VECTOR DIFFERENTIATION

Vector function: Let s be the set of real numbers for scalar t ∈ s, if there exists a unique vector $\overline{f}$ then $\overline{f}$ is said to be a vector function.

The vector function of a scalar variable 't' denoted by $\overline{f}$(t) which can be expressed in terms of mutually orthogonal unit vectors $\overline{i}, \overline{j}, \overline{k}$ as

$$\overline{f}(\text{t}) = f_1(\text{t})\overline{i} + f_2(\text{t})\overline{j} + f_3(\text{t})\overline{k}$$

Derivative of a vector function:

If $\lim_{t\to a} \frac{\overline{f}(t)-\overline{f}(a)}{t-a}$, t ≠ a and a ∈ I exists then the limit is called as derivative of $\overline{f}$ at t = a and is denoted by $\left(\frac{d\overline{f}}{dt}\right)_{t=a}$ or f '(a)

Properties:

If $\overline{A}$ and $\overline{B}$ are differentiable vector functions of a scalar variable t , then

i) $\frac{d}{dt}(\overline{A} \pm \overline{B}) = \frac{d\overline{A}}{dt} \pm \frac{d\overline{B}}{dt}$

ii) $\frac{d}{dt}(\overline{A}.\overline{B}) = \frac{d\overline{A}}{dt}.\ \overline{B} \pm \overline{A}.\frac{d\overline{B}}{dt}$

iii) $\frac{d}{dt}(\overline{A} \text{ x } \overline{B}) = \frac{d\overline{A}}{dt} \text{ x } \overline{B} + \overline{A} \text{ x } \frac{d\overline{B}}{dt}$

iv) $\frac{d}{dt}(\emptyset\overline{A}) = \frac{d\emptyset}{dt}\overline{A} + \emptyset\,\frac{d\overline{A}}{dt}$

v) If $\overline{c}$ is any constant vector , then $\frac{d}{dt}(\overline{c}) = 0$

Note: If $\overline{f} = f_1(\text{t})\overline{i} + f_2(t)\overline{j} + f_3(t)\overline{k}$ then

$$\frac{d\overline{f}}{dt} = \overline{i}\,\frac{df_1}{dt} + \overline{j}\,\frac{df_2}{dt} + \overline{k}\,\frac{df_3}{dt}$$

Partial derivative:

If $\overline{f}$ is a vector depending on more than one similar variable then the partial derivative of $\overline{f}$ w.r.t x is defined as

$$\frac{\partial\overline{t}}{\partial x} = \lim_{\Delta x\to 0} \frac{\overline{f}(x+\Delta x,y,z)-\overline{f}(x,y,z)}{\Delta x}$$

Similarly $\frac{\partial\overline{f}}{\partial y} = \lim_{\Delta y\to 0} \frac{\overline{f}(x,y+\Delta y,z)-\overline{f}(x,y,z)}{\Delta y}$

Properties:

If $\overline{f}$ and $\overline{g}$ are vector functions, $\emptyset$ is a scalar function of several variables then

i) $\frac{\partial}{\partial t}(\overline{f} \pm \overline{g}) = \frac{\partial \overline{f}}{\partial t} \pm \frac{\partial \overline{g}}{\partial t}$

ii) $\frac{\partial}{\partial t}(\overline{f}.\overline{g}) = \overline{f}.\frac{\partial \overline{g}}{\partial t} + \overline{g}.\frac{\partial \overline{f}}{\partial t}$

iii) $\frac{\partial}{\partial t}(\overline{f} \times \overline{g}) = \overline{f} \times \frac{\partial \overline{g}}{\partial t} + \frac{\partial \overline{f}}{\partial t} \times \overline{g}$

iv) $\frac{\partial}{\partial t}(\emptyset \overline{f}) = \emptyset \frac{\partial \overline{f}}{\partial t} + \frac{\partial \overline{\emptyset}}{\partial t}\overline{f}$

v) If $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$ then

$$\frac{\partial \overline{f}}{\partial t} = \frac{\partial f_1}{\partial t}\overline{i} + \frac{\partial f_2}{\partial t}\overline{j} + \frac{\partial f_3}{\partial t}\overline{k}$$

Vector Differential Equation:

The vector differential operator $\nabla(del)$ is defined as

$$\nabla = \overline{i}\frac{\partial}{\partial x} + \overline{j}\frac{\partial}{\partial y} + \overline{k}\frac{\partial}{\partial z}$$

Gradient: Let $\emptyset(x, y, z)$ be a scalar point function then the vector function $\overline{i}\frac{\partial \emptyset}{\partial x} + \overline{j}\frac{\partial \emptyset}{\partial y} + \overline{k}\frac{\partial \emptyset}{\partial z}$ is called gradient of a function and is denoted by $\nabla\emptyset$ or $grad\emptyset$

Directional derivative:

Let $\emptyset(x, y, z)$ be a scalar point function, the function defined on the direction of a vector then the directional derivative can be defined as $\nabla\emptyset.\overline{e}$

where $\overline{e}$ is the unit normal vector.

