Unit – III

Integer Programming Models :-

Introduction.

A linear programming problem in which some or all of the variables in the optimal solution are restricted to assume non-negative integer values is called an integer programming problem [or I.P.P or integer linear programming].

In a linear programming problem, if all the variables in the optimal solution are restricted to assume non-negative integer values, then it is called the pure (all) integer programming problem.

In a linear programming problem, if only some of the variables in the optimal solution are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a mixed integer programming problem [Mixed I.P.P]

The general integer programming problem is given by $\quad \text{Max } Z = CX$

Subject to the constraints

$$AX \leq b$$

$X \geq 0$ and some or all variables are integers.

Importance of Integer programming.

In linear programming problem, all the decision variables were allowed to take any non-negative real (continuous or fractional) values, as it is quite possible and appropriate to have fractional values in many situations. For example, it is quite possible to use 6.38 kg of raw material, or 5.62 machine hours etc. However in many situations, especially in business and industry, these decision variables make sense only if they have integer values in the optimal solution.

For example, it is meaningless to produce 8.13 chairs or 6.85 tables, or to open 3.83 branches of a bank or to run 9.6 cars etc. Hence a new procedure has been developed in this direction for the case of LPP subjected to the additional restriction that the decision variables must have integer values.

Methods of integer programming

Integer programming methods can be categorized as

i) Cutting methods

ii) Search methods.

Cutting methods:

A systematic procedure for solving pure integer programming problem was first developed by R. E. Gomory in 1958. Later on he extended the procedure to solve mixed I.P.P named as cutting plane algorithm. The method consists in first solving the L.P.P as ordinary LPP by ignoring the integrality restriction and then introducing additional constraints one after the other, to cut (eliminate) certain part of the solution space until an integral solution is obtained.

Search method:- It is an enumeration method in which all feasible integer points are enumerated. The widely used search method is the branch and bound technique.

Gomory's fractional cut algorithm (or) Cutting plane method for pure (all) I.P.P.

Step1: Convert the minimization I.P.P into an equivalent maximization I.P.P and all the co-efficients and constants should be integers. Ignore the integerality condition.

Step2: Find the optimum solution of the resulting maximization L.P.P by using simplex method.

Step3: Test the integrality of the optimum solution.

i) If all $x_{BL} \geq 0$ and are integers, an optimum integer solution is obtained.

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iii) If all $x_{Bi} \geq 0$ and atleast one x_{Bi} is not an integer, then go to the next step.

Step 4: Rewrite each x_{Bi} as $x_{Bi} = [x_{Bi}] + f_i$, where $[x_{Bi}]$ is the integral part of x_{Bi} and f_i is the positive fractional part of x_{Bi}, $0 \leq f_i \leq 1$.

Choose the largest fraction of x_{Bi}, s,i,c, choose $\max\{f_i\}$. In case of a tie, select arbitrarily.

Let $\max\{f_i\} = f_k$ corresponding to x_{Bk}.

Step 5: Express each of the negative fractions if any, in the k^{th} row of the optimum simplex table as the sum of a negative integer and a non-negative fraction.

Step 6: Find the fractional cut constraint

$$\sum_{j=1}^{n} f_{kj}\, x_j \geq f_k$$

(or) $$-\sum_{j=1}^{n} f_{kj}\, x_j \leq -f_k$$

(or) $$-\sum_{j=1}^{n} f_{kj} + s_1 = -f_k$$

where s_1 is the Gomorian slack.

Step 7 Add the fractional cut constraint obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step 8 Go to step 3 and repeat the procedure until an optimum integer solution is obtained.

Problem 1

Find the optimum integer solution to the following L.P.P

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t.c} \quad 3x_1 + 2x_2 \leq 5$$

$$x_2 \leq 2$$

and $x_1 \geq 0$, $x_2 \geq 0$ and are integers.

→

Soln $\text{Max } Z = x_1 + x_2$

s.t.c

$$3x_1 + 2x_2 + s_1 = 5$$

$$x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

where s_1, s_2 are slack variables.

