$

When $\theta = 0^\circ$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 0$$

$$\boxed{\sigma_n = \sigma_1}$$

When $\theta = 90^\circ$

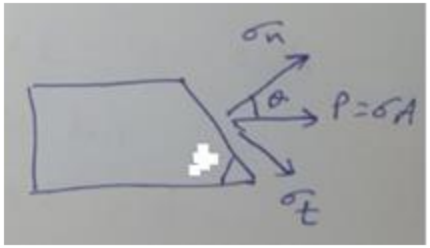
$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180$$

$$\boxed{\sigma_n = \sigma_2}$$

- Topics covered in previous class:

- Stresses on inclined plane

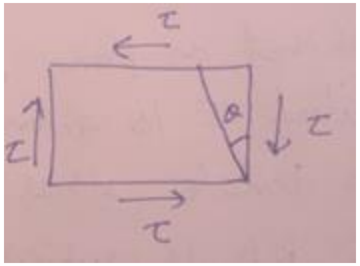
$$\sigma_n = \sigma \cos^2\theta \qquad \sigma_t = \frac{\sigma}{2}\sin 2\theta$$




- Stresses on inclined plane when subjected to simple shear stress

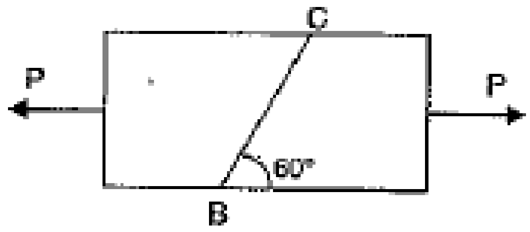
Normal stress $\sigma_n = \tau \sin 2\theta$

Shear stress $\sigma_t = \tau \cos 2\theta$




- Principal planes and Principal stresses
- Methods of determining stresses on oblique plane
  - Analytical and Graphical

- A rectangular bar of cross-sectional area of 11000mm2 is subjected to a tensile load P as shown. The permissible normal and shear stresses on the oblique plane BC are given as 7N/mm$^2$ and 3.5N/mm$^2$, respectively. Determine the safe value of P.

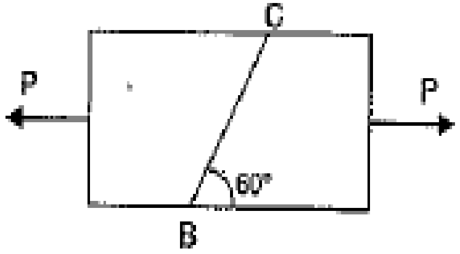


Area of cross-section, $A = 11000\ \text{mm}^2$

Normal stress, $\sigma_n = 7\ \text{N/mm}^2$

Shear stress, $\sigma_t = 3.5\ \text{N/mm}^2$

Angle of oblique plane with the axis of bar = 60°.




∴ Angle of oblique plane $BC$ with the normal cross-section of the bar,

$$\theta = 90° - 60° = 30°$$

Let $P$ = Safe value of axial pull

$\sigma$ = Safe stress in the member.

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30°$$

$$= \sigma (0.866)^2. \qquad (\because \cos 30° = 0.866)$$

$$\therefore \quad \sigma = \frac{7}{0.866 \times 0.866} = 9.334\ \text{N/mm}^2$$

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or

$$3.5 = \frac{\sigma}{2} \sin 2 \times 30° = \frac{\sigma}{2} \sin 60° = \frac{\sigma}{2} \times 0.866$$

$\therefore$

$$\sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

The safe stress is the least of the two, *i.e.*, 8.083 N/mm$^2$.

$\therefore$ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$

$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913 \text{ kN}}. \quad \textbf{Ans.}$$

- Topics covered in previous class:

- Stresses on inclined plane in 2-D stress system

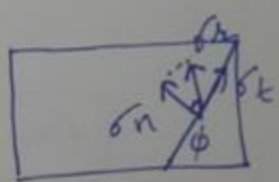
$$\sigma_\theta \; (\sigma_n) = \left(\frac{\sigma_1 + \sigma_2}{2}\right) + \left(\frac{\sigma_1 - \sigma_2}{2}\right) \cos 2\theta$$

$$\tau_\theta \; (\sigma_t) = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta$$

$$\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

Obliquity

$$\tan \phi = \frac{\sigma_t}{\sigma_n}$$



## Max. Shear stress :

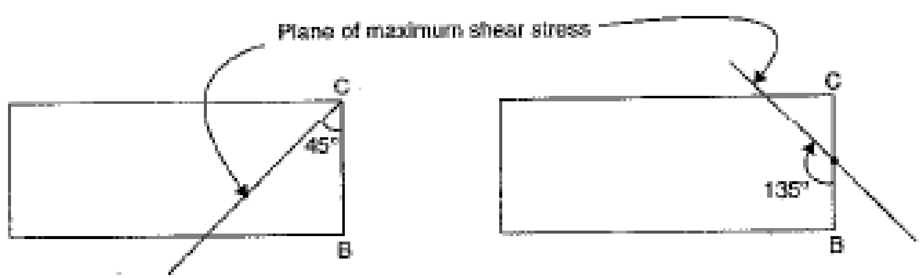
$$\tau_\theta = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta$$

Shear stress is max. when $\sin 2\theta = 1$

or $2\theta = 90^\circ$ or $270^\circ$

i.e. $\theta = 45^\circ$ or $135^\circ$

And $$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}}$$



## Principal Planes:

For principal planes, $\tau_\theta = 0$

i.e. $\left(\frac{\sigma_1 - \sigma_2}{2}\right) \sin 2\theta = 0$

$$\sin 2\theta = 0$$

$$\Rightarrow 2\theta = 0 \text{ or } 180^\circ$$

$$\boxed{\theta = 0 \text{ or } 90^\circ}$$

When $\theta = 0^\circ$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 0$$

$$\boxed{\sigma_n = \sigma_1}$$

When $\theta = 90^\circ$

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180$$

$$\boxed{\sigma_n = \sigma_2}$$

# FORMULAS DERIVED

Normal Stress $\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$

Shear Stress $\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$

Resultant Stress $\sigma_r = \sqrt{\sigma_n{}^2 + \sigma_t{}^2}$

Maximum Shear Stress $(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2}$

Obliquity $\text{Tan}\ \phi = \frac{\sigma_t}{\sigma_n}$

- The tensile stresses at a point across two mutually perpendicular planes are 120N/mm$^2$ and 60N/mm$^2$. Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of minor stress.

Major principal stress, $\sigma_1 = 120 \text{ N/mm}^2$

Minor principal, $\sigma_2 = 60 \text{ N/mm}^2$

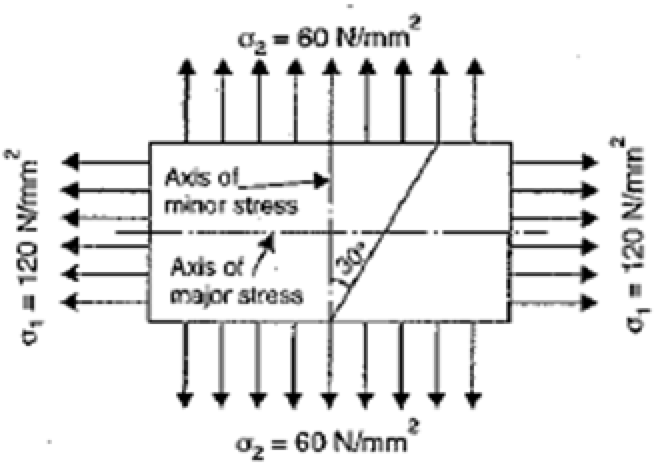
Angle of oblique plane with the axis of minor principal stress,

$$\theta = 30°.$$

*Normal stress*

The normal stress ($\sigma_n$) is given by equation

$$\therefore \quad \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$




$$= \frac{120 + 60}{2} + \frac{120 - 60}{2} \cos 2 \times 30°$$

$$= 90 + 30 \cos 60° = 90 + 30 \times \frac{1}{2}$$

$$= \mathbf{105 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

*Tangential stress*

The tangential (or shear stress) $\sigma_t$ is given by equation

$$\therefore \quad \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$$= \frac{120 - 60}{2} \sin (2 \times 30°)$$

$$= 30 \times \sin 60° = 30 \times 0.866$$

$$= \mathbf{25.98\ N/mm^2}. \quad \textbf{Ans.}$$

*Resultant stress*

The resultant stress ($\sigma_R$) is given by equation

$$\therefore \quad \sigma_R = \sqrt{{\sigma_n}^2 + {\sigma_t}^2} = \sqrt{105^2 + 25.98^2}$$

$$= \sqrt{11025 + 674.96} = \mathbf{108.16\ N/mm^2}. \quad \textbf{Ans.}$$

- The stresses at a point in a bar are 200N/mm2(tensile) and 100N/mm2 (compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60° to the axis of the major stress. Also determine the maximum intensity of shear stress in the material at the point.

Major principal stress, $\sigma_1 = 200\ N/mm^2$

Minor principal stress, $\sigma_2 = -100\ N/mm^2$

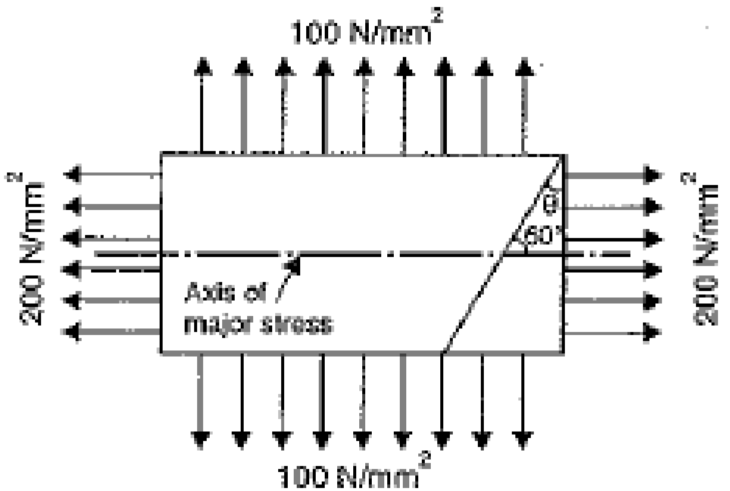
(Minus sign is due to compressive stress)

Angle of the plane, which it makes with the major principal stress = 60°

$\therefore$ Angle $\theta = 90° - 60° = 30°$.

*Resultant stress in magnitude and direction*

First calculate the normal and tangential stresses.



$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{200 + (-100)}{2} + \frac{200 - (-100)}{2} \cos (2 \times 30^\circ) \quad (\because \theta = 30^\circ)$$

$$= \frac{200 - 100}{2} + \frac{200 + 100}{2} \cos 60^\circ$$

$$= 50 + 150 \times \tfrac{1}{2} \qquad (\because \cos 60^\circ = \tfrac{1}{2})$$

$$= 50 + 75 = 125 \text{ N/mm}^2.$$

Using equation (3.7) for tangential stress,

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \frac{200 - (-100)}{2} \sin (2 \times 30^\circ)$$

$$= \frac{200 + 100}{2} \sin 60^\circ = 150 \times 0.866 = 129.9 \text{ N/mm}^2.$$

for resultant stress,

$$\sigma_R = \sqrt{\sigma_n^{\ 2} + \sigma_t^{\ 2}} = \sqrt{125^2 + 129.9^2}$$

$$= \sqrt{15625 + 16874} = 180.27 \text{ N/mm}^2.$$

The inclination of the resultant stress with the normal of the inclined plane is given by

$$\tan\phi = \frac{\sigma_t}{\sigma_n} = \frac{129.9}{125} = 1.04$$

$$\phi = \tan^{-1} 1.04 = 46^\circ\ 6'.$$

*Maximum shear stress*

Maximum shear stress is given by equation

$$\therefore \qquad (\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{200 - (-100)}{2} = \frac{200 + 100}{2} = 150 \text{ N/mm}^2.$$

- At a point in a strained material the principal tensile stresses across two perpendicular planes, are 80N/mm2 and 40N/mm2. Determine normal stress, shear stress and the resultant stress on a plane inclined at 20° with the major principal plane. Determine also the obliquity.
- What will be the intensity of stress, which acting alone will produce the same maximum strain if poisson's ratio is ¼.

Major principal stress, $\sigma_1 = 80 \text{ N/mm}^2$

Minor principal stress, $\sigma_2 = 40 \text{ N/mm}^2$

The plane $CE$ is inclined at angle 20° with major principal plane (*i.e.*, plane $BC$).

$$\therefore \qquad \theta = 20°$$

Poisson's ratio, $\mu = \dfrac{1}{4}$

Let $\sigma_n$ = Normal stress on inclined plane $CE$

$\sigma_t$ = Shear stress and

$\sigma_R$ = Resultant stress.

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta = \frac{80 + 40}{2} + \frac{80 - 40}{2} \cos (2 \times 20°)$$

$$= 60 + 20 \times \cos 40° = 75.32 \text{ N/mm}^2. \quad \textbf{Ans.}$$

The shear stress is given by equation

$$\therefore \qquad \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \frac{80 - 40}{2} \sin (2 \times 20°) = 20 \sin 40°$$

$$= 12.865 \text{ N/mm}^2. \quad \textbf{Ans.}$$

The resultant stress is given by equation

$$\therefore \quad \sigma_R = \sqrt{\sigma_n^{\,2} + \sigma_t^{\,2}}$$

$$= \sqrt{75.32^2 + 12.856^2} = \mathbf{76.4\ N/mm^2.}$$

$$\tan\phi = \frac{\sigma_t}{\sigma_n} = \frac{12.856}{75.32}$$

$$\therefore \quad \phi = \tan^{-1}\frac{12.856}{75.32} = \mathbf{9^\circ\ 41'.}$$

Let σ = stress which acting alone will produce the same maximum strain. The maximum strain will be in the direction of major principal stress.

$$\therefore \quad \text{Maximum strain} = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} = \frac{1}{E}(\sigma_1 - \mu\sigma_2)$$

$$= \frac{1}{E}\left(80 - \frac{40}{4}\right) = \frac{70}{E}$$

The strain due to stress $\sigma = \frac{\sigma}{E}$

Equating the two strains, we get $\frac{70}{E} = \frac{\sigma}{E}$

$\therefore \quad \sigma = 70 \text{ N/mm}^2$. **Ans.**

# Case 3: A member subjected to Direct Stresses in two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress

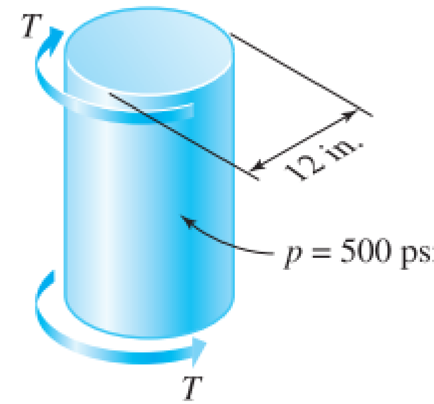


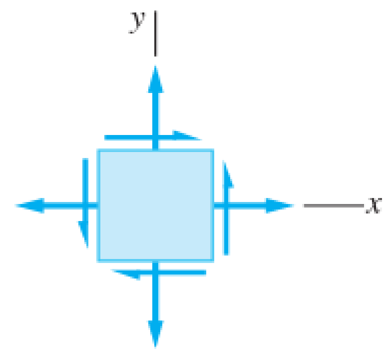
Figure shows a rectangular bar ABCD of uniform cross sectional area A and of unit thickness. This bar is subjected to

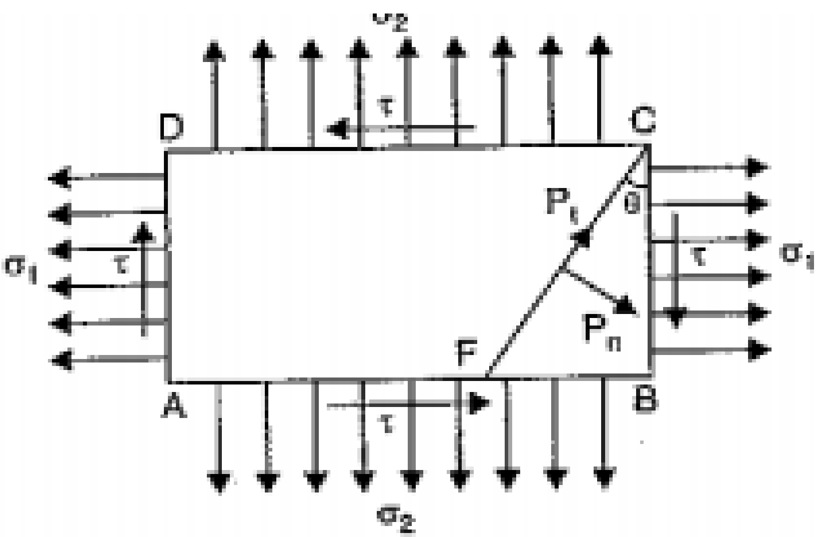
a) Tensile stress $\sigma_1$ on face BC and AD

b) Tensile stress $\sigma_2$ on face AB and CD

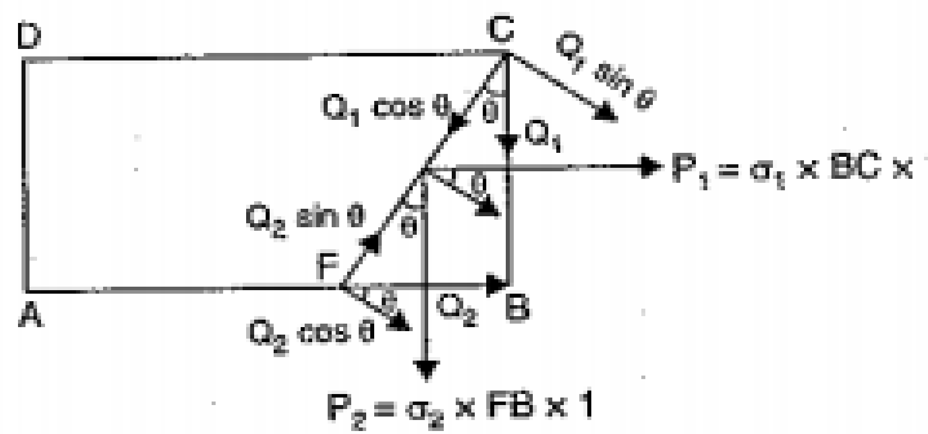
c) A simple shear stress $\tau$ on face BC and AD




(a)




(b)

We want to calculate normal and tangential stresses on oblique section $FC$, which is inclined at an angle $\theta$ with the normal cross-section $BC$. The given stresses are converted into equivalent forces.

