1 Matrices

1.1 Introduction

The theory of matrices was first introduced by the well known mathematician, Arthur Cayley in the second half of the nineteenth century. For more than 65 years after its introduction, the application of it was never felt by any of the mathematicians. It's validity was first felt by mathematicians when Heisenberg used the theory of matrices in quantum mechanics. Its applications are plenty in the fields of almost all branches of science and engineering. One can notice the use of matrices in Astronomy, Mechanics, Nuclear physics, Electrical circuits, Differential equations, Computer graphics, Graph theory, Optimization techniques and so on.

Many research articles are noticed on the applications of the inverse of the matrices and eigen value properties on the study of equilibrium analysis of industrial organizations. A recent research paper in the year 2010 in one of the most popular research journals reveals a special class of M_k matrices which arise from strategic behaviour in industrial organizations and by suggesting a new inverse, lead directly to closed form expressions of strategic equilibria for arbitrary coalition structures in linear oligopolies, which include four classes of oligopoly equilibria, Bertrand equilibria for an arbitrary coalition structure, Cournot equilibria for an arbitrary coalition structure, Bertrand and Cournot equilibria with multi product firms which are previously unknown and now it will be useful to scholars in industrial organization.

Many applications of matrices can be cited on recent studies, for example in environmental analysis, Thermosetting composite fibres and Polyester via matrices. As matrices are widely used by engineers in industry, it is essential for every engineering graduate to have a thorough knowledge about matrices. As you are exposed already to the basic concepts of operations on matrices like addition, subtraction, scalar multiplication, multiplication of the matrices, let us now concentrate on further analysis on matrices. In the lower classes you are aware of the concepts of finding the inverse of a square matrix, solution of simultaneous equations using matrices, rank of a matrix. Let us concentrate on the evaluation of eigen values and its applications and move further to know more on matrix theory.

1.2 Eigen values and Eigen vectors of a Real Matrix

Definition

Let A be a square matrix of order n. A number λ is called an eigen value of A if there exists a nonzero column matrix X such that $AX = \lambda X$. Then X is called an eigen vector of A corresponding to λ.

Note

If λ is an eigen value and X is an eigen vector corresponding to λ, then $AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0$.

This is a homogeneous system of equations. It will have a nontrivial solution if $|A - \lambda I| = 0$. This equation is called the characteristic equation.
$|A - \lambda I|$ is called the characteristic polynomial of A. It is an n^{th} degree polynomial in λ. The roots of the characteristic equation are called the eigen values of A.
(Eigen in German means characteristic)
The following are simple ways to find the characteristic equation of a given square matrix.

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Results

1. If A is a 2×2 matrix, then the characteristic equation takes the form $\lambda^2 - s_1\lambda + s_2 = 0$ where s_1 = sum of the diagonal elements of A = trace of A $(tr(A))$ and $s_2 = |A|$.

2. If A is a square matrix of order 3, then the characteristic equation $|A-\lambda I| = 0$ takes the form $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where
   s_1 = sum of the main diagonal elements of A.
   s_2 = sum of the minors of the elements of the main diagonal.
   $s_3 = |A|$.

3. Sum of the eigen values of a matrix A is equal to the trace of the matrix.

4. Product of the eigen values $= |A|$.

Worked Examples

Example 1.1. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 2×2 matrix, the characteristic equation takes the form

$$\lambda^2 - s_1\lambda + s_2 = 0$$

where s_1 = sum of the diagonal elements of $A = 1+4 = 5$.

$$s_2 = |A| = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6.$$

$\therefore$ (1) becomes

$$\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda-6)(\lambda+1) = 0 \Rightarrow \lambda = -1, \lambda = 6.$$

$\therefore$ The eigen values are -1 and 6.

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Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector of A corresponding to λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{1}$$

when $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 2 & -2 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2}$$

$$2x_1 - 2x_2 = 0 \Rightarrow x_1 - x_2 = 0.$$

$$-5x_1 + 5x_2 = 0 \Rightarrow x_1 - x_2 = 0.$$

Both the equations are same. They are reduced to one single equation namely $x_1 = x_2 \Longrightarrow \frac{x_1}{1} = \frac{x_2}{1} \Longrightarrow x_1 = 1, x_2 = 1.$

$\therefore$ The eigen vector X_1 corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$

$$-5x_1 - 2x_2 = 0 \Rightarrow 5x_1 + 2x_2 = 0$$

$$-5x_1 - 2x_2 = 0 \Rightarrow 5x_1 + 2x_2 = 0.$$

We have the same situation as above. Both the equations are reduced to $5x_1 = -2x_2$

$$\Longrightarrow \frac{x_1}{2} = \frac{x_2}{-5}$$

$$\Longrightarrow x_1 = 2, x_2 = -5.$$

Hence, the eigen vector X_2 corresponding to the eigen value $\lambda = 6$ is $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

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Example 1.2. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

[Jan 1997]

Solution. Let $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 2×2 matrix, the characteristic equation takes the form

$$\lambda^2 - s_1\lambda + s_2 = 0$$

where, s_1 = sum of the main diagonal elements of $A = 4 + 2 = 6$.

