## MC9242 RESOURCE MANAGEMENT TECHNIQUES L T P C 3 0 0 3

### UNIT I LINEAR PROGRAMMING MODELS 9

Mathematical Formulation - Graphical Solution of linear programming models – Simplex method – Artificial variable Techniques- Variants of Simplex method

### UNIT II TRANSPORTATION AND ASSIGNMENT MODELS 9

Mathematical formulation of transportation problem- Methods for finding initial basic feasible solution – optimum solution - degeneracy – Mathematical formulation of assignment models – Hungarian Algorithm – Variants of the Assignment problem

### UNIT III INTEGER PROGRAMMING MODELS 9

Formulation – Gomory's IPP method – Gomory's mixed integer method – Branch and bound technique.

### UNIT IV SCHEDULING BY PERT AND CPM 9

Network Construction – Critical Path Method – Project Evaluation and Review Technique – Resource Analysis in Network Scheduling

### UNIT V QUEUEING MODELS 9

Characteristics of Queuing Models – Poisson Queues - (M / M / 1) : (FIFO / ∞ /∞), (M / M / 1) : (FIFO / N / ∞), (M / M / C) : (FIFO / ∞ / ∞), (M / M / C) : (FIFO / N / ∞) models.

**Total No. of Periods : 45**

### TEXT BOOKS

1. Taha H.A., "Operations Research : An Introduction " 7th Edition, Pearson Education, 2004.

### REFERENCES

1. A.M.Natarajan, P.Balasubramani, A.Tamilarasi, "Operations Research", Pearson Education, Asia, 2005.
2. Prem Kumar Gupta, D.S. Hira, "Operations Research", S.Chand & Company Ltd, New Delhi, 3rd Edition , 2003.

# MC9242 – Resource Management Techniques

Introduction:- The first formal activities of Operation Research (OR) were initiated in England during world war II, when a team of British scientists set out to make scientifically based decisions regarding the best utilization of war material. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

## Scope or uses or Applications of O.R

O.R is useful for solving

1, Resource allocation problems
2, Inventory control problems
3, maintenance and Replacement problems
4, sequencing and scheduling problems
5) Assignment of Jobs to applicants to maximise total profit, or minimize total cost
6, Transportation problems etc.

Defn:-

1, Operations Research is a scientific approach to problem solving for executive management.

2, Operations Research is a scientific ~~model~~ method of providing executive departments with a quantitative basis for decisions regarding the operations under their control

## Unit - I

## Linear programming models

Linear programming deals with optimization of a function of variables known as objective functions. It is subject to a set of linear equalities and / or inequalities known as constraints. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality

## Mathematical Formulation of LPP

If $x_j$ $(j=1,2,\ldots,n)$ are the $n$ decision variables of the problem and if the system is subject to $m$ constraints, the general mathematical model can be written in the form:

Optimize $z = f(x_1, x_2, \ldots, x_n)$

subject to $g_i(x_1, x_2, \ldots, x_n) \le, =, \ge b_i, \; (i=1,2,\ldots,m)$

(called structural constraints)

and $x_1, x_2, \ldots, x_n \ge 0$.

(called the non-negativity restrictions or constraints)

## Procedure for forming a LPP model:

Step 1: Identify the unknown decision variables to be determined and assign symbols to them.

Step 2: Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.

Step 3: Identify the objective or aim and represent it also as a linear function of decision variables

Step 4: Express the complete formulation of LPP as a general mathematical model.

Note:

We consider only those situations where this will help the reader to put proper inequalities in the formulation.

i, usage of manpower, time, raw materials etc are always less than or equal to the availability of manpower, time, raw materials etc.

2. Production is always greater than or equal to the requirement so as to meet the demand.

(1) A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines $M_1$ and $M_2$. Type A requires 1 minute of processing time on $M_1$ and 2 minutes on $M_2$. Type B requires 1 minute on $M_1$ and 1 minute on $M_2$. Machine $M_1$ is available for not more than 6 hours 40 minutes while machine $M_2$ is available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the profit.

Soln

Let the firm decide to produce $x_1$ units of product A and $x_2$ units of product B to maximize its profit.

To produce these units of type A and type B products, it requires $x_1 + x_2$ processing minutes on $M_1$

$2x_1 + x_2$ processing minutes on $M_2$

Since machine $M_1$ is available for not more than 6 hours and 40 minutes and machine B is available for 10 hours doing any working day, the contraints are

$$x_1 + x_2 \le 400$$

$$2x_1 + x_2 \le 600$$

and $x_1, x_2 \ge 0$.

Since the profit from type A is Rs. 2 and profit from type B is Rs 3, the total profit is $2x_1 + 3x_2$.

As the objective is to maximize the profit, the objective function is maximize $Z = 2x_1 + 3x_2$.

∴ The complete formulation of the LPP is

maximize $Z = 2x_1 + 3x_2$

~~Sto~~. Subject to the constraints

$$x_1 + x_2 \le 400$$

$$2x_1 + x_2 \le 600$$

and $x_1, x_2 \ge 0$.

2. Reddy mikks produces both, interior and exterior paints from two raw materials, $M_1$ and $M_2$. the following table provides, the basic data of the problem.

| | Tons of raw material per ton of Exterior paint | Interior paint | maximum daily availability |
|---|---|---|---|
| Raw material, $M_1$ | 6 | 4 | 24 |
| Raw material, $M_2$ | 1 | 2 | 6 |
| profit per ton | | | |

A Market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily of interior paint is, 2 tons.

Reddy mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Soln

$x_1$ = Tons produced daily of exterior paint

$x_2$ = Tons produced daily of interior paint.

$$\max z = 5x_1 + 4x_2$$

s.t.c
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0.$$

Graphical Solution:

The graphical procedure includes two steps:

1, Determination of the feasible solution space.

2, Determination of the optimum solution from among all the feasible points in the solution space.