The forces acting on the wedge $FBC$ are :

$P_1$ = Tensile force on face $BC$ due to tensile stress $\sigma_1$

$= \sigma_1 \times$ Area of $BC$

$= \sigma_1 \times BC \times 1$ $\quad (\because \text{ Area} = BC \times 1)$

$= \sigma_1 \times BC$

$P_2$ = Tensile force on face $FB$ due to tensile stress $\sigma_2$

$= \sigma_2 \times$ Area of $FB = \sigma_2 \times FB \times 1$

$= \sigma_2 \times FB$

$Q_1$ = Shear force on face $BC$ due to shear stress $\tau$

$= \tau \times$ Area of $BC$

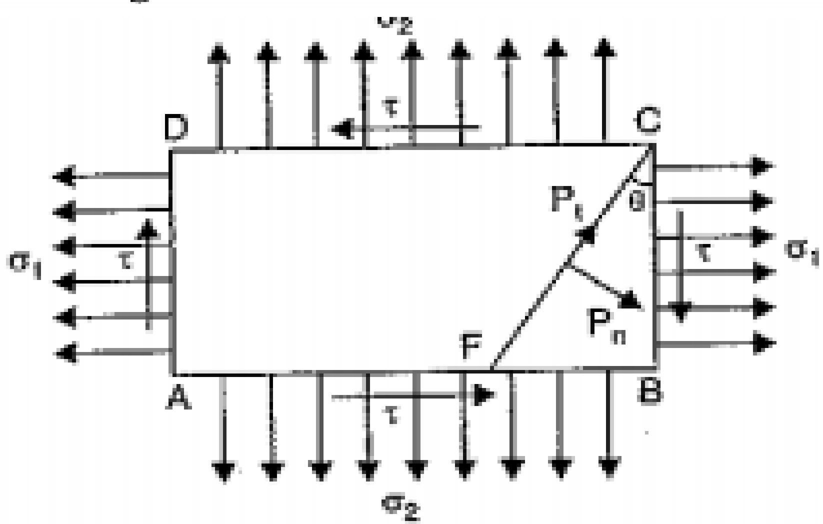
$= \tau \times BC \times 1 = \tau \times BC$

$Q_2$ = Shear force on face $FB$ due to shear stress $\tau$

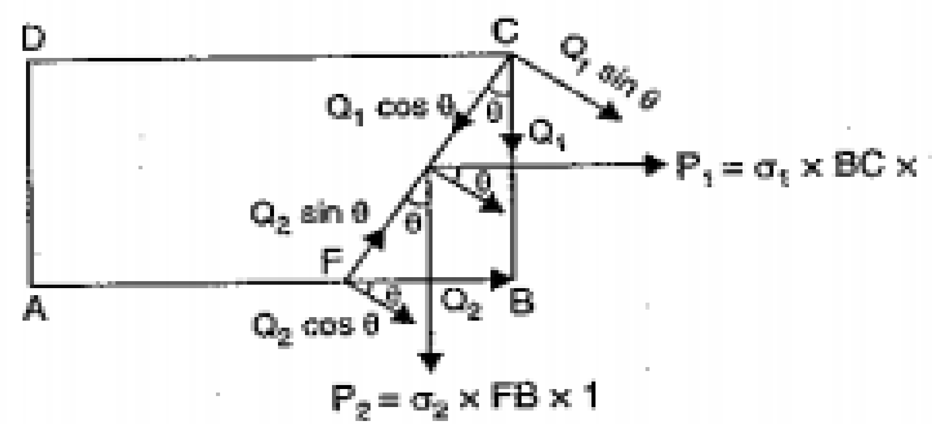
$= \tau \times$ Area of $FB$

$= \tau \times FB \times 1 = \tau \times FB.$

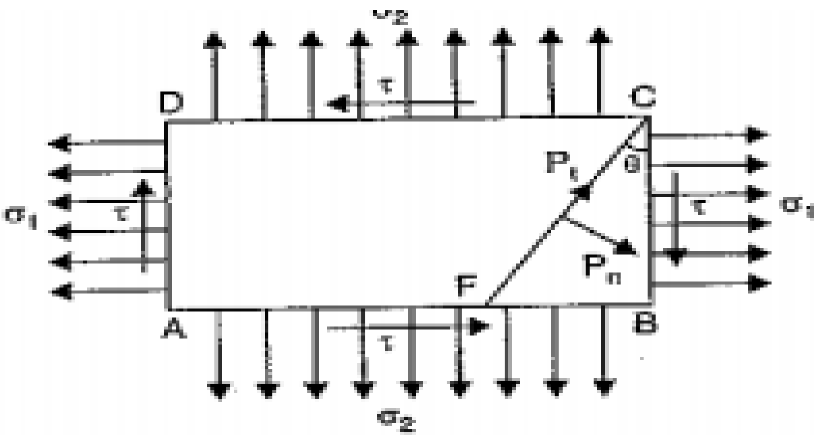
Resolving the above four forces (*i.e.*, $P_1$, $P_2$, $Q_1$ and $Q_2$) normal to the *oblique section FC*, we get



(a)

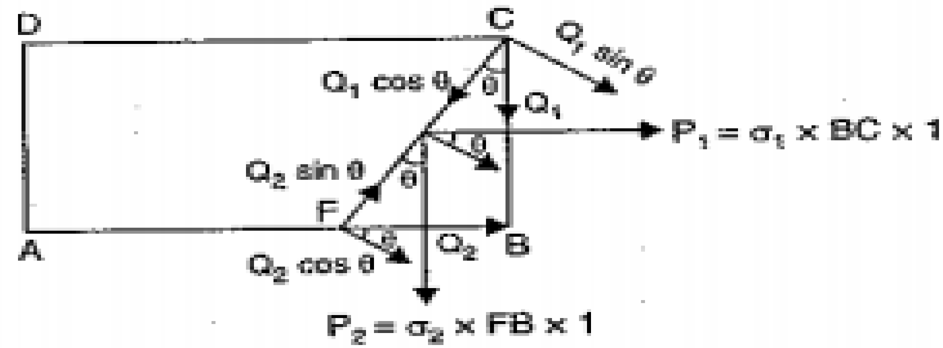


(b)



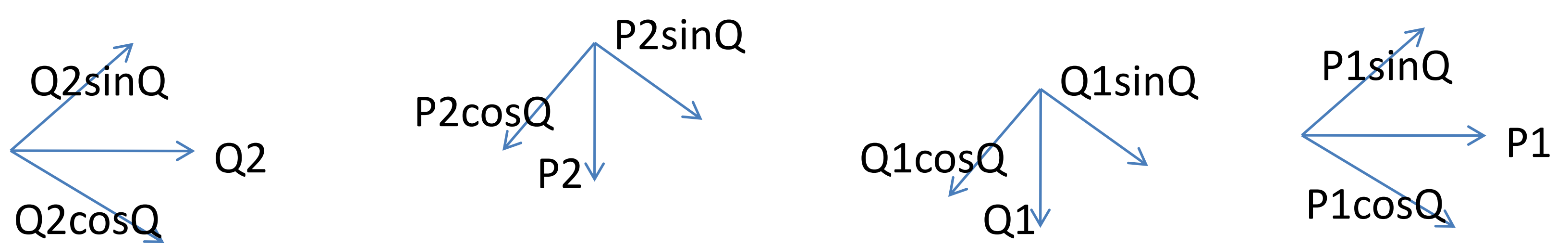
D
C
τ
σ1
P_t
θ
P_n
F
A
B
σ2


(a)



D
C
Q1 sin θ
Q1 cos θ
θ
Q1
P1 = σ1 × BC × 1
Q2 sin θ
F
A
Q2
B
Q2 cos θ
P2 = σ2 × FB × 1


(b)



Q2sinQ
Q2
Q2cosQ
P2sinQ
P2cosQ
P2
Q1sinQ
Q1cosQ
Q1
P1sinQ
P1
P1cosQ

Total normal force,

$$P_n = P_1 \cos\theta + P_2 \sin\theta + Q_1 \sin\theta + Q_2 \cos\theta$$

Substituting the values of $P_1$, $P_2$, $Q_1$ and $Q_2$, we get

$$P_n = \sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta$$

Similarly, the total tangential force ($P_t$) is obtained by resolving $P_1$, $P_2$, $Q_1$ and $Q_2$ along the oblique section $FC$.

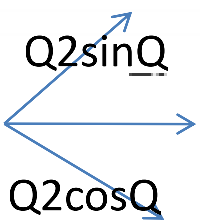
$\therefore$ Total tangential force,

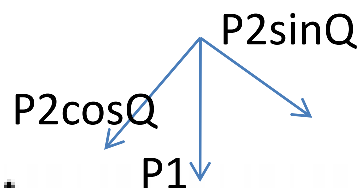
$$P_t = P_1 \sin\theta - P_2 \cos\theta - Q_1 \cos\theta + Q_2 \sin\theta$$
$$= \sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta$$

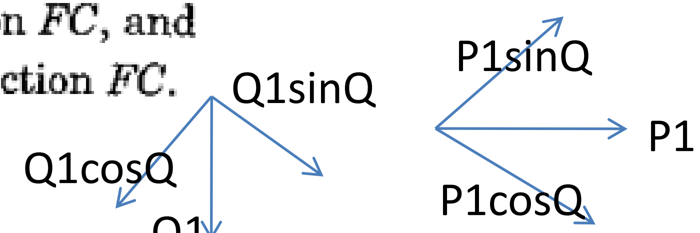
(substitute the values of $P_1$, $P_2$, $Q_1$ and $Q_2$)

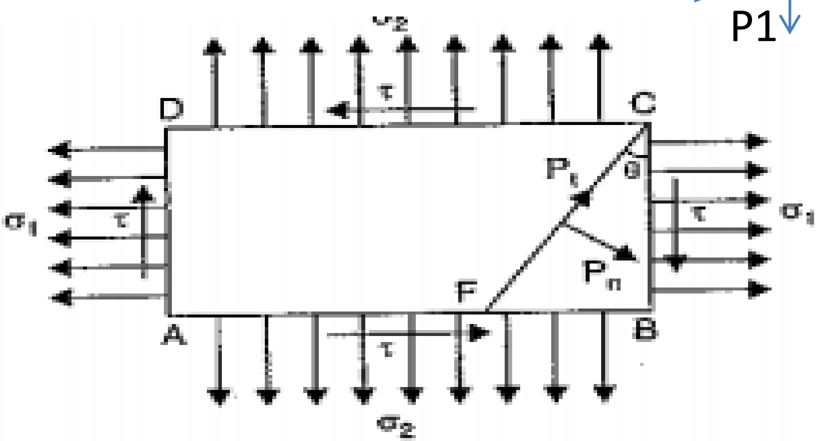
Now, Let $\sigma_n$ = Normal stress across the section $FC$, and

$\sigma_t$ = Tangential stress across the section $FC$.

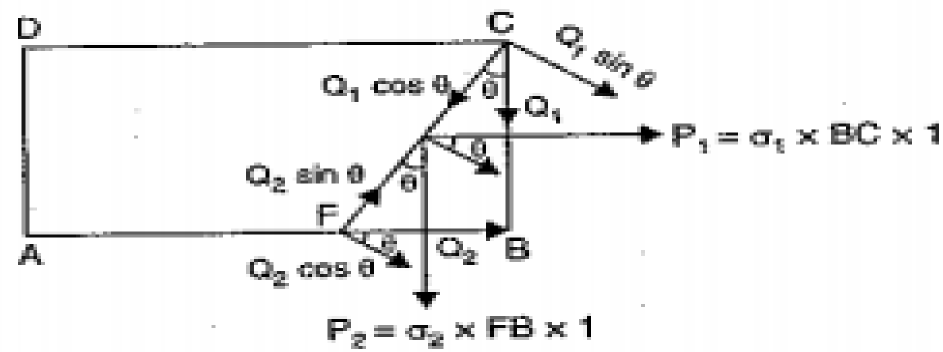



(a)




(b)

(Substitute the values of $P_1$, $P_2$, $Q_1$ and $Q_2$)



Now, Let $\sigma_n$ = Normal stress across the section $FC$, and

$\sigma_t$ = Tangential stress across the section $FC$.

Then normal stress across the section $FC$,

$$\sigma_n = \frac{\text{Total normal force across section } FC}{\text{Area of section } FC} = \frac{P_n}{FC \times 1}$$

$$= \frac{\sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta}{FC \times 1}$$

$$= \sigma_1 . \frac{BC}{FC} . \cos\theta + \sigma_2 . \frac{FB}{FC} . \sin\theta + \tau . \frac{BC}{FC} . \sin\theta + \tau . \frac{FB}{FC} . \cos\theta$$

$$= \sigma_1 . \cos\theta . \cos\theta + \sigma_2 \sin\theta . \sin\theta + \tau . \cos\theta . \sin\theta + \tau \sin\theta . \cos\theta$$

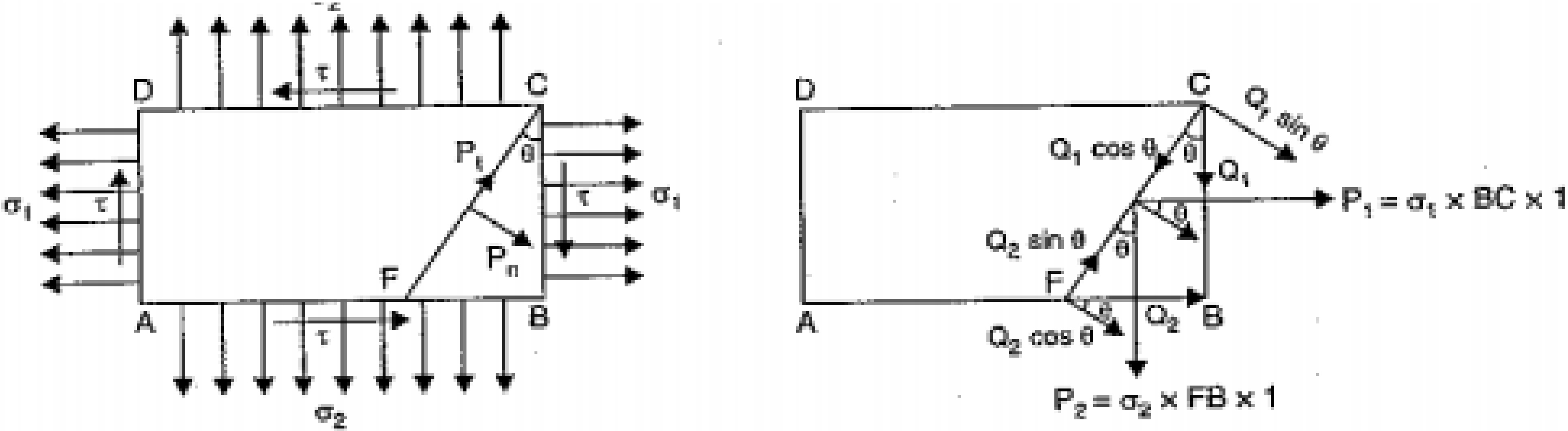
$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau \cos\theta \sin\theta$$

$$= \sigma_1 \left(\frac{1 + \cos 2\theta}{2}\right) + \sigma_2 \left(\frac{1 - \cos 2\theta}{2}\right) + \tau \sin 2\theta$$

$$\left(\because \cos^2\theta = \frac{1 + \cos 2\theta}{2}, \sin^2\theta = \frac{1 - \cos 2\theta}{2} \text{ and } 2\cos\theta \sin\theta = \sin 2\theta\right)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta \qquad ...(3.12)$$