Initial iteration:-

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	-1	-1	0	0	0	Ratio
s_1	0	3	2	1	0	5	5/3
s_2	0	0	1	0	1	2	-

First iteration:-

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	0	$-1/3$	$1/3$	0	$5/3$	Ratio
x_1	0	1	$2/3$	$1/3$	0	$5/3$	$5/2$
s_2	0	0	1	0	1	2	$2 \rightarrow$

Second iteration:-

Basic	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	$1/3$	$1/3$	$7/3$
x_1	0	1	0	$1/3$	$-2/3$	$1/3$
x_2	0	0	1	0	1	2

Since all z-row ≥ 0, the current basic feasible

Solution is optimal and non-integer

i.e., max $z = 7/3$, $x_1 = \frac{1}{3}$, $x_2 = 2$.

Since $x_1 = \frac{1}{3}$, from the source row (first row)

we have $\frac{1}{3} = x_1 + \frac{1}{3} s_1 - \frac{2}{3} s_2$

$$\frac{1}{3} = x_1 + \frac{1}{3} s_1 + \left(-1 + \frac{1}{3}\right) s_2$$

The fractional cut (Gomorian) constraint is given by

$$\frac{1}{3} s_1 + \frac{1}{3} s_2 \geq \frac{1}{3}$$

$$-\frac{1}{3} s_1 - \frac{1}{3} s_2 \leq -\frac{1}{3}$$

$$\Rightarrow -\frac{1}{3} s_1 - \frac{1}{3} s_2 + G_1 = -\frac{1}{3}$$

where G_1 is Gomorian slack.

Add this fractional cut constraint at the bottom of the above optimum simplex table, we have the new simplex table.

Basic	z	x_1	x_2	s_1	s_2	a_1	RHS
z	1	0	0	$1/3$	$\frac{1}{3}$	0	$7/3$
x_1	0	1	0	$1/3$	$-2/3$	0	$1/3$
x_2	0	0	1	0	1	0	2
← a_1	0	0	0	$-1/3$	$-1/3$	1	$-1/3$

Use the dual simplex method.

To find the entering variable.

$$= \max \left\{ \frac{\frac{1}{3}}{-1/3}, \frac{\frac{1}{3}}{-1/3} \right\}$$

$= \max\{-1, -1\} = -1$, which corresponds to both s_1 and s_2. We choose s_1 as the entering variable arbitrarily.

Drop a_1 and introduce s_1

Basic	z	x_1	x_2	s_1	s_2	a_1	RHS
z	1	0	0	0	0	1	2
x_1	0	1	0	0	-1	1	0
x_2	0	0	1	0	1	1	2
s_1	0	0	0	1	1	-3	1

$\therefore$ z-row ≥ 0, RHS ≥ 0, the current solution is feasible and optimum and integer.

$\therefore$ The optimum integer solution is

$\max z = 2,$

$x_1 = 0$

$x_2 = 2$

2, using Gomory's cutting plane method.

$\max z = 2x_1 + 2x_2$

s.t.c $5x_1 + 3x_2 \le 8$

$2x_1 + 4x_2 \le 8$

and $x_1, x_2 \ge 0$ and are all integers.

Soln:

Standard form:-

$\max z = 2x_1 + 2x_2$

s.t.c $5x_1 + 3x_2 + s_1 = 8$

$2x_1 + 4x_2 + s_2 = 8$

and $x_1, x_2, s_1, s_2 \ge 0$.

Initial iteration:-

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	-2	-2	0	0	0	Ratio
s_1	0	5	3	1	0	8	8/5
s_2	0	2	4	0	1	8	8/2

1st Iteration:

Basic	z	x_1	x_2	s_1	s_2	RHS	Ratio
z	1	0	$-4/5$	$2/5$	0	$16/5$	
x_1	0	1	$3/5$	$1/5$	0	$8/5$	$8/3$
s_2	0	0	$14/5$	$-2/5$	1	$24/5$	$12/7$

2nd iteration:

Basic	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	$2/7$	$2/7$	$32/7$
x_1	0	1	0	$2/7$	$-3/4$	$4/7$
x_2	0	0	1	$-1/7$	$5/14$	$12/7$

Since z-row ≥ 0.

$$x_1 = 4/7 = 0 + 4/7 = [x_{B1}] + f_1$$

$$x_2 = 12/7 = 1 + 5/7 = [x_{B2}] + f_2$$

$$\text{Max}\{f_1, f_2\} = \max\{4/7, 5/7\} = 5/7$$

$$\frac{12}{7} = x_2 - \frac{1}{7}s_1 + \frac{5}{14}s_2$$

$\therefore$ The fractional cut [Gomorian] constraint

Is given by

$$\frac{6}{7} s_1 + \frac{5}{14} s_2 \geq 5/7$$

$$-6/7\, s_1 - \frac{5}{14} s_2 \leq -5/7$$

$$\Rightarrow -6/7\, s_1 - \frac{5}{14} s_2 + G_1 = -5/7$$

where G_1 is the Gomorian slack.

