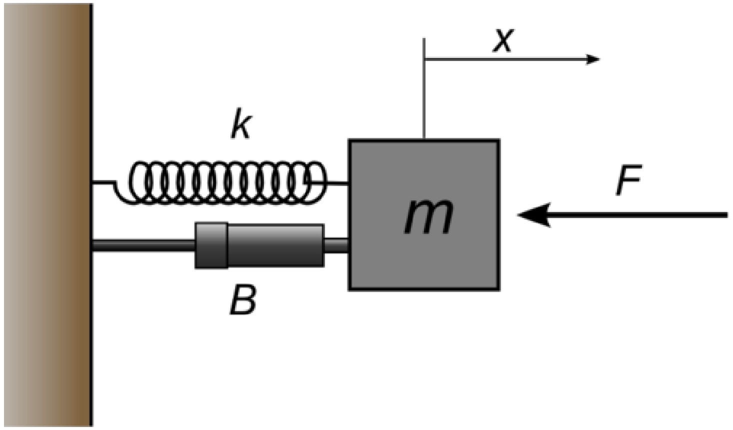
Control Systems

Mathematical Modeling of Mechanical Systems

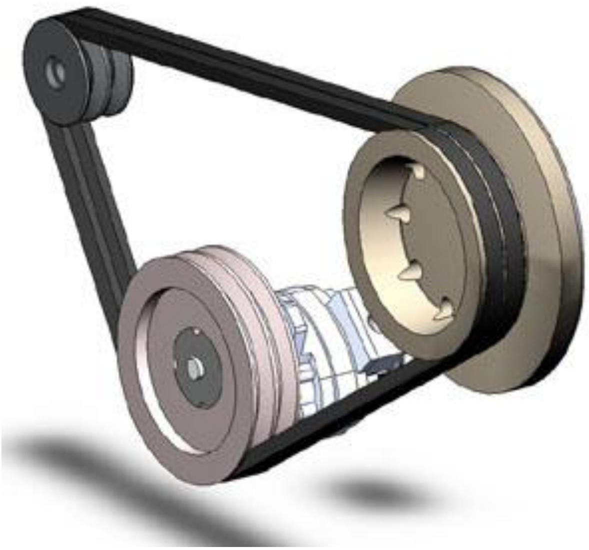
(Translational and Rotational Mechanical Systems)

Basic Types of Mechanical Systems

- Translational
  - Linear Motion



- Rotational
  - Rotational Motion



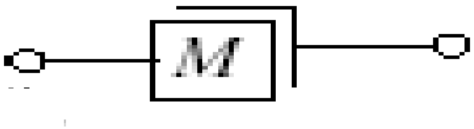
Basic Elements of Translational Mechanical Systems

Translational Spring

i)

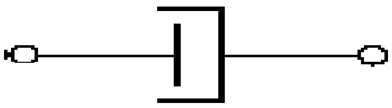
Translational Mass

ii)



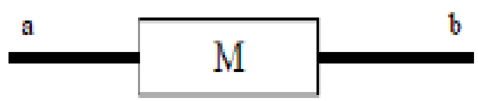
Translational Damper

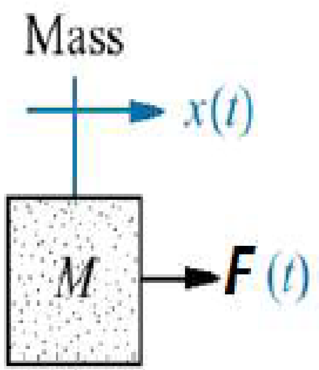
iii)



- **Mechanical Translational system (mass-spring-dashpot)**

- **Mass:** The Mass is an inertial element
  - Force (F) ⟶ Acceleration ⟶ Reaction force



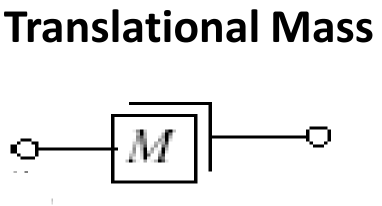


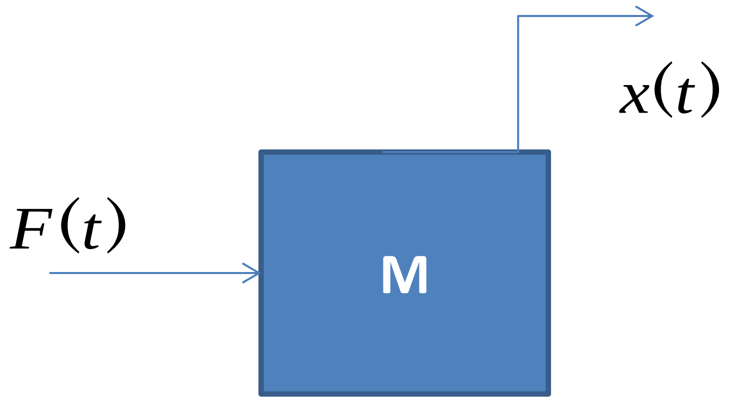
$$F(t) = M\frac{dv(t)}{dt} \qquad F(t) = M\frac{d^2x(t)}{dt^2}$$

Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.
- F=ma

ii)





Translational Damper

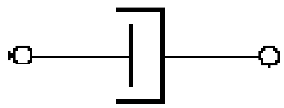
- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

- All the materials exhibit the property of damping to some extent.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

iii)

Translational Damper



Common Uses of Dashpots

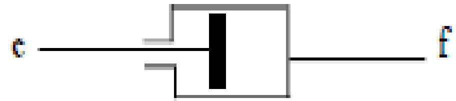
Door Stoppers

Vehicle Suspension

Bridge Suspension

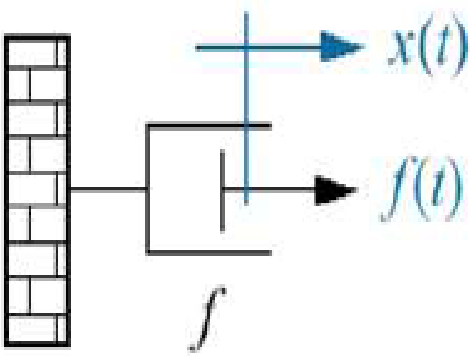
Flyover Suspension

- **Dashpot (damper)**



  - The reaction damping force *F* is approximated by the product of damping *f* and relative velocity if any.
  - $F(t) = f\,(v_1 - v_2) = fv$

Viscous damper



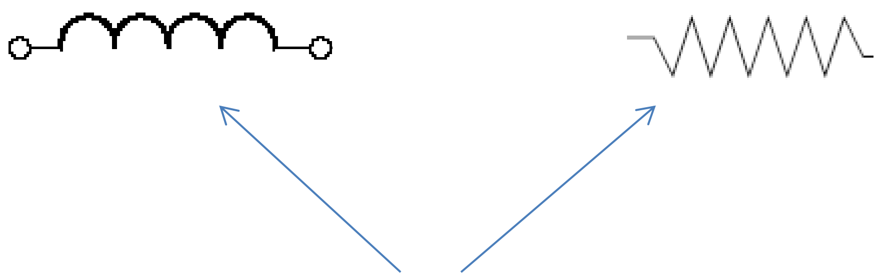
$$\mathbf{F}(t) = f\,v(t) \qquad \mathbf{F}(t) = f\,\frac{dx(t)}{dt}$$

Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)



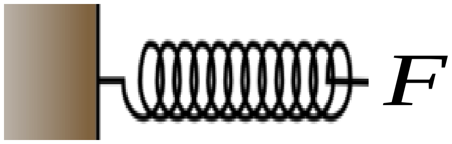
Circuit Symbols

Translational Spring

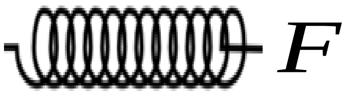
Translational Spring

- If *F* is the applied force

- Then x_1 is the deformation if $x_2 = 0$



- Or $(x_1 - x_2)$ is the deformation.