(a) (b)

and tangential stress (*i.e.*, shear stress) across the section $FC$,

$$\sigma_t = \frac{\text{Total tangential force across section } FC}{\text{Area of section } FC} = \frac{P_t}{FC \times 1}$$

$$= \frac{\sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta}{FC \times 1}$$

$$= \sigma_1 . \frac{BC}{FC} . \sin\theta - \sigma_2 . \frac{FB}{FC} . \cos\theta - \tau . \frac{BC}{FC} . \cos\theta + \tau . \frac{FB}{FC} . \sin\theta$$

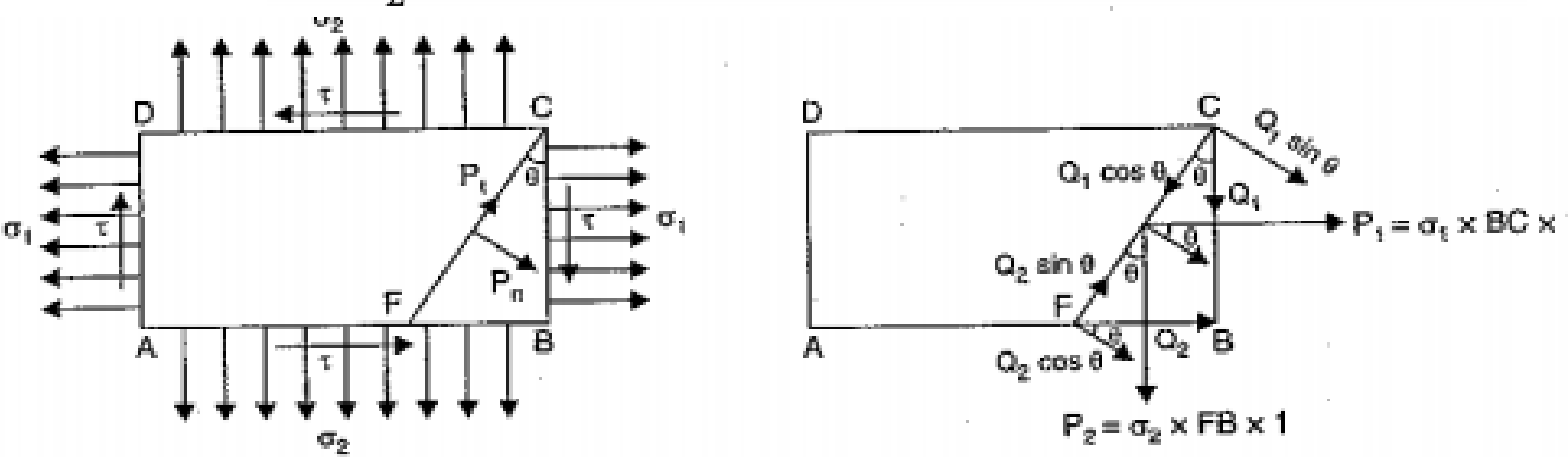
$$= \sigma_1 . \cos\theta . \sin\theta - \sigma_2 . \sin\theta . \cos\theta - \tau . \cos\theta . \cos\theta + \tau . \sin\theta . \sin\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= (\sigma_1 - \sigma_2) . \cos\theta \sin\theta - \tau \cos^2\theta + \tau \sin^2\theta$$

$$= \left(\frac{\sigma_1 - \sigma_2}{2}\right) . 2\cos\theta \sin\theta - \tau(\cos^2\theta - \sin^2\theta)$$

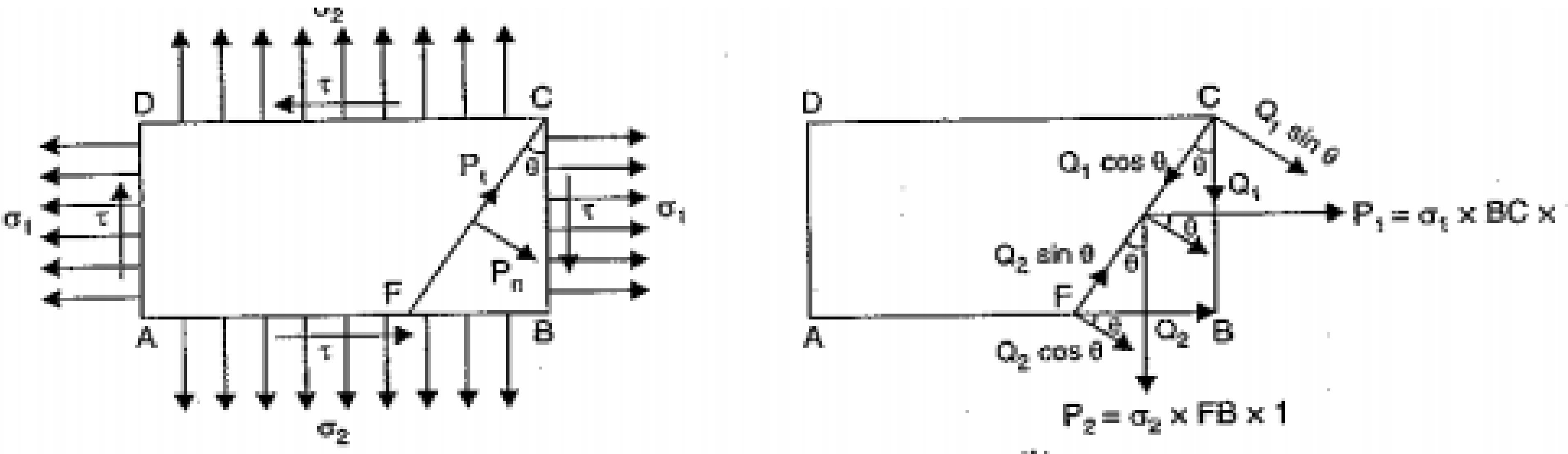
$$= \frac{\sigma_1 - \sigma_2}{2} . \sin 2\theta - \tau \cos 2\theta \qquad (\because \cos^2\theta - \sin^2\theta = \cos 2\theta) \quad ...(3.13)$$




(a) (b)

Normal Stress: $\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$

Tangential Stress $\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$



(a) (b)

Position of Principal Planes: The planes on which shear stress is zero are known as principal planes. The stresses acting on principal planes are known as principal stresses.

The position of principal planes are obtained by equating the tangential stress [given by equation (3.13)] to zero.

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

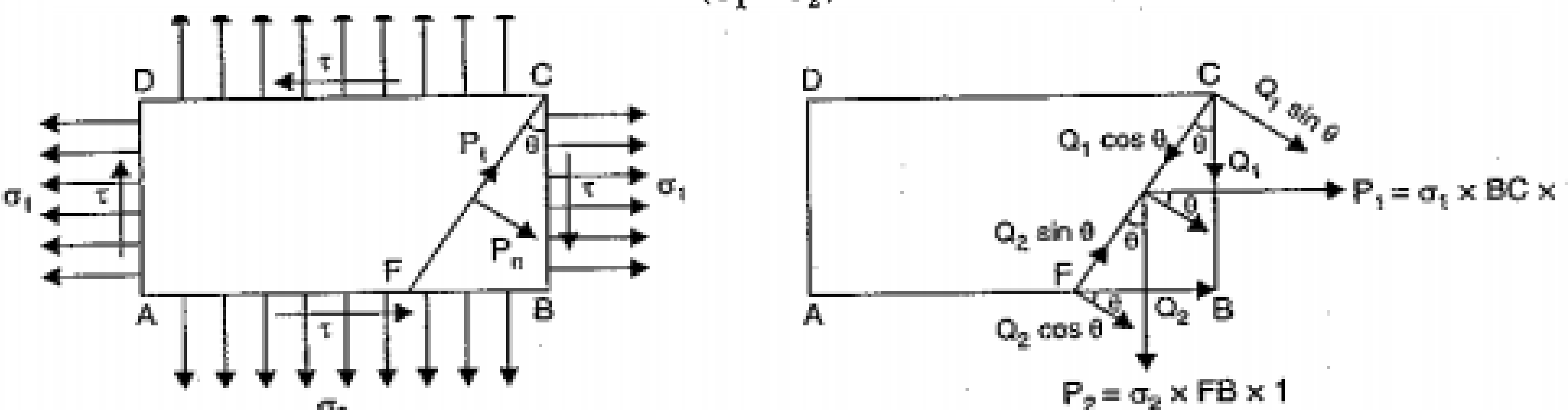
$\therefore$ For principal planes, $\sigma_t = 0$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \tau \cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\tau}{\frac{(\sigma_1 - \sigma_2)}{2}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

or $$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)} \qquad \ldots(3.14)$$




(a) (b)

$$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)} \qquad \text{...(3.14)}$$

But the tangent of any angle in a right angled triangle

$$= \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}}$$

$$\therefore \quad \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

$\therefore$ Height of right angled triangle = $2\tau$

Base of right angled triangle = $(\sigma_1 - \sigma_2)$.

Now diagonal of the right angled triangle

$$= \pm\sqrt{(\sigma_1 - \sigma_2)^2 + (2\tau)^2} = \pm\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad \text{and} \quad -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

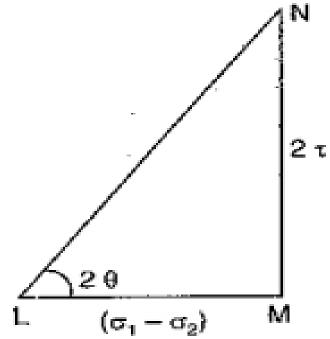


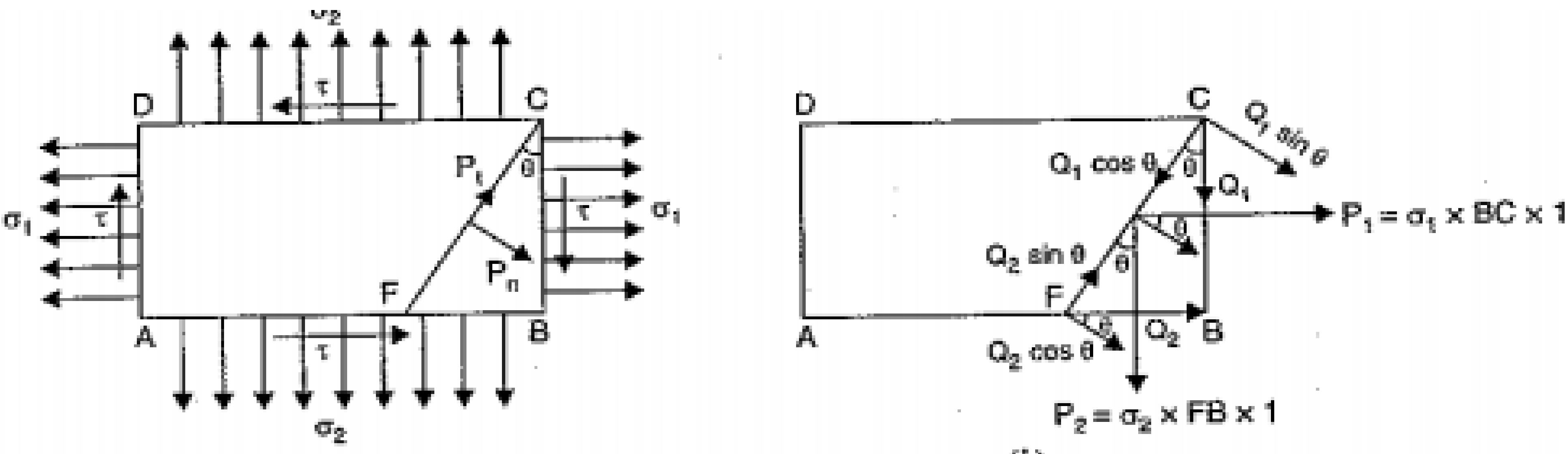

Fig. 3.11

**1st Case.** Diagonal = $\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then

$$\sin 2\theta = \frac{\text{Height}}{\text{Diagonal}} = \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$\cos 2\theta = \frac{\text{Base}}{\text{Diagonal}} = \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}.$$



The value of major principal stress is obtained by substituting the values of $\sin 2\theta$ and $\cos 2\theta$ in equation (3.12).

$\therefore$ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2} \frac{(\sigma_1 - \sigma_2)^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

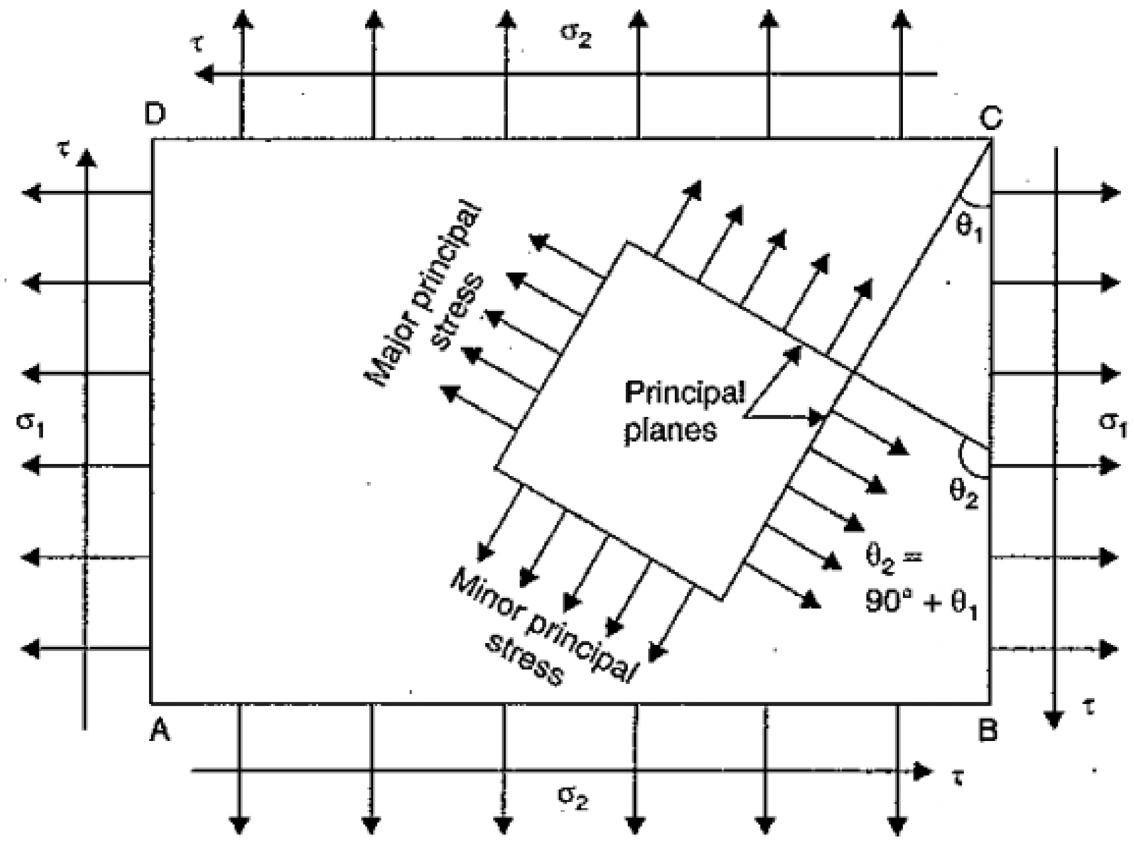
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

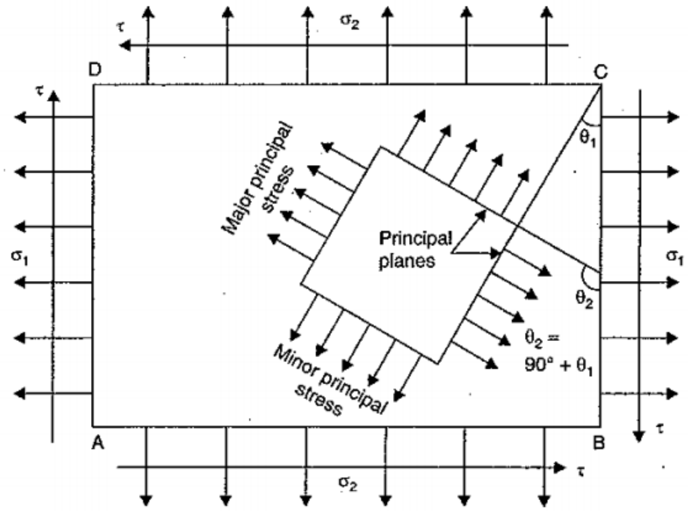
$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2} \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad ...(3.15)$$