Unit normal vector:

If $\overline{f} = f_1\overline{i} + f_2\overline{j} + f_3\overline{k}$ then the unit normal vector can be defined as

$$\overline{e} = \frac{\overline{f}}{|\overline{f}|}$$

Problems

1. Find grad $\emptyset$ when e
   i) $\emptyset = x^3 + y^3 + 3xyz$ at (1,1,-2)
   ii) $\emptyset = x^2y + y^2x + z$ at (1,1,1)

Sol: i) Given $\emptyset = x^3 + y^3 + 3xyz$

$$\text{Grad}\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^3 + y^3 + 3xyz) + \bar{j}\frac{\partial}{\partial y}(x^3 + y^3 + 3xyz) + \bar{k}\frac{\partial}{\partial z}(x^3 + y^3 + 3xyz)$$

$$= \bar{i}(3x^2 + 3yz) + \bar{j}(3y^2 + 3xz) + \bar{k}(3xy)$$

$$\text{Grad}\emptyset_{(1,1,-2)} = \bar{i}\big(3(1^2) + 3(1)(-2)\big) + \bar{j}\big(3(1^2) + 3(1)(-2)\big) + \bar{k}\big(3(1)(1)\big)$$

$$= \bar{i}(3-6) + \bar{j}(3-6) + 3\bar{k}$$

$$\text{Grad}\emptyset = -3\bar{i} - 3\bar{j} + 3\bar{k}$$

ii) Given $\emptyset = x^2y + y^2x + z$

$$\text{grad}\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^2y + y^2x + z) + \bar{j}\frac{\partial}{\partial y}(x^2y+y^2x + z) + \bar{k}\frac{\partial}{\partial z}(x^2y + y^2x + z)$$

$$= \bar{i}(2xy + y^2)+\bar{j}(x^2 + 2xy) + \bar{k}(1)$$

$$\text{Grad}\emptyset_{(1,1,1)} = \bar{i}(2+1) + \bar{j}(2+1) + \bar{k}$$

$$= 3\bar{i}+3\bar{j}+\bar{k}$$

2. Find $\nabla\emptyset$,where $\emptyset = \log(x^2 + y^2+z^2)$

Sol: Given $\emptyset = \log(x^2 + y^2 + z^2)$

$$\nabla\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(\log(x^2 + y^2 + z^2))+\bar{j}(\log(x^2 + y^2 + z^2))+ \bar{k}(\log(x^2 + y^2 + z^2))$$

$$=\bar{i}\frac{1}{x^2+y^2+z^2}(2x)+\bar{j}\frac{1}{x^2+y^2+z^2}(2y)+\bar{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$=\frac{1}{x^2+y^2+z^2}\left(2x\bar{i}+2y\bar{j}+2z\bar{k}\right)$$

3. Prove that $\nabla(r^n)=nr^{n-2}\bar{r}$

Sol: Let $\bar{r}$=x$\bar{i}$+y$\bar{j}+z\bar{k}$

$$\mathrm{r}=|\bar{r}|=\sqrt{x^2+y^2+z^2}$$

$$\mathrm{r}=(x^2+y^2+z^2)^{1/2}$$

diff w.r.t x partially we get

$$2\mathrm{r}\frac{\partial r}{\partial x}=2x$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

We have, $\nabla r^n=\bar{i}\frac{\partial}{\partial x}(r^n)+\bar{j}\frac{\partial}{\partial y}(r^n)+\bar{k}\frac{\partial}{\partial z}(r^n)$

$$=\sum\bar{i}\frac{\partial}{\partial x}(r^n)$$

$$=\sum\bar{i}\,n.r^{n-1}.\frac{\partial r}{\partial x}$$

$$=\sum\bar{i}\,n.r^{n-1}\frac{x}{r}$$

$$=\sum\bar{i}\,n.r^{n-2}x$$

$$=n.r^{n-2}\sum x\bar{i}$$

$$=n.r^{n-2}\left(x\bar{i}+y\bar{j}+z\bar{k}\right)$$

$$=n.r^{n-2}\bar{r}$$

4. If a=x+y+z , b=$x^2+y^2+z^2$, $c=xy+yz+zx$ then prove that $[grad\ a\ ,grad\ b,grad\ c]=0$

Sol: Given a=x+y+z b=$x^2+y^2+z^2$ $c=xy+yz+zx$

$$grad\ a=\bar{i}+\bar{j}+\bar{k} \qquad grad\ b=2x\bar{i}+2y\bar{j}+2z\bar{k}$$

$c = xy + yz + zx$

$= (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

then $[grad\ a\ , grad\ b, grad\ c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+z \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} \quad R_3 \to R_3 + R_2$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0$$

5. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at (1,-2,-1) in the direction of a vector $2\bar{i}$-$\bar{j}$-$2\bar{k}$