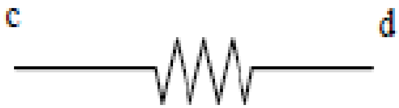


- The equation of motion is given as

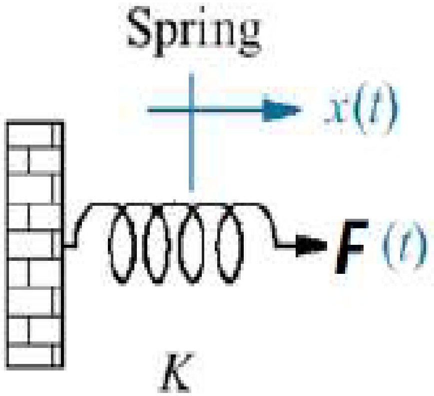
$$F = k(x_1 - x_2)$$

- Where k is stiffness of spring expressed in N/m

- **Spring**



  - Restoring force → Reaction Force on each end is same
  - The spring element force equation, in accordance with Hooke's Law is given by:
  - $$F_k = K\,(x_c - x_d)$$

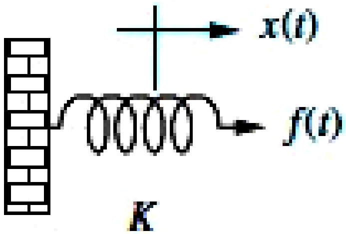
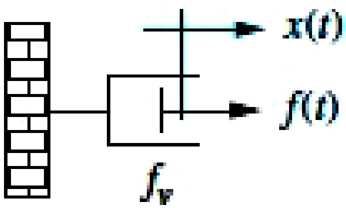
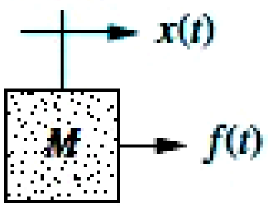


$$\mathbf{F}(t) = K\int_0^t v(\tau)\,d\tau \qquad \mathbf{F}(t) = Kx(t)$$

Basic Translational Mechanical System Properties and Their Units

Parameter	Symbol Used	SI Units	Other Units	Conversion Factors
Mass	M	kilogram (kg)	slug ft/sec^2	1 kg = 1000 g = 2.2046 lb(mass) = 35.274 oz(mass) = 0.06852 slug
Distance	y	meter (m)	ft in	1 m = 3.2808 ft = 39.37 in 1 in. = 25.4 mm 1 ft = 0.3048 m
Velocity	v	m/sec	ft/sec in/sec	
Acceleration	a	m/sec^2	ft/sec^2 in/sec^2	
Force	f	Newton (N)	pound (lb force) dyne	1 N = 0.2248 lb(force) = 3.5969 oz(force) 1 N = 1 kg–m/s^2 1 dyn = 1 g–cm/s^2
Spring Constant	K	N/m	lb/ft	
Viscous Friction Coefficient	B	N/m/sec	lb/ft/sec	

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

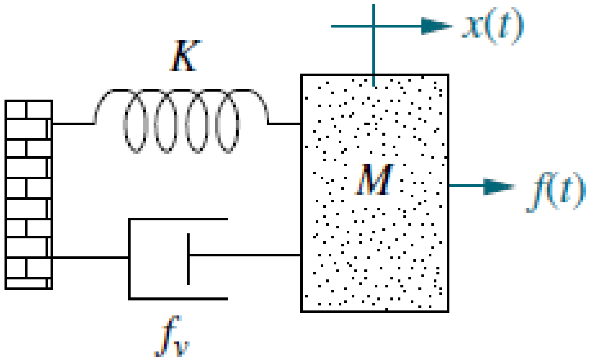
Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

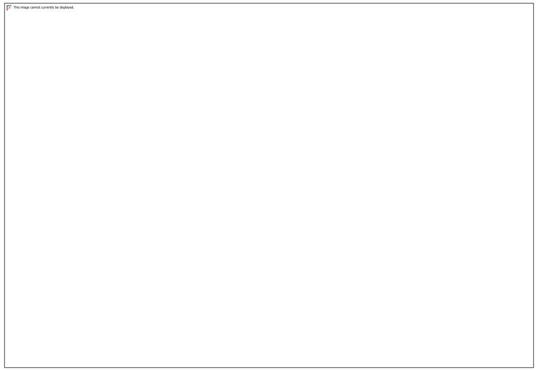
Steps to Obtain the Transfer Function of Mechanical System.

- The mechanical system requires just one differential equation, called the equation of motion, to describe it.
- **First,** draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it.
- **Second,** use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.
- **Finally,** assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function.

Example-1: Find the transfer function, *X(s)/F(s),* of the system.



Free Body Diagram (FBD)



- First step is to draw the free-body diagram.
- Place on the mass all forces felt by the mass.
- We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

- Second step is to write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass.

$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

Example-1: Continue.

- Third step is to take the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

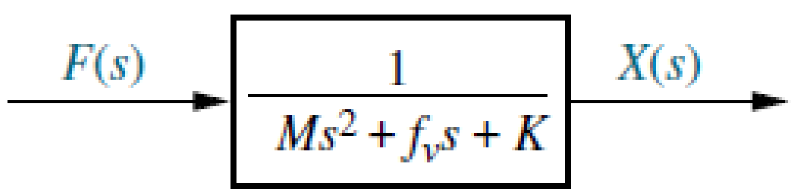
or

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

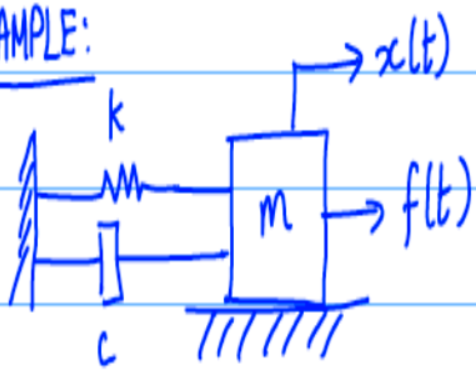
- Finally, solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Block Diagram



EXAMPLE:



The governing equation of motion is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t).$$

$m\ddot{x}(t)$: Inertia; $c\dot{x}(t)$: Viscous dissipation; $kx(t)$: Compliance

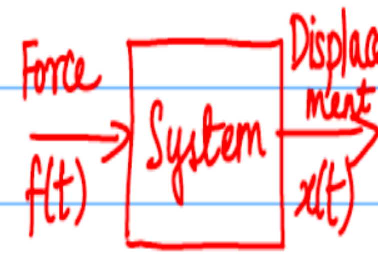
LTI Causal SISO system.

$\dot{x}(t) = \dfrac{dx(t)}{dt}$.

$\ddot{x}(t) = \dfrac{d^2x(t)}{dt^2}$.

Note that the governing eqn. is a 2nd order linear inhomogeneous ODE with constant coefficients.

System response < non-zero initial conditions (ICs). / input provided.



Let us consider a scenario where the spring constant changes with time.

$$m\ddot{x}(t) + c\dot{x}(t) + k(t)\,x(t) = f(t).$$

Linear Time Varying system (LTV)

The governing equation may be written as

$$m\ddot{x}(t) + c\dot{x}(t) + \underbrace{kx^2(t)}_{} = f(t).$$ NONLINEAR

2nd order nonlinear inhomogeneous ODE with constant coefficients.

Return to Laplace Transform:

Properties:

1). Transform of Derivatives: If $\mathcal{L}[f(t)] = F(s)$,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0). \quad \text{Here, } f(0) = f(t)\Big|_{t=0}. \rightarrow \text{INITIAL CONDITION.}$$

$$\dot{f}(t) = \frac{df(t)}{dt}.$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \frac{df(t)}{dt}\Big|_{t=0}.$$

$$\ddot{f}(t) = \frac{d^2f(t)}{dt^2}.$$

$$m\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + c\left[sX(s) - x(0)\right] + kX(s) = F(s).$$

$$\Rightarrow \left[ms^2 + cs + k\right]X(s) = (ms + c)\,x(0) + m\dot{x}(0) + F(s).$$

$$\Rightarrow X(s) = \underbrace{\frac{(ms+c)\,x(0) + m\dot{x}(0)}{(ms^2+cs+k)}}_{\text{Due to non-zero initial conditions}} + \underbrace{\left(\frac{1}{ms^2+cs+k}\right)F(s)}_{\text{Due to input}}$$

Due to non-zero initial conditions — 'FREE RESPONSE'

Due to input — 'FORCED RESPONSE'

Consider ALL initial conditions to be zero. That is, $x(0) = 0$, $\dot{x}(0) = 0$.

$$\Rightarrow X(s) = \left(\frac{1}{ms^2+cs+k}\right)F(s).$$

$$\Rightarrow \boxed{\frac{X(s)}{F(s)} = \frac{1}{(ms^2+cs+k)}.}$$ Transfer function of the system/plant. $P(s)$

ROTATIONAL SYSTEM: The rotational motion of a body can be defined as the motion of a body about a fixed axis. There are three types of torques resists the rotational motion.

1. Inertia Torque: Inertia(J) is the property of an element that stores the kinetic energy of rotational motion. The inertia torque TI is the product of moment of inertia J and angular acceleration α(t).

$$T_I(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

Where ω(t) is the angular velocity and θ(t) is the angular displacement.

2. Damping torque: The damping torque $T_D(t)$ is the product of damping coefficient B and angular velocity ω. Mathematically

$$T_D(t) = B\omega(t) = B\frac{d\theta(t)}{dt}$$

3. Spring torque: Spring torque $T_\theta(t)$ is the product of torsional stiffness and angular displacement.

Unit of 'K' is N-m/rad

$$T_\theta(t) = K\theta(t)$$

Torsional Springs

Consider the torsional spring shown in Figure 3-2 (a), where one end is fixed and a torque τ is applied to the other end. The ***angular displacement*** of the free end is θ. The torque T in the torsional spring is

$$T = k_T\,\theta \tag{3-3}$$

where θ is ***the angular displacement*** and k_T is the ***spring constant*** for torsional spring and has units of [Torque/angular displacement]=[N-m/rad] in SI units.