$\therefore$ Minor principal stress

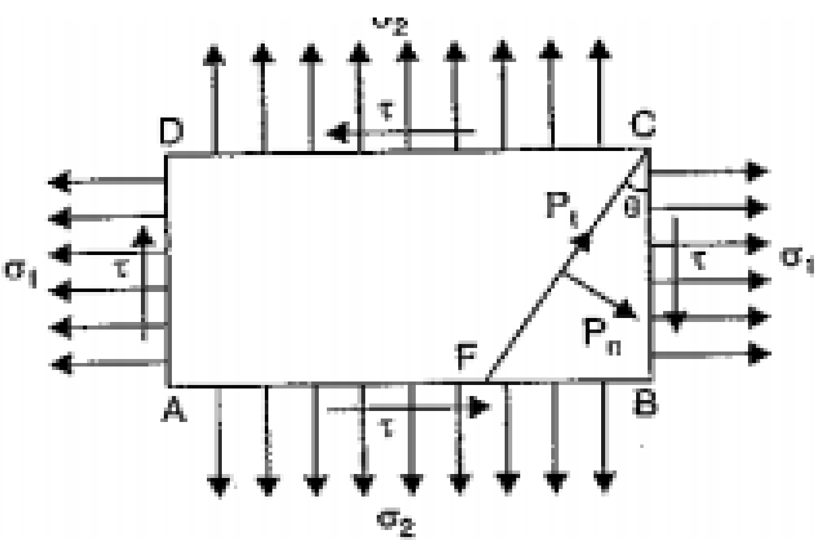
$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{(\sigma_1 - \sigma_2)}{2}\right)^2 + \tau^2}$$


τ
σ2
D
C
τ
Major principal stress
Principal planes
θ1
σ1
σ1
θ2
θ2 = 90° + θ1
Minor principal stress
τ
A
B
τ
σ2

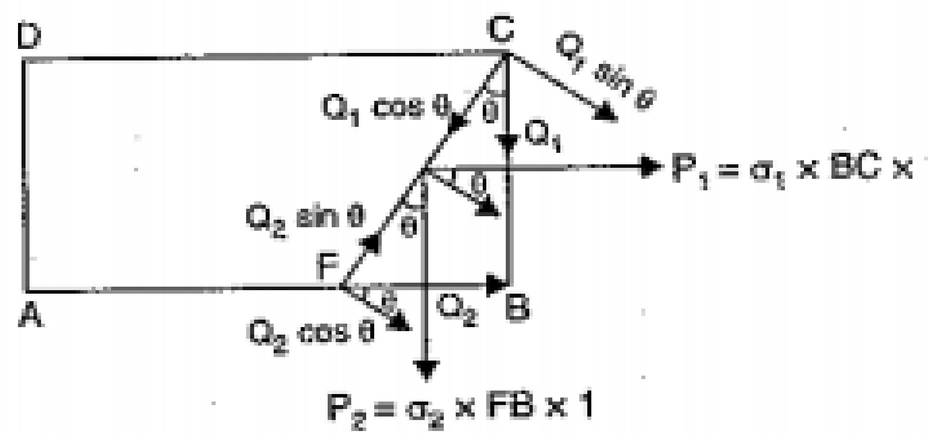



$$\text{Major Principal Stress} = \frac{\sigma_1+\sigma_2}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

$$\text{Minor Principal Stress} = \frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$




(a)




(b)

# Maximum Shear Stress:

**Maximum shear stress.** The shear stress is given by equation (3.13). The shear stress will be maximum or minimum when

$$\frac{d}{d\theta}(\sigma_t) = 0$$

or
$$\frac{d}{d\theta}\left[\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau\cos 2\theta\right] = 0$$

or
$$\frac{\sigma_1 - \sigma_2}{2}(\cos 2\theta) \times 2 - \tau(-\sin 2\theta) \times 2 = 0$$

$$(\sigma_1 - \sigma_2) \cdot \cos 2\theta + 2\tau \sin 2\theta = 0$$

or
$$2\tau\sin 2\theta = -(\sigma_1 - \sigma_2)\cos 2\theta$$
$$= (\sigma_2 - \sigma_1)\cos 2\theta$$

or
$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\sigma_2 - \sigma_1}{2\tau}$$

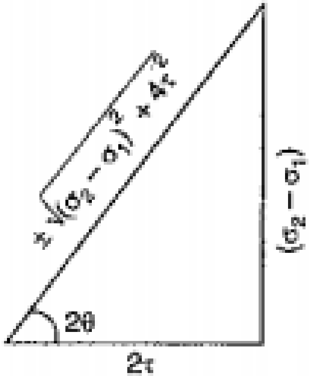
or
$$\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau} \qquad \text{...(3.17)}$$

Equation (3.17) gives condition for maximum or minimum shear stress.

If $\tan 2\theta = \dfrac{\sigma_2 - \sigma_1}{2\tau}$

Then
$$\sin 2\theta = \pm\frac{\sigma_2 - \sigma_1}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

and
$$\cos 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$



Substituting the values of sin 2θ and cos 2θ in equation (3.13), the maximum and minimum shear stresses are obtained.

∴ Maximum shear stress is given by

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \pm \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_2 - \sigma_1)}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \tau \times \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

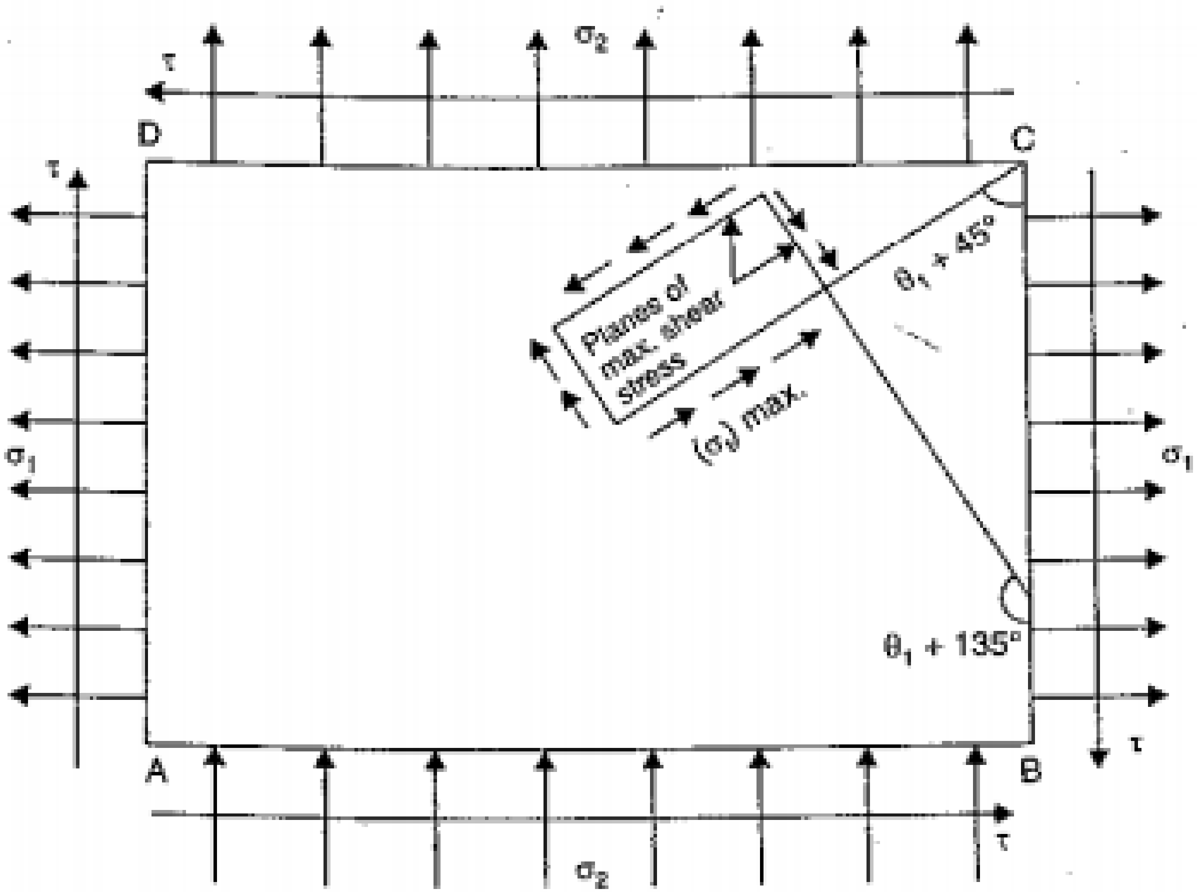
$$= \pm \frac{(\sigma_2 - \sigma_1)^2 + 4\tau^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} = \pm \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$\therefore \quad (\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \qquad \text{...(3.18)}$$

The planes on which maximum shear stress is acting, are obtained after finding the two values of θ from equation (3.17). These two values of θ will differ by 90°.

The second method of finding the planes of maximum shear stress is to find first principal planes and principal stresses. Let $\theta_1$ is the angle of principal plane with plane *BC* of Fig. 3.11 (*a*). Then the planes of maximum shear will be at $\theta_1 + 45°$ and $\theta_1 + 135°$ with the plane *BC* as shown in Fig. 3.12 (*a*).



# FORMULA'S DERIVED

**Normal Stress** $\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$

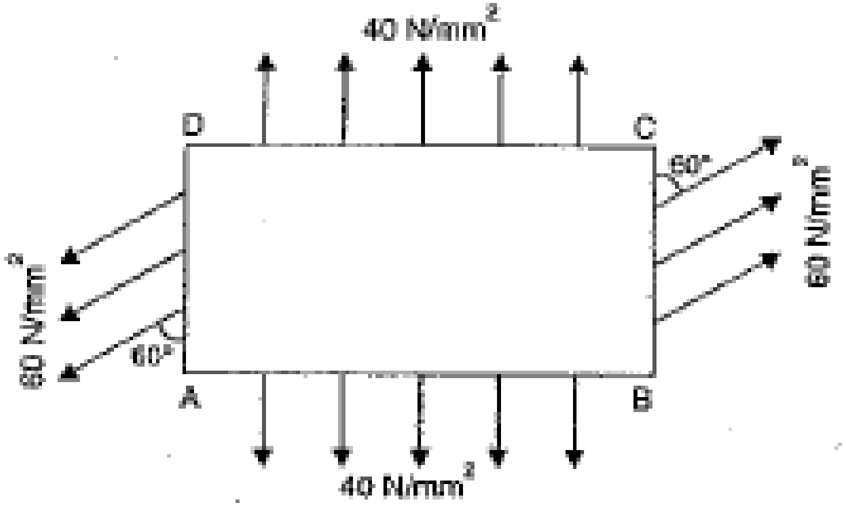
**Shear Stress** $\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$

**Major Principal Stress** $\frac{\sigma_1+\sigma_2}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$

**Minor Principal Stress** $\frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$

**Maximum Shear Stress** $(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

- A point in a strained material is subjected to the stresses a shown. Locate the principal planes and evaluate the principal stresses.



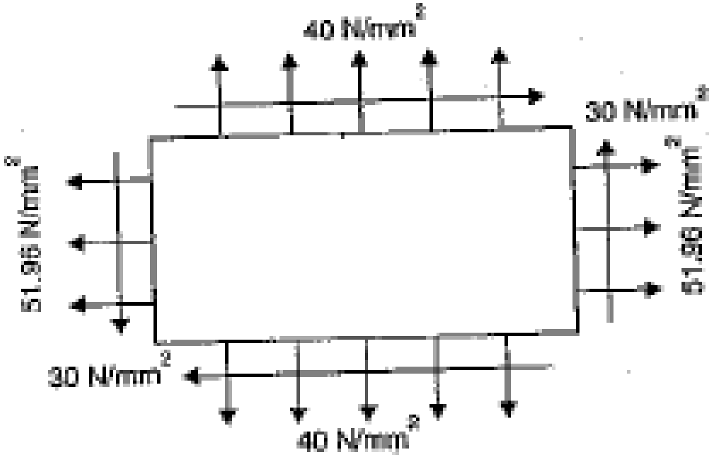
The stress on the face $BC$ or $AD$ is not normal. It is inclined at an angle of 60° with face $BC$ or $AD$. This stress can be resolved into two components *i.e.*, normal to the face $BC$ (or $AD$) and along the face $BC$ (or $AD$).

∴ Stress normal to the face $BC$ or $AD$

$$= 60 \times \sin 60° = 60 \times 0.866 = 51.96 \text{ N/mm}^2$$

Stress along the face $BC$ or $AD$

$$= 60 \times \cos 60° = 60 \times 0.5 = 30 \text{ N/mm}^2$$



Location of principal planes

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 30}{51.96 - 40} = 4.999$$

$$2\theta = \tan^{-1} 4.999 = 78^\circ\ 42' \text{ or } 258^\circ\ 42'$$

$$\theta = \mathbf{39^\circ\ 21' \text{ or } 129^\circ\ 21'}.\quad \textbf{Ans.}$$

Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96 + 40}{2} + \sqrt{\left(\frac{51.96 - 40}{2}\right)^2 + 30^2}$$

$$= 45.98 + 30.6$$

$$= \mathbf{76.58\ N/mm^2}.\quad \textbf{Ans.}$$

The minor principal stress is given by equation (3.16).

∴ Minor principal stress

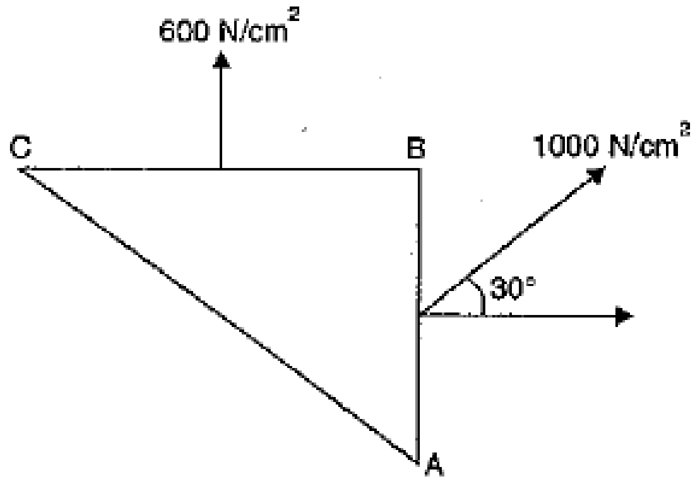
$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96 + 40}{2} - \sqrt{\left(\frac{51.96 - 40}{2}\right)^2 + 30^2}$$

$$= 45.98 - 30.6$$

$$= \mathbf{15.38\ N/mm^2}.\quad \textbf{Ans.}$$

*At a certain point in a material under stress the intensity of the resultant stress on a vertical plane is 1000 N/cm² inclined at 30° to the normal to that plane and the stress on a horizontal plane has a normal tensile component of intensity 600 N/cm² as shown in Fig. Find the magnitude and direction of the resultant stress on the horizontal plane and the principal stresses.*



Resultant stress on vertical plane $AB$ = 1000 N/cm²

Inclination of the above stress = 30°

Normal stress on horizontal plane $BC$ = 600 N/cm²

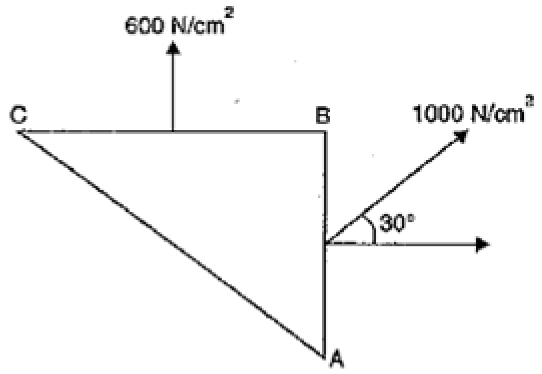
The resultant stress on plane $AB$ is resolved into normal and tangential component.