Problems:

1. Solve the following LPP by the graphical method.

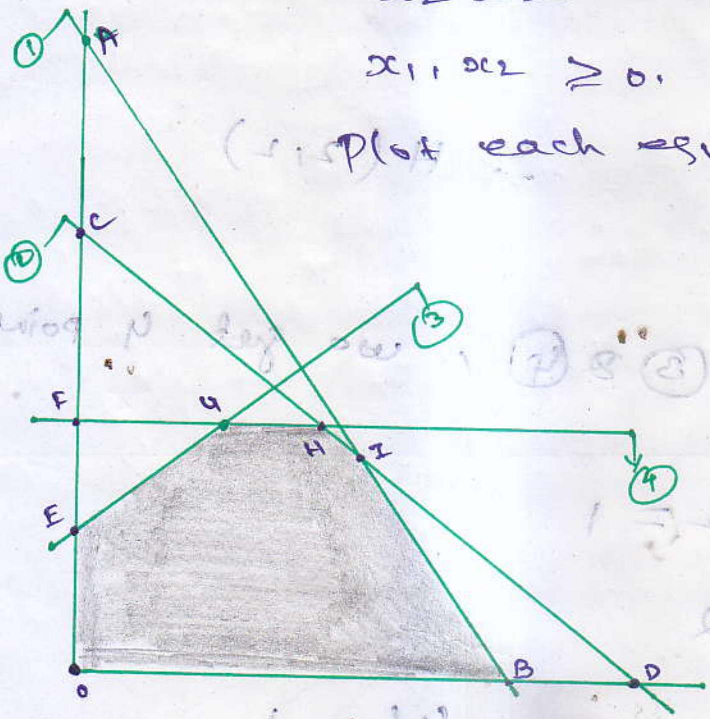
$$\text{Max } Z = 5x_1 + 4x_2$$

s.t.c
$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Soln:
$$\max Z = 5x_1 + 4x_2$$

s.t.c
$$6x_1 + 4x_2 = 24 \quad \text{— (1)}$$
$$x_1 + 2x_2 = 6 \quad \text{— (2)}$$
$$-x_1 + x_2 = 1 \quad \text{— (3)}$$
$$x_2 = 2 \quad \text{— (4)}$$
$$x_1, x_2 \ge 0.$$

(i) Plot each equation on the graph.




Solving the equations (1) & (2), we get point I

(1) × 1 $\quad 6x_1 + 4x_2 = 24$

(2) × 6 $\quad \overset{(-)}{6x_1} \overset{(-)}{+} 12x_2 \overset{(-)}{=} 36$

$$-8x_2 = -12$$

$$x_2 = -12/ \qquad x_2 = 3/ = 1.5$$

Sub in ②

$x_1 + \cancel{2}(3/\cancel{2}) = 6$

$x_1 + 3 = 6$

$x_1 = 6 - 3$

$x_1 = 3$

$I(3, 1.5)$

Solving the equations ② & ④, we get H points

$x_1 + 2x_2 = 6$

$x_2 = 2$

$x_1 + 2(2) = 6$

$x_1 + 4 = 6$

$x_1 = 6 - 4$ $\therefore H(2, 2)$

$x_1 = 2$

Solving the equation ③ & ④, we get G points

$-x_1 + x_2 = 1$

$x_2 = 2$

$-x_1 + 2 = 1$ $G(1, 2)$

$-x_1 = 1 - 2$

$-x_1 = -1$

$x_1 = 1$

| Corner point | $(x_1, x_2)$ | Z |
|---|---|---|
| O | (0,0) | 0 |
| B | (4,0) | 20 |
| I | (3, 1.5) | 21 (optimum) |
| H | (2,2) | 18 |
| E | (0,1) | 4 |
| G | (1,2) | 13 |

Since the problem is of maximization type, the optimum solution is

$$\text{Max } Z = 21, \quad x_1 = 3, \quad x_2 = 1.5.$$

2. Solve the following LPP by graphical method.

$$\text{minimize } Z = 20x_1 + 10x_2$$

s.t.c
$$x_1 + 2x_2 \le 40$$
$$3x_1 + x_2 \ge 30$$
$$4x_1 + 3x_2 \ge 60$$
$$x_1, x_2 \ge 0.$$

Soln

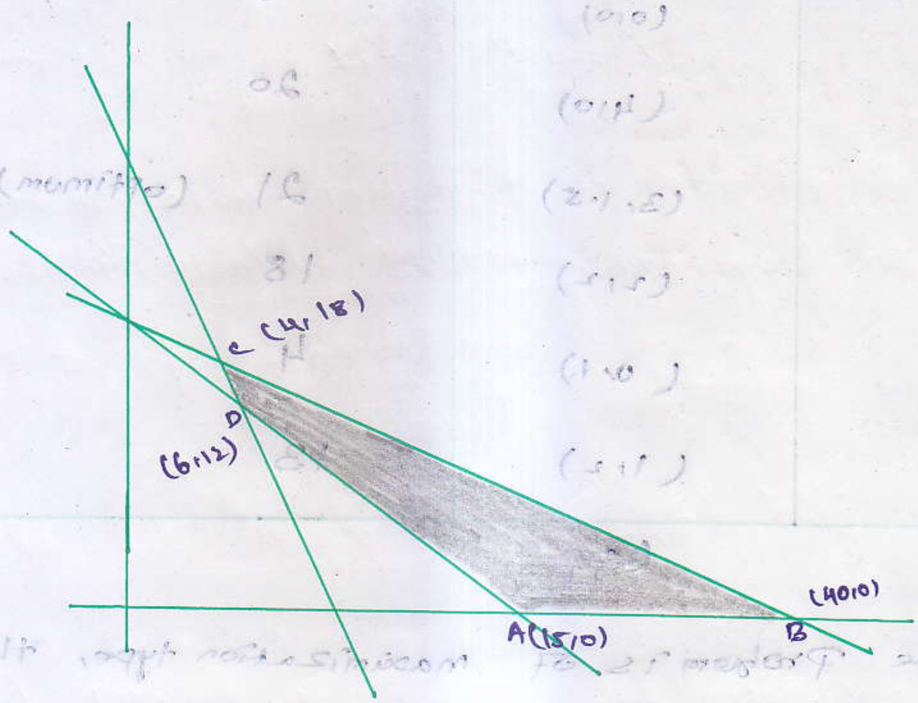
Replace all the inequalities of the constraints by equation

$x_1 + 2x_2 = 40$ passes through (0,20) (40,0)

$3x_1 + x_2 = 30$ passes through (0,30) (10,0)

$4x_1 + 3x_2 = 60$ passes through (0,20) (15,0)

Plot each equation on the graph.