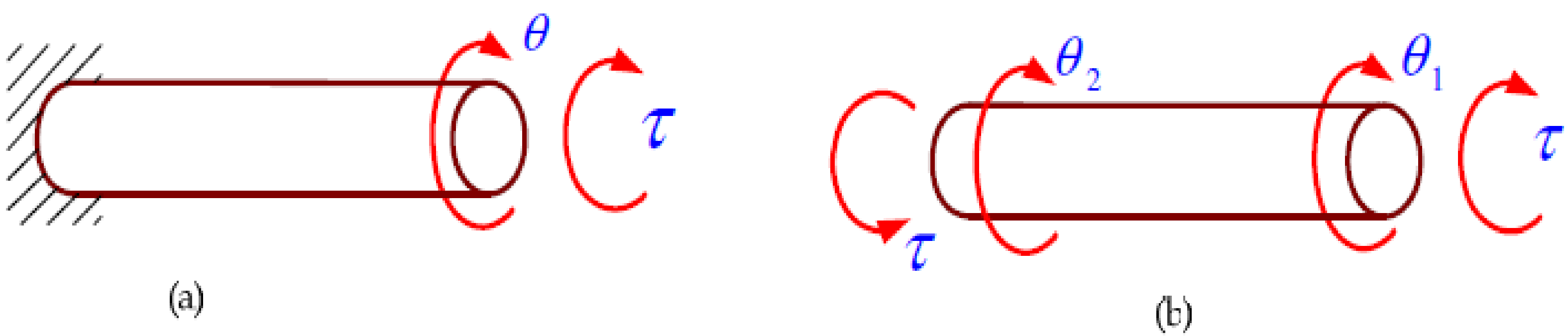


Figure 3-2 (a) A torque τ is applied at one end of torsional spring, and the other end is fixed; (b) a torque τ is applied at one end, and a torque τ, in the opposite direction, is applied at the other end.

Rotational Spring

Rotational Damper

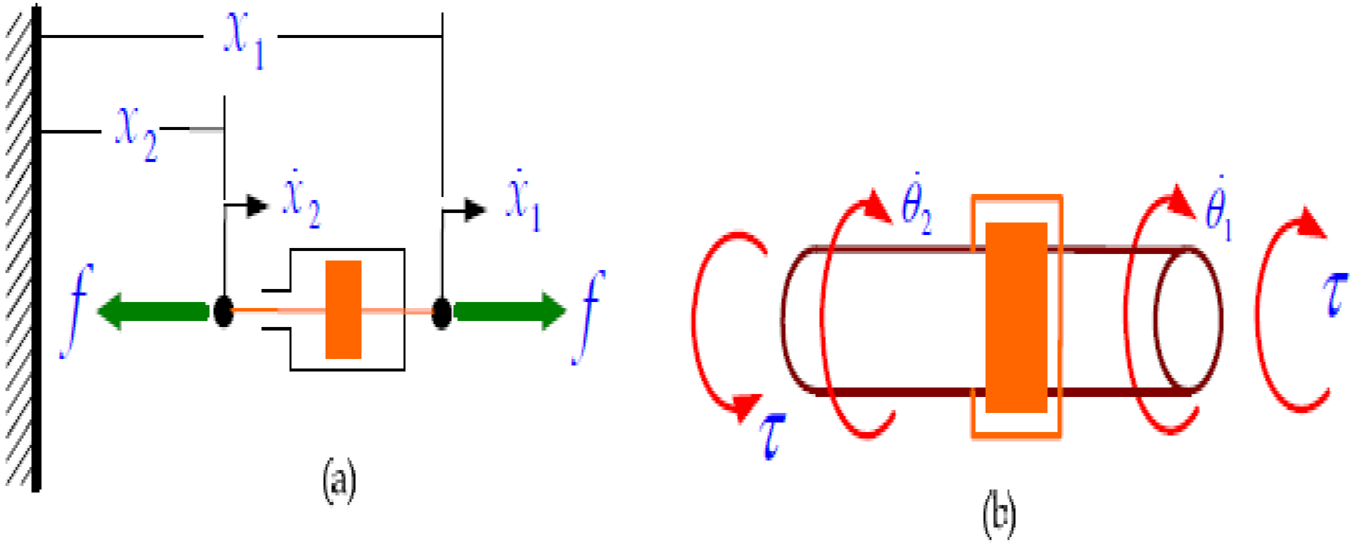


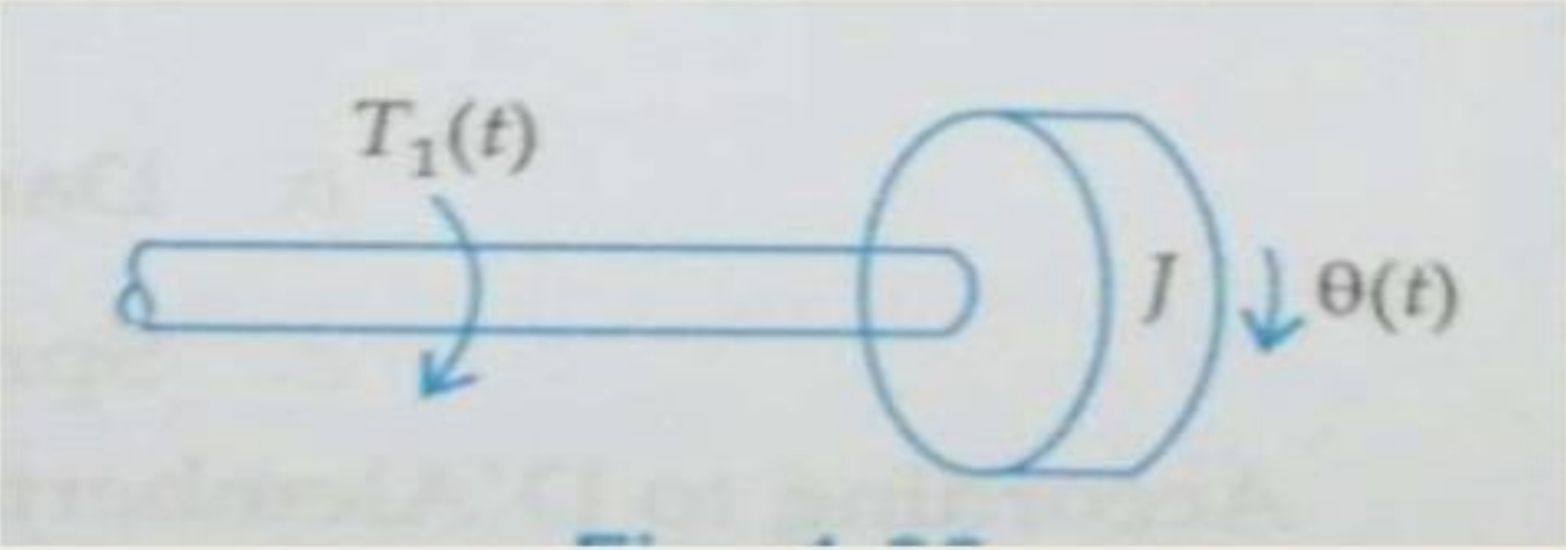
Figure 3-4 (a) Translational damper; (b) torsional (or rotational) damper.

