**Discrete Mathematical Structures (BCS405A)**

**Question Bank**

**Module 1: Fundamentals of Logics -2**

1. Define (i) open proposition (ii) quantifiers (iii) bound variables and (iv) free variables. Identify the bound variables and the free variables in each of the following statements. The Universe is comprising of all real numbers. (a) $\exists x \exists y\left[x^2 - y^2 = z\right]$ (b) $\forall x \forall y\left[e^{3x+y} = 3^z\right]$ (c) $\forall y \exists z\left[\cos(x+y) = \sin(z-x)\right]$
2. Let $p(x)$, $q(x)$ and $r(x)$ denote the following open statements

   $p(x): x^2 - 8x + 15 = 0$ $\quad q(x): x$ is odd $\quad r(x): x > 0$

   For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counter-example

   i) $\forall x\left[p(x) \to q(x)\right]$ ii) $\exists x\left[p(x) \to q(x)\right]$ iii) $\forall x\left[q(x) \to p(x)\right]$ iv) $\exists x\left[q(x) \to p(x)\right]$

   v) $\exists x\left[r(x) \to p(x)\right]$ vi) $\forall x\left[\neg q(x) \to \neg p(x)\right]$ vii) $\exists x\left[p(x) \to (q(x) \wedge r(x))\right]$

   viii) $\forall x\left[(p(x) \vee q(x)) \to r(x)\right]$
3. Let $p(x)$, $q(x)$, $r(x)$ and $s(x)$ denote the following open statements

   $p(x): x \geq 0$ $\quad q(x): x^2 \geq 0$ $\quad r(x): x^2 - 3x - 4 = 0$ $\quad s(x): x^2 - 3 > 0$

   For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counter-example

   i) $\exists x\ p(x)$ ii) $\exists x\ \neg p(x)$ iii) $\forall x\ r(x)$ iv) $\forall x\ \neg r(x)$ v) $\exists x\left[p(x) \wedge q(x)\right]$ vi) $\forall x\left[p(x) \to q(x)\right]$

   vii) $\forall x\left[q(x) \to s(x)\right]$ viii) $\forall x\left[r(x) \vee s(x)\right]$ ix) $\exists x\left[p(x) \wedge r(x)\right]$ x) $\forall x\left[r(x) \to p(x)\right]$
4. Let $p(x)$, $q(x)$ and $r(x)$ denote the following open statements

   $p(x): x^2 - 7x + 10 = 0$ $\quad q(x): x^2 - 2x - 3 = 0$ $\quad r(x): x < 0$

   (a) For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counter-example.

   i) $\forall x\left[p(x) \to \neg r(x)\right]$ ii) $\forall x\left[q(x) \to r(x)\right]$ iii) $\exists x\left[q(x) \to r(x)\right]$ iv) $\exists x\left[p(x) \to r(x)\right]$

   (b) Find the answers to part (a) when the universe consists of all positive integers.

   (c) Find the answers to part (a) when the universe contains only the integers 2 and 5.
5. Consider the universe of all polygons with three or four sides, and define the following open statements for this universe

   $a(x)$: all interior angles of $x$ are equal $\quad e(x)$: $x$ is an equilateral triangle

   $h(x)$: all sides of $x$ are equal $\quad i(x)$: $x$ is an isosceles triangle

   $p(x)$: $x$ has an interior angle that exceeds $180^\circ$ $\quad q(x)$: $x$ is a quadrilateral

   $t(x)$: $x$ is a triangle $\quad r(x)$: $x$ is a rectangle $\quad s(x)$: $x$ is a square

   Translate each of the following statements into an English sentence and determine its truth value.

   a) $\forall x\left[q(x) \veebar t(x)\right]$ b) $\exists x\left[t(x) \wedge p(x)\right]$ c) $\exists x\left[q(x) \to p(x)\right]$ d) $\exists x\left[r(x) \wedge \neg s(x)\right]$

   e) $\forall x\left[s(x) \leftrightarrow (a(x) \wedge h(x))\right]$ f) $\forall x\left[t(x) \to (a(x) \leftrightarrow p(x))\right]$ g) $\exists x\left[(a(x) \wedge t(x)) \leftrightarrow e(x)\right]$

   write the following statements symbolically and determine their truth values

   h) every isosceles triangle is an equilateral triangle

   i) If all sides of $x$ are equal then it is equilateral triangle.

   j) If $x$ is a triangle then it has no angle that exceeds $180^0$ .