Sol: Given $\emptyset = x^2yz + 4xz^2$

$$\text{Grad}\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= \bar{i}(2xyz + 4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y + 8xz)$$

$$grad\emptyset_{(1,-2,-1)} = \bar{i}(2(1)(-2)(-1) + 4(1^2)) + \bar{j}\big(1(-1)\big) + \bar{k}((1^2)(-2) + 8(-1)(1))$$

$$= 8\bar{i} - \bar{j} - 10\bar{k}$$

and also the given vector $\bar{f} = 2\bar{i}$-$\bar{j}$-$2\bar{k}$

unit normal vector $\bar{e} = \frac{\bar{f}}{|\bar{f}|} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{\sqrt{2^2+(-1)^2+(-2)^2}}$

$$= \frac{2\bar{i}-\bar{j}-2\bar{k}}{\sqrt{9}} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{3}$$

The required directional derivative along the vector $= \nabla\emptyset.\bar{e}$

$$= (8\bar{i} - \bar{j} - 10\bar{k}).(\frac{2\bar{i}-\bar{j}-2\bar{k}}{3})$$

$$= \frac{37}{3}$$

6. Find the directional derivative of $\emptyset = xy + yz + zx$ at A in the directional of $\overline{AB}$ where A(1,2,-1) and B(1,2,3)

Sol: Given $\emptyset = xy + yz + zx$

The position vectors of A and B w.r.t origin $\overline{OA} = \bar{i} + 2\bar{j} - \bar{k}$ and $\overline{OB} = \bar{i} + 2\bar{j} + 3\bar{k}$

$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$= (\bar{i} + 2\bar{j} + 3\bar{k}) - (\bar{i} + 2\bar{j} - \bar{k})$$

$$= 4\bar{k}$$

Let $\bar{e}$ be the unit normal vector

$$\bar{e} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{4\bar{k}}{4} = \bar{k}$$

and also given $\emptyset = xy + yz + zx$ at A(1,2,-1)

$$\text{grad}\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(x+y)$$

$$grad\emptyset_{(1,2,-1)} = \bar{i}(2\text{-}1) + \bar{j}(1-1) + \bar{k}(1+2)$$

$$\nabla\emptyset = \bar{i} + 3\bar{k}$$

Directional derivative of $\emptyset$ at A in the direction of $\overline{AB}$ is $\nabla\emptyset.\bar{e}$

$$= (\bar{i} + 3\bar{k}).\bar{k} = 3$$

7. Find the directional derivative of the function $\emptyset = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the normal to the surface $xlogz - y^2 + 4 = 0$ at (-1,2,1)

Sol: Given $\emptyset = xy^2 + yz^3$

$$\nabla\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= (y^2)\bar{i} + (2xy + z^3)\bar{j} + (3yz^2)\bar{k}$$

$$\nabla\emptyset_{(2,-1,1)} = \bar{i} + (2(2)(-1) + 1^3)\bar{j} + (3(-1)(1))\bar{k}$$

$$= \bar{i} - 3\bar{j} - 3\bar{k}$$

and also s= $xlogz - y^2 + 4$

$$\nabla s = \bar{i}\frac{\partial s}{\partial x} + \bar{j}\frac{\partial s}{\partial y} + \bar{k}\frac{\partial s}{\partial z}$$

$$= logz\bar{i} - 2y\bar{j} + \frac{x}{z}\bar{k}$$

$$(\nabla s)_{(-1,2,1)} = -4\bar{j} - \bar{k}$$

$$\bar{e} = \frac{\nabla s}{|\nabla s|} = \frac{-4\bar{j} - \bar{k}}{|-4\bar{j} - \bar{k}|} = \frac{-4\bar{j} - \bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j} - \bar{k}}{\sqrt{17}}$$

Directional derivative of $\emptyset$ is $\nabla\emptyset.\bar{e}$

$$= (\bar{i} - 3\bar{j} - 3\bar{k}).\left(\frac{-4\bar{j} - \bar{k}}{\sqrt{17}}\right)$$

$$= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Angle between two surfaces:

Let p be a point of intersection to the level surfaces f(x,y,z)=0, s(x,y,z)=0 then the angle between the normal to the surfaces f(x,y,z)=0, s(x,y,z)=0 at p is called the angle between surfaces at p.

Note:The vectors $\overline{n_1}$ and $\overline{n_2}$ are along the normal to the two surfaces at the point p, then the angle between two sufaces is $cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}$