$s_2 = |A| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5.$

The characteristic equation is $\lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = 1, \lambda = 5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector of A corresponding to λ.

$$\therefore (A - \lambda I)X = 0$$

$$\Longrightarrow \begin{pmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad (1)$$

When $\lambda = 1$, the equation (1) becomes $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$i.e., 3x_1 + x_2 = 0 \Rightarrow x_2 = -3x_1 \Rightarrow \frac{x_2}{-3} = \frac{x_1}{1}$$

$$\Longrightarrow x_1 = 1, x_2 = -3.$$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

When $\lambda = 5$,(1) becomes $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

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$$i.e., -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2 = 1$$

Hence, the eigen vector X_2 corresponding to the eigen value $\lambda = 5$ is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note. In the above example in case (i), if we take $x_1 = k$, we get $x_2 = -3k$.
The corresponding eigen vector is $\begin{pmatrix} k \\ -3k \end{pmatrix} = k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.
i.e., we get infinite number of eigen vectors corresponding to an eigen value.
i.e., the eigen vector corresponding to an eigen value is not unique.

Type-I: Evaluation of eigen vectors when all the eigen values are different.

Example 1.3. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$s_1 = tr(A) = 3 - 2 + 3 = 4.$

s_2 = sum of the minors of the main diagonal elements of A.

$= \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$

$s_2 = -6 + 4 + 9 - 4 - 6 + 4 = -2 + 5 - 2 = 1.$

—

$$s_3 = |A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 3(-2) + 4(-1) + 4 = -6 - 4 + 4 = -6.$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

i.e., $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$.

$\lambda = -1$ is a root.

By synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & 0 & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2, \lambda = 3.$$

The eigen values are $-1, 2, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to λ.

$$\therefore \qquad (A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$4x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - x_2 + 4x_3 = 0$$

—

$$x_1 - x_2 + 4x_3 = 0.$$

Taking the first two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -4 & \times & 4 & \times & 4 & \times & -4 \\ -1 & & 4 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{-12} = \frac{x_2}{-12} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 0.$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, (1) becomes

$$\begin{pmatrix} 1 & -4 & 4 \\ 1 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - x_2 + x_3 = 0.$$

Taking the last two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -4 & \times & 4 & \times & 1 & \times & -4 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{3}$$
$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$
$$\Rightarrow x_1 = 0, x_2 = 1, x_3 = 1.$$
$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 0 & -4 & 4 \\ 1 & -5 & 4 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$
$$-4x_2 + 4x_3 = 0 \Rightarrow x_2 = x_3$$
$$x_1 - 5x_2 + 4x_3 = 0 \Rightarrow x_1 - x_2 = 0$$
$$x_1 = x_2 \quad \Rightarrow x_1 = x_2 = x_3 = 1.$$
$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Example 1.4. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

[Jan 2013]

Solution. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$.

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Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where s_1 = sum of the main diagonal elements of A

$$s_1 = 1 - 1 - 1 = -1.$$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} -1 & 0 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 6 & -1 \end{vmatrix}$$

$$= 1 - 0 + (-1) + 1 + (-1 - 12)$$

$$= 1 + 0 - 13$$

$$= -12.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{vmatrix}$$

$$= 1(1 - 0) - 2(-6 - 0) + 1(-12 - 1)$$

$$= 1 + 12 - 13$$

$$= 0.$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 12\lambda = 0$$

$$\lambda(\lambda^2 + \lambda - 12) = 0$$

$$\lambda(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda = 0, -4, 3.$$

The eigen values are $0, -4, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e., \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\text{i.e., } x_1 + 2x_2 + x_3 = 0 \quad (2)$$

$$6x_1 - x_2 = 0 \quad (3)$$

$$-x_1 - 2x_2 - x_3 = 0. \quad (4)$$

(3) can be written as

$$6x_1 = x_2$$

$$x_1 = \frac{x_2}{6}$$

$$\Rightarrow x_1 = 1, x_2 = 6.$$

Substituting in (2) we obtain, $1 + 12 + x_3 = 0$

$$\Rightarrow x_3 = -13.$$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$.