Basic	z	x_1	x_2	s_1	s_2	G_1	RHS
z	1	0	0	2/7	2/7	0	32/7
x_1	0	1	0	2/7	-3/4	0	4/7
x_2	0	0	1	-1/7	5/14	0	12/7
G_1	0	0	0	-6/7	-5/14	1	-5/7

Use dual simplex method

$$= \max \left\{ \frac{2/7}{-6/7}, \frac{\frac{2}{7}}{-\frac{5}{14}} \right\}$$

$$= \max \left\{ -\frac{1}{3}, -4/5 \right\} = -1/3$$ which

Corresponds to s_1.

3rd iteration:-

Basic	Z	x_1	x_2	S_1	S_2	a_1	RHS
Z	1	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	$13/3$
x_1	0	1	0	0	$-1/3$	$\frac{1}{3}$	$\frac{1}{3}$
x_2	0	0	1	0	$5/12$	$-1/6$	$11/6$
S_1	0	0	0	1	$5/12$	$-7/6$	$5/6$

Z - row ≥ 0, RHS ≥ 0

∴ The obtain the optimum integer solution, we have to construct a fractional cut constraint.

$\max Z = 2, \; x_1 = 0, \; x_2 = 2.$

Now $x_1 = \frac{1}{3} = 0 + \frac{1}{3} = [XB_1] + f_1$

$x_2 = 11/6 = 1 + 5/6 = [XB_2] + f_2$

$x_3 = 5/6 = 0 + 5/6 = [XB_3] + f_3$

$\therefore \max \{ f_1, f_2, f_3 \}$

$= \max \{ \frac{1}{3}, 5/6, 5/6 \} = 5/6$ which corresponds to both 2nd and 3rd rows. we select the 2nd row arbitrarily as the source row.

$$\frac{11}{6} = x_2 + \frac{5}{12} s_2 + \frac{-1}{6} G_1$$

$$1 + \frac{5}{6} = x_2 + \frac{5}{12} s_2 + \left(-1 + \frac{5}{6}\right) G_1$$

$\therefore$ The fractional cut (Gomorian) construct is given by

$$\frac{5}{12} s_2 + \frac{5}{6} G_1 \geq 5/6$$

$$-5/12\, s_2 - 5/6\, G_1 \leq -5/6$$

$$-5/12\, s_2 - 5/6\, G_1 + G_2 = -5/6$$

Basic	z	x_1	x_2	s_1	s_2	G_1	G_2	RHS
z	1	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	0	$13/3$
x_1	0	1	0	0	$-1/3$	$+1/3$	0	$\frac{1}{3}$
x_2	0	0	1	0	$5/12$	$-1/6$	0	$11/6$
s_1	0	0	0	1	$5/12$	$-7/6$	0	$5/6$
$\leftarrow G_2$	0	0	0	0	$-5/12$	$-5/6$	1	$-5/6$?

$G_2 = -5/6$

$$= \max \left\{ \frac{\frac{1}{6}}{\frac{-5}{12}}, \frac{\frac{1}{3}}{-5/6} \right\}$$

$$= \max \left\{ -2/5, -2/5 \right\} = -2/5$$

we select S_2 arbitrarily as the entering variable.

4th Iteration:-

Basic	Z	x_1	x_2	S_1	S_2	u_1	u_2	RHS
Z	1	0	0	0	0	0	2/5	4
x_1	0	1	0	0	0	1	-4/5	1
x_2	0	0	1	0	0	-1	1	1
S_1	0	0	0	1	0	-2	1	0
S_2	0	0	0	0	1	2	-12/5	2

All Z-row ≥ 0 and RHS ≥ 0, the current solution is feasible integer optimal.