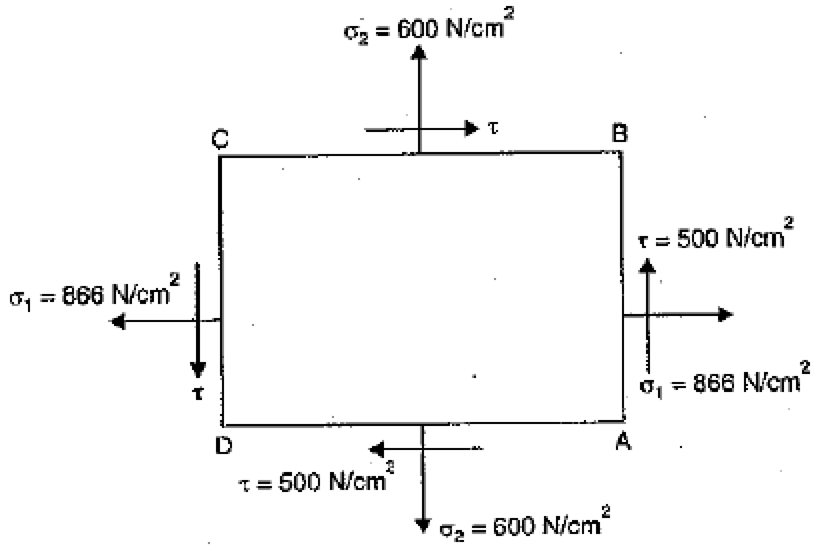
The normal component

$$= 1000 \times \cos 30° = 866 \text{ N/cm}^2$$

Tangential component

$$= 1000 \times \sin 30° = 500 \text{ N/cm}^2.$$




Free body diagram of block ABCD showing the effect of Normal and Shear stress

- Resultant stress on horizontal plane:

$\therefore$ Resultant stress $= \sqrt{\sigma_2^2 + \tau^2}$

$= \sqrt{600^2 + 500^2} = \mathbf{781.02\ N/cm^2}$. **Ans.**

The direction of the resultant stress with the horizontal plane $BC$ is given by,

$$\tan\theta = \frac{\sigma_2}{\tau} = \frac{600}{500} = 1.2$$

$$\theta = \tan^{-1} 1.2 = \mathbf{50.19°}. \quad \textbf{Ans.}$$

- Principal stresses

$\therefore$ Principal stresses $= \dfrac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\left(\dfrac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$

$= \dfrac{866 + 600}{2} \pm \sqrt{\left(\dfrac{866 - 600}{2}\right)^2 + 500^2}$

$= 733 \pm 517.38$

$= (733 + 517.38)$ and $(733 - 517.38)$

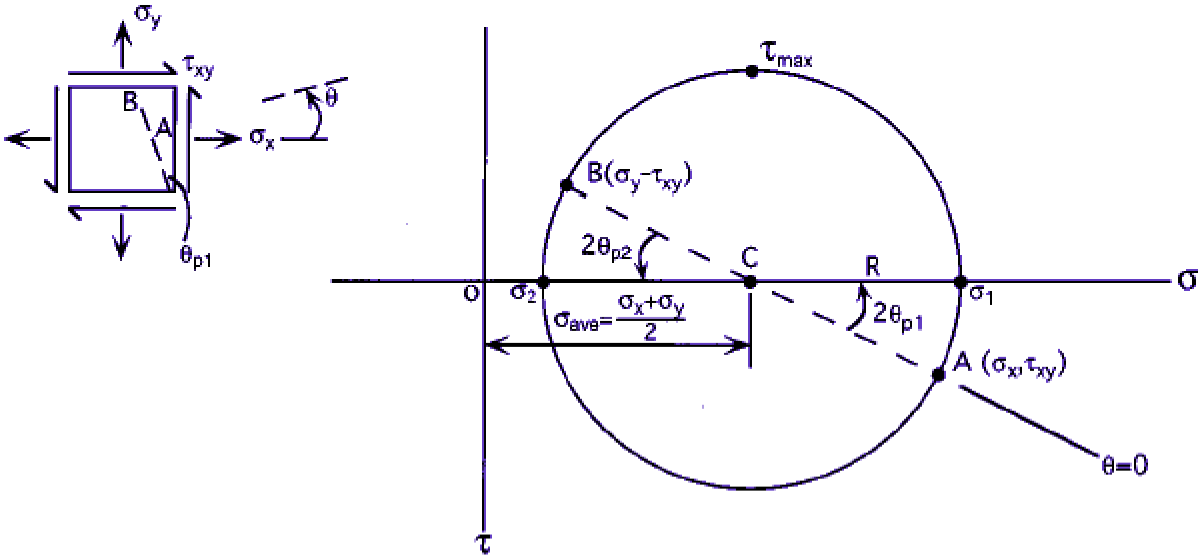
$= 1250.38$ and $215.62\ N/cm^2$.

$\therefore$ Major principal stress $= \mathbf{1250.38\ N/cm^2}$. **Ans.**

$\therefore$ Minor principal stress $= \mathbf{215.62\ N/cm^2}$. **Ans.**

# GRAPHICAL METHOD : MOHR'S CIRCLE

Mohr's circle is a graphical method of finding normal, tangential and resultant stresses on an oblique plan. Principal planes and Principal stresses will also be evaluated.



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- Sign Conventions:
  - Normal stress and shear stress are represented along x-y coordinate, Normal along abscissa and Shear along ordinate.
  - Tensile normal stress taken along +ve x direction and compressive normal stress along –ve x direction.
  - Positive shear stress taken along +ve y direction and negative shear stress along –ve y direction.
  - Principal stresses are normal stresses, so obtained along x axis.

Mohr's circle will be drawn for the following cases:

a) A body subjected to two mutually perpendicular principal tensile stresses of unequal intensities.
b) A body subjected to two mutually perpendicular principal stresses which are unequal and unlike (i.e One tensile and other compressive)
c) A body subjected to two mutually perpendicular principal tensile stresses accompanied by a simple shear stress.

Q. The tensile stresses at a point across two mutually perpendicular planes are 120 $N/mm^2$ and 60 $N/mm^2$. Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of minor stress.

Q. The tensile stresses at a point across two mutually perpendicular planes are 120 N/mm$^2$ and 60 N/mm$^2$. Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of minor stress.

Sol. $\sigma_1$ = 120 N/mm$^2$ (Tensile)  $\sigma_2$ = 60 N/mm$^2$ (Tensile)

$\Theta = 30°$

Step 1: Take a scale 1 cm = 10 N/mm$^2$.

Step 2:

$$\sigma_1 = \frac{120}{10} = 12 \text{ cm}, \qquad \sigma_2 = \frac{60}{10} = 6 \text{ cm}$$

Step 3: Take any point A and draw the horizontal line through A.

Step 4: Take AB = $\sigma_1$ = 12 cm and AC = $\sigma_2$ = 6 cm.

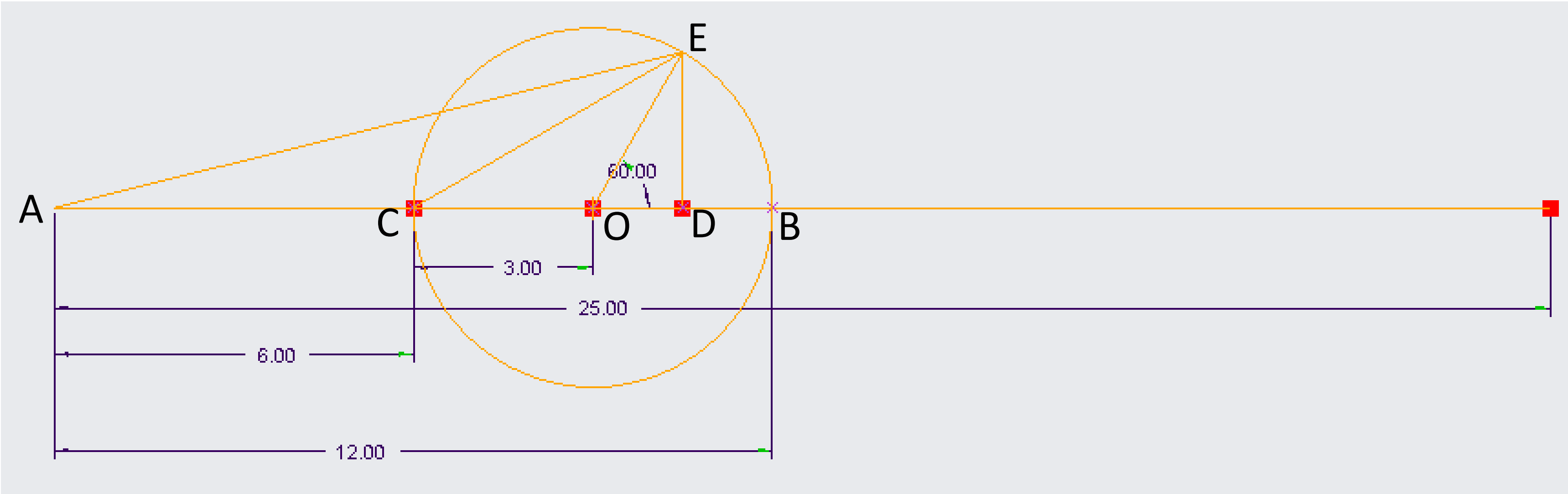
Step 5: Draw circle by taking BC as diameter. Let O is the Centre o the circle.

Step 6: Draw a line OE making an angle 2θ (i.e 2 x 30 = 60° with OB.

Step 7: Draw ED perpendicular to CB.

Step 8: Join AE

Step 9 : Measure lengths AD, ED and AE



By Measurement :

Length AD = 10.50 cm

Length ED = 2.60 cm

Length AE = 10.82 cm

Normal Stress σn = Length AD X Scale

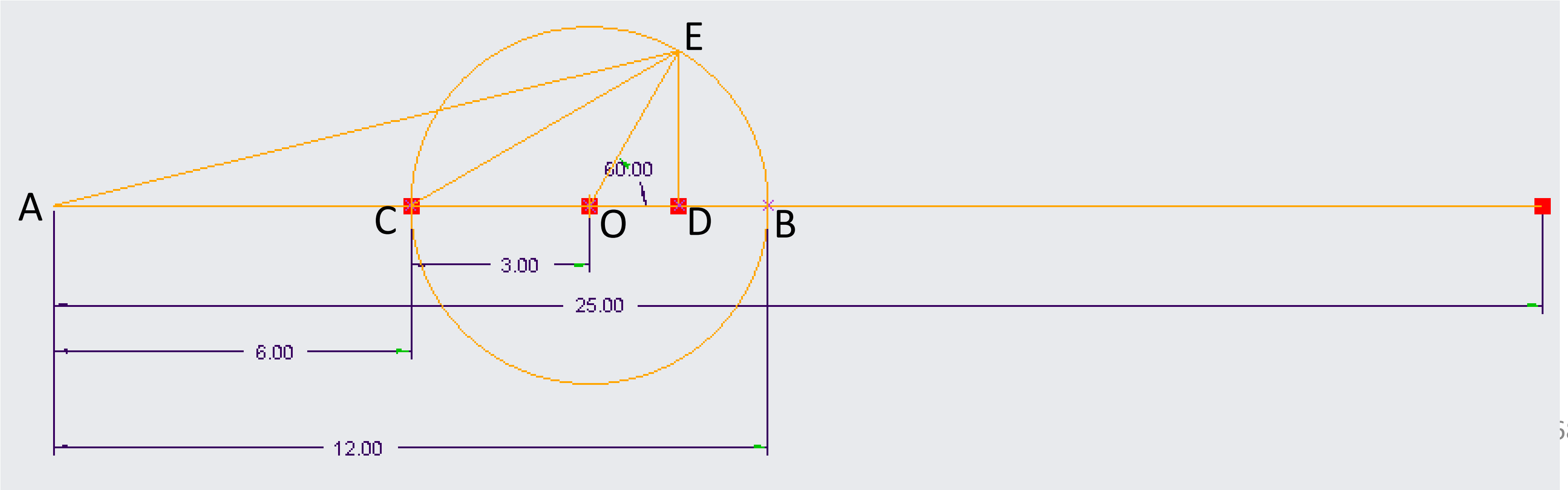
= 10.50 X 10 = 105 N/mm$^2$

Shear Stress σt = Length ED X Scale

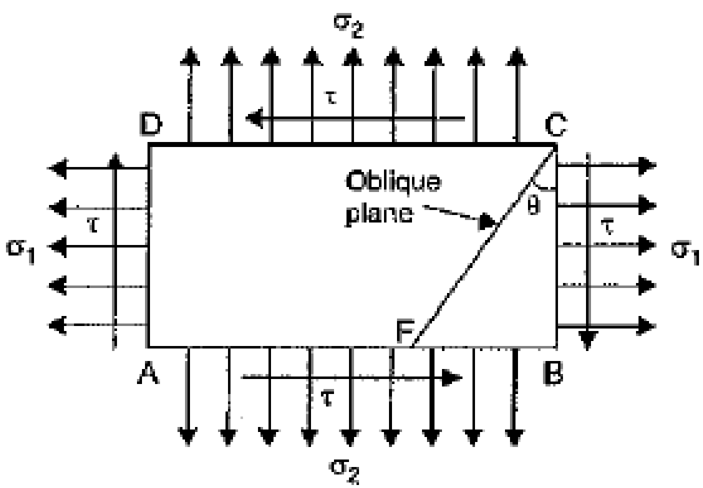
= 2.60 X 10 = 26 N/mm$^2$

Resultant Stress = Length AE X scale

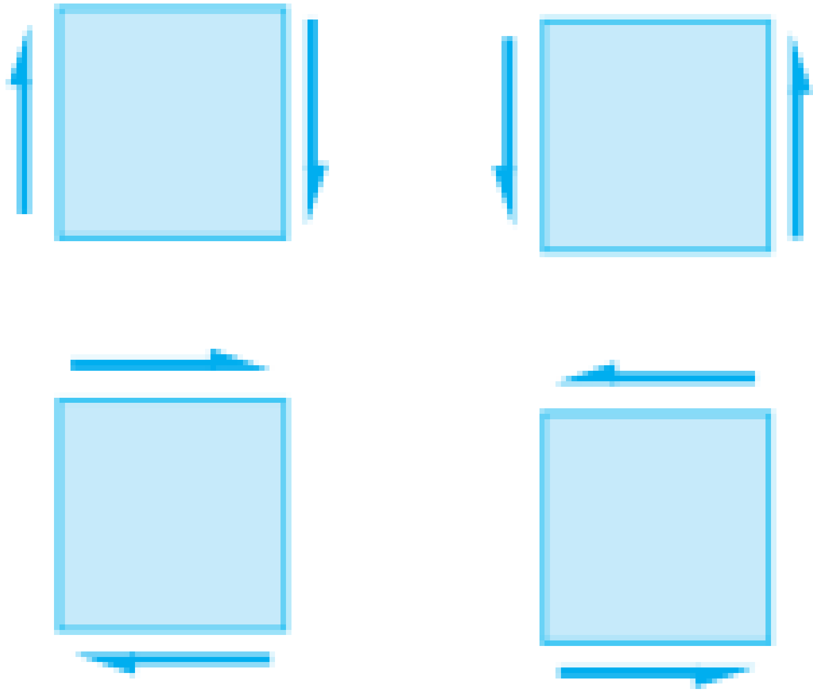
= 10.82 X 10 = 108.2 N/mm$^2$



- Mohr’s circle when a body is subjected to two mutually perpendicular principal tensile stresses accompanied by a simple shear stress



# Sign conventions.



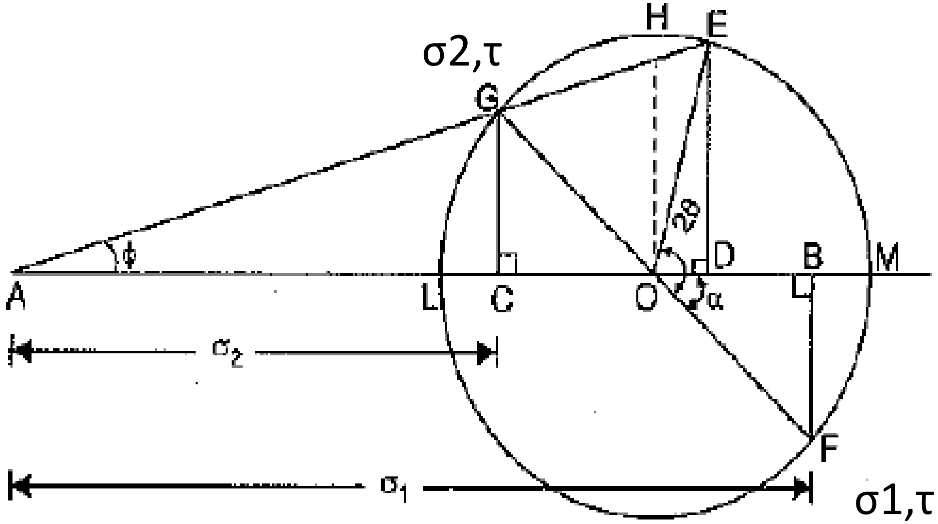
a) Shear Plotted Down Clockwise (-ve)

b) Shear Plotted UP Anticlockwise (+ve)

Take any point $A$ and draw a horizontal line through $A$.

Take $AB = \sigma_1$ and $AC = \sigma_2$ towards right of $A$ to some suitable scale. Draw perpendiculars at $B$ and $C$ and cut off $BF$ and $CG$ equal to shear stress $\tau$ to the same scale. Bisect $BC$ at $O$. Now with $O$ as centre and radius equal to $OG$ or $OF$ draw a circle. Through $O$, draw a line $OE$ making an angle of $2\theta$ with $OF$ as shown in Fig.

From $E$, draw $ED$ perpendicular to $CB$. Join $AE$. Then length $AE$ represents the resultant stress on the given oblique plane. And lengths $AD$ and $ED$ represents the normal stress and tangential stress respectively.