1. Find the angle between the surfaces $x^2 + y^2+z^2 = 9$ and $\overline{z} = x^2 + y^2 - 3$ at the point (2,-1,2)

Sol: Let $\emptyset_1 = x^2 + y^2+z^2 - 9 = 0$

$\emptyset_2 = x^2 + y^2 - 3 = 0$ be two surfaces

$$\nabla\emptyset_1 = \overline{i}\frac{\partial\emptyset_1}{\partial x} + \overline{j}\frac{\partial\emptyset_1}{\partial y} + \overline{k}\frac{\partial\emptyset_1}{\partial z}$$

$$= 2x\overline{i} + 2y\overline{j} - 2z\overline{k}$$

$$(\nabla\emptyset_1)_{(2,-1,2)} = 4\overline{i} - 2\overline{j} + 4\overline{k}$$

$$\nabla\emptyset_2 = \overline{i}\frac{\partial\emptyset_2}{\partial x} + \overline{j}\frac{\partial\emptyset_2}{\partial y} + \overline{k}\frac{\partial\emptyset_2}{\partial z}$$

$$= 2x\overline{i} + 2y\overline{j} - \overline{k}$$

$$(\nabla\emptyset_2)_{(2,-1,2)} = 4\overline{i} - 2\overline{j} - \overline{k}$$

then the unit normal vectors are

$$\overline{n_1} = \frac{\nabla\emptyset_1}{|\nabla\emptyset_1|} = \frac{4\overline{i}-2\overline{j}+4\overline{k}}{\sqrt{4^2+2^2+4^2}} = \frac{4\overline{i}-2\overline{j}+4\overline{k}}{\sqrt{36}}$$

$$= \frac{4\overline{i}-2\overline{j}+4\overline{k}}{6}$$

$$\overline{n_2} = \frac{\nabla\emptyset_2}{|\nabla\emptyset_2|} = \frac{4\overline{i}-2\overline{j}-\overline{k}}{\sqrt{4^2+2^2+1^2}} = \frac{4\overline{i}-2\overline{j}-\overline{k}}{\sqrt{21}}$$

Let θ be the angle between two surfaces then $cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}$

$$=\frac{4\overline{i}-2\overline{j}+4\overline{k}}{6}.\frac{4\overline{i}-2\overline{j}-\overline{k}}{\sqrt{21}}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

2. Find the angle between the normal to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3)

Sol: Given $\emptyset = xy - z^2 = 0$

$$\nabla\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$\nabla\emptyset = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$\overline{n_1} = (\nabla\emptyset)_{(4,1,2)} = \bar{i} + 4\bar{j} - 4\bar{k}$$

$$\overline{n_2} = (\nabla\emptyset)_{(3,3,-3)} = 3\bar{i} + 3\bar{j} + 6\bar{k}$$

Angle between two surfaces is $cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}$

$$= \frac{(\bar{i}+4\bar{j}-4\bar{k}).(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1^2+4^2+(-4)^2}\sqrt{3^2+3^2+6^2}}$$

$$= \frac{3+12-24}{\sqrt{33}\sqrt{54}}$$

$$= \frac{-9}{\sqrt{33}\sqrt{54}}$$

3. Find the maximum value of the directional derivative of $\emptyset = 2x^2 - y - z^4$ at (2,-1,1)

Sol: Given $\emptyset = 2x^2 - y - z^4$

$$\nabla\emptyset = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$= 4x\bar{i} - \bar{j} - 4z^3\bar{k}$$

$$(\nabla\emptyset)_{(2,-1,1)} = 8\bar{i} - \bar{j} - 4\bar{k}$$

Maximum value of the directional derivative value in $|\nabla\emptyset|$

$$= \sqrt{8^2 + (-1)^2 + (-4)^2}$$

$$= 9$$

Divergence of a vector:

If $\overline{f}$ is a continuously differentiable vector point function then $\overline{i}\frac{\partial f}{\partial x}+\overline{j}\frac{\partial f}{\partial y}+\overline{k}\frac{\partial f}{\partial z}$ is called the divergence of a vector $\overline{f}$ and it is denoted by $\nabla\overline{f}=\overline{i}\frac{\partial f}{\partial x}+\overline{j}\frac{\partial f}{\partial y}+\overline{k}\frac{\partial f}{\partial z}$

Note: If $\overline{f}=f_1\overline{i}+f_2\overline{j}+f_3\overline{k}$ then

div $\overline{f}$ or $\nabla.\overline{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

If div$\overline{f}=0$ then $\overline{f}$ is said to be solenoidal vector.

1. Find div$\overline{f}$ where $\overline{f}=x\overline{i}+y\overline{j}+z\overline{k}$

Sol: div$\overline{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

$$=1+1+1$$

$$=3$$

$\therefore$ div$\overline{f}=3$

2. Find div$\overline{f}$,where $\overline{f}=grad(x^3+y^3+z^3-3xyz)$

Sol: Given $\overline{f}=grad(x^3+y^3+z^3-3xyz)$

$$=\overline{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+$$
$$\overline{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)+$$
$$\overline{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\overline{f}=\overline{i}(3x^2-3yz)+\overline{j}(3y^2-3xz)+\overline{k}(3z^2-3xy)$$

$$\text{div}\,\overline{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$$=\frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$=6x+6y+6z$$

3. Show that $3y^4z^2\overline{i}+z^3x^2\overline{j}-3x^2y^2\overline{k}$ in solenoidal

Sol: Given $\overline{f} = 3y^4z^2\overline{i} + z^3x^2\overline{j} - 3x^2y^2\overline{k}$

$$\text{div}\,\overline{f} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(z^3x^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