$\therefore$ The optimal solution to the new problem is $\max Z = 4$

$x_1 = 1$

$x_2 = 1$

Gomory's mixed integer method:-

In mixed integer programming problem only some of the variables are integer constrained, while the other variables may take integer or other real values. Like the pure integer problem, the mixed integer problem should be of the maximization type and all the co-efficients and constants should be integers.

$$\sum_{j=1}^{n} f_{kj} x_j \geq f_k$$

$$\sum_{j=1}^{n} f_{kj} x_j \geq f_k$$

ie, $$\sum_{j=1} f_{kj} x_j + \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj} x_j \geq f_k$$

ie, $$-\sum_{j\in J^+} f_{kj} x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj} x_j \leq -f_k$$

ie $$-\sum_{j\in J^+} f_{kj} x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j\in J^-} f_{kj} x_j + S_k = -f_k$$

where S_k : Gomorian slack

$$J^+ = \{ j \mid f_{kj} \geq 0 \}$$

$$J^- = \{ j \mid f_{kj} < 0 \}$$

(i) Solve the following mixed integer programming problem:

$$\max z = x_1 + x_2$$

S.T.C $2x_1 + 5x_2 \le 16$

$6x_1 + 5x_2 \le 30.$

$x_2 \ge 0,\ x_1$, non-negative integer.

<u>Soln</u>

Standard form:.

$$\max z = x_1 + x_2$$

S.T.C $2x_1 + 5x_2 + s_1 = 16$

$6x_1 + 5x_2 + s_2 = 30$

$s_1, s_2 \ge 0,\ x_1, x_2 \ge 0$

<u>Initial Iteration:</u>

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	-1	-1	0	0	0	Ratio
s_1	0	2	5	1	0	16	8
s_2	0	(6)	5	0	1	30	5

<u>1st Iteration</u>

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	0	$-1/6$	0	$\frac{1}{6}$	5	Ratio
s_1	0	0	$(10/3)$	1	$-1/3$	6	$9/5$
x_1	0	1	$5/6$	0	$\frac{1}{6}$	5	6

2nd Iteration:

Basic	Z	x_1	x_2	s_1	s_2	RHS
Z	1	0	0	$1/20$	$3/20$	$53/10$
x_2	0	0	1	$3/10$	$-1/10$	$18/10$
x_1	0	1	0	$-1/4$	$1/4$	$7/2$

Since all Z-row ≥ 0, the current basic feasible solution is optimal.

$$\frac{7}{2} = x_1 + 0x_2 - \frac{1}{4}s_1 + \frac{1}{4}s_2$$

$$3+\frac{1}{2} = x_1 + 0x_2 - \frac{1}{4}s_1 + \frac{1}{4}s_2$$

The Gomorian constraint is given by

$$\left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right)\left(-\frac{1}{4}\right)s_1 + \frac{1}{4}s_2 \geq \frac{1}{2}$$

$$\frac{1}{4}s_1 + \frac{1}{4}s_2 \geq 1/2$$

$$-1/4\, s_1 - 1/4\, s_2 \leq -1/2$$

$$-1/4\, s_1 - \frac{1}{4}s_2 + G_1 = -1/2$$

where G_1 is Gomorian slack.

Basic	z	x_1	x_2	s_1	s_2	G_1	RHS
z	1	0	0	$\frac{1}{20}$	$3/20$	0	$53/10$
x_2	0	0	1	$3/10$	$-1/10$	0	$9/5$
x_1 ←	0	1	0	$-1/4$	$1/4$	0	$7/2$
G_1	0	0	0	$-1/4$	$-1/4$	1	$-1/2$

$G_1 = -1/2$, G_1 is leave variable

$$\max\left\{\frac{\frac{1}{20}}{-1/4}, \frac{\frac{3}{20}}{-1/4}\right\}$$

$$= \max\left\{\frac{-4}{20}, \frac{-12}{20}\right\} = \max\left\{-1/5, -3/5\right\}$$

$= -1/5$ which corresponds to the variables S_1 So S_1 entering variables.

3rd Iteration:

Basic	z	x_1	x_2	s_1	s_2	G_1	RHS
z	1	0	0	0	$1/10$	$1/5$	$26/5$
x_2	0	0	1	0	$-2/5$	$6/5$	$6/5$
x_1	0	1	0	0	$1/2$	-1	4
s_1	0	0	0	1	1	-4	2

Since all z-row ≥ 0, the current solution is feasible and optimal.

$\max z = \frac{26}{5}$

$x_1 = 4$.

$x_2 = 6/5$

H.W.

2. Solve the following mixed integer programming problem by Gomory's cutting plane algorithm.

$\max z = x_1 + x_2$

s.t.c $3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

and $x_1, x_2 \geq 0$ and x_1 an integer.

Ans:- $\max z = 2$, $x_1 = 0$, $x_2 = 2$.