Hence from Fig. we have

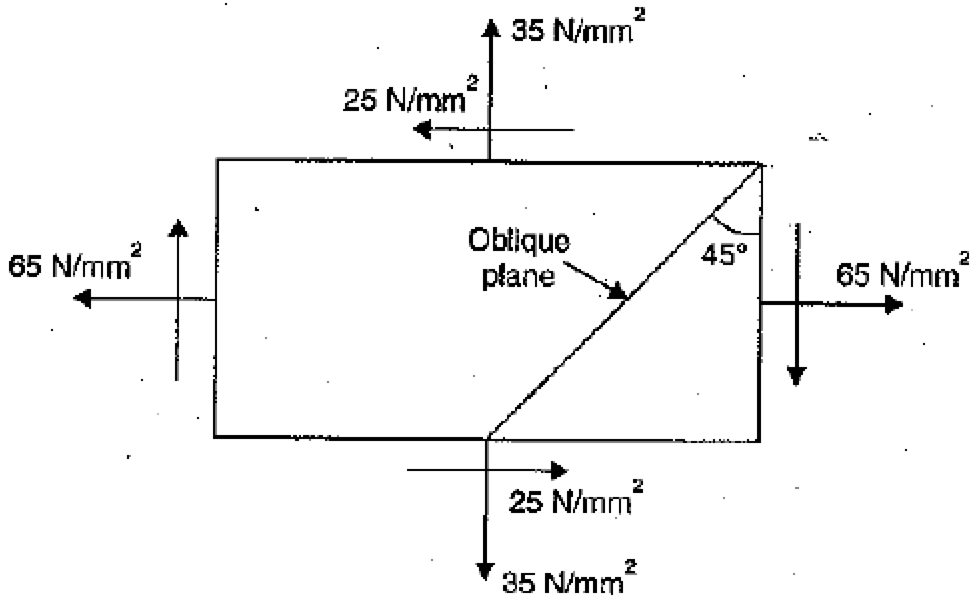
Length $AE$ = Resultant stress on the oblique plane

Length $AD$ = Normal stress on the oblique plane

Length $ED$ = Shear stress on the oblique plane.

- AM and AL are max and min principal stresses
- Angle EAM is obliquity
- Angle FOB ($\alpha$) and $\alpha$+180 are Principal plane angles

A point in a strained material is subjected to stresses shown in Fig. Using Mohr's circle method, determine the normal and tangential stresses across the oblique plane. Check the answer analytically.



Sol. Given : $\sigma_1 = 65$ N/mm$^2$, $\sigma_2 = 35$ N/mm$^2$

Shear Stress $\tau = 25$ N/mm$^2$.

Angle of oblique Plane, $\theta = 45°$

Let 1 cm = 10 N/mm$^2$

$\sigma_1 = \frac{65}{10} = 6.5$ cm $\quad \sigma_2 = \frac{35}{10} = 3.5$ cm , $\tau = \frac{25}{10} = 2.5$ cm

Step 1: Take any point A and draw a horizontal line through A. Take AB = $\sigma_1$ = 6.5 cm and AC = $\sigma_2$ = 3.5 cm towards right of A.

Step 2: Draw perpendicular at B and C cut off BF and CG equal to shear stress $\tau$ = 2.5 cm.

equal to shear stress $\tau$ = 2.5 cm. Bisect *BC* at *O*. Now with *O* as centre and radius equal to *OF* (or *OG*) draw a circle. Through *O*, draw a line *OE* making an angle of 2θ (*i.e.*, 2 × 45° = 90°) with *OF* as shown in Fig. 3.29. From *E*, draw *ED* perpendicular to *AB* produced. Join *AE*. Then length *AD* represents the normal stress and length *ED* represents the shear stress.

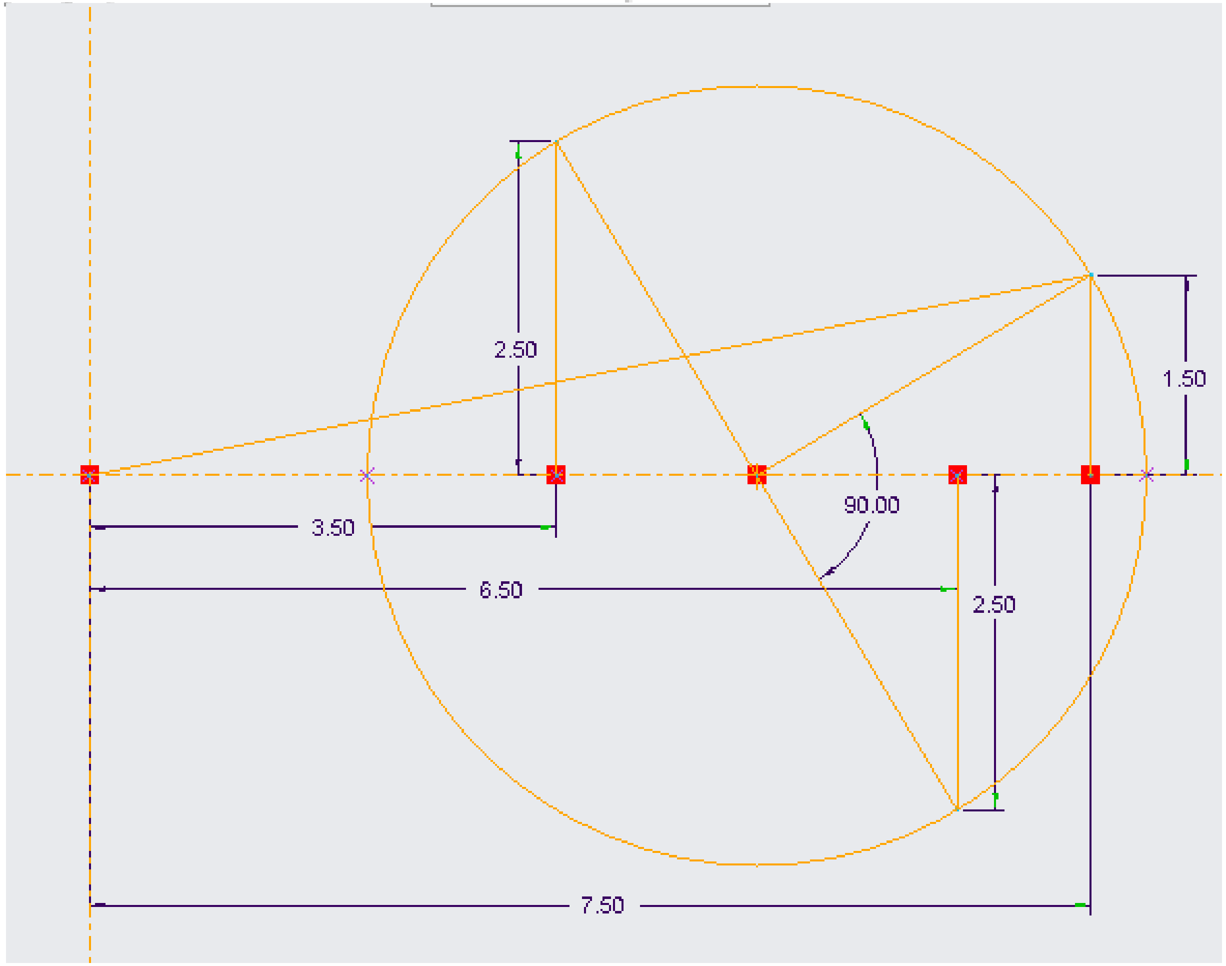
By measurements, length *AD* = 7.5 cm and

length *ED* = 1.5 cm.

∴ Normal stress ($\sigma_n$) = Length *AD* × Scale = 7.5 × 10 = **75 N/mm².** **Ans.**

(∵ 1 cm = 10 N/mm²)

And tangential stress ($\sigma_t$) = Length *ED* × Scale = 1.5 × 10 = **15 N/mm².** **Ans.**



Q. An elemental cube is subjected to tensile stresses of 30 $N/mm^2$ and 10 $N/mm^2$ acting on two mutually perpendicular planes and a shear stress of 10 $N/mm^2$ on these planes. Draw the Mohr's circle of stresses and hence or otherwise determine the magnitudes and directions of principal stresses and also the greatest shear stress.

Sol. $\sigma_1 = 30\ N/mm^2$, $\sigma_2 = 10\ N/mm^2$

Shear Stress $\tau = 10\ N/mm^2$.

Let 1 cm = 2 $N/mm^2$

$\sigma_1 = \frac{30}{2} = 15$ cm   $\sigma_2 = \frac{10}{2} = 5$ cm ,  $\tau = \frac{10}{2} = 5$ cm

By measurements, we have

$$\text{Length } AM = 17.1 \text{ cm}$$
$$\text{Length } AL = 2.93 \text{ cm}$$
$$\text{Length } OH = \text{Radius of Mohr's circle} = 7.05 \text{ cm}$$
$$\angle FOB \text{ (or } 2\theta) = 45°.$$

∴ Major principal stress

$= \text{Length } AM \times \text{Scale}$

$= 17.1 \times 2$ ($\because$ 1 cm = 2 N/mm²)

$=$ **34.2 N/mm².** **Ans.**

Minor principal stress $= \text{Length } AL \times \text{Scale}$

$= 2.93 \times 2$ ($\because$ 1 cm = 2 N/mm²)

$=$ **5.86 N/mm².** **Ans.**

$$\angle FOB \text{ or } 2\theta = 45°$$

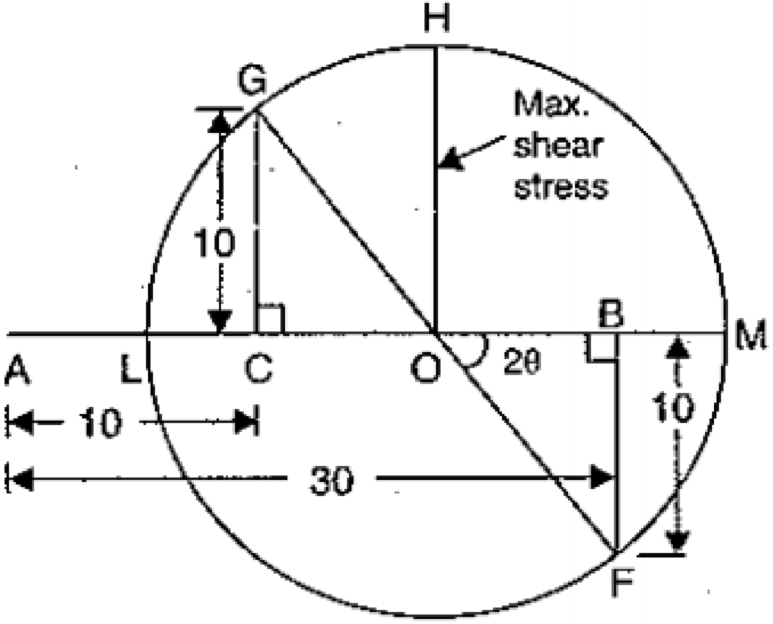
∴ $\theta = \frac{45}{2} =$ **22.5°.** **Ans.**

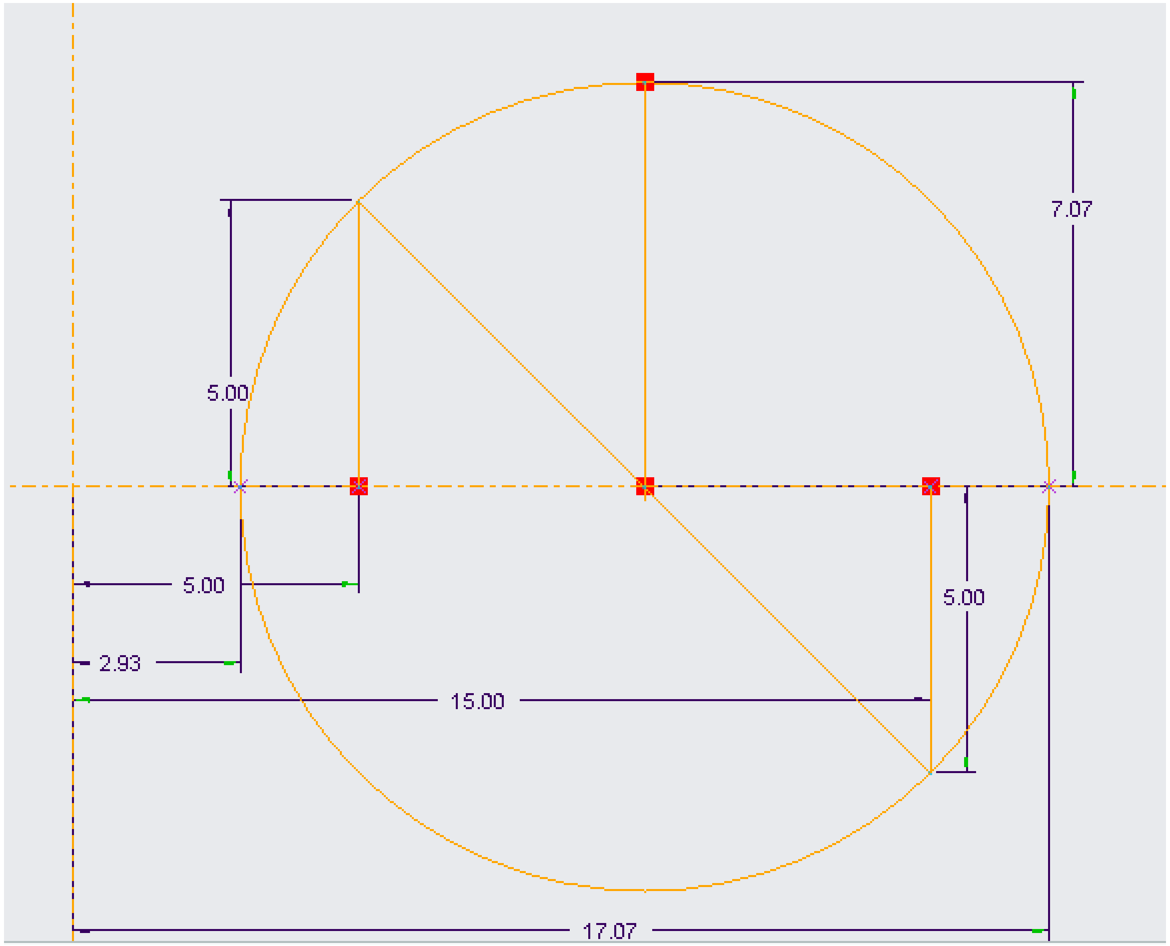
The second principal plane is given by $\theta + 90°$.

∴ Second principal plane $= 22.5 + 90 =$ **112.5°.** **A**

The greatest shear stress $= \text{Length } OH \times \text{Scale}$

$= 7.05 \times 20 =$ **14.1 N/mn**




7.07
5.00
5.00
5.00
2.93
15.00
17.07

# Thin Cylinders and Spheres

# THIN CYLINDER AND SPHEREICAL VESSEL

# Thin Cylinders

- If the thickness of walls of the cylindrical vessel is less than 1/15 to 1/20 of its internal diameter. Then vessel is a thin vessel.