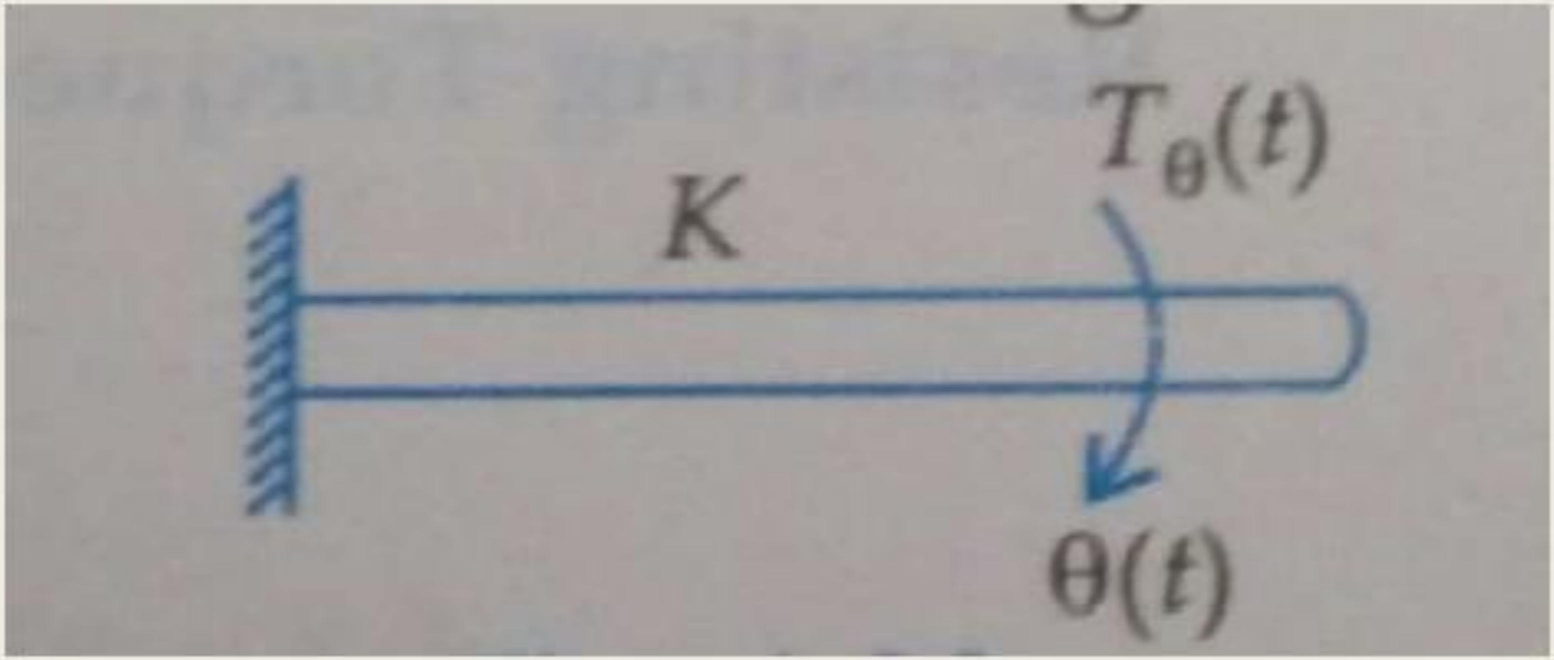
Rotational Damper

TORSIONAL DAMPER

For the torsional damper shown in Figure 3-4(b), the torques τ applied to the ends of the damper are of equal magnitude, but opposite in direction. The angular velocities of the ends of the damper are $\dot{\theta}_1$ and $\dot{\theta}_2$ and they are taken relative to the same frame of reference. The damping torque T that arises in the damper is proportional to the angular velocity differences $\dot{\theta}_1 - \dot{\theta}_2$ of the ends, or

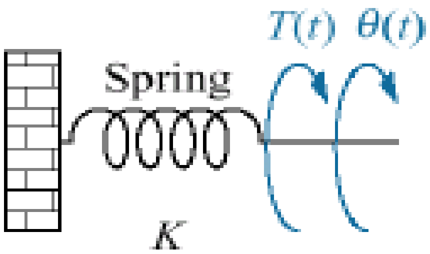
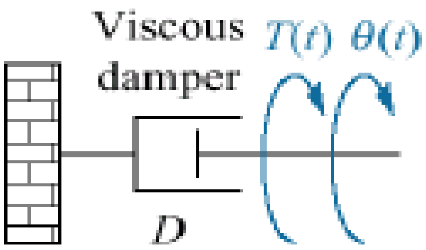
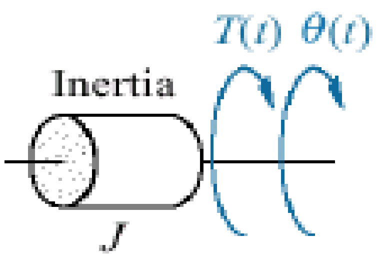
$$T = b_T\left(\dot{\theta}_1 - \dot{\theta}_2\right) = b_T\dot{\theta} \qquad (3\text{-}6)$$


$T_1(t)$
J
$\theta(t)$


$T_\theta(t)$
K
$\theta(t)$

Basic Rotational Mechanical System Properties and Their Units

Parameter	Symbol Used	SI Units	Other Units	Conversion Factors
Inertia	J	kg-m^2	slug-ft^2 lb-ft-sec^2 oz-in.-sec^2	1 g-cm = 1.417×10^{-5} oz-in.-sec^2 1 lb-ft-sec^2 = 192 oz-in.-sec^2 = 32.2 lb-ft^2 1 oz-in.-sec^2 = 386 oz-in^2 1 g-cm-sec^2 = 980 g-cm^2
Angular Displacement	T	Radian	Radian	$1\ \text{rad} = \frac{180}{\pi} = 57.3\ \text{deg}$
Angular Velocity	O	radian/sec	radian/sec	$1\ \text{rpm} = \frac{2\pi}{60}$ $= 0.1047\ \text{rad/sec}$ 1 rpm = 6 deg/sec
Angular Acceleration	A	radian/sec^2	radian/sec^2	
Torque	T	(N-m) dyne-cm	lb-ft oz-in.	1 g-cm = 0.0139 oz-in. 1 lb-ft = 192 oz-in. 1 oz-in. = 0.00521 lb-ft
Spring Constant	K	N-m/rad	ft-lb/rad	
Viscous Friction Coefficient	B	N-m/rad/sec	ft-lb/rad/sec	
Energy	Q	J (joules)	Btu Calorie	1 J = 1 N-m 1 Btu = 1055 J 1 cal = 4.184 J

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Practical Examples.

Pictures of various examples of real-world dampers are found below.

Mechanical Systems

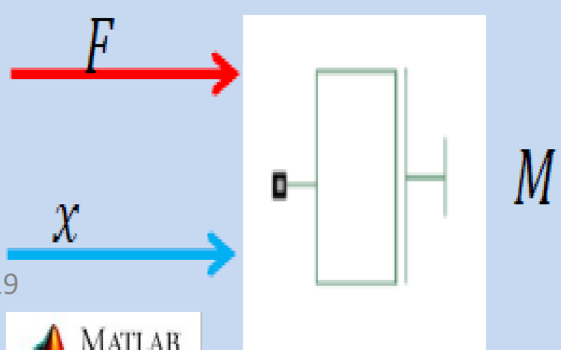
Classification based on type of motion:

- **Translational systems** having linear motion
- **Rotational systems** having angular motion about a fixed axis

Translational	Rotational
Basic System Elements	
Mass (M)	Inertia (J)
Damper (B)	Damper (D)
Linear spring (K)	Torsional spring (K)
Basic System Variables	
Force (F)	Torque (T)
Displacement (x)	Angular displacement (θ)

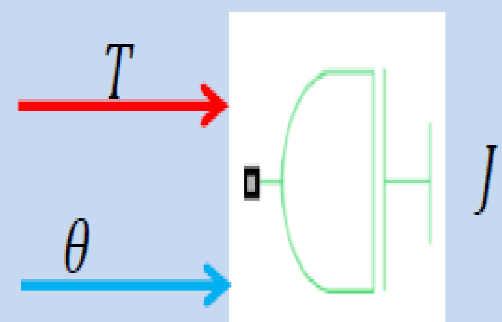
Mass

- Property of an element that stores the kinetic energy due to translational motion
- When a force is acting on a body of mass M causing displacement x, then:
- $F = \frac{dP}{dt} = M\frac{d^2x}{dt^2} = M\ddot{x}$



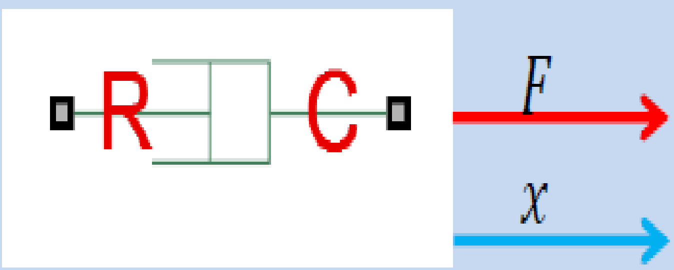
Inertia

- Property of an element that stores the kinetic energy due to rotational motion
- When a torque is acting on a body of inertia J causing displacement θ, then:
- $T = J\frac{d^2\theta}{dt^2} = J\ddot{\theta}$



- Damper is an element that generates force which acts opposite to the direction of motion, translational or rotational
- Damper resists motion
- Friction or dashpot are examples of dampers

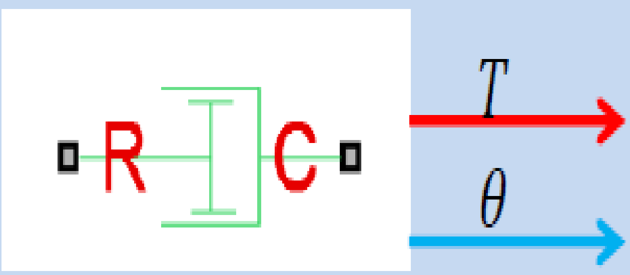
Translational



B

$$F = B\frac{dx}{dt} = B\dot{x}$$

Rotational



D

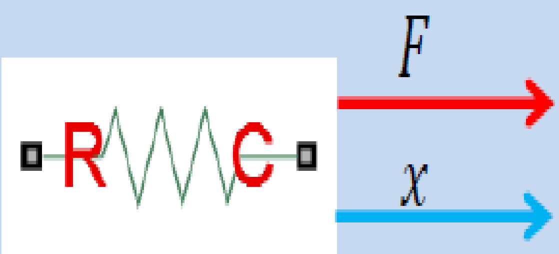
$$T = D\frac{d\theta}{dt} = D\dot{\theta}$$

Linear Spring

Property of an element that stores the potential energy due to translational motion

When a spring of spring constant K is applied a force F causing an elastic displacement x, then:

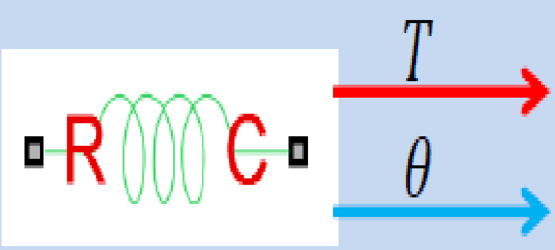
$F = Kx$



K

Torsional spring

- Property of an element that stores the potential energy due to rotational motion
- When a torsional spring of constant K is applied a torque T causing an angular displacement θ, then:
- $T = K\theta$