③ Solve the following mixed integer programming problem.

$\min z = x_1 - 3x_2$

s.t.c $x_1 + x_2 \leq 5$

$-2x_1 + 4x_2 \leq 11$

$x_1, x_2 \geq 0$, & x_2 is an integer.

$\min z = -\frac{17}{2}$, $x_1 = 1/2$, $x_2 = 3$.

Branch and Bound method:

This method is applicable to both pure (all) as well as mixed integer programming problems and involves the continuous version of the problem.

$$\max Z = cx$$

$$\text{S.T } Ax \leq b$$

$$x \geq 0 \text{ and integers.}$$

In this method also the given problem is first solved as a continuous LPP by ignoring the integrality condition. If in the optimal solution some one of the variables say x_r is not an integer, then

$x_r^* < x_r < x_r^* + 1$ where x_r^* and $x_r^* + 1$

are consecutive non-negative integers.

Hence any feasible integer value of x_r must satisfy one of the two conditions.

$$x_r \leq x_r^* \quad \text{or} \quad x_r \geq x_r^* + 1$$

By adding these two conditions separately to the continuous LPP. we form two different

Sub — problems.

Sub problem 1	Sub problem 2
$\max Z = Cx$	$\max Z = Cx$
s.t. $Ax \leq b$	s.t. $Ax \leq b$
$x_r \leq x_r^*$	$x_r \geq x_r^* + 1$
and $x \geq 0$	and $x \geq 0$.

Thus we have branched or partitioned the original problem in to two sub-problems. Geometrically it means that the branching process eliminates that portion of the feasible region that contains no feasible - Integer solution. Each of these sub-problems is then solved separately as a LPP.

2. Use Branch and Bound method to solve the following.

$$\max z = 2x_1 + 2x_2$$

s.t.c $\quad 5x_1 + 3x_2 \le 8$

$\quad x_1 + 2x_2 \le 4$

and $x_1, x_2 \ge 0$ and integer.

Soln

$$\max z = 2x_1 + 2x_2$$

s.t.c $\quad 5x_1 + 3x_2 + s_1 = 8$

$\quad x_1 + 2x_2 + s_2 = 4$

$\quad x_1, x_2, s_1, s_2 \ge 0$

Initial iteration:

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	−2	−2	0	0	0	Ratio
s_1	0	5	3	1	0	8	$8/5$
s_2	0	1	2	0	1	4	4

1st iteration:

Basic	z	x_1	x_2	s_1	s_2	RHS	
z	1	0	$-4/5$	$2/5$	0	$16/5$	Ratio
x_1	0	1	$3/5$	$1/5$	0	$8/5$	$8/3$
s_2	0	0	$\boxed{7/5}$	$-1/5$	1	$12/5$	$12/7$

Second iteration:

Basic	z	x_1	x_2	s_1	s_2	RHS	~~Ratio~~
z	1	0	0	$2/7$	$4/7$	$32/7$	
x_1	0	1	0	$2/7$	$-3/7$	$4/7$	
x_2	0	0	1	$-1/7$	$5/7$	$12/7$	

Since all z- row ≥ 0, the current basic feasible solution is optimal but non-integer.

$$\max z = 32/7$$

$$x_1 = 4/7$$

$$x_2 = 12/7$$

We have to branch this problem into two sub-problem.

Now from $x_2 = 12/7$

$$1 < x_2 < 2.$$

$$x_2 \leq 1 \text{ or } x_2 \geq 2$$

Applying these two conditions separately in the continuous LPP, we have two sub-problems.

Sub-problem (1)

$$\max z = 2x_1 + 2x_2$$

s.t.c $5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_2 \le 1$

$x_1, x_2 \ge 0$

Its optimal solution is $\max z = 4$, $x_1 = 1$, $x_2 = 1$.

So this sub-problem is fathomed. The lower bound of the objective function is 4.

Sub-problem (2)

$$\max z = 2x_1 + 2x_2$$

s.t.c $5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_2 \ge 2$

and $x_1, x_2 \ge 0$

Its optimal solution is $\max Z = 4$, $x_1 = 0$, $x_2 = 2$.

So this sub-problem is also fathomed.

Hence from both the sub-problems (1) and (2) the integer optimum solution is given by

$$\max z = 4$$

with $x_1 = 1$, $x_2 = 1$, or $x_1 = 0$, $x_2 = 2$

Original problem

$\max z = 2x_1 + 2x_2$

s.t.c $\quad 5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

$\max z = 32/7$, $x_1 = 4/7$, $x_2 = 12/7$.