- Examples of vessels
- Boilers
- Tanks
- Compressed Air Receivers.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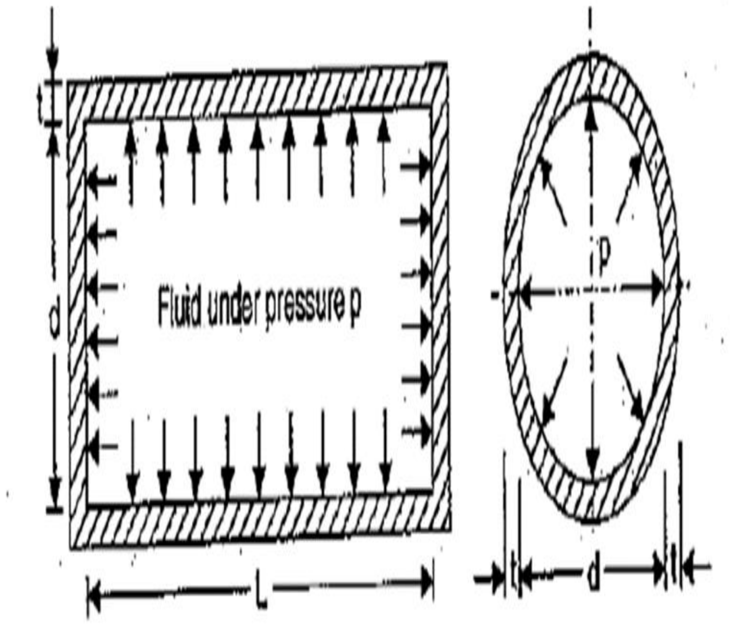
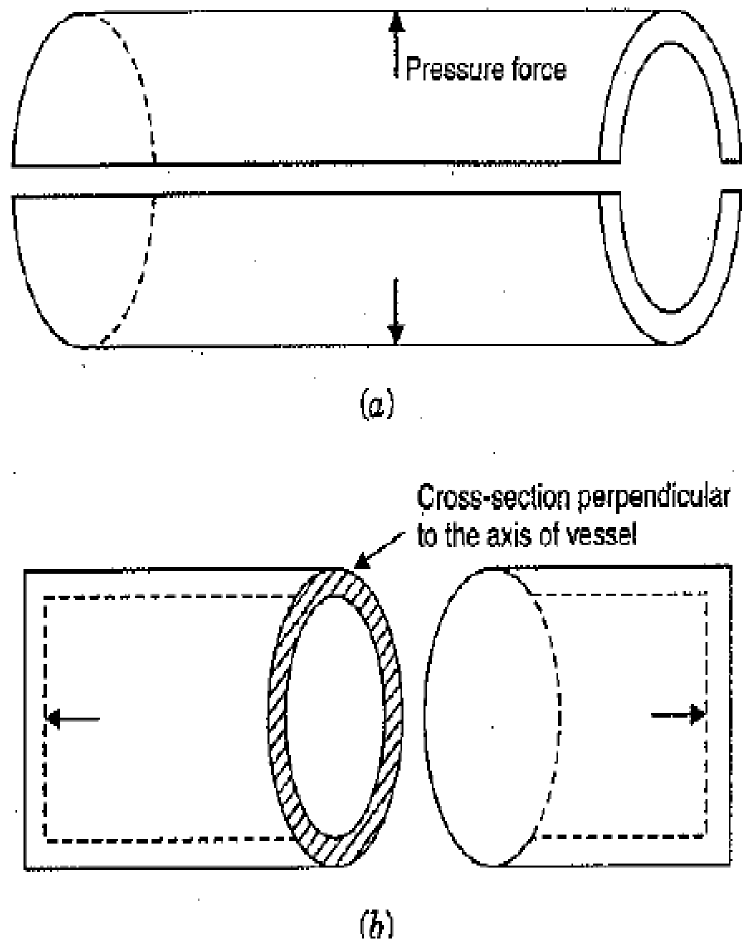
# Thin Vessels Subjected to Internal Pressure

Cylindrical Shells




- p= Internal Fluid Pressure
- L= Length of the cylinder
- D= Internal Diameter
- t= Thickness of the wall of cylinder.
- Due to this fluid pressure the cylinder tends to split up into two parts.

# Failure Modes of Cylinder



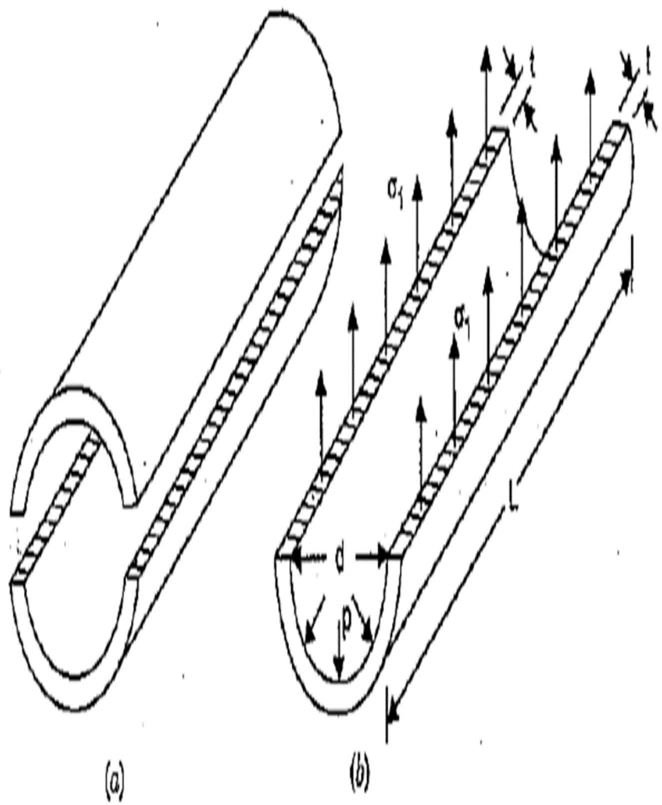

**Two Types of failure**

1. **Along the Circumference**
2. **Along the length.**

**Stresses devolved in thin cylinder**

1. **Circumferential or Hoop Stress**
2. **Longitudinal Stress**

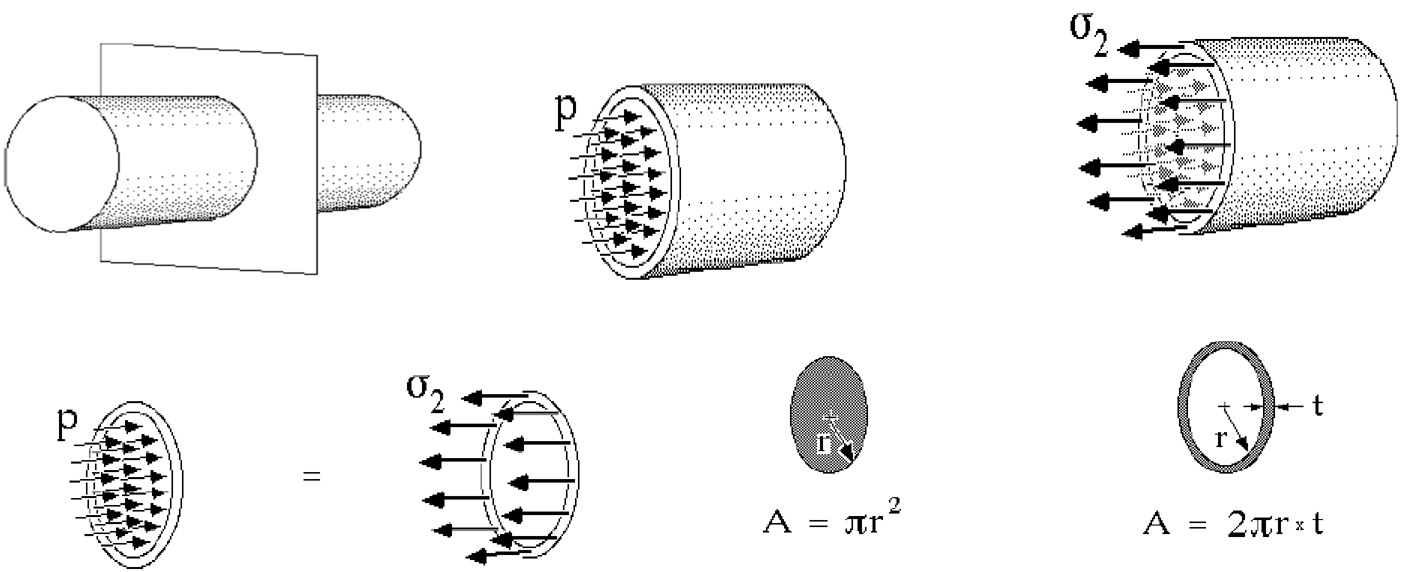
# Circumferential Stress or Hoop Stress




- Force due to fluid pressure= Þx area on which pressure is acting.
- $F_P$ = Þx dx L.
- Force due to circumferential stress= $\sigma_H$ x (Lxt+ Lxt)= 2x $\sigma_H$ xLxt.
- Therefore
- Þx dx L= 2x $\sigma_H$ x L x t
- Or $\sigma_H = \frac{Þxd}{2t}$

To avoid bursting, force due to fluid pressure must be equal to resisting force

# Longitudinal Stresses




Force due to fluid pressure = Force due to Longitudinal stress

$$\text{Þ} \times \pi \times r^2 = \sigma_L \times 2 \times \pi \times r \times t$$

To avoid bursting, force due to fluid pressure must be equal to resisting force

$$\sigma_L = \frac{\text{Þ}r}{2t}$$

or

$$\sigma_L = \frac{\text{Þ}d}{4t}$$

# Important Formulae

- Hoop Stress $\sigma = \frac{pd}{2t}$
- Longitudinal stress $\sigma = \frac{pd}{4t}$
- Therefore
- Longitudinal stress= **Half** of Circumferential stress.
- Maximum Shear Stress = $\frac{pd}{8t}$

- A cylindrical pipe of diameter 1.5m and thickness 1.5cm is subjected to an internal fluid pressure of 1.2N/mm2. Calculate a) Longitudinal stress developed in the pipe, b) Circumferential stress developed in the pipe.

## Longitudinal Strain or axial strain

$$\epsilon_L = \epsilon_2 = \frac{1}{E}(\sigma_L - \mu\sigma_H) = \frac{1}{E}\left[\frac{Pd}{4t} - \mu\frac{Pd}{2t}\right]$$

$$\frac{\Delta L}{L} = \varepsilon_L = \frac{Pd}{4tE}[1 - 2\mu]$$

## Hoop strain or Circumferential strain -

$$\varepsilon_1 = \varepsilon_H = \frac{1}{E}(\sigma_H - \mu\sigma_L) = \frac{1}{E}\left[\frac{Pd}{2t} - \mu\frac{Pd}{4t}\right]$$

$$\frac{\Delta d}{d} = \varepsilon_H = \frac{Pd}{4tE}[2 - \mu]$$

## Ratio of Hoop Strain to Longitudinal Strain

$$\frac{\epsilon_H}{\epsilon_L} = \frac{\text{circumeferentialstrain}}{\text{longitudinalstrain}} = \frac{\frac{Pd}{4tE}(2-\mu)}{\frac{Pd}{4tE}(1-2\mu)} = \frac{(2-\mu)}{(1-2\mu)}$$

Strains/Deformation in the cylinderical shell

Hoop/Circumferential strain $= \frac{pD}{4tE}(2-\nu)$

Longitudinal strain $= \frac{pD}{4tE}(1-2\nu)$

Change in diameter of cylinder

$$\delta D = \epsilon_c \times D$$

$$= \frac{pD}{4tE}(2-\nu) \cdot D$$

$$= \frac{pD^2}{4tE}(2-\nu)$$

change in length of cylinder

$$\delta L = \epsilon_L \times L$$

$$= \frac{pD}{4tE}(1-2\nu) \cdot L$$

$$= \frac{pDL}{4tE}(1-2\nu)$$

## Volumetric strain :

$$\text{Volume of cylinder } V = \frac{\pi}{4} D^2 . L \quad \text{— (1)}$$

Taking Partial derivatives

$$\delta V = \frac{\pi}{4} 2 D \delta D L + \frac{\pi}{4} D^2 . \delta L \quad \text{— (2)}$$

Divide (2) by (1)

$$\frac{\delta V}{V} = \frac{2 \delta D}{D} + \frac{\delta L}{L}$$

$$= 2 \epsilon_c + \epsilon_L \qquad \text{[Sum of 3 strains]}$$

$$= 2 \frac{pD}{4tE}(2-\nu) + \frac{pD}{4tE}(1-2\nu)$$

$$\left[ \text{Volumetric strain } \epsilon_v = \frac{\delta V}{V} = \frac{pD}{4tE}(5-4\nu) \right]$$

Note:

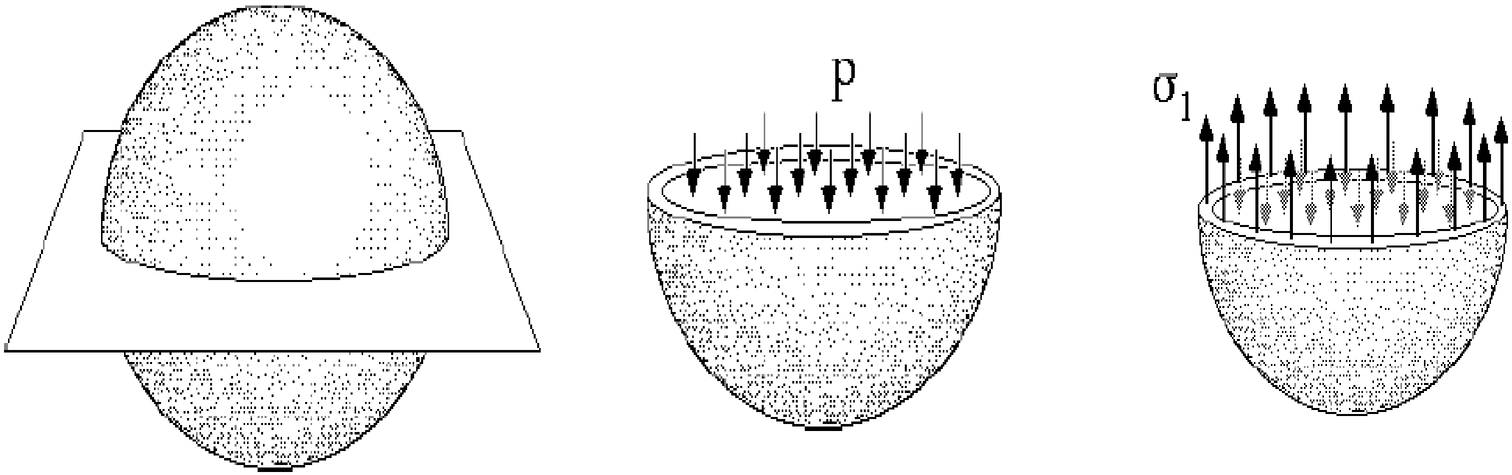
In case additional fluid is pumped inside cylinder, Then

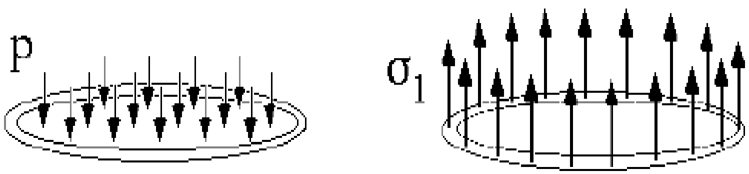
Total change in volume $\delta V$ = change in vol. of cylinder $(\delta V_1)$ + change in vol. of fluid $(\delta V_2)$

$$\delta V = \delta V_1 + \delta V_2$$
$$= \epsilon_v \times V + \epsilon_v' \times V$$
$$= (\epsilon_v + \epsilon_v') \times V$$

Here $\epsilon_v = \frac{pD}{4tE}(5-4\nu)$ and $\epsilon_v' = \frac{p}{k}$

## Spherical Shells




$A = \pi r^2$ $\qquad$ $A = 2\pi r \times t$

$$p(\pi r^2) = \sigma_1 (2\pi r t)$$

$$\boxed{\sigma_1 = \frac{pr}{2t}} \quad \text{OR} \quad \frac{pD}{4t}$$

- A thin cylindrical shell of thickness 5mm and diameter 350mm is subjected to an internal pressure which produces a strain of 1/2500 in diameter. Find this internal pressure and the consequent hoop and longitudinal stress. Take E=2x10^5 N/mm2 and v=0.3

Strain of diameter = Strain in circumference

$= \frac{1}{2500}$

$\frac{pD}{4tE}(2-\nu) = \frac{1}{2500}$

$\frac{p \times 350}{4 \times 5 \times 2 \times 10^5}(2-0.3) = \frac{1}{2500}$

$p = 2.69 \, N/mm^2$

Hoop stress, $\sigma_1 = \frac{pD}{2t} = 94.15 \, N/mm^2$

Longi stress, $\sigma_2 = \frac{pD}{4t} = 47.08 \, N/mm^2$

- A cylindrical thin shell 800mm in diameter and 3m long is having 10mm metal thickness. If the shell is subjected to an internal pressure of 2.5x10^6 N/m2, determine a) change in length, b) change in diameter, c) change in volume.

Take E = 200GN/m2 and v = 0.25

$D = 800\,mm \quad L = 3\,m$

$t = 10\,mm \quad p = 2.5 \times 10^6\,N/m^2$

$E = 200\,GN/m^2 \quad \nu = 0.25$

$\delta l = ? \quad \delta d = ? \quad \delta v = ?$

Change in length $\delta l = \frac{pDL}{4tE}(1-2\nu)$

$$= \frac{2.5 \times 800 \times 3000}{4 \times 10 \times 200 \times 10^3}(1 - 2 \times 0.25)$$

$$= 0.375\,mm$$

Change in diameter $\delta D = \frac{pD^2}{4tE}(2-\nu)$

$$= \frac{2.5\times 800\times 800}{4\times 10\times 200\times 10^3}(2-0.25)$$

$$\delta D = 0.35\ mm$$

Change in volume $\delta V = \epsilon_v \times V$

$$= \frac{pD}{4tE}(5-4\nu)\times\frac{\pi}{4}D^2 L$$

$$= \frac{2.5\times 800}{4\times 10\times 200\times 10^3}(5-0.25)\times\frac{\pi}{4}\times 800^2\times 3000$$

$$= 150000\ mm^3$$

- A thin cylindrical pressure vessel 2.5m diameter and 18mm thick is subjected to an internal pressure of 1.2N/mm2. In addition, the vessel is also subjected to an axial tensile load of 2800kN. Calculate the principal stresses, Normal and Shear stresses on a plane at an angle of 60° to the axis of vessel. Find also the maximum shear stress.