The feasible region is ABCD.

C and D are points of intersection of lines

C intersect $x_1 + 2x_2 = 40$ $3x_1 + x_2 = 30$

D intersect $4x_1 + 3x_2 = 60$; $x_1 + x_2 = 30$

$C = (4, 18)$

$D = (6, 12)$

| Corner points | $(x_1, x_2)$ | Z |
|---|---|---|
| A (15, 0) | (15, 0) | 300 |
| B (40, 0) | (40, 0) | 800 |
| C (4, 18) | (4, 18) | 260 |
| D (6, 12) | (6, 12) | 240 (minimum value) |

Since the problem is of minimum type, the optimum soln. is

3, Use graphical method to solve the LPP.

maximize $z = 3x_1 + 2x_2$

S.T.C $5x_1 + x_2 \geq 10$

$x_1 + x_2 \geq 6$

$x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0.$

Soln

$\max z = 3x_1 + 2x_2$

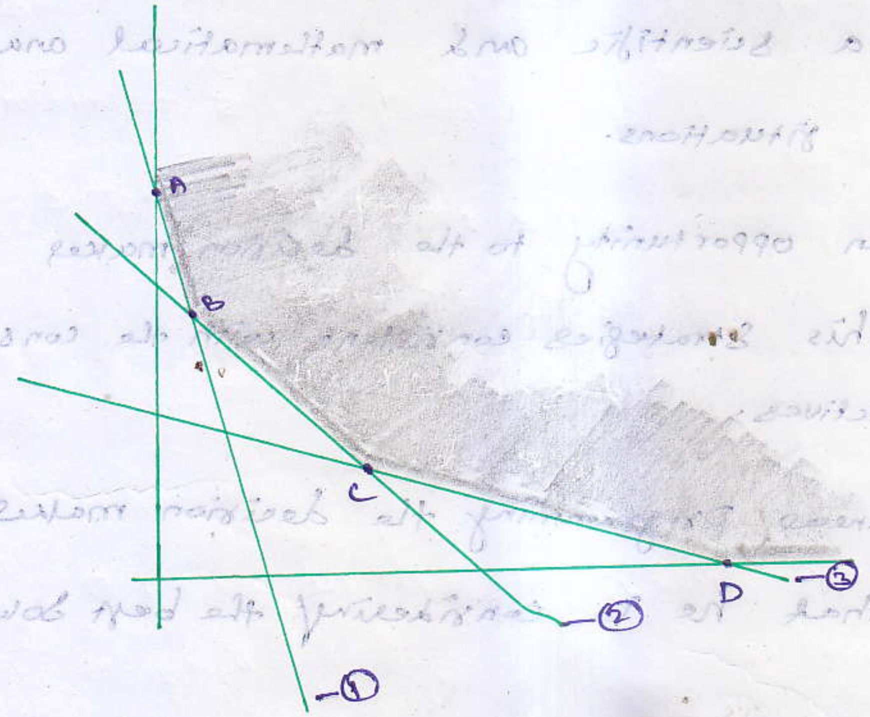
S-T-C $5x_1 + x_2 = 10$ — ①

$x_1 + x_2 = 6$ — ②

$x_1 + 4x_2 = 12$ — ③

$x_1, x_2 \geq 0.$

Plot each equation on the graph.




| Corner points | Value of $z = 3x_1 + 2x_2$ |
|---|---|
| A (0, 10) | 20 |
| B (1, 5) | 13 (minimum value) |
| C (4, 2) | 16 |
| D (12, 0) | 36 |

Since the minimum value is attained at B (1,5) the optimum solution is $x_1 = 1, x_2 = 5$

**Note** In the above problem if the objective function is maximization, then the solution is unbounded as maximum value of Z occurs at infinity.

## Advantage of Linear programming.

1. It provides an insight and perspective in to the problem environment. this generally results in clear picture of the true problem.

2. It makes a scientific and mathematical analysis of the problem situations.

3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.

4. By using Linear programming the decision maker makes sure that he is considering the best solution.

## Simplex Algorithm:

General Linear programming problem

The linear programming involving more than two variables may be expressed as follows.

maximize (or) minimize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to the constraints.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le \text{ or } = \text{ or } \ge b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le \text{ or } = \text{ or } \ge b_2$$

$$\vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le \text{ or } = \text{ or } \ge b_n$$

**Defn:** A set of values $x_1, x_2, \ldots x_n$ which satisfies the constraints of the LPP is called its Solution.

**Defn:** Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution.

**Defn:** Any feasible solution which optimizes (maximizes or minimizes) the objective function of the LPP is called its optimum solution or optimal solution.

**Defn:** If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i = 1, 2, 3, \ldots k) \quad \text{①}$$

then the non-negative variables $S_i$ which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ji} x_j + s_i = b_i \quad (i = 1, 2, 3, \ldots k)$$

are called slack variables.

Defn: If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad (i = k, k+1, \ldots) \quad \text{— (1)}$$

then the non-negative variables $s_i$ which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ji} x_j - s_i = b_i \quad (i = k, k+1, \ldots)$$

are called surplus variables.

Characteristics of the canonical form:

i) The objective function is of maximization type.

ii) All constraints are of ($\leq$) type.

iii) All variables $x_j$ are non-negative.

Characteristics of the standard form:

i) The objective function is of maximization type.

ii) All constraints are expressed as equations.

iii) RHS of each constraints is non-negative

iv) All variables are non-negative.

1) Express the following LPP in standard form

minimize $z = 5x_1 + 7x_2$

s.t.c $x_1 + x_2 \leq 8$

$3x_1 + 4x_2 \geq 3$

$6x_1 + 7x_2 \geq 5$, & $x_1, x_2 \geq 0$.