$x_2 \leq 1$ $\qquad\qquad$ $x_2 \geq 2$

Sub problem (1)

$\max z = 4$

$x_1 = 1 \quad x_2 = 1$

Fathomed

Sub problem (2)

$\max z = 4$.

$x_1 = 0 \quad x_2 = 2$.

Fathomed

Hence the integer optimum solution is

$$\max z = 4$$

with $x_1 = 1$, $x_2 = 1$ (or) $x_1 = 0$, $x_2 = 2$.

(2) Use Branch and Bound technique to solve the following.

$$\max z = x_1 + 4x_2$$

s.t.c $2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ and integers.

Solution:

Ignoring the integrality condition, the continuous LPP becomes

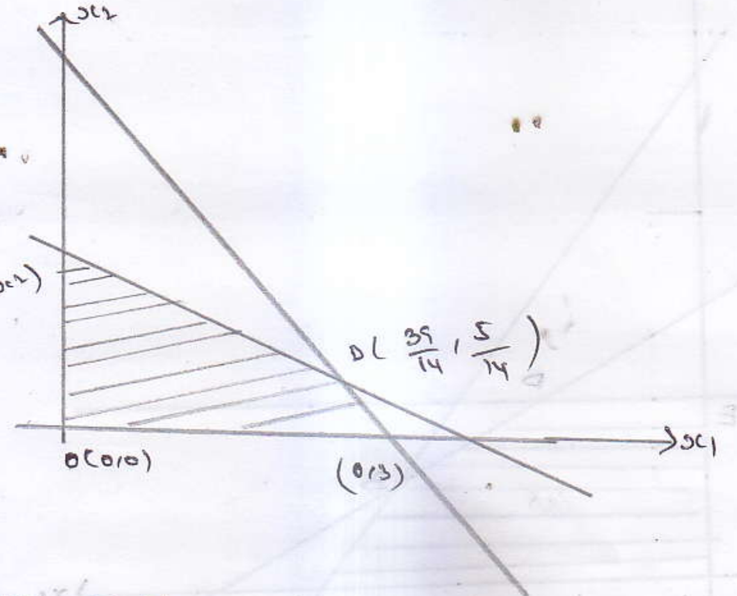
$$\max z = x_1 + 4x_2$$

s.t.c $2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

and $x_1, x_2 \ge 0$

By using graphical method, the solution space is given by the region OABC.



$\max z = 7$, $x_1 = 0$, $x_2 = 7/4$

Since $x_2 = 7/4$, this problem should be branched in to two problems.

For $x_2 = 7/4 \Rightarrow 1 < x_2 < 2$

$\Rightarrow x_2 \leq 1$ or $x_2 \geq 2$

Applying these two conditions separately in the continuous LPP we have two sub problems.

Sub-problem (1)

$\max z = x_1 + 4x_2$

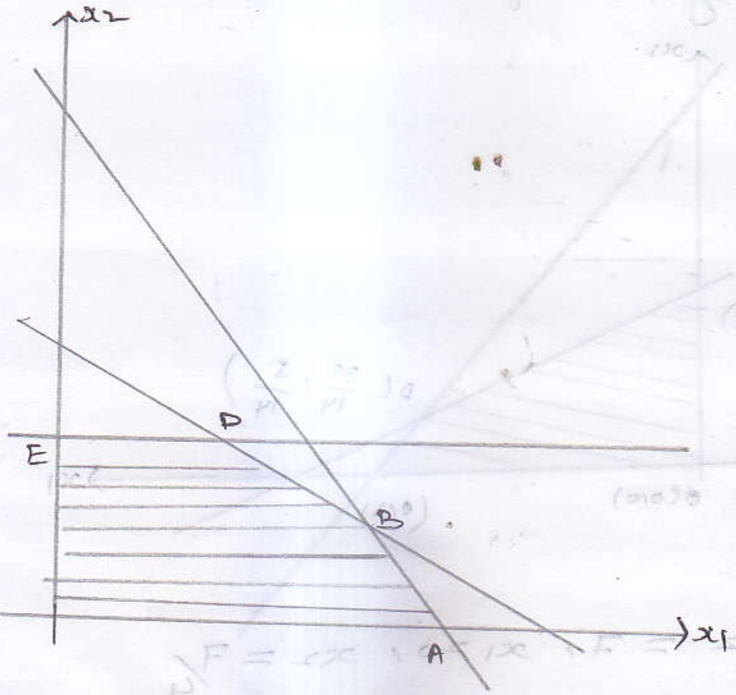
$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

& $x_1, x_2 \geq 0$

its solution space is given by the region OABDE



and its optimal solution is

$\max z = 11/2$, $x_1 = 3/2$, $x_2 = 1$

Since $x_1 = 3/2$, this sub-problem is branched again.