   k) For any triangle if all the interior angles are not equal, then all its sides are not equal.
6. Write the converse, inverse and contrapositive of the following statements. Also determine the truth values of the statement and its converse, inverse and contrapositive

a) If $x$ is a square, then it is equilateral [Universe: set of all quadrilaterals]

b) The universe consists of all real numbers

i) $\forall x\left[(x>3)\rightarrow(x^2>9)\right]$. ii) If $x^2+4x-21>0$, then $x>3$ or $x<-7$

c) The universe consists of all integers

i) If $m>n$ then $m^2>n^2$. ii) if $m$ divides $n$ and $n$ divides $p$, then $m$ divides $p$

d) The universe consists of all real numbers, let $p(x):|x|>3$ $q(x):x>3$ $r(x):x<-3$

i) $\forall x\left[p(x)\rightarrow q(x)\right]$ ii) $\forall x\left[p(x)\rightarrow\left(q(x)\vee r(x)\right)\right]$

7. Given $R(x,y): x+y$ is even, where the variables $x$ & $y$ represent integers. Write down the following in words and determine their truth values: i) $\forall x\ \exists y\, R(x,y)$ ii) $\exists x\ \forall y\ R(x,y)$.

8. For the following statements, the universe comprises all non-zero integers. Determine truth values

i) $\exists x\exists y[xy=1]$ ii) $\exists x\forall y[xy=1]$ iii) $\forall x\exists y[xy=1]$ iv) $\exists x\exists y[xy=2]$ v) $\exists x\forall y[xy=2]$

vi) $\forall x\exists y[xy=2]$ vii) $\exists x\exists y\left[(3x+y=8)\wedge(2x-y=7)\right]$ viii) $\exists x\exists y\left[(4x+2y=3)\wedge(x-y=1)\right]$

ix) $\exists x\exists y\left[(2x+y=5)\wedge(x-3y=-8)\right]$ viii) $\exists x\exists y\left[(3x-y=17)\wedge(2x+4y=3)\right]$

9. Let $p(x,y): x$ divides(exactly) $y$.

(a) Determine the truth values of the following with the universe is set of all integers:

i) $\forall x\ p(x,x)$ ii) $\forall y\ p(1,y)$ iii) $\forall x\ p(x,0)$ iv) $\forall x\ \forall y\, p(x,y)$ v) $\exists x\ \exists y\, p(x,y)$ vi) $\forall x\ \exists y\, p(x,y)$

vii) $\exists x\ \forall y\ p(x,y)$ viii) $\exists y\forall x\ p(x,y)$ ix) $\forall y\ \exists x\ p(x,y)$ x) $\forall x\ \forall y\left[\left(p(x,y)\wedge p(y,x)\right)\rightarrow(x=y)\right]$

xi) $\forall x\ \forall y\left[p(x,y)\vee p(y,x)\right]$

(b) Determine the truth values of the statements from (vi) to (ix) in (a) with universe for $x$ is set of all integers and for $y$ is set of all positive integers.

(c) Determine the truth values of the statements from (vi) to (ix) in (a) with universe for both $x$ and $y$ is set of all positive integers.

10. Negate and simplify the following

a) $\exists x\left[p(x)\vee q(x)\right]$ b) $\forall x\left[p(x)\wedge\neg q(x)\right]$ c) $\forall x\left[p(x)\rightarrow q(x)\right]$ d) $\exists x\left[\left(p(x)\vee q(x)\right)\rightarrow r(x)\right]$

e) For all $x$, if $x$ is odd, then $x^2-1$ is even f) No even integer is divisible by 7.

g) all integers are rational numbers and some rational numbers are not integers.

h) If all triangles are right angled, then no triangle is equiangular.

i) for the universe consists of integers, $\exists x\left[r(x)\wedge s(x)\right]$, where $r(x): 2x+1=5$ and $s(x): x^2=9$

11. Write the negation of the following:

i) $\forall x\ \exists y\left[\left(p(x,y)\wedge q(y,x)\right)\rightarrow r(x,y)\right]$ ii) $\exists x\ \forall y\left[p(x,y)\rightarrow\left(q(x,y)\vee r(x,y)\right)\right]$

iii) $\exists x\ \forall y\left[(x<y)\wedge\{(x-y)\le 0\}\right]$ iv) $\forall x\ \forall y\left[(x^2+y^2=0)\vee(x=0\wedge y=0)\right]$

v) $\forall x\ \forall y\left[(|x|=|y|)\rightarrow(y=\pm x)\right]$ vi) $\left[\forall x\ \forall y\left((x<0)\wedge(y>0)\right)\right]\rightarrow\left[\exists z\,(xz>y)\right]$