$\therefore$ div $\overline{f} = 0$

$\Rightarrow \overline{f}$ is said to be solenoid vector

4. Show that $\frac{\overline{r}}{r^3}$ is solenoidal

Sol: Let $\overline{r}$=x$\overline{i}$+y$\overline{j} + z\overline{k}$

$$\text{r} = |\overline{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

then $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

Given $\frac{\overline{r}}{r^3} = \frac{x}{r^3}\overline{i} + \frac{y}{r^3}\overline{j} + \frac{z}{r^3}\overline{k}$

$$\text{div}\left(\frac{\overline{r}}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$= \sum \left[\frac{r^3 - 3r^2(x/r)x}{r^6}\right]$$

$$= \sum \frac{r^3 - 3x^2r}{r^6}$$

$$= \frac{1}{r^6}[r^3 - 3rx^2 + r^3 - 3ry^2 + r^3 - 3rz^2]$$

$$= \frac{1}{r^6}[3r^3 - 3r(x^2 + y^2 + z^2)]$$

$$= \frac{1}{r^6}[3r^3 - 3r^3]$$

$$= 0$$

$\therefore \frac{\overline{r}}{r^3}$ is solenoidal

Curl of a vector:

If $\overline{f}$ is continuously differentiable vector point function then $\bar{i}\, X\, \frac{\partial \overline{f}}{\partial x} + \bar{j}\, X\, \frac{\partial \overline{f}}{\partial y} + \overline{k}\, X\, \frac{\partial \overline{f}}{\partial z}$ is called curl of $\overline{f}$ and it is denoted by $\nabla x \overline{f}$ or curl $\overline{f}$

i.e curl $\overline{f} = \bar{i}\, X\, \frac{\partial \overline{f}}{\partial x} + \bar{j}\, X\, \frac{\partial \overline{f}}{\partial y} + \overline{k}\, X\, \frac{\partial \overline{f}}{\partial z}$ or

$$\text{curl}\, \overline{f} = \begin{vmatrix} \bar{i} & \bar{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

If curl $\overline{f} = 0$ then $\overline{f}$ is said to be irrotational vector

If $\overline{f}$ is irrotational , there will exists a scalar function $\emptyset(x, y, z)$ such that, $\overline{f}$= grad∅ and this ∅ is called a scalar potential of $\overline{f}$

1. If $\overline{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\overline{k}$ find curl $\overline{f}$ at A(1,-1,1)

Sol: Given $\overline{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\overline{k}$

$$\text{curl}\, \overline{f} = \begin{vmatrix} \bar{i} & \bar{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \overline{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \bar{i}(-3z^2 - 2x^2y) - 0 + \overline{k}(4xyz - 2xy)$$

$$curl\overline{f}_{(1,-1,1)} = \bar{i}\big(-3(1^2) - 2(1^2)(-1)\big) + \overline{k}(4(1)(-1)(1) - 2(1)(-1))$$

$$= -\bar{i} - 2\overline{k}$$

2. Prove that $\text{curl}\bar{r} = 0$

Sol: Let $\bar{r} = \text{x}\bar{i} + \text{y}\bar{j} + z\bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= 0$$

$\therefore curl\bar{r} = 0$

$\therefore \bar{r}$ is irrotational

3. Show that the vectors $(\text{x}^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find its scalar potential

Sol: Let $\bar{f} = (\text{x}^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{x}^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \bar{i}(-x + x) - \bar{j}(-y + y) + \bar{k}(-z + z)$$

$$= 0$$

$\therefore curl\bar{f} = 0$

$\therefore \bar{f}$ is irrotational

If $\bar{f}$ is irrotational , there exists a scalar $\emptyset$ such that $\bar{f} = \nabla\emptyset$

$$(\text{x}^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k} = \bar{i}\frac{\partial\emptyset}{\partial x} + \bar{j}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

$$\frac{\partial\emptyset}{\partial x} = \text{x}^2 - yz \qquad \frac{\partial\emptyset}{\partial y} = y^2 - zx \qquad \frac{\partial\emptyset}{\partial z} = z^2 - xy$$

By total derivative formula,

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$= (x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz$$

Integrate on both sides

$$\int d\emptyset = \int(x^2 - yz)dx + \int(y^2 - zx)dy + \int(z^2 - xy)dz$$

$$\emptyset = \frac{x^3}{3} - xyz + \frac{y^3}{3} - xyz + \frac{z^3}{3} - xyz + c$$

$$\emptyset = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + c$$

repeated terms are considered as single term

4. Find the constants a,b,c so that $\overline{A} = (x + 2y + az)\overline{i} + (bx - 3y - z)\overline{j} + (4x + cy + 2z)\overline{k}$ is irrotational. Also find $\emptyset$ such that $\overline{A} = \nabla\emptyset$

Sol: Given $\overline{A} = (x + 2y + az)\overline{i} + (bx - 3y - z)\overline{j} + (4x + cy + 2z)\overline{k}$

$$\text{curl } \overline{A} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= \overline{i}(c + 1) - \overline{j}(a - 4) + \overline{k}(b - 2)$$