K

D'ALEMBERT PRINCIPLE

This principle states that "for any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero"

D'ALEMBERT PRINCIPLE contd……

External Force: F(t)

Resisting Forces :

1. Inertia Force:

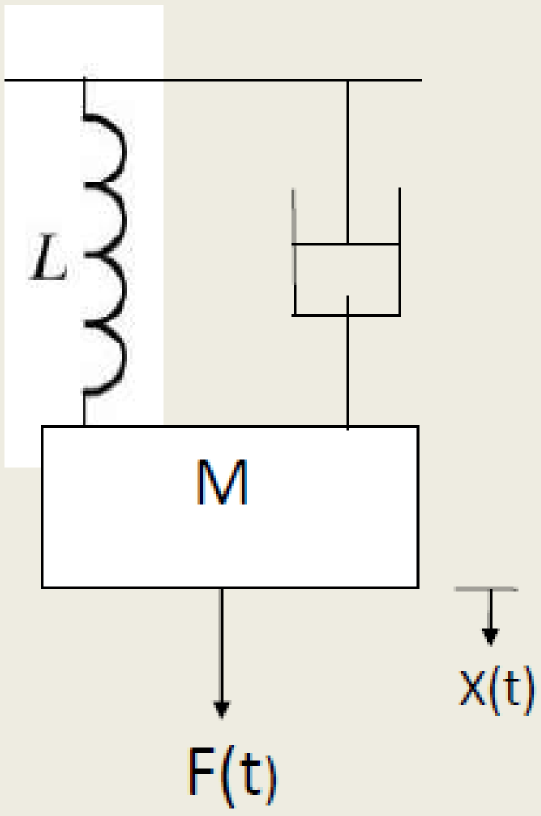
$$F_M(t) = -M\frac{d^2x(t)}{dt^2}$$

2. Damping Force:

$$F_D(t) = -B\frac{dx(t)}{dt}$$

3. Spring Force:

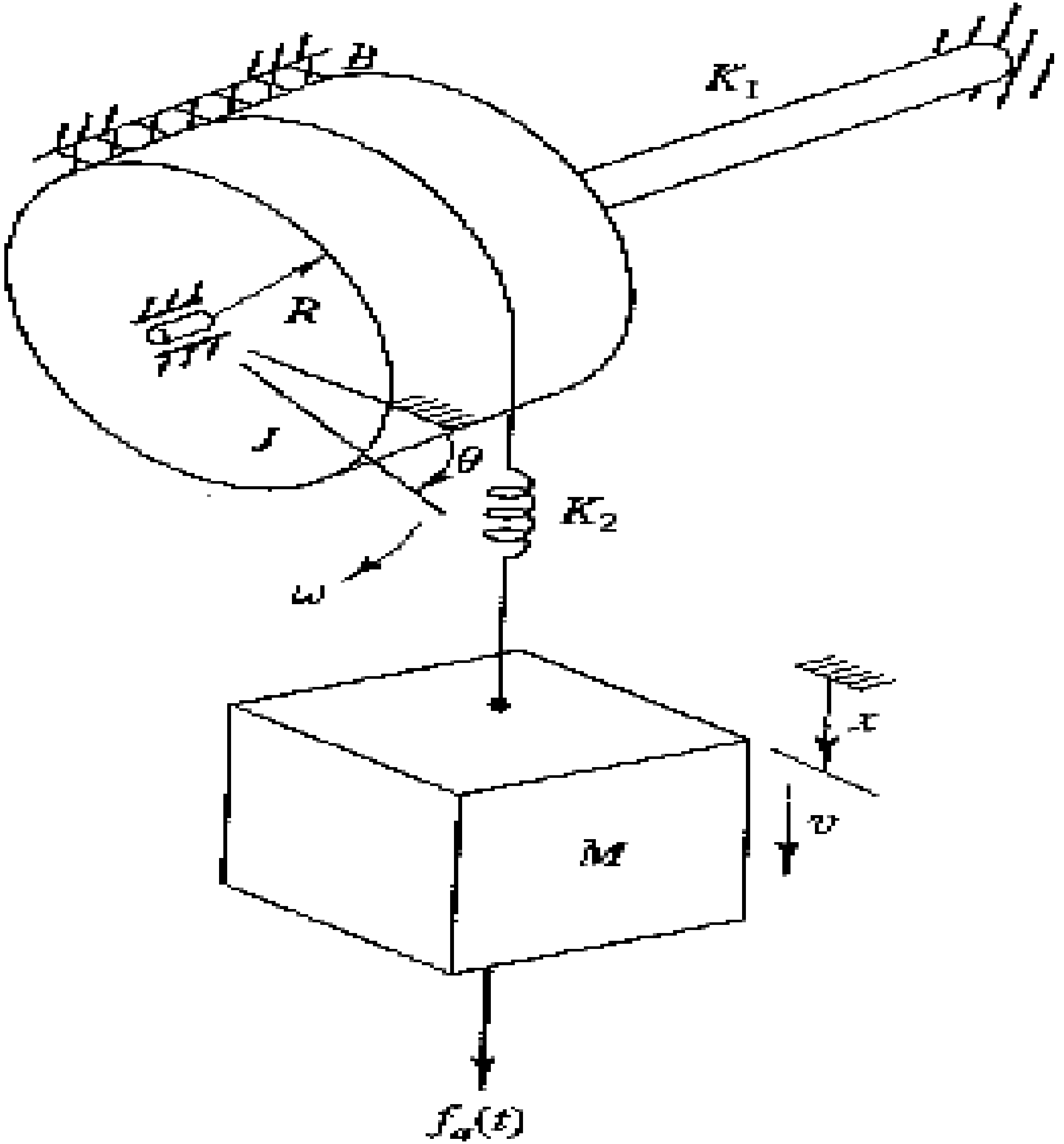
$$F_K(t) = -Kx(t)$$



According to D'Alembert Principle

$$F(t) - M\frac{d^2x(t)}{dt^2} - B\frac{dx(t)}{dt} - Kx(t) = 0$$

$$F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t)$$



B
K_1
R
J
θ
K_2
ω
x
v
M
$f_a(t)$

Consider rotational system:

External torque: T(t)

Resisting Torque:

(i) Inertia Torque: $T_I(t) = -J\dfrac{d\omega(t)}{dt}$

(ii) Damping Torque: $T_D(t) = -B\dfrac{d\theta(t)}{dt}$

(iii) Spring Torque: $T_K(t) = -K\theta(t)$

According to D'Alembert Principle:

$$T(t) + T_I(t) + T_D(t) + T_K(t) = 0$$

$$T(t) - J\frac{d\omega(t)}{dt} - B\frac{d\theta(t)}{dt} - K\theta(t) = 0$$

$$T(t) = J\frac{d\omega(t)}{dt} + B\frac{d\theta(t)}{dt} + K\theta(t)$$

D'Alembert Principle for rotational motion states that

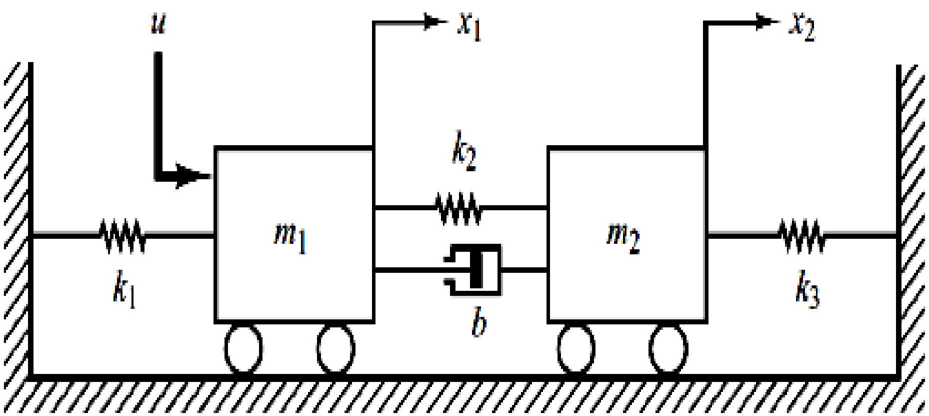
"For anybody, the algebraic sum of externally applied torques and the torque resisting rotation about any axis is zero."