Soln

Since $\min z = -\max(-z) = -\max Z$

Given LPP becomes $\max Z = -5x_1 - 7x_2$

$x_1 + x_2 \leq 8$

$3x_1 + 4x_2 \geq 3$

$6x_1 + 7x_2 \geq 5$, & $x_1, x_2 \geq 0$

By introducing slack variable $s_1$ and surplus variables $s_2, s_3$ the standard form of the LPP is given by

$\max Z = -5x_1 - 7x_2$

s.t.c $x_1 + x_2 + s_1 = 8$

$3x_1 + 4x_2 - s_2 = 3$

$6x_1 + 7x_2 - s_3 = 5$

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

2, Express the following LPP in the canonical form.

$\max Z = 2x_1 + 3x_2 + x_3$

s.t.c $4x_1 - 3x_2 + x_3 \leq 6$

$x_1 + 5x_2 - 7x_3 \geq -4$

and $x_1, x_3 \geq 0$, $x_2$ is unrestricted

Soln

As $x_2$ is unrestricted

$x_2 = x_2' - x_2''$

$$\max z = 2x_1 + 3x_2' - 3x_2'' + x_3$$

s.t.c $\quad 4x_1 - 3x_2' + 3x_2'' + x_3 \le 6$

$\quad -x_1 - 5x_2' + 5x_2'' + 7x_3 \le 4$

$\quad \& \quad x_1, x_2', x_2'', x_3 \ge 0.$

which is in the canonical form.

The steps of the simplex method are

Step 1. Determine a starting basic feasible solution.

Step 2. Select an entering variable using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else, go to step 3.

Step 3. Select a leaving variable using the feasibility condition.

Step 4. Determine the new basic solution by using the appropriate Gauss – Jordan computations. Go to step 2.

Problem

① Use simplex method to solve the LPP

$$\max z = 4x_1 + 10x_2$$

s.t.c $\quad 2x_1 + x_2 \le 50$

$\quad 2x_1 + 5x_2 \le 100$

$\quad 2x_1 + 3x_2 \le 90$

Soln

By introducing the slack variables $s_1, s_2, \& s_3$, the problem in standard form becomes

$$\text{Max } z = 4x_1 + 10x_2$$

S.T.C

$$2x_1 + x_2 + s_1 = 50$$

$$2x_1 + 5x_2 + s_2 = 100$$

$$2x_1 + 3x_2 + s_3 = 90$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Initial Simplex table

| | Basic | z | $x_1$ | $x_2$ ↓ | $s_1$ | $s_2$ | $s_3$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| R1 | z | 1 | −4 | −10 | 0 | 0 | 0 | 0 | Ratio |
| R2 | $s_1$ | 0 | 2 | 1 | 1 | 0 | 0 | 50 | $50/1 = 50$ |
| R3 | ← $s_2$ | 0 | 2 | $\boxed{5}$ | 0 | 1 | 0 | 100 | $100/5 = 20$ |
| R4 | $s_3$ | 0 | 2 | 3 | 0 | 0 | 1 | 90 | $90/3 = 30$ |

$R_1' \to R_1 + 10 \times R_3'$ $\qquad R_3' \to R_3/5$

$R_2' \to R_2 - 1 \times R_3'$ $\qquad R_4' \to R_4 - 3 \times R_3'$

1st Iteration: $x_2$ is Entering variable, $s_2$ is Leaving variable

| Basic | z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| z | 1 | 0 | 0 | 0 | 2 | 0 | 200 |
| $s_1$ | 0 | 8/5 | 0 | 1 | −1/5 | 0 | 30 |
| $x_2$ | 0 | 2/5 | 1 | 0 | 1/5 | 0 | 20 |
| $s_3$ | 0 | 4/5 | 0 | 0 | −3/5 | 1 | 30 |

Since all Z-row $\geq 0$ the current basic feasible solution is optimal

$\therefore$ The optimal solution is

| |
|---|
| $\max Z = 200$ |
| $x_1 = 0$ <br> $x_2 = 20$ |

(2) Solve the following LPP by simplex method

minimize $Z = 8x_1 - 2x_2$

S.T.C $-4x_1 + 2x_2 \leq 1$

$5x_1 - 4x_2 \leq 3$

& $x_1, x_2 \geq 0$.

Soln

Since the given objective function is of minimization type, we shall convert it in to a maximization type as follows:

Maximize $(z) = -$minimize $Z = -8x_1 + 2x_2$

S.T.C $-4x_1 + 2x_2 \leq 1$

$5x_1 - 4x_2 \leq 3$

$x_1, x_2 \geq 0$.

Introducing slack variables $s_1, s_2$, the standard form

$\max Z = -8x_1 + 2x_2$

S.T.C

$-4x_1 + 2x_2 + s_1 = 1$

$5x_1 - 4x_2 + s_2 = 3$

## Initial iteration:

| | Basic | $Z$ | $x_1$ | $x_2$ ↓ | $s_1$ | $s_2$ | RHS | |
|---|---|---|---|---|---|---|---|---|
| $R_1$ | $Z$ | 1 | +8 | -2 | 0 | 0 | 0 | Ratio |
| $R_2$ | ← $S_1$ | 0 | -4 | $\boxed{2}$ | 1 | 0 | 1 | $\frac{1}{2}$ → |
| $R_3$ | $s_2$ | 0 | 5 | -4 | 0 | 1 | 3 | -ve |

$x_2$ is Entering variables, $s_1$ is leaving variable

$R_1 \to R_1 + 2 \times R_2'$ $\qquad R_3 \to R_3 + 4 \times R_2'$

$R_2' \to R_2/2$

## 1st Iteration:

| Basic | $Z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|---|
| $Z$ | 1 | 4 | 0 | 1 | 0 | 1 |
| $x_2$ | 0 | -2 | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| $s_2$ | 0 | -3 | 0 | 2 | 1 | 5 |

Since all $Z$ row $\geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

$\boxed{\max z = 1, \; x_1 = 0, \; x_2 = \frac{1}{2}}$ But $\boxed{\min z = -1, \; x_1 = 0, \; x_2 = \frac{1}{2}}$

# Artificial Variable Techniques

To solve a LPP by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems, we see that the slack variable readily provided the initial basic feasible solution.

However, in some problems, the slack variables can not provide the initial basic feasible solution. In these problems atleast one of the constraints is of $=$ or $\geq$ type. To solve such Linear programming problems, there are two methods available.