<u>Sub - problem 2</u>

$\max z = x_1 + 4x_2$

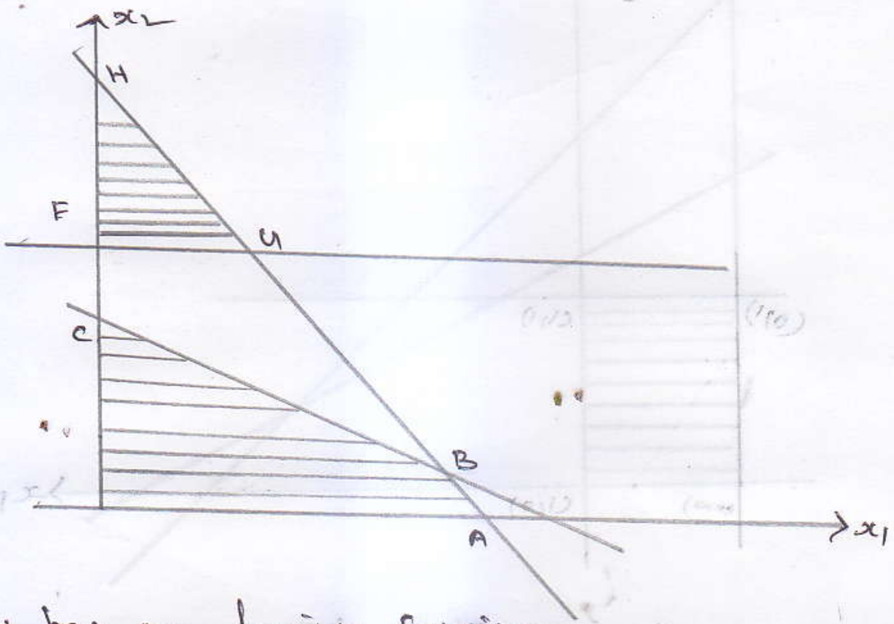
$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \geq 2$

and $x_1, x_2 \geq 0$

Its solution space is given by the region OABC and FGH.



and it has no feasible solution.

Hence this sub-problem is fathomed.

In sub-problem (1), since $x_1 = 3/2$ we have $1 \leq x_1 \leq 2$.

$\Rightarrow x_1 \leq 1$ or $x_1 \geq 2$

Applying these two conditions separately in the sub-problem (1) we have two sub-problems.

Sub-problems (3)

$$\max z = x_1 + 4x_2$$

$$2x_1 + 4x_2 \le 7$$

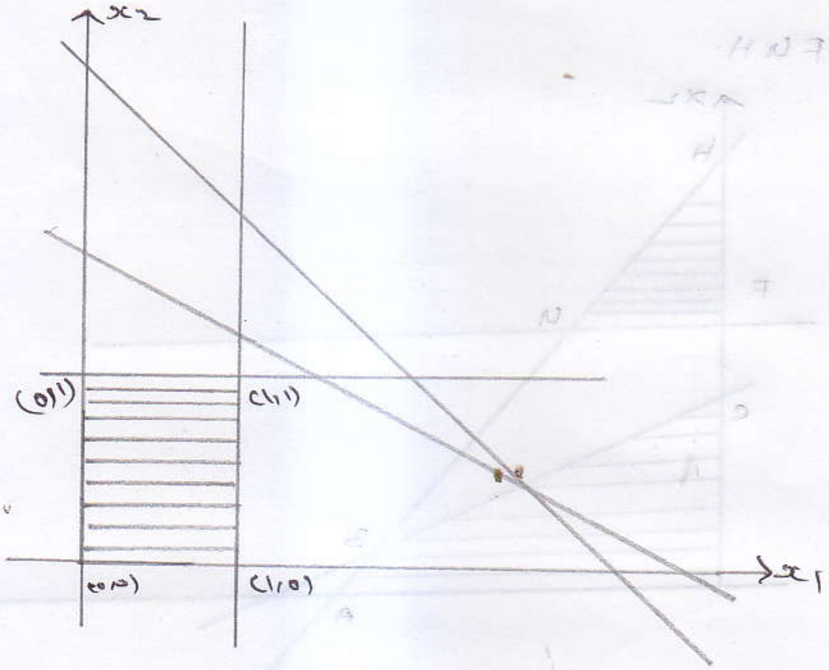
$$5x_1 + 3x_2 \le 15$$

$$x_2 \le 1$$

$$x_1 \le 1$$

and $x_1, x_2 \ge 0$.