When $\lambda = -4$, (1) becomes

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

—

$$\text{i.e., } 5x_1 + 2x_2 + x_3 = 0 \qquad (5)$$

$$6x_1 + 3x_2 = 0 \qquad (6)$$

$$-x_1 - 2x_2 + 3x_3 = 0. \qquad (7)$$

(6) can be written as

$$6x_1 = -3x_2$$

$$\frac{x_1}{-3} = \frac{x_2}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{-2}$$

$$\therefore x_1 = 1, x_2 = -2.$$

Substituting in (5) we obtain

$$5 - 4 + x_3 = 0$$

$$\Rightarrow x_3 = -1.$$

$\therefore$ The eigen vector is $X_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\text{i.e., } -2x_1 + 2x_2 + x_3 = 0 \qquad (8)$$

$$6x_1 - 4x_2 = 0 \qquad (9)$$

$$-x_1 - 2x_2 - 4x_3 = 0. \qquad (10)$$

(9) can be written as

$$6x_1 = 4x_2$$

$$\Rightarrow 3x_1 = 2x_2$$

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$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{3}$$

$$\therefore x_1 = 2, x_2 = 3.$$

Substituting in (8) we obtain

$$-4 + 6 + x_3 = 0$$

$$2 + x_3 = 0$$

$$\Rightarrow x_3 = -2.$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

Type-II: Evaluation of Eigen vectors when two of the eigen values are equal.

Example 1.5. Find the eigen values and eigen vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.[Jan 2009]

Solution. Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

where $s_1 = tr(A) = -2 + 1 + 0 = -1$.

s_2 = sum of the minors of the main diagonal elements of A.

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$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$s_2 = 0 - 12 + 0 - 3 - 2 - 4 = -21.$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= -2(0 - 12) - 2(0 - 6) - 3(-4 + 1) = 24 + 12 + 9 = 45.$$

The characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

$\lambda = -3$ is a root. By synthetic division,

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$$\Longrightarrow (\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$$

$$(\lambda + 3)(\lambda - 5)(\lambda + 3) = 0 \Rightarrow \lambda = -3, \lambda = -3, \lambda = 5.$$

The eigen values are $-3, -3, 5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to λ.

$$\therefore \quad (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -3$, (1) becomes $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above three equations are reduced to

$$x_1 + 2x_2 - 3x_3 = 0. \tag{2}$$

Since two of the eigen values are equal, from (2) we have to get two eigen vectors by assigning arbitrary values for two variables. First, choosing $x_3 = 0$, we obtain $x_1 + 2x_2 = 0$ which implies $x_1 = -2x_2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} \Rightarrow x_1 = 2, x_2 = -1$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Also in (2) assign $x_2 = 0$, we obtain $x_1 - 3x_3 = 0$,
which implies $x_1 = 3x_3 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1} \Rightarrow x_1 = 3, x_3 = 1$.

$$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 5$, (1)$\Longrightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The equations are

$$\left. \begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 5x_3 &= 0. \end{aligned} \right\} \tag{A}$$

Taking the first two equations and on solving by the method of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & \times & -3 & \times & -7 & \times & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

—

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}.$$

$$\implies x_1 = 1, x_2 = 2, x_3 = -1.$$

The values of x_1, x_2, x_3 satisfy the last equation in (A).

$$\therefore\ X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Example 1.6. Find the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

[Jun 2010, Jan 2007]

Solution. Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements of A

$= 2 + 3 + 2 = 7.$

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s_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$= (6 - 2) + (4 - 1) + (6 - 2)$

$= 4 + 3 + 4$

$= 11.$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$= 2(6 - 2) - 2(2 - 1) + 1(2 - 3)$

$= 8 - 2 - 1$

$= 5.$

The characterstic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

$\lambda = 1$ is a root.

By synthetic division we have

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$$\lambda^3 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1, 5.$$

The given eigen values are 1, 1, 5.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

All the three equations are reduced to one single equation

$$x_1 + 2x_2 + x_3 = 0 \quad (2)$$

Since two of the eigen values are equal, from(2), we have to get two eigen vectors by assigning arbitrary values for two variables.