I) Big m-method or m-technique or method of penalties.

II) Two phase method.

# Big M- method

(1) Use Penalty (or Big-M) method to

minimise $z = 4x_1 + 3x_2$

S.T.C $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0.$

Soln

Convert the problem into maximisation type

then objective function becomes.

$$\max z = -4x_1 - 3x_2$$

S.T.C $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

Introducing surplus, slack, Artificial variable,

Standard form.

$$\max z = -4x_1 - 3x_2 - MA_1 - MA_2$$

S.T.C $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 - S_3 + A_2 = 6$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

where $S_1, S_3$ are surplus variables

$S_2$ are slack variable

$A_1, A_2$ are artificial variable

## Initial Iteration:-

| | Basic | Z | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $R_1$ | Z | 1 | $-3M+4$ | $-2M+3$ | M | 0 | M | 0 | 0 | $-16M$ | |
| $R_2$ | $A_1$ | 0 | $\boxed{2}$ | 1 | $-1$ | 0 | 0 | 1 | 0 | 10 | $10/2$ |
| $R_3$ | $S_2$ | 0 | $-3$ | 2 | 0 | 1 | 0 | 0 | 0 | 6 | $-ve$ |
| $R_4$ | $A_2$ | 0 | 1 | 1 | 0 | 0 | $-1$ | 0 | 1 | 6 | $6/1$ |

$x_1$ is Entering variable, $A_1$ is Leaving variables

$R_1' \longrightarrow R_1 + 3M-4\,(R_2')$  $R_3' \longrightarrow R_3 + 3 \times R_2'$

$R_2' \longrightarrow R_2/2$  $R_4' \longrightarrow R_4 - R_2'$

## 1st Iteration:

| | Basic | Z | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_2$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_1'$ | Z | 1 | 0 | $\frac{-M+2}{2}$ | $\frac{-M+4}{2}$ | 0 | M | 0 | $-M-20$ | |
| $R_2'$ | $x_1$ | 0 | 1 | $1/2$ | $-1/2$ | 0 | 0 | 0 | 5 | 10 |
| $R_3'$ | $S_2$ | 0 | 0 | $7/2$ | $-3/2$ | 1 | 0 | 0 | 21 | $42/7$ |
| $R_4'$ | $A_2$ | 0 | 0 | $\boxed{1/2}$ | $1/2$ | 0 | $-1$ | 1 | 1 | 2 |

$x_2$ is Entering Variable, $A_2$ is Leaving variables

$R_1'' \longrightarrow R_1 + \frac{M-2}{2} \times R_4''$  $R_2'' \longrightarrow R_2 - \frac{1}{2} R_4$

$R_3'' \rightarrow R_3' - \frac{7}{2} \times R_4''$

$R_4'' \rightarrow 2R_4'$.

2nd Iteration:

| Basic | z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| z | 1 | 0 | 0 | 1 | 0 | 2 | -22 |
| $x_1$ | 0 | 1 | 0 | -1 | 0 | 1 | 4 |
| $s_2$ | 0 | 0 | 0 | -5 | 1 | 7 | 14 |
| $x_2$ | 0 | 0 | 1 | 1 | 0 | -2 | 2 |

Since all z-row $\geq 0$, the current basic feasible solution is optimal.

$\boxed{\begin{array}{l} Max\ z = -22 \\ x_1 = 4 \\ x_2 = 2 \end{array}}$ But $\boxed{\begin{array}{l} Min\ z = 22 \\ x_1 = 4 \\ x_2 = 2 \end{array}}$

(2) Use penalty method to

$$Max\ z = 2x_1 + x_2 + x_3$$

$$\text{s.t.c} \quad 4x_1 + 6x_2 + 3x_3 \leq 8$$

$$3x_1 - 6x_2 - 4x_3 \leq 1$$

$$2x_1 + 3x_2 - 5x_3 \geq 4$$

$$x_1, x_2, x_3 \leq 0.$$

Soln

By introducing slack variables $s_1, s_2$, and

Surplus variable $s_3$, and artificial variable $A_1$.

Standard form:

$$\max z = 2x_1 + x_2 + x_3 - M A_1$$

s.t.c
$$4x_1 + 6x_2 + 3x_3 + s_1 = 8$$

$$3x_1 - 6x_2 - 4x_3 + s_2 = 1$$

$$2x_1 + 3x_2 - 5x_3 - s_3 + A_1 = 4$$

$$x_1, x_2, x_3, s_1, s_2, s_3, A_1 \geq 0$$

Initial Iteration:

| | Basic | z | $x_1$ | $x_2$ ↓ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | $A_1$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_1$ | z | 1 | $-2M-2$ | $-3M-1$ | $5M-1$ | 0 | 0 | M | 0 | $-4M$ |
| $R_2$ | $s_1$ | 0 | 4 | 6 | 3 | 1 | 0 | 0 | 0 | 8 |
| $R_3$ | $s_2$ | 0 | 3 | $-6$ | $-4$ | 0 | 1 | 0 | 0 | 1 |
| $R_4$ | ← $A_1$ | 0 | 2 | $\boxed{3}$ | $-5$ | 0 | 0 | $-1$ | 1 | 4 |

$x_2$ is Entering variable, $A_1$ is Leaving variable

$R_1' \rightarrow R_1 + 3M+1 \times R_4'$ $\quad$ $R_3' \rightarrow R_3 + 6 \times R_4'$

$R_2' \rightarrow R_2 - 6 \times R_4'$ $\quad$ $R_4 \rightarrow R_4 - 3 \times R_4'$

## 1st Iteration:-

| Basic | $z$ | $x_1$ | $x_2$ | $x_3$ ↓ | $s_1$ | $s_2$ | $s_3$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | $-4/3$ | 0 | $-8/3$ | 0 | 0 | $-1/3$ | $4/3$ | ratio |
| ← $s_1$ | 0 | 0 | 0 | $\boxed{13}$ | 1 | 0 | 2 | 0 | 0 ✓ |
| $s_2$ | 0 | 7 | 0 | $-14$ | 0 | 1 | $-2$ | 9 | -ve |
| $x_2$ | 0 | $2/3$ | 1 | $-5/3$ | 0 | 0 | $-1/3$ | $4/3$ | -ve |