TANSLATIONAL-ROTATIONAL COUNTERPARTS

S.NO.	TRANSLATIONAL	ROTATIONAL
1.	Force, F	Torque, T
2.	Acceleration, a	Angular acceleration, α
3.	Velocity, v	Angular velocity, ω
4.	Displacement, x	Angular displacement, θ
5.	Mass, M	Moment of inertia, J
6.	Damping coefficient, B	Rotational damping coefficient, B
7.	Stiffness	Torsional stiffness

Mathematical Modeling of Mechanical Systems:

Example 2:



(1) Equation of motion:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

(2) Simplifying,

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

(3) Laplace transform,

$$[m_1s^2 + bs + (k_1 + k_2)]X_1(s) = (bs + k_2)X_2(s) + U(s)$$

$$[m_2s^2 + bs + (k_2 + k_3)]X_2(s) = (bs + k_2)X_1(s)$$

(4) Substitute by $X_2(s)$,

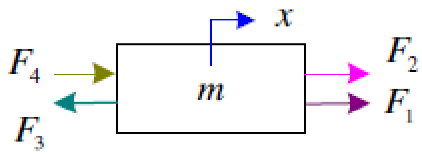
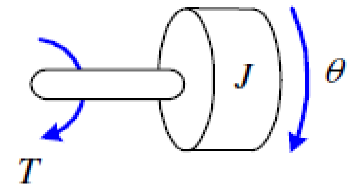
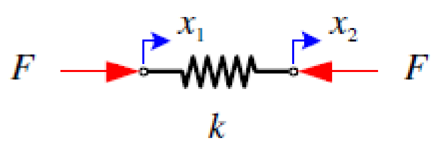
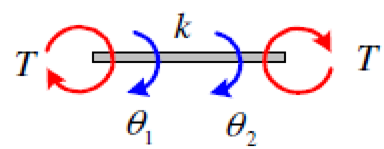
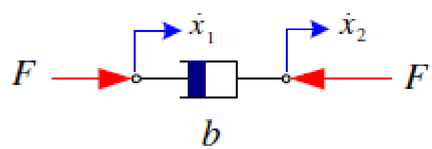
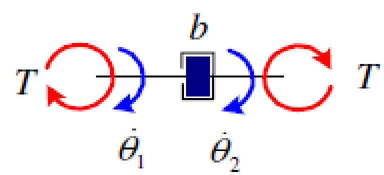
$$[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]X_1(s)$$
$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

(5) Finally,

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

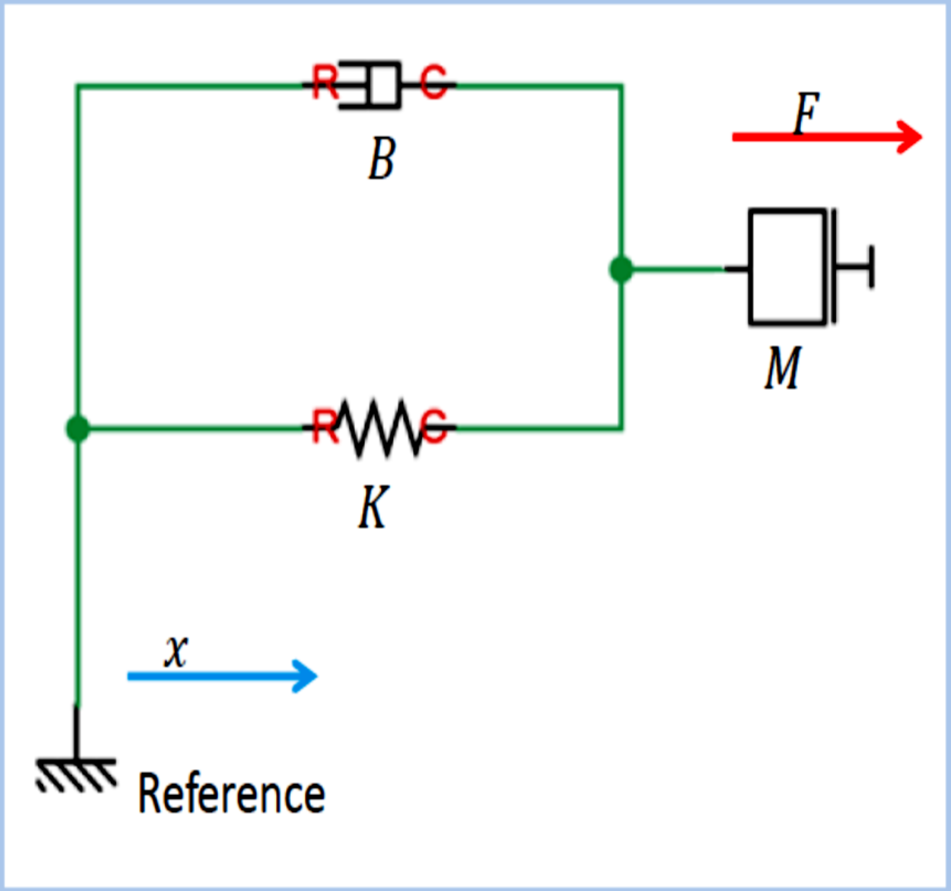
TABLE 1. SUMMARY OF ELEMENTS INVOLVED IN LINEAR MECHANICAL SYSTEMS

Element	Translation	Rotation
Inertia	 $\sum F = m\,a$	 $\sum T = J\,\alpha$
Spring	 $F = k(x_1 - x_2) = kx$	 $T = k(\theta_1 - \theta_2) = k\theta$
Damper	 $F = b(\dot{x}_1 - \dot{x}_2) = b\dot{x}$	 $T = b(\dot{\theta}_1 - \dot{\theta}_2) = b\dot{\theta}$

Analogous Systems

- Mechanical systems can be represented using electrical elements by the following analogies
- Two types of analogies:
  - Force (Torque) - Voltage analogy (F-V analogy)
    - Force is analogous to voltage
  - Force (Torque) - Current Analogy (F-I analogy)
    - Force is analogous to current

Mass – Spring - Damper



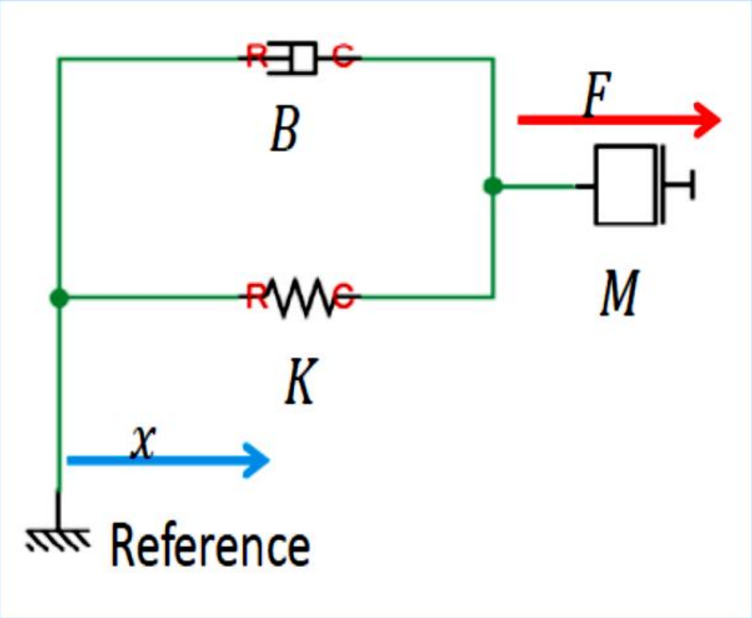
Based on Newton's 2nd law,

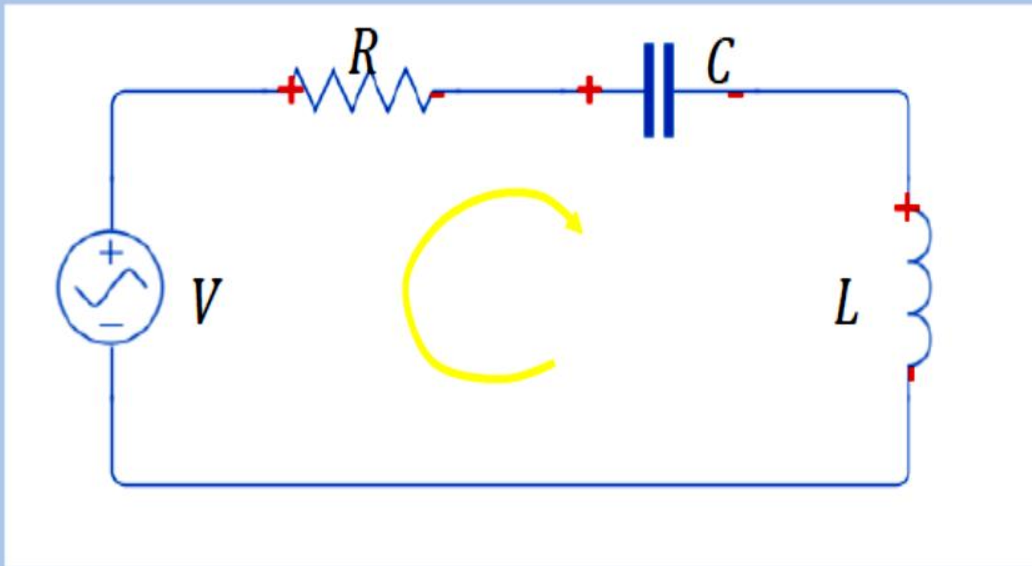
$$F = M\ddot{x} + B\dot{x} + Kx$$

Similarly for a rotational system,

$$T = J\ddot{\theta} + D\dot{\theta} + K\theta$$

F-V Analogy of MSD





$$F \rightarrow V$$
$$M \rightarrow L$$
$$B \rightarrow R$$
$$K \rightarrow \frac{1}{C}$$
$$x \rightarrow q$$