First, choosing $x_3 = 0$, we obtain $\quad x_1 + 2x_2 = 0$

$$x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1}$$

$$\Rightarrow x_1 = -2, x_2 = 1$$

$\therefore$ One eigen vector is $X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Assign $x_2 = 0$, (2) becomes

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

$$\Rightarrow x_1 = 1, x_3 = -1.$$

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$\therefore$ The second eigen vector is $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

When $\lambda = 5$, (1) becomes

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$\left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0. \end{aligned}\right\} \qquad \text{(A)}$$

All the three equations are different.

Taking the first two equations and applying the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & \times & 1 & \times & -3 & \times & 2 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\Rightarrow x_1 = 1, x_2 = 1, x_3 = 1.$$

The eigen vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\therefore$ The eigen vectors are $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

—

Example 1.7. Find the eigen values and eigen vectors of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$.

[May 2000]

Solution. Given $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Now, S_1 = sum of the main diagonal elements

$= 3 - 3 + 7 = 7.$

S_2 = sum of the minors of the main diagonal elements of A.

$$= \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$= -21 + 20 + 21 - 15 + (-9 + 20)$

$= -1 + 6 + 11$

$= 16.$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$= 3(-21 + 20) - 10(-14 + 12) + 5(-10 + 9)$

$= -3 + 20 + (-5)$

$= 20 - 8$

$= 12.$

—

The characteristic equation is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$.

$\lambda = 2$ is a root.

By synthetic division, we have

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3.$$

The eigen values are 2, 2, 3.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 2$, (1) becomes

$$\begin{pmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations becomes

$$x_1 + 10x_2 + 5x_3 = 0$$

$$2x_1 + 5x_2 + 4x_3 = 0$$

$$3x_1 + 5x_2 + 5x_3 = 0.$$

—

Since all the three equations are different, we will get only one eigen vector. Since we need two eigen vectors corresponding to the twice repeated eigen value 2, the two eigen vectors are the same.

Taking the first two equations and using the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 10 & \times & 5 & \times & 1 & \times & 10 \\ 5 & & 4 & & 2 & & 5 \end{array}$$

$$\frac{x_1}{40-25} = \frac{x_2}{10-4} = \frac{x_3}{5-20}$$

$$\frac{x_1}{15} = \frac{x_2}{6} = \frac{x_3}{-15}$$

$$\frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5}$$

$$\therefore x_1 = 5, \quad x_2 = 2, \quad x_3 = -5$$

The two eigen vectors are $X_1 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}.$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$10x_2 + 5x_3 = 0$$

$$2x_1 + 6x_2 + 4x_3 = 0$$

$$3x_1 + 5x_2 + 4x_3 = 0.$$

From $10x_2 + 5x_3 = 0$ we obtain

$$10x_2 = -5x_3$$

$$2x_2 = -x_3$$

—

$$\frac{x_2}{1} = \frac{x_3}{-2}$$

$$\therefore x_2 = 1, x_3 = -2.$$

Substituting this in the second equation we obtain

$$2x_1 + 6 - 8 = 0$$

$$2x_1 - 2 = 0$$

$$2x_1 = 2$$

$$x_1 = 1.$$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

The eigen vectors are $\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Type-III: Evaluation of eigen vectors when all the eigen values are equal.

Example 1.8. Find the eigen values and eigen vectors of $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$.

[Dec 1998]

Solution. Given $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

—

Now, s_1 = sum of the main diagonal elements

$= 6 - 13 + 4$

$= -3.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$

$= -52 + 60 + 24 - 35 + (-78 + 84)$

$= 8 - 11 + 6$

$= 3.$

$$s_3 = |A| = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix}$$

$= 6(-52 + 60) + 6(56 - 70) + 5(-84 + 91)$

$= 6(8) + 6(-14) + 5(7) = 48 - 84 + 35$

$= 83 - 84$

$= -1.$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0.$$

$$(\lambda + 1)^3 = 0$$

$$\therefore \lambda = -1, -1, -1.$$

The eigen values are $-1, -1, -1$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

—

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad (1)$$

when $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above three equations are reduced to one single equation

$$7x_1 - 6x_2 + 5x_3 = 0.$$

With this single equation, we must get three different eigen vectors, which can be achieved by assigning arbitrary values to x_1, x_2 & x_3.

$$x_1 = 0 \;\Rightarrow\; -6x_2 + 5x_3 = 0 \;\Rightarrow\; \frac{x_2}{5} = \frac{x_3}{6} \;\therefore\; X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}.$$

$$x_2 = 0 \;\Rightarrow\; 7x_1 + 5x_3 = 0 \;\Rightarrow\; \frac{x_1}{-5} = \frac{x_3}{7} \;\therefore\; X_2 = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix}.$$

$$x_3 = 0 \;\Rightarrow\; 7x_1 - 6x_2 = 0 \;\Rightarrow\; \frac{x_1}{6} = \frac{x_2}{7} \;\therefore\; X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}.$$

Example 1.9. Find the eigen values and eigen vectors of $A = \begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}$.