$x_3$ is Entering variable, $s_1$ is Leaving variable

$R_1'' \to R_1' + 8/3 \times R_2''$ $\qquad$ $R_3'' \to R_3' + 14 \times R_2''$

$R_2'' \to R_2'/13$ $\qquad$ $R_4'' \to R_4' + 5/3 \times R_2''$

## 2nd Iteration

| Basic | $z$ | $x_1$ ↓ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | $-4/3$ | 0 | 0 | $8/39$ | 0 | $1/13$ | $4/3$ | 0 |
| $x_3$ | 0 | 0 | 0 | 1 | $1/13$ | 0 | $2/13$ | 0 | – |
| ← $s_2$ | 0 | $\boxed{7}$ | 0 | 0 | $4/13$ | 1 | $2/13$ | 9 | $9/7$ |
| $x_2$ | 0 | $2/3$ | 1 | 0 | $5/39$ | 0 | $-1/13$ | $4/3$ | 2 |

$x_1$ is Entering variable, $s_2$ is Leaving variable

$R_1''' \to R_1'' + 4/3\ R_3'''$ $\qquad$ $R_3''' \to R_3''/7$

$R_2''' \to R_2''$ $\qquad$ $R_4''' \to R_4'' - 2/3 \times R_3'''$

3rd Iteration :-

| Basic | Z | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | 0 | 0 | $16/39$ | $4/21$ | $\frac{29}{273}$ | $64/21$ |
| $x_3$ | 0 | 0 | 0 | 1 | $1/13$ | 0 | $2/13$ | 0 |
| $x_1$ | 0 | 1 | 0 | 0 | $2/13$ | $\frac{1}{7}$ | $2/91$ | $9/7$ |
| $x_2$ | 0 | 0 | 1 | 0 | $1/39$ | $-2/21$ | $\frac{-25}{273}$ | $10/21$ |

Since all Z-row $\geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is

$$\text{Max } Z = \frac{64}{21}$$

$$x_1 = 9/7$$

$$x_2 = 10/21$$

$$x_3 = 0$$

## Two - Phase Simplex Method :-

The two phase method is another method to solve a given problem in which some artifical variables are involved. The solution is obtained in two phases as follows.

**Phase I** Put the problem in equation form, and add the necessary artificial variables to the constraints to secure a starting basic solution. Next, find a basic solution of the resulting equations that, regardless of whether the LP is maximization or minimization, always minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process. otherwise, proceed to phase II.

**Phase II** use the feasible solution from phase I as a starting basic feasible solution for the original problem.

(1) Use Two - phase simplex method to solve

maximize $z = 5x_1 + 8x_2$

S.T.C

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0.$

Soln

By introducing the non-negative slack, surplus and artifical variables, the standard form of the LPP becomes

Max $z = 5x_1 + 8x_2$

s.t.c

$$3x_1 + 2x_2 - s_1 + R_1 = 3$$

$$x_1 + 4x_2 - s_2 + R_2 = 4$$

$$x_1 + x_2 + s_3 = 5$$

$$x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0.$$

Phase-I :-

Assigning a cost $-1$ to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } z = -R_1 - R_2$$

Subject to the given constraints

Initial iteration

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $R_1$ | $R_2$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | $-4$ | $-6$ | 1 | 1 | 0 | 0 | 0 | $-7$ |
| $R_1$ | 0 | 3 | 2 | $-1$ | 0 | 0 | 1 | 0 | 3 |
| $R_2$ | 0 | 1 | $\boxed{4}$ | 0 | $-1$ | 0 | 0 | 1 | 4 |
| $s_3$ | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 5 |

$x_2$ is Entering variable

$R_2$ is Leaving variable

$R_1' \longrightarrow R_1 + 6R_3'$ $\qquad$ $R_2' \longrightarrow R_2 - 2R_3'$

$R_3' \to R_3/4$ $\qquad$ $R_4' \longrightarrow R_4 - R_3'$

1st iteration

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $R_1$ | $R_2$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | $-5/2$ | 0 | 1 | $-1/2$ | 0 | 0 | $3/2$ | $-1$ | Ratio |
| $R_1$ | 0 | $\boxed{5/2}$ | 0 | $-1$ | $1/2$ | 0 | 1 | $-1/2$ | 1 | $2/5 \to$ |
| $x_2$ | 0 | $1/4$ | 1 | 0 | $-1/4$ | 0 | 0 | $1/4$ | 1 | 4 |
| $s_3$ | 0 | $3/4$ | 0 | 0 | $1/4$ | 1 | 0 | $-1/4$ | 4 | $16/3$ |

$x_1$ is Entering variable . $R_1$ is Leaving variable

$R_1'' \longrightarrow R_1' + 5/2 \times R_2''$ $\qquad$ $R_3'' \longrightarrow R_3' - \frac{1}{4} \times R_2''$

$R_2'' \longrightarrow 2/5 \times R_2'$ $\qquad$ $R_4'' \longrightarrow R_4' - 3/4 \times R_2''$

2nd Iteration

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $R_1$ | $R_2$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| $x_1$ | 0 | 1 | 0 | $-2/5$ | $1/5$ | 0 | $2/5$ | $-1/5$ | $2/5$ |
| $x_2$ | 0 | 0 | 1 | $1/10$ | $-3/10$ | 0 | $-1/10$ | $3/10$ | $9/10$ |
| $s_3$ | 0 | 0 | 0 | $3/10$ | $1/10$ | 1 | $-3/10$ | $-1/10$ | $\frac{37}{10}$ |

Since all Z-row $\geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so we proceed to phase-II.

Phase-II

Here, we consider the actual costs associated with the original variables.

The new objective function then becomes

$$\max z = 5x_1 + 8x_2$$

The initial basic feasible solution for this phase is the one obtained at the end of phase-I.