12. Establish the validity of the following

a)
$$\begin{array}{l}(x)\left[p(x)\rightarrow q(x)\right]\\ (x)\left[q(x)\rightarrow r(x)\right]\\ \hline \therefore (x)\left[p(x)\rightarrow r(x)\right]\end{array}$$

b)
$$\begin{array}{l}\forall x\left[p(x)\wedge s(x)\right]\\ \forall x\left[p(x)\rightarrow\left(q(x)\wedge r(x)\right)\right]\\ \hline \therefore \forall x\left[r(x)\wedge s(x)\right]\end{array}$$

c)
$$\begin{array}{l}\forall x\left[p(x)\vee q(x)\right]\\ \forall x\left[\left(\neg p(x)\wedge q(x)\right)\rightarrow r(x)\right]\\ \hline \therefore \forall x\left[\neg r(x)\rightarrow p(x)\right]\end{array}$$

$\forall x\left[p(x)\vee q(x)\right]$

$\exists x\ \neg p(x)$

d) $\forall x\left[\neg q(x)\vee r(x)\right]$

$\forall x\left[s(x)\to\neg r(x)\right]$

$\therefore \exists x\ \neg s(x)$

13. Determine which of the following arguments are valid or not
    a) [Universe is presently residing people of Bengaluru]. All people concerned about the environment, recycle their plastic containers. Rahul is not concerned about the environment. Therefore, Rahul does not recycle his plastic containers.
    b) [Universe is set of all triangles]. In triangle XYZ, there is no pair of angles of equal measure. If the triangle has 2 sides of equal length, then it is isosceles. If the triangle is isosceles, then it has two angles of equal measure. Therefore triangle XYZ has no two sides of equal length.
    c) [Set of all students studying under VTU]. No engineering student is bad in studies. Anil is not bad in studies. Therefore Anil is an engineering student.
    d) [Universe is presently studying students of BIT]. No engineering student of first and second semester studies logic. Anil is an engineering student who studies logic. Therefore Anil is not in second semester.
    e) [Universe is presently residing people of India]. All law-abiding citizens pay their taxes. Ms. Menakshi pays her taxes. Therefore Ms Menakshi is a law-abiding citizen.
    f) [Universe is presently residing people of India]. All librarian know the library of congress classification system. Mr. Gangadhar is a librarian. Therefore Gangadhar knows the library of congress classification system.
    g) [Universe –all adults who are presently residing in the city of cruces (New Mexico). Two of these individuals are Roxe and Imogene]. All credit union employees must know COBOL. All credit union employees who write loan applications must know Excel. Roxe works for the credit union, but she doesn’t know Excel. Imogene knows excel but she doesn’t know COBOL. Therefore Roxe doesn’t write loan applications and Imogene doesn’t work for the credit union.
14. Give the i) an indirect proof and ii) proof by contradiction for the following:
    a) For all real numbers $x$ and $y$, if $x+y>100$, then $x>50$ or $y>50$.
    b) The product of two even integers is even
    c) For an integer $n$, if $n^2$ is odd, then $n$ is odd.
15. Give the direct proof of the following:
    a) The square of an odd integer is odd integer
    b) For all integers $k$ and $l$, if $k$ and $l$ are both odd, then $k+l$ is even and $kl$ is odd.
    c) For all integers $k$ and $l$, if $k$ and $l$ are both even, then $k+l$ is even and $kl$ is even.
    d) For all integers $m$ and $n$, if $m$ and $n$ are perfect squares, then $mn$ is also perfect square.
    e) For positive integers $a,b$ and $c$, if $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$.
    f) If an integer $a$ is such that $a-2$ is divisible by 3, then $a^2-1$ is divisible by 3.
16. Give: i) a direct proof ii) an indirect proof and iii) proof by contradiction for the following:
    a) If $m$ is an even integer, then $m+7$ is odd. b) If $m$ is an odd integer, then $m+9$ is even.
17. a) Prove that for every integer $n$, $n^2$ is even if and only if $n$ is even.
    b) Prove that for every integer $n$, $n$ is odd if and only if $7n+8$ is odd.
18. a) Prove that every even integer $m$ with $2\le m\le 26$ can be written as a sum of at most three perfect squares.
    b) Prove that every even integer $m$ with $4\le m\le 38$ can be written as a sum of two primes.
19. a) Prove that there exists a positive integers $m$ and $n$ such that $m$, $n$ and $m+n$ are all perfect squares.
    b) Prove that there exists an integer $n$ such that $n^2=n$.
    c) Prove that there exists a real number $x$ such that $x^2-3x+2=0$.

20. Disprove the following by contradiction:
    a) The square of an odd integer is even
    b) The sum of two odd integers is an odd integer.
21. Disprove the following by counter example
    a) For all integers $m$ and $n$, if $m$ and $n$ are perfect squares, then $m+n$ is also perfect square.
    b) For all integers $m$ and $n$, $m^2 = n^2$ if and only if $m = n$.
    c) There is no function which is equal to its derivative.