and its solution space is given by



Its optimal solution is given by $\max z = 5$

$x_1 = 1$, $x_2 = 1$.

Since this solution is integer valued, this sub-problem cannot be further branched and the

Lower bound of the objective function is 5.

Sub - problem (4)

$\max z = x_1 + 4x_2$

$2x_1 + 4x_2 \le 7$

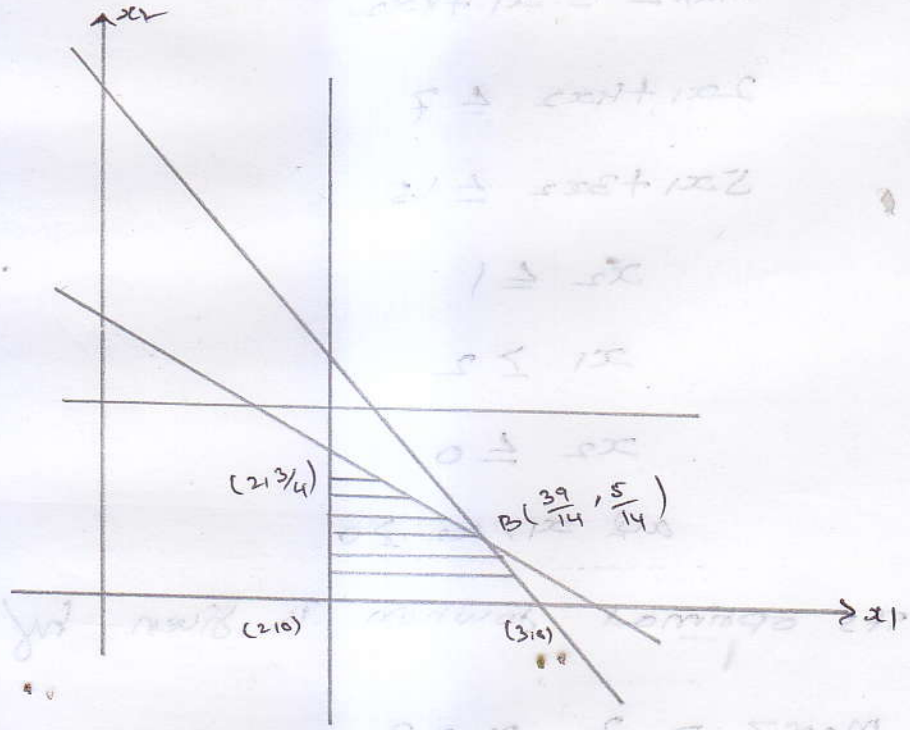
$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \ge 2$

and $x_1, x_2 \ge 0$

and its solution space is given by



Its optimal solution is given by

$\max z = 5, \quad x_1 = 2, \quad x_2 = 3/4$

Since $x_2 = 3/4$, this sub-problem is branched further.

In Sub-problem (4), since $x_2 = 3/4$, we have

$0 \leq x_2 \leq 1$.

$x_2 \leq 0$ or $x_2 \geq 1$.

Applying these two conditions one by one

In the sub-problem (4), we have two sub-problems.

Sub-problem (5)

$\max Z = x_1 + 4x_2$

$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \geq 2$

$x_2 \leq 0$

and $x_1, x_2 \geq 0$

and its optimal solution is given by

$\max Z = 3$, $x_1 = 3$, $x_2 = 0$

This sub-problem is fathomed

Sub-problem (6)

$\max Z = x_1 + 4x_2$

$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_2 \leq 1$

$x_1 \geq 2$

$x_2 \geq 1$

$x_1, x_2 \geq 0$

Thus sub-problem has no feasible solution. Hence this sub-problem is also fathomed.

$\therefore$ The optimum integer solution is

$\max z = 5, \quad x_1 = 1 \quad x_2 = 1.$