Initial iteration

| Basic | $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS | Ratio |
|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 0 | $-6/5$ | $-7/5$ | 0 | $46/5$ | |
| $x_1$ | 0 | 1 | 0 | $-2/5$ | $\boxed{1/5}$ | 0 | $2/5$ | 2 |
| $x_2$ | 0 | 0 | 1 | $1/10$ | $-3/10$ | 0 | $9/10$ | |
| $s_3$ | 0 | 0 | 0 | $3/10$ | $1/10$ | 1 | $37/10$ | 37 |

$s_2$ is Entering variable

$x_1$ is Leaving variable

$R_1' \to R_1 + 7/5 \times R_2'$ $\qquad$ $R_3' \to R_3 + 3/10 \times R_2'$

$R_2' \to 5 \times R_2$ $\qquad$ $R_4' \to R_4 - 1/10 \times R_2'$

## 1st Iteration :-

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS | Ratio |
|---|---|---|---|---|---|---|---|---|
| Z | 1 | 7 | 0 | -4 | 0 | 0 | 12 | |
| $s_2$ | 0 | 5 | 0 | -2 | 1 | 0 | 2 | |
| $x_2$ | 0 | $3/2$ | 1 | $-1/2$ | 0 | 0 | $3/2$ | — |
| $s_3$ | 0 | $-1/2$ | 0 | $\boxed{1/2}$ | 0 | 1 | $7/2$ | 7 |

$R_1'' \rightarrow R_1' + 4R_4''$ $\qquad R_3'' \rightarrow R_3' + \frac{1}{2} \times R_4''$

$R_2'' \rightarrow R_2' + 2R_4''$ $\qquad R_4'' \rightarrow 2R_4'$

$s_1$ is Entering variable , $s_3$ is Leaving variable

## 2nd Iteration :-

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| Z | 1 | 3 | 0 | 0 | 0 | 8 | 40 |
| $s_2$ | 0 | 3 | 0 | 0 | 1 | 4 | 16 |
| $x_2$ | 0 | 1 | 1 | 0 | 0 | 1 | 5 |
| $s_1$ | 0 | -1 | 0 | 1 | 0 | 2 | 7 |

Since all Z-row $\geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is

$Max Z = 40, \quad x_1 = 0, \quad x_2 = 5.$

## Variants of the Simplex Method

Here we present certain complications and variations encountered in the application of the simplex method and how they are resolved. These are called the variants of the simplex method. The following variants are being considered

1. Degeneracy and cycling
2. Unbounded solution
3. Multiple solution
4. Non-existing feasible solution
5. Unrestricted variables.

## Degeneracy and cycling:-

The concept of obtaining a degenerate basic feasible solution in a LPP is known as Degeneracy.

As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as cycling or circling.

① Solve the following LPP by simplex method:

$$\max Z = x_1 + 2x_2 + x_3$$

s.t.c $\quad 2x_1 + x_2 - x_3 \leq 2$

$$-2x_1 + x_2 - 5x_3 \geq -6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Soln

$$\max Z = x_1 + 2x_2 + x_3$$

s.t.c $\quad 2x_1 + x_2 - x_3 \leq 2$

$$2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0.$$

By Introducing slack variables, the standard form of LPP becomes.

$$\max Z = x_1 + 2x_2 + x_3$$

s.t.c $\quad 2x_1 + x_2 - x_3 + s_1 = 2$

$$2x_1 - x_2 + 5x_3 + s_2 = 6$$

$$4x_1 + x_2 + x_3 + s_3 = 6$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

where $s_1, s_2, s_3$ are slack variables.

## Initial Iteration:

| Basic | Z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | -1 | -2 | -1 | 0 | 0 | 0 | 0 | Ratio |
| $s_1$ | 0 | 2 | $\boxed{1}$ | -1 | 1 | 0 | 0 | 2 | 2 |
| $s_2$ | 0 | 2 | -1 | 5 | 0 | 1 | 0 | 6 | — |
| $s_3$ | 0 | 4 | 1 | 1 | 0 | 0 | 1 | 6 | 6 |

$x_2$ is Entering variable, $s_1$ is Leaving variables

## 1st iteration:

| Basic | Z | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | 3 | 0 | -3 | 2 | 0 | 0 | 4 | Ratio |
| $x_2$ | 0 | 2 | 1 | -1 | 1 | 0 | 0 | 2 | — |
| $s_2$ | 0 | 4 | 0 | 4 | 1 | 1 | 0 | 8 | 2 |
| $s_3$ | 0 | 2 | 0 | 2 | -1 | 0 | 1 | 4 | 2 |

$x_3$ Entering variable. Since both the basic variable $s_2$ and $s_3$ having same minimum ratio 2, there is a tie in selecting the leaving variable. To resolve this degeneracy, we divide each entry corresponding to basic variables $s_2$ & $s_3$ and then corresponding to non-basic variables $x_1, x_2, x_3$ and $S_1$.

| | $S_2$ | $S_3$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ |
|---|---|---|---|---|---|---|
| Row 2: | $1/4$ | $0/4$ | $4/4$ | $0/4$ | $4/4$ | $1/4$ |
| Row 3 | $0/2$ | $1/2$ | $2/2$ | $0/2$ | $2/2$ | $-1/2$ |

The column wise comparison of quotients starting with basic variables $S_2$ and $S_3$, we find that column $S_2$ gives algebraically smaller ratio: 0 for Row 3 and as such Row 3 is selected as key row. So the basic variable $S_3$ leaves the basis.

2nd iteration.

| Basic | Z | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| Z | 1 | 6 | 0 | 0 | $1/2$ | 0 | $3/2$ | 10 |
| $x_2$ | 0 | 3 | 1 | 0 | $1/2$ | 0 | $1/2$ | 4 |
| $S_2$ | 0 | 0 | 0 | 0 | 3 | 1 | $-2$ | 0 |
| $x_3$ | 0 | 1 | 0 | 1 | $-1/2$ | 0 | $1/2$ | 2 |

Since all Z-row $\geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is $\boxed{\max Z = 10}$

$x_1 = 0$

$x_2 = 4$

## Unbounded Solution:-

In Some Linear Programming Problems, the Solution Space becomes unbounded so that the value of the objective function also can be increased Indefinitely without a limit. But It is wrong to conclude that Just because the Solution Space is unbounded the Solution also is unbounded. The Solution Space may be unbounded but the Solution may be finite.