Based on Newton's 2nd law,

$$F = M\ddot{x} + B\dot{x} + Kx$$

Velocity to Current

Based on KVL around the loop,

$$V = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

F-V Analogy

F(t)=$F_m + F_b + F_k$

F(t)=M$\frac{d^2x(t)}{dt^2}$+B$\frac{dx(t)}{dt} + Kx(t)$

Take LT with Initial conditions zero

F(s)=Ms^2x(s)+Bsx(s)+kx(s)

Conversion Techniques(F-V)

Force to Voltage (F-V)

Mass to Inductance (M-L)

Friction to Resistance (B-R)

Spring to Reciprocal of Capacitance (K - -$\frac{1}{C}$)

Displacement to Charge (X-Q)

Velocity to Current

V(t)=Ri(t)+L$\frac{di(t)}{dt}$+$\frac{1}{C}\int i(t)dt$

Take Laplace

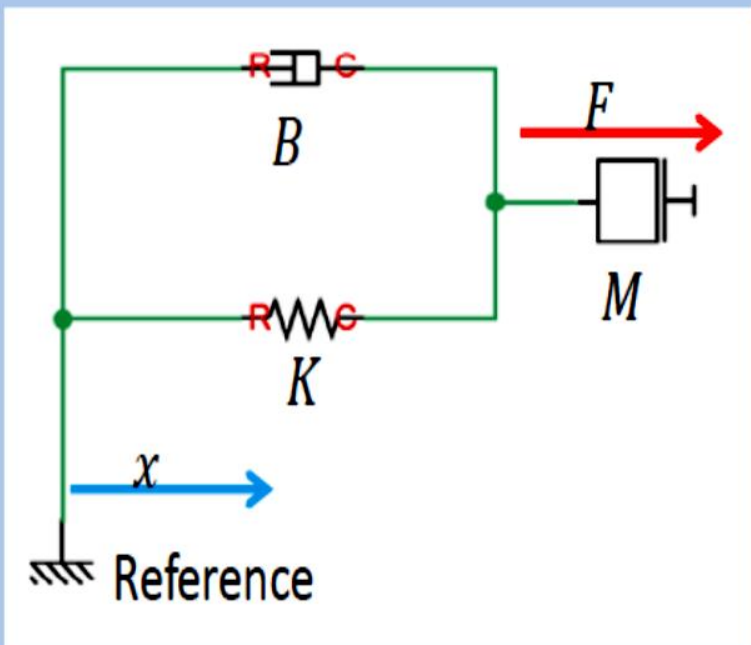
RI(s)+LsI(s)+$\frac{I(s)}{CS}$ =V(S)

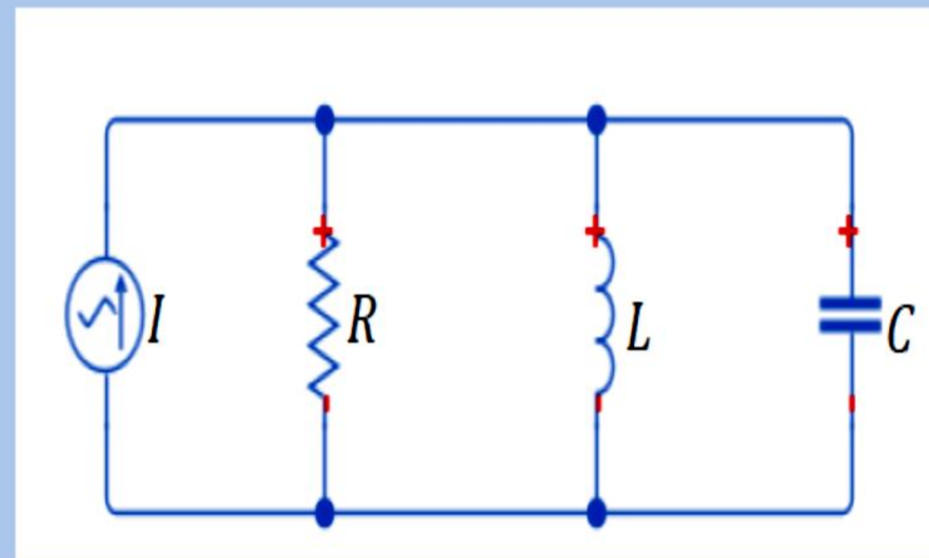
i(t)=$\frac{dq(t)}{dt}$ I(s)= SQ(s)

R[SQ(s)]+Ls[SQ(s)]+$\frac{\text{SQ(s)}}{Cs}$ = V(s)

V(s)=L s^2Q(s)+RSQ(s)+$\frac{Q(s)}{C}$

F-I Analogy of MSD





$$F \rightarrow I$$
$$M \rightarrow C$$
$$B \rightarrow \frac{1}{R}$$
$$K \rightarrow \frac{1}{L}$$
$$x \rightarrow \phi$$

Based on Newton's 2nd law,

$$F = M\ddot{x} + B\dot{x} + Kx$$

Based on KVL around the loop,

$$I = C\ddot{\phi} + \frac{\dot{\phi}}{R} + \frac{\phi}{L}$$

F-I Analogy

Mass-Spring-Damper

- F(s)=Ms^2x(s)+Bsx(s)+kx(s)

Conversion Techniques(F-I)

Force to Current (F-I)
Mass to Capacitance (M-C)
Friction to Mho (B- $\frac{1}{R}$)
Stiffness to Reci.Procal of L (K- $\frac{1}{L}$)
Displacment to Mag.Flux (X- Φ)
Velocity to Voltage

R-L-C Parallel

I = $I_R + I_l + I_R$

I(t)=$\frac{1}{L}\int V(t)\,dt + \frac{V(t)}{R(t)} + C\frac{dV(t)}{dt}$

Take LT with initial conditions Zero

I(S)= $\frac{V(s)}{SL} + \frac{V(s)}{R(s)}$+CsV(s)

WKT v(t) = $\frac{d\Phi}{dt}$ → V(s)=SΦ(s)

I(S)= $\frac{S\Phi(s)}{SL} + \frac{S\Phi(s)}{R(s)}$+Cs[SΦ(s)]

I(s)=CS^2 Φ(s) + $\frac{S\Phi(s)}{R(s)} + \frac{\Phi(s)}{L}$

Summary: Analogous Systems

- Following table shows the analogue between the elements of mechanical and electrical systems:

Mechanical System		Electrical System	
Translational	**Rotational**	**F-V Analogy**	**F-I Analogy**
Force (F)	Torque (T)	Voltage (V)	Current (I)
Mass (M)	Inertia (J)	Inductor (L)	Capacitor (C)
Friction (B)	Friction (D)	Resistor (R)	Conductor ($1/R$)
Linear spring (K)	Torsional spring (K)	Capacitor ($1/C$)	Inductor ($1/L$)
Displacement (x)	Displacement (θ)	Charge (q)	Flux (ϕ)

Transfer Function of Mechanical (Translational)System

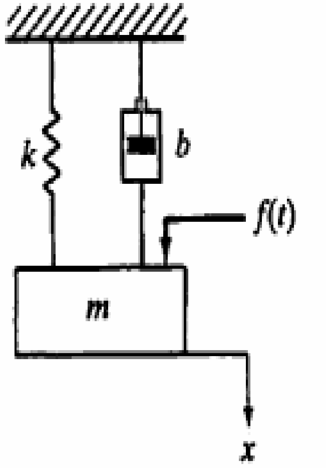


Figure 4-1 Mechanical system.

The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

Taking the Laplace transform of both sides of this equation and assuming that all initial conditions are zero yields

$$(ms^2 + bs + k)X(s) = F(s)$$

where $X(s) = \mathcal{L}[x(t)]$ and $F(s) = \mathcal{L}[f(t)]$. From Equation (4–1), the transfer function for the system is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

The equations of motion for the system are

$$m\ddot{x} + k_1 x + k_2(x - y) = p$$
$$k_2(x - y) = b_2\dot{y}$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(ms^2 + k_1 + k_2)X(s) = k_2Y(s) + P(s)$$
$$k_2X(s) = (k_2 + b_2s)Y(s)$$

Solving Equation for $Y(s)$ and substituting the result into Equation we get

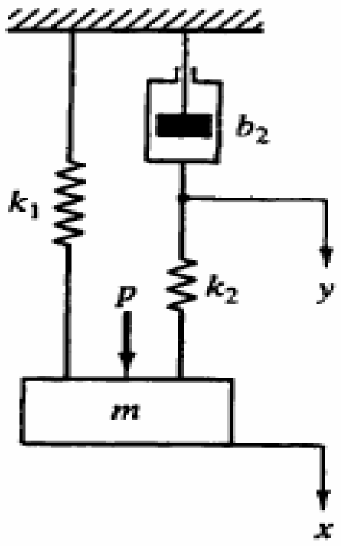
$$(ms^2 + k_1 + k_2)X(s) = \frac{k_2^2}{k_2 + b_2s}X(s) + P(s)$$

or

$$[(ms^2 + k_1 + k_2)(k_2 + b_2s) - k_2^2]X(s) = (k_2 + b_2s)P(s)$$

from which we obtain the transfer function

$$\frac{X(s)}{P(s)} = \frac{b_2s + k_2}{mb_2s^3 + mk_2s^2 + (k_1 + k_2)b_2s + k_1k_2}$$



Mechanical system.