$\therefore$ given $\overline{A}$ is irrotational such that curl$\overline{A}$=0

$$(c + 1)\overline{i} + (a - 4)\overline{j} + (b - 2)\overline{k} = 0\overline{i} + 0\overline{j} + 0\overline{k}$$

$$\Rightarrow a = 4, b = 2, c = -1$$

$$\therefore \overline{A} = (x + 2y + 4z)\overline{i} + (2x - 3y - z)\overline{j} + (4x - y + 2z)\overline{k}$$

then $\overline{A} = \nabla\emptyset$

$$(x + 2y + 4z)\overline{i} + (2x - 3y - z)\overline{j} + (4x - y + 2z)\overline{k} = \overline{i}\frac{\partial\emptyset}{\partial x} + \overline{j}\frac{\partial\emptyset}{\partial y} + \overline{k}\frac{\partial\emptyset}{\partial z}$$

$$\frac{\partial\emptyset}{\partial x} = x + 2y + 4z, \quad \frac{\partial\emptyset}{\partial y} = 2x - 3y - z, \quad \frac{\partial\emptyset}{\partial z} = 4x - y + 2z$$

$$d\emptyset = (x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y + 2z)dz$$

Integrate on both sides

$$\int d\emptyset = \int(x + 2y + 4z)dx + \int(2x - 3y - z)dy + \int(4x - y + 2z)dz$$

$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + c$

5. Prove that div$(\overline{r}\ X\ \overline{a}) = 0$ where $\overline{a}$ is a constant vector

Sol: let $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ and $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$

$$\overline{r}\ X\ \overline{a} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \overline{i}(a_3y - a_2z) - \overline{j}(a_3x - a_1z) + \overline{k}(a_2x - a_1y)$$

$$\text{div}(\overline{r}\ X\ \overline{a}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(a_3y - a_2z) + \frac{\partial}{\partial y}\big(-(a_3x - a_1z)\big) + \frac{\partial}{\partial z}(a_2x - a_1y)$$

$$= 0$$

6. Show that if $\overline{f}$ is conservative then $\overline{f}$ is irrotational ?

Sol: The def of conservative vector function there exists a scalar function $\emptyset$ such that $\overline{f} = \nabla\emptyset$

$$\text{curl}\overline{f} = \nabla\ X\ (\nabla\emptyset)$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \overline{i}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial y\partial z}\right) - \overline{j}\left(\frac{\partial^2\emptyset}{\partial x\partial z} - \frac{\partial^2\emptyset}{\partial x\partial z}\right) + \overline{k}\left(\frac{\partial^2\emptyset}{\partial x\partial y} - \frac{\partial^2\emptyset}{\partial x\partial y}\right)$$

$$= 0$$

$\therefore \overline{f}$ is irrotational

Vector identities:

1. Let $\overline{f}$, $\emptyset$ be differentiable vector and scalar functions respectively then $\nabla.\left(\emptyset\overline{f}\right) = (\nabla\emptyset).\overline{f} + \emptyset(\nabla.\overline{f})$

Sol: Given $\nabla.\left(\emptyset\overline{f}\right) = \sum \overline{i}.\frac{\partial}{\partial x}\left(\emptyset\overline{f}\right)$

$$= \sum \overline{i}.\left(\frac{\partial\emptyset}{\partial x}.\overline{f} + \emptyset\frac{\partial\overline{f}}{\partial x}\right)$$

$$= \sum\left(\overline{i}.\frac{\partial\emptyset}{\partial x}\right)\overline{f} + \sum\left(\overline{i}.\frac{\partial\overline{f}}{\partial x}\emptyset\right)$$

$$= \sum\left(\overline{i}.\frac{\partial\emptyset}{\partial x}\right)\overline{f} + \sum\left(\overline{i}.\frac{\partial\overline{f}}{\partial x}\right)\emptyset$$

$$= \nabla\emptyset.\overline{f} + (\nabla.\overline{f})\emptyset$$

2. Prove that $\text{curl}\left(\emptyset\overline{f}\right) = \nabla\emptyset\, X\, \overline{f} + \emptyset(\nabla\, X\, \overline{f})$

Sol: $curl\left(\emptyset\overline{f}\right) = \nabla\, X\, \left(\emptyset\overline{f}\right)$

$$= \sum \overline{i}\, X\, \frac{\partial}{\partial x}\left(\emptyset\overline{f}\right)$$

$$= \sum \overline{i}\, X\, \left[\frac{\partial\emptyset}{\partial x}\overline{f} + \emptyset\frac{\partial\overline{f}}{\partial x}\right]$$

$$= \sum \overline{i}\, X\, \frac{\partial\emptyset}{\partial x}\overline{f} + \sum \overline{i}\, X\, \frac{\partial\overline{f}}{\partial x}\emptyset$$

$$= \sum\left(\overline{i}\, X\, \frac{\partial\emptyset}{\partial x}\right)\overline{f} + \sum\left(\overline{i}\, X\frac{\partial\overline{f}}{\partial x}\right)\emptyset$$

$$= \nabla\emptyset\, X\, \overline{f} + (\nabla\, X\, \overline{f})\emptyset$$

3. Prove that $\text{div}(\overline{A}\, X\, \overline{B}) = \overline{B}.curl\overline{A} - \overline{A}.curl\overline{B}$