(1) Solve the following LPP by Simplex method.

$$\max z = 2x_1 + x_2$$

S.T.C

$$x_1 - x_2 \le 10$$

$$2x_1 - x_2 \le 40$$

$$\& \; x_1, x_2 \ge 0$$

Soln/ Std. form

$$\max z = 2x_1 + x_2$$

S.T.C

$$x_1 - x_2 + s_1 = 10$$

$$2x_1 - x_2 + s_2 = 40$$

$$s_1, s_2, x_1, x_2 \ge 0$$

Initial iteration:-

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS | |
|---|---|---|---|---|---|---|---|
| Z | 1 | -2 | -1 | 0 | 0 | 0 | Ratio |
| $s_1$ | 0 | $\boxed{1}$ | -1 | 1 | 0 | 10 | $10/1 = 10$ |
| $s_2$ | 0 | 2 | -1 | 0 | 1 | 40 | $40/2 = 20$ |

$x_1$ is Entering variable

$s_1$ is Leaving variable.

1st iteration:-

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS | |
|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | -3 | 2 | 0 | 20 | Ratio |
| $x_1$ | 0 | 1 | -1 | 1 | 0 | 10 | — |
| $s_2$ | 0 | 0 | $\boxed{1}$ | -2 | 1 | 20 | $20/1 = 20$ |

$x_2$ is Entering variable, $s_2$ is Leaving variable

2nd iteration:-

| Basic | Z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS | |
|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | 0 | -4 | 3 | 80 | Ratio |
| $x_1$ | 0 | 1 | 0 | -1 | 1 | 30 | -ve |
| $x_2$ | 0 | 0 | 1 | -2 | 1 | 20 | -ve |

Since Z-row = -4 < 0, the current basic feasible solution is not optimal.

Also, since all ratio < 0, it is not possible to find the positive ratio.

ie, It is not possible to find the leaving variable.

The solution of this problem is unbounded.

3. Multiple solutions

In some linear programming problems the optimal solution need not be unique. There ~~are~~ may be alternative or infinite number of solutions ie, with different product mixes, we have the same value of the objective function.

4 Non - existing feasible solution :-

In an LPP, when there is no point belonging to the solution space satisfying all the constraints, then the problem is said to have no feasible solution. In other words, in the optimum simplex table, if atleast one artificial variable appears in the basis at non-zero level and the optimality condition is satisfied, then the problem is said to have no feasible solution.

5, Unrestricted variables :-

In an LPP, If any variable is unconstrained (without specifying its sign) it can be expressed as the difference between two non-negative variables. The problem can be converted in to an equivalent one involving only non-negative variables.

① Solve the LPP.

$$\text{Maximize } Z = 8x_2$$

S.T.C

$$x_1 - x_2 \geq 0$$

$$2x_1 + 3x_2 \leq -6$$

and $x_1, x_2$ are unrestricted

**Soln** Since $x_1, x_2$ are unrestricted, we put $x_1 = x_1' - x_1''$

and $x_2 = x_2' - x_2''$ so that $x_1', x_1'', x_2', x_2'' \geq 0$

$$\max z = 8x_2' - 8x_2''$$

S.T.C

$$x_1' - x_1'' - x_2' + x_2'' \geq 0$$

$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' \geq 6$$

$$x_1', x_1'', x_2', x_2'' \geq 0.$$

Standard form:-

$$\max z = 8x_2' - 8x_2'' - MR_1 - MR_2$$

S.T.C

$$x_1' - x_1'' - x_2' + x_2'' - S_1 + R_1 = 0$$

$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - S_2 + R_2 = 6$$

$$x_1', x_1'', x_2', x_2'', S_1, S_2 \geq 0, R_1, R_2 \geq 0$$

Initial Iteration:-

| Basic | $z$ | $x_1'$ | $x_1''$ | $x_2'$ | $x_2''$ | $S_1$ | $S_2$ | $R_1$ | $R_2$ | RHS |
|---|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | $M$ | $-M$ | $4M-8$ | $-M+8$ | $M$ | $M$ | 0 | 0 | $-6M$ |
| $R_1$ | 0 | 1 | $-1$ | $-1$ | $\boxed{1}$ | $-1$ | 0 | 1 | 0 | 0 |
| $R_2$ | 0 | $-2$ | 2 | $-3$ | 3 | 0 | $-1$ | 0 | 1 | 6 |

$x_2''$ is Entering variable

$R_1$ is Leaving variable

1st iteration:

| Basic | z | $x_1'$ | $x_1''$ | $x_2'$ | $x_2''$ | $s_1$ | $s_2$ | $R_2$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| z | 1 | 5M-8 | -5M+8 | 0 | 0 | -3M+8 | M | 0 | -6M |
| $x_2''$ | 0 | 1 | -1 | -1 | 1 | -1 | 0 | 0 | 0 |
| $R_2$ | 0 | -5 | $\boxed{5}$ | 0 | 0 | 3 | -1 | 1 | 6 |

$x_1''$ is Entering variable

$R_2$ is Leaving variable.

2nd iteration.

| Basic | z | $x_1'$ | $x_1''$ | $x_2'$ | $x_2''$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|---|---|---|
| z | 1 | 0 | 0 | 0 | 0 | 16/5 | 8/5 | $\frac{-48}{5}$ |
| $x_2''$ | 0 | 0 | 0 | -1 | 1 | -2/5 | -1/5 | 6/5 |
| $x_1'$ | 0 | -1 | 1 | 0 | 0 | 3/5 | -1/5 | 6/5 |

Since all z-row $\geq 0$, and no artificial variable appears in the basis, the current basic feasible solution is optimal.

$\therefore$ optimal solution $Max\ z = \frac{-48}{5}$ $x_1' = 0$

$x_2' = 0$, $x_1' = 6/5$, $x_2'' = 6/5$

But $x_1 = x_1' - x_1''$

$= 0 - 6/5 = -6/5$ $\quad \therefore x_1 = -6/5$

$$x_2 = x_2' - x_2''$$

$$= 0 - 6/5$$

$$x_2 = -6/5$$

$$\therefore \boxed{\begin{array}{c} \text{Max } z = -\frac{48}{5} \\ \hline x_1 = -6/5,\ x_2 = -6/5 \end{array}}$$