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$D = 2.5 m = 2500 mm$

$t = 18 mm \quad p = 1.2 N/mm^2$

$P = 2800 kN = 2800 \times 10^3 N$

$\theta = 60^\circ$

Hoop / Circum stress, $\sigma_1 = \frac{pD}{2t} = \frac{1.2 \times 2500}{2 \times 18}$

$= 83.33 N/mm^2$

Longitud stress, $\sigma_2 = \frac{pD}{4t} = \frac{1.2 \times 2500}{4 \times 18}$

$= 41.67 N/mm^2$

Longitud stress due to tensile load,

$\sigma_2' = \frac{P}{A} = \frac{2800 \times 10^3}{\pi D t}$

$= 19.81 N/mm^2$

2020.09.01

So, the Principal stresses are :

$$p_1 = \sigma_1 = 83.33\ N/mm^2$$

$$p_2 = \sigma_2 + \sigma_2' = 61.48\ N/mm^2$$

Now,

$$\sigma_n = \frac{p_1 + p_2}{2} + \frac{p_1 - p_2}{2} \cos 60$$

$$= 94.2\ N/mm^2$$

$$\sigma_t = \frac{p_1 - p_2}{2} \sin 2\theta = 9.46\ N/mm^2$$

$$\tau_{max} = \frac{p_1 - p_2}{2} = 10.92\ N/mm^2$$

- A cylindrical shell 90cm long and 20cm internal diameter having thickness of metal as 8mm is filled with fluid at atmospheric pressure. If an additional 20cm3 of fluid is pumped into the cylinder, Find a) the pressure exerted by the fluid on the cylinder and b) the hoop stress developed. Take E = 2x10^5 N/mm2 and v=0.3

Sol$^n$

$$V = \frac{\pi}{4} D^2 . L = \frac{\pi}{4} \times 20^2 \times 90 = 28274.33 \text{ cm}^3$$

Increase in volume = Additional fluid added
= 20 cm$^3$

a) Let $p$ = Pressure exerted by fluid on cylinder

Now, $\frac{\delta V}{V} = 2e_1 + e_2$

$$e_1 = \frac{pD}{4tE}(2-\nu) \qquad e_2 = \frac{pD}{4tE}(1-2\nu)$$

$$e_1 = \frac{pD}{4tE}(2-\nu) \qquad e_2 = \frac{pD}{4tE}(1-2\nu)$$

$$\therefore \frac{20}{28274.33} = \frac{pD}{4tE}(5-4\nu)$$

$$= \frac{p \times 200}{4 \times 8 \times 2 \times 10^5}(5 - 4 \times 0.3)$$

$$0.000707 = \frac{1.05\,p}{8000}$$

$$p = 5.386 \text{ N/mm}^2$$

b) Hoop stress $= \frac{pD}{2t} = 67.33 \text{ N/mm}^2$

- A thin cylindrical shell made of 5mm thick steel plate is filled with water under a pressure of 3N/mm2. The internal diameter of the cylinder is 200mm and its length is 1m. Calculate the additional volume of water pumped inside the cylinder to develop the required pressure. Given for steel, E=208kN/mm2, v=0.3 and for water, K=2200N/mm2.

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$K = 2200\ N/mm^2$

$P = 3\ N/mm^2$, $E = 208\ kN/mm^2 = 208 \times 10^3\ N/mm^2$

$D = 200\ mm$  $\nu = 0.3$  $t = 5\ mm$  $L = 1\ m = 1000\ mm$

~~volume~~ volume $= \frac{\pi}{4} D^2 L = 3.14 \times 10^7\ mm^3$

Additional volume pumped in the shell

$$\delta V = \delta V_1 + \delta V_2$$

$\delta V_1$ = change in volume of cylinder

$$= \epsilon_v \times V$$

$$= \frac{PD}{4tE}(5 - 4\nu) \times \frac{\pi}{4} D^2 L$$

$$\delta V_1 = \frac{3 \times 2000}{4 \times 5 \times 208 \times 10^3} (5 - 4 \times 0.3) \times 3.14 \times 10^7$$

$$= 1.72 \times 10^4 \ mm^3$$

$\delta V_2$ = change in volume of water

$$= \epsilon_v' \times V = \frac{p}{K} \times V$$

$$= \frac{3}{2200} \times 3.14 \times 10^7 = 4.284 \times 10^4 \ mm^3$$

$$\therefore \delta V = \delta V_1 + \delta V_2$$

$$= 1.72 \times 10^4 + 4.284 \times 10^4$$

$$= 6.004 \times 10^4 \ mm^3$$ or $60 \ cm^3$ of water

# MCQ QUESTIONS.

- In a thin cylinder, the stress which acts along the circumference of the cylinder is known as

a. Longitudinal b. hoop c. normal d. tangential.

- A cylindrical pipe of diameter 2m and thickness 2cm is subjected to an internal fluid pressure of 1.5N/mm$^2$.What is the longitudinal stress developed.

a. 70MPa b. 50MPa c. 60MPa. d. None of theses

- A thin spherical shell an inner diameter 400mm is subjected to an internal pressure of 2.5N/mm2. if the hoop stress is not to exceed 100MPa, What is the thickness of shell ?

a.2.5mm b. 5mm c. 10mm d. None of theses

- Circumferential stress is __ of the longitudinal stress.

a. one third b. half c. twice d. thrice.

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Shear stress on a principal plane is

a. Maximum

b. Minimum

c. Zero

d. A non-zero value

Principal planes are those on which normal stress is

a. Zero

b. Maximum

c. Minimum

d. Either maximum or minimum

Shear stress on a principal plane is

a. Maximum
b. Minimum
c. Zero
d. A non-zero value

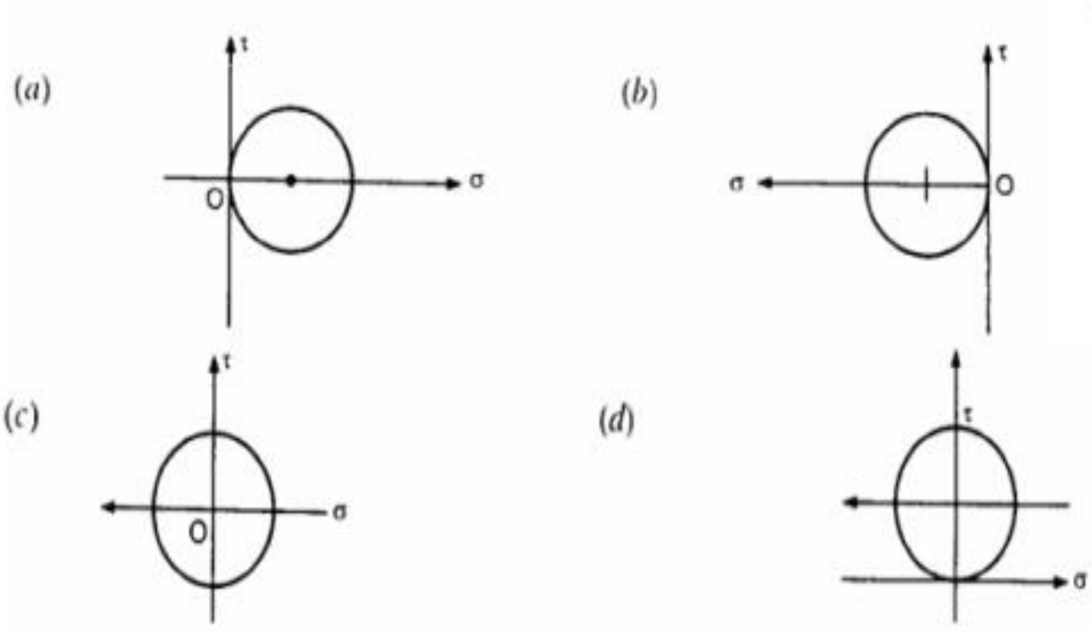
C

Principal planes are those on which normal stress is

a. Zero
b. Maximum
c. Minimum
d. Either maximum or minimum

D

Which one of the following Mohr's circles represents the state of pure shear?



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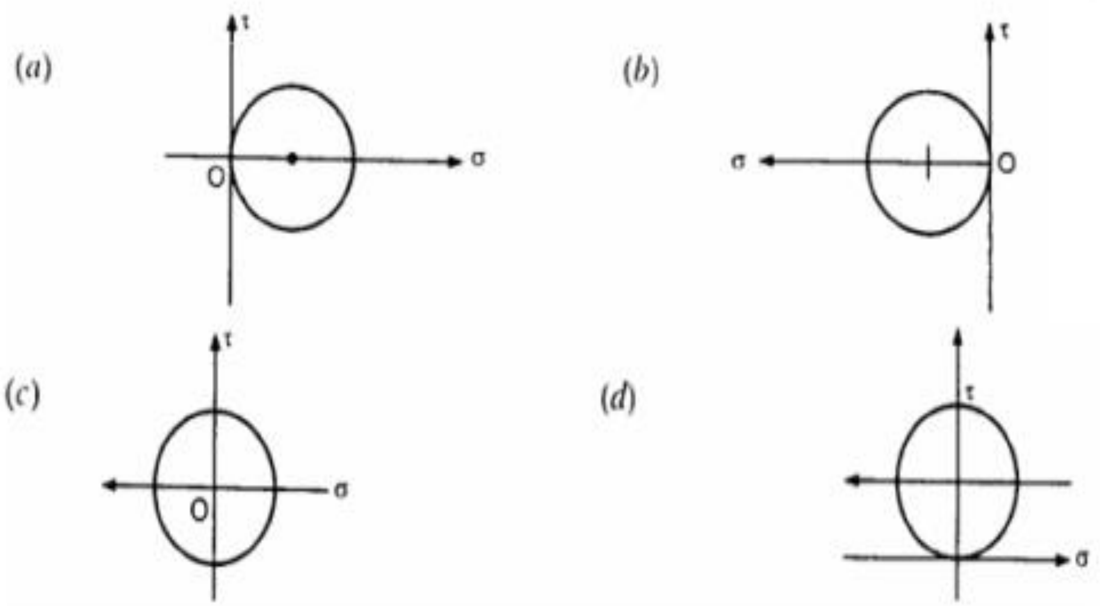
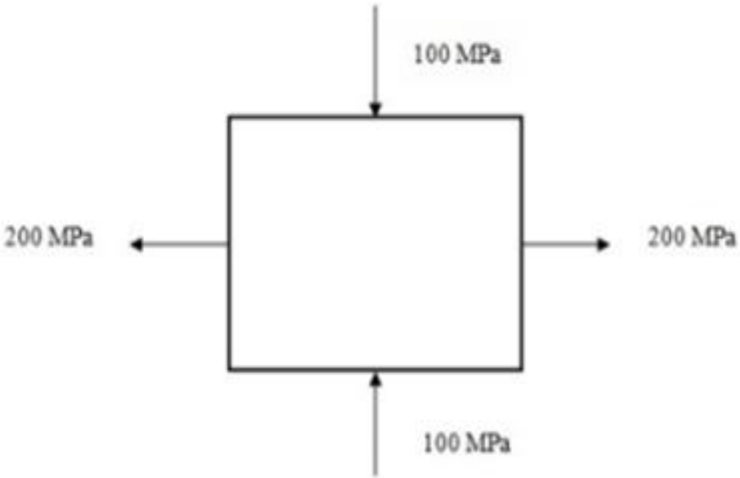



C

At a point two mutually perpendicular stresses are 120MPa and 60MPa and shear stress is 40MPa. The maximum shear stress developed is:

a) 60MPa
b) 50MPa
c) Zero
d) None of the above

Consider a two-dimensional state of stress given for an element as shown in the figure given below:




What are the coordinates of the centre of Mohr's Circle?

a. (0, 0)
b. (100, 200)
c. (200, 100)
d. (50, 0)

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**B**

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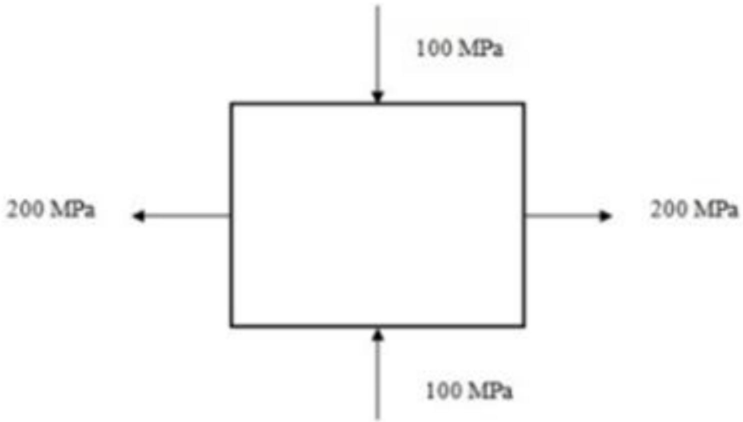


What are the coordinates of the centre of Mohr's Circle?

a. (0, 0)
b. (100, 200)
c. (200, 100)
d. (50, 0)

**D**

At a point in two-dimensional stress system $\sigma_x = 100$ N/mm$^2$, $\sigma_y = \tau_{xy} = 40$ N/mm$^2$. What is the radius of the Mohr circle for stress drawn with a scale of 1 cm = 10 N/mm$^2$?

a. 3 cm
b. 4 cm
c. 5 cm
d. 6 cm

A body is subjected to a pure tensile stress of 100 units. What is the maximum shear produced in the body at some oblique plane due to the above?

100 units b) 75 units c) 50 units d) 0 units

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**B**

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**C**
**(Max shear is at 45 deg)**

Major principal stress at a point is 220MPa and radius of Mohr's circle is 70MPa. Then Minor Principal stress is given by

a. 80Mpa b. 150 Mpa c. 20 Mpa d.200 Mpa.

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**A**

$$p_1 = \frac{\sigma_1+\sigma_2}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau^2} = 220 \quad \text{— (1)}$$

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2+4\tau^2} = 70 \quad \text{— (2)}$$

Put equ (2) in (1)

$$\frac{\sigma_1+\sigma_2}{2} + 70 = 220$$

$$\frac{\sigma_1+\sigma_2}{2} = 220-70 = 150$$

Now,

$$p_2 = \frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau^2}$$

$$= 150 - 70 = 80\ MPa$$

- Practice Question

*A cylindrical vessel is 1.5 m diameter and 4 m long is closed at ends by rigid plates. It is subjected to an internal pressure of 3 N/mm$^2$. If the maximum principal stress is not to exceed 150 N/mm$^2$, find the thickness of the shell. Assume $E = 2 \times 10^5$ N/mm$^2$ and Poisson's ratio = 0.25. Find the changes in diameter, length and volume of the shell.*

Dia., $d = 1.5 \text{ m} = 1500 \text{ mm}$

Length, $L = 4 \text{ m} = 4000 \text{ mm}$

Internal pressure, $p = 3 \text{ N/mm}^2$

Max. principal stress $= 150 \text{ N/mm}^2$

Max. principal stress means the circumferential stress

$\therefore$ Circumferential stress, $\sigma_1 = 150 \text{ N/mm}^2$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$.

Poisson's ratio, $\mu = 0.25$

Let $t$ = thickness of the shell,

$\delta d$ = change in diameter,

$\delta L$ = change in length, and

$\delta V$ = change in volume.

$$\sigma_1 = \frac{p \times d}{2t}$$

$$t = \frac{p \times d}{2 \times \sigma_1} = \frac{3 \times 1500}{2 \times 150}$$

$$= 15 \text{ mm.} \quad \textbf{Ans.}$$

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$$\delta d = \frac{pd^2}{2t \times E}\left(1 - \frac{1}{2} \times \mu\right)$$

$$= \frac{3 \times 1500^2}{2 \times 15 \times 2 \times 10^5}\left(1 - \frac{1}{2} \times 0.25\right) = \mathbf{0.984\ mm.}\quad \textbf{Ans.}$$

$$\delta L = \frac{p \times d \times L}{2t \times E}\left(\frac{1}{2} - \mu\right)$$

$$= \frac{3 \times 1500 \times 4000}{2 \times 15 \times 2 \times 10^5}\left(\frac{1}{2} - 0.25\right)$$

$$= \mathbf{0.75\ mm.}\quad \textbf{Ans.}$$

$$\frac{\delta V}{V} = \frac{p \times d}{2E \times t}\left(\frac{5}{2} - 2 \times \mu\right)$$

$$= \frac{3 \times 1500}{2 \times 2 \times 10^5 \times 15}\left(\frac{5}{2} - 2 \times 0.25\right) = \frac{3 \times 1500 \times 2}{4 \times 10^5 \times 15}$$

$$\delta V = \frac{3}{2000} \times V = \frac{3}{2000} \times \left(\frac{\pi}{4} \times d^2 \times L\right)$$

$$= \frac{3}{2000} \times \left(\frac{\pi}{4} \times 1500^2 \times 4000\right) = \mathbf{10602875\ mm^3.}\quad \textbf{Ans.}$$