Obtain the transfer function $X(s)/U(s)$ of the system shown in Figure 4–25, where u is the force input. The displacement x is measured from the equilibrium position.

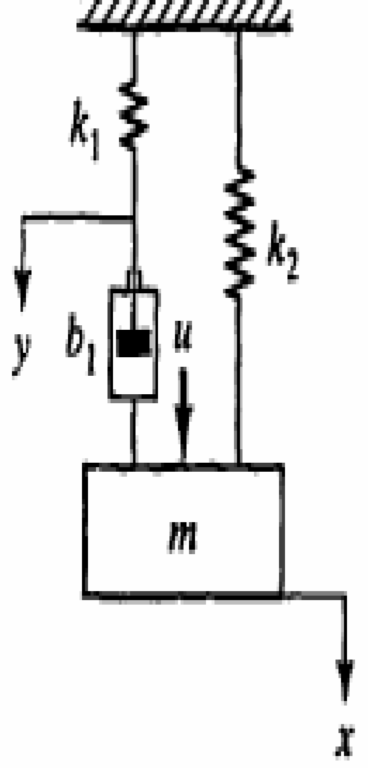


Figure 4–25 Mechanical system.

Solution The equations of motion for the system are

$$m\ddot{x} = -k_2 x - b_1(\dot{x} - \dot{y}) + u$$

$$b_1(\dot{x} - \dot{y}) = k_1 y$$

Laplace transforming these two equations and assuming initial conditions equal to zero, we obtain

$$ms^2X(s) = -k_2X(s) - b_1sX(s) + b_1sY(s) + U(s)$$

$$b_1sX(s) - b_1sY(s) = k_1Y(s)$$

Eliminating $Y(s)$ from the last two equations yields

$$(ms^2 + b_1s + k_2)X(s) = b_1s\frac{b_1s}{b_1s + k_1}X(s) + U(s)$$

Simplifying, we obtain

$$[(ms^2 + b_1s + k_2)(b_1s + k_1) - b_1^2s^2]X(s) = (b_1s + k_1)U(s)$$

from which we get the transfer function $X(s)/U(s)$ as

$$\frac{X(s)}{U(s)} = \frac{b_1s + k_1}{mb_1s^3 + mk_1s^2 + b_1(k_1 + k_2)s + k_1k_2}$$

B–3–6. Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–35.

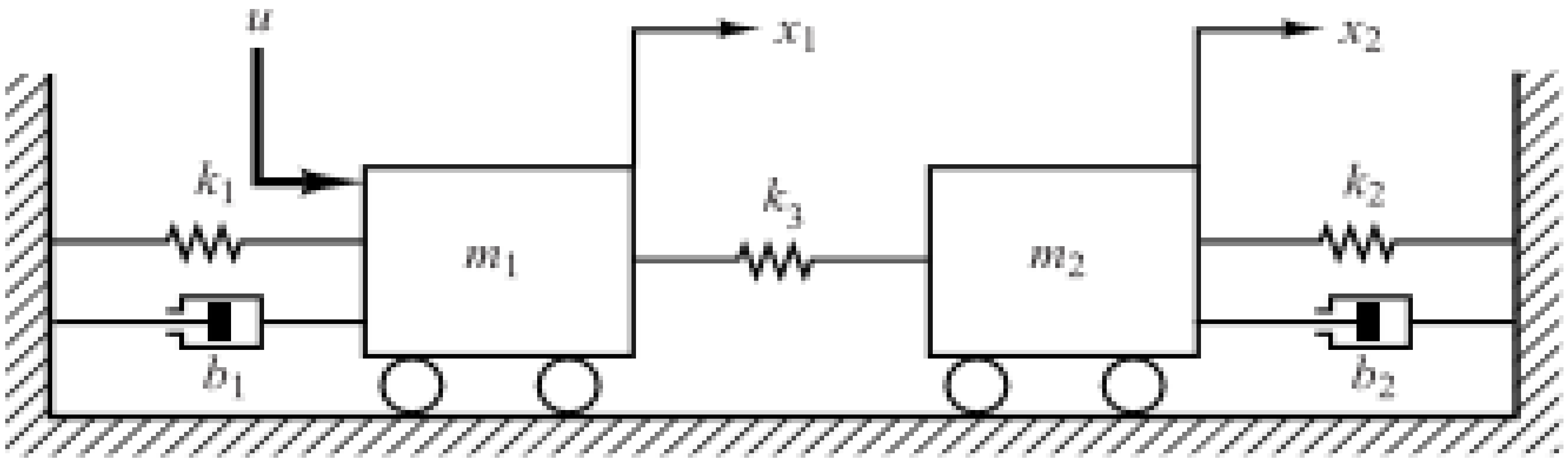


Figure 3–35 Mechanical system.

B-3-6. The equations for the system are

$$m_1 \ddot{x}_1 = -k_1 x_1 - b_1 \dot{x}_1 - k_3 (x_1 - x_2) + u$$

$$m_2 \ddot{x}_2 = -k_2 x_2 - b_2 \dot{x}_2 - k_3 (x_2 - x_1)$$

Rewriting, we have

$$m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 + k_3 x_1 = k_3 x_2 + u$$

$$m_2 \ddot{x}_2 + b_2 \dot{x}_2 + k_2 x_2 + k_3 x_2 = k_3 x_1$$

Assuming the zero initial condition and taking the Laplace transforms of these two equations, we obtsin

$$(m_1 s^2 + b_1 s + k_1 + k_3) X_1(s) = k_3 X_2(s) + U(s) \qquad (1)$$

$$(m_2 s^2 + b_2 s + k_2 + k_3) X_2(s) = k_3 X_1(s) \qquad (2)$$

By eliminating $X_2(s)$ from Equations (1) and (2), we get

$$(m_1 s^2 + b_1 s + k_1 + k_3) X_1(s) = \frac{k_3^2}{m_2 s^2 + b_2 s + k_2 + k_3} X_1(s) + U(s)$$

Hence

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + b_2 s + k_2 + k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

From Equation (2), we obtain

$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{m_2 s^2 + b_2 s + k_2 + k_3}$$

Hence

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

Rotational Systems

Dr V S Krushnasamy

EXAMPLE 1.5

Write the differential equations governing the mechanical rotational system shown in fig 1. Obtain the transfer function of the system.

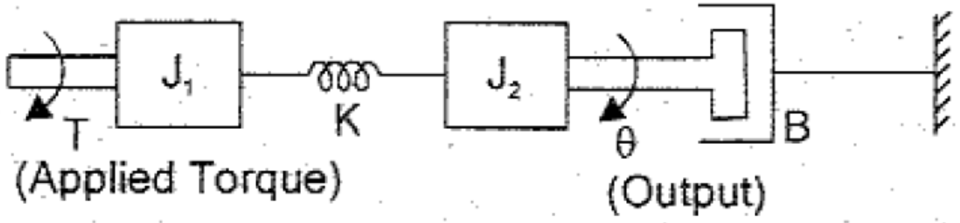


Fig 1.

SOLUTION

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

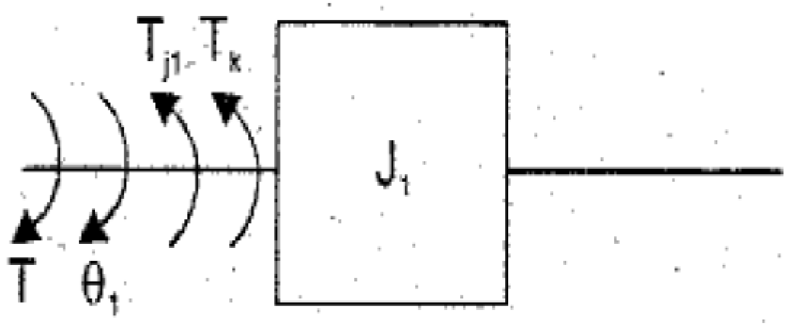


***Fig 2 :** Free body diagram of mass with moment of inertia J_1.*

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\, \theta_1(s) - K\, \theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

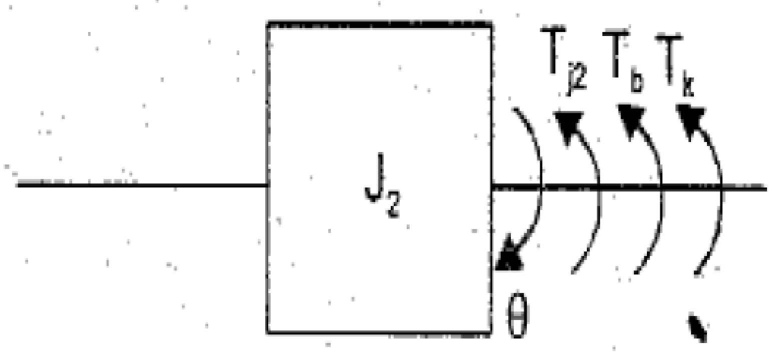


Fig 3 : Free body diagram of mass with moment of inertia J_2.

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) + B s \theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad \text{.....(3)}$$

Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are,

1. $$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$$

2. $$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$