Sol: $\text{div}(\overline{A}\, X\, \overline{B}) = \nabla.(\overline{A}\, X\, \overline{B})$

$$= \sum \overline{i}.\frac{\partial}{\partial x}(\overline{A}\, X\, \overline{B})$$

$$= \sum \overline{i}.\left(\frac{\partial\overline{A}}{\partial x}\, X\, \overline{B} + \overline{A}\, X\, \frac{\partial\overline{B}}{\partial x}\right)$$

$$= \sum \overline{i}.\left(\frac{\partial\overline{A}}{\partial x}\, X\, \overline{B}\right) - \sum \overline{i}.\left(\frac{\partial\overline{B}}{\partial x}\, X\, \overline{A}\right)$$

$$= \sum\left(\overline{i}\, X\, \frac{\partial\overline{A}}{\partial x}\right)\overline{B} - \sum\left(\overline{i}\, X\, \frac{\partial\overline{B}}{\partial x}\right)\overline{A}$$

$$= (\,curl\overline{A}).\overline{B} - (\,curl\overline{B}).\overline{A}$$

4. Prove that $curl(\overline{A}\,X\,\overline{B}) = \overline{A}div\overline{B} - \overline{B}div\overline{A} + (\overline{B}.\nabla)\overline{A} - (\overline{A}.\nabla)\overline{B}$

Sol: $curl(\overline{A}\,X\,\overline{B}) = \sum \bar{i}\,X\,\frac{\partial}{\partial x}(\overline{A}\,X\,\overline{B})$

$$= \sum \bar{i}\,X\,\left(\frac{\partial\overline{A}}{\partial x}\,X\,\overline{B} + \overline{A}\,X\,\frac{\partial\overline{B}}{\partial x}\right)$$

$$= \sum \bar{i}\,X\,\left(\frac{\partial\overline{A}}{\partial x}\,X\,\overline{B}\right) - \sum \bar{i}\,X\,\left(\frac{\partial\overline{B}}{\partial x}\,X\,\overline{A}\right)$$

$$= \sum\left[(\bar{i}.\overline{B})\frac{\partial\overline{A}}{\partial x} - (\bar{i}\,.\frac{\partial\overline{A}}{\partial x})\overline{B}\right] + \sum\left[\left(\bar{i}\,.\frac{\partial\overline{B}}{\partial x}\right)\overline{A} - (\bar{i}.\overline{A})\frac{\partial\overline{B}}{\partial x}\right]$$

$$\therefore \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$$

$$= \sum(\overline{B}.\bar{i})\frac{\partial\overline{A}}{\partial x} - \sum\left(\bar{i}.\frac{\partial\overline{A}}{\partial x}\right)\overline{B} + \sum\left(\bar{i}\,X\,\frac{\partial\overline{B}}{\partial x}\right)\overline{A} - \sum(\bar{i}.\overline{A})\frac{\partial\overline{B}}{\partial x}$$

$$= \sum(\overline{B}.\bar{i}.\frac{\partial\overline{A}}{\partial x})\,\overline{A} - \left(\sum\bar{i}.\frac{\partial\overline{A}}{\partial x}\right)\overline{B} + \sum\left(\bar{i}\,.\frac{\partial\overline{B}}{\partial x}\right)\overline{A} - \sum\left(\overline{A}.\bar{i}\frac{\partial}{\partial x}\right)\overline{B}$$

$$= (\overline{B}.\nabla)\overline{A} - (\nabla.\overline{A})\overline{B} + (\nabla.\overline{B})\overline{A} - (\overline{A}.\nabla)\overline{B}$$

$$= \overline{A}div - \overline{B}div\overline{A} + (\overline{B}.\nabla)\overline{A} - (\overline{A}.\nabla)\overline{B}$$

5. Prove that $\nabla f\ X\ \nabla g$ is solenoidal

Sol: w.k.t div$(\overline{A}\,X\,\overline{B}) = \overline{B}.curl\overline{A} - \overline{A}.curl\overline{B}$

Let $\overline{A}$=∇f and $\overline{B} = \nabla g$

$$div(\nabla f - \nabla g) = \nabla g.curl\nabla f - \nabla f.curl\nabla g$$

$$= 0 - 0$$

$$= 0$$

$$\therefore curl(grad\ g) = 0 = curl(grad\ f)$$

6. Let $\emptyset$, φ be two scalar functions then $div(\emptyset\nabla\varphi) = \emptyset\nabla^2\varphi + \nabla\emptyset.\nabla\varphi$

Sol: we have $\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$

$$\emptyset\nabla\varphi = \emptyset\frac{\partial\varphi}{\partial x}\bar{i} + \emptyset\frac{\partial\varphi}{\partial y}\bar{j} + \emptyset\frac{\partial\varphi}{\partial z}\bar{k}$$

$$div(\emptyset\nabla\varphi) = \frac{\partial}{\partial x}\left(\emptyset\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\emptyset\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\emptyset\frac{\partial\varphi}{\partial z}\right)$$

$$= \emptyset \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \emptyset}{\partial x}.\frac{\partial \varphi}{\partial x} + \emptyset \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial \emptyset}{\partial y}.\frac{\partial \varphi}{\partial y} + \emptyset \frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial \emptyset}{\partial z}.\frac{\partial \varphi}{\partial z}$$

$$= \emptyset \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}\right] + \left[\frac{\partial \emptyset}{\partial x}.\frac{\partial \varphi}{\partial x} + \frac{\partial \emptyset}{\partial y}.\frac{\partial \varphi}{\partial y} + \frac{\partial \emptyset}{\partial z}.\frac{\partial \varphi}{\partial z}\right]$$

$$= \emptyset \nabla^2 \varphi + \nabla \emptyset . \nabla \varphi$$

Second order differential operator:

The divergence of gradient of a function is called Laplacian operator and is denoted by ∇^2

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator

If $\nabla^2 \emptyset = 0$ then $\emptyset$ is called harmonic function

1. Prove that $\nabla^2 (r^n) = n(n+1) r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r^2 = x^2 + y^2 + z^2$$

then $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2 (r^n) = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(r^n)\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n.r^{n-1}.\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}[n.r^{n-2}.x]$$

$$= \sum n.\frac{\partial}{\partial x}[x r^{n-1}]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}\right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2}\right]$$

$$= \sum n[x^2(n-2)r^{n-4} + r^{n-2}]$$

$$= n\big((n-2)x^2 r^{n-4} + r^{n-2} + (n-2)y^2 r^{n-4} + r^{n-2} + (n-2)z^2 r^{n-4} + r^{n-2}\big)$$

$$= n[3r^{n-2} + (n-2)(x^2 + y^2 + z^2)r^{n-4}]$$

$$= n[3r^{n-2} + (n-2)r^{n-4}r^2]$$

$$= n[3r^{n-2} + (n-2)r^{n-2}]$$

$$= n(r^{n-2})[3 + n - 2]$$

$$\therefore\ \nabla^2(r^n) = n(n+1)r^{n-2}$$

2. Prove that $\nabla\left(\nabla.\frac{\overline{r}}{r}\right) = \frac{-2}{r^3}\overline{r}$

Sol: Let $\overline{r}$=x$\overline{i}$+y$\overline{j}$ $+ z\overline{k}$

$$r^2 = x^2 + y^2 + z^2$$

then $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla.\frac{\overline{r}}{r} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \sum\left[\frac{r.1 - x.\frac{\partial}{\partial x}}{r^2}\right]$$

$$= \sum\left[\frac{r - \frac{x^2}{r}}{r^2}\right]$$

$$= \sum\frac{1}{r^2}\left[r - \frac{x^2}{r}\right]$$

$$= \frac{1}{r^2}\left[r - \frac{x^2}{r} + r - \frac{y^2}{r} + r - \frac{z^2}{r}\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{1}{r}(x^2 + y^2 + z^2)\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{r^2}{r}\right]$$

$$= \frac{2r}{r^2}$$

$$= \frac{2}{r}$$

$$\nabla\left(\nabla.\frac{\overline{r}}{r}\right) = \overline{i}\frac{\partial}{\partial x}\left(\frac{2}{r}\right) + \overline{j}\frac{\partial}{\partial y}\left(\frac{2}{r}\right) + \overline{k}\frac{\partial}{\partial z}\left(\frac{2}{r}\right)$$

$$= \sum\overline{i}\left[\frac{-2}{r^2}\frac{\partial r}{\partial x}\right]$$

$$= \sum\overline{i}\left[\frac{-2x}{r^3}\right]$$

$$= \frac{-2}{r^3} \sum x\bar{i}$$

$$= \frac{-2}{r^3} [\mathrm{x}\bar{i} + \mathrm{y}\bar{j} + z\bar{k}]$$

$$= \frac{-2}{r^3} \bar{r}$$

3. Show that $\nabla\emptyset$ is both solenoidal and irrotational if $\nabla^2\emptyset = 0$

Sol: Given that $\nabla^2\emptyset = 0$

$div(\nabla\emptyset) = 0 \quad \Rightarrow \nabla\emptyset$ is solenoidal

$$curl(\nabla\emptyset) = \nabla\, X\, \nabla\emptyset$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial y\partial z}\right) - \bar{j}\left(\frac{\partial^2\emptyset}{\partial x\partial z} - \frac{\partial^2\emptyset}{\partial x\partial z}\right) + \bar{k}\left(\frac{\partial^2\emptyset}{\partial x\partial y} - \frac{\partial^2\emptyset}{\partial x\partial y}\right)$$

$$= 0$$

$\therefore$ $\nabla\emptyset$ is irrotational