Solution. Given $A = \begin{pmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{pmatrix}$.

—

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements.

$$= -5 + 9 - 7$$

$$= 9 - 12 = -3.$$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 9 & 18 \\ -3 & -7 \end{vmatrix} + \begin{vmatrix} -5 & -9 \\ -2 & -7 \end{vmatrix} + \begin{vmatrix} -5 & -5 \\ 8 & 9 \end{vmatrix}$$

$$= -63 + 54 + 35 - 18 + (-45 + 40)$$

$$= -9 + 17 - 5 = 3.$$

$$s_3 = |A| = \begin{vmatrix} -5 & -5 & -9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{vmatrix}$$

$$= -5(-63 + 54) + 5(-56 + 36) - 9(-24 + 18)$$

$$= -5(-9) + 5(-20) - 9(-6)$$

$$= 45 - 100 + 54 = -1.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

$$(\lambda + 1)^3 = 0$$

$$\lambda = -1, -1, -1.$$

The eigen values are $-1, -1, -1$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} -5-\lambda & -5 & -9 \\ 8 & 9-\lambda & 18 \\ -2 & -3 & -7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -1$, (1) becomes

$$\begin{pmatrix} -4 & -5 & -9 \\ 8 & 10 & 18 \\ -2 & -3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The equations are

$$-4x_1 - 5x_2 - 9x_3 = 0$$

$$8x_1 + 10x_2 + 18x_3 = 0$$

$$-2x_1 - 3x_2 - 6x_3 = 0.$$

The above three equations are reduced to two equations

$$4x_1 + 5x_2 + 9x_3 = 0$$

$$2x_1 + 3x_2 + 6x_3 = 0.$$

Using the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 5 & \times & 9 & \times & 4 & \times & 5 \\ 3 & & 6 & & 2 & & 3 \end{array}$$

$$\frac{x_1}{30-27} = \frac{x_2}{18-24} = \frac{x_3}{12-10}$$

$$\frac{x_1}{3} = \frac{x_2}{-6} = \frac{x_3}{2}$$

The only eigen vector is $X = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

In this case $X_1 = X_2 = X_3 = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

Exercise I(A)

I. Find the characteristic equation, eigen values and eigen vectors of the following matrices.

1. $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

2. $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

II. Find the eigen values and eigen vectors of the following matrices.

1. $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

6. $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}$

7. $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$

8. $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

10. $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 4 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 5 & 4 \end{pmatrix}$.

—

1.3 Properties of Eigen values and Eigen vectors

Property 1.1. A square matrix A and its transpose A^T have the same eigen values.

Proof. The eigen values of the matrix A are the roots of its characteristic equation $|A - \lambda I| = 0$.

Now, $(A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$.

Also, $|(A - \lambda I)^T| = |A - \lambda I| \Rightarrow |A^T - \lambda I| = |A - \lambda I|$.

This implies that the characteristic polynomial of A and A^T are equal.

$\Rightarrow$ The characteristic equations of A and A^T are equal.

$\Rightarrow$ A and A^T have the same eigen values. □

Worked Examples

Example 1.10. 1, $\sqrt{5}$ and $-\sqrt{5}$ are the eigen values of the matrix $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

Write down the eigen values of the matrix $\begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

B is the transpose of A.

By the above property, A & B have the same eigen values.

$\therefore$ The eigen values of B are 1, $\sqrt{5}$ and $-\sqrt{5}$.

Example 1.11. If 1, 3 and −4 are the eigen values of the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$, what

—

are the eigen values of the matrix $\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$?

Solution. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$.

B is the transpose of A.

By the above property, A and B have the same eigen values.

$\therefore$ The eigen values of B are 1, 3 and -4.

Property 1.2. Sum of the eigen values of a square matrix A is equal to the sum of the elements on its main diagonal.

Proof. Let A be a square matrix of order n.

The characteristic equation is $|A - \lambda I| = 0$.

It is of the form $\lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + \cdots + (-1)^n s_n = 0$ (1)

where, s_1 = sum of the elements of the leading diagonal.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of (1).

From the theory of equations,

Sum of the roots $= -\dfrac{\text{coeff. of } \lambda^{n-1}}{\text{coeff. of } \lambda^n}$.

i.e., $\lambda_1 + \lambda_2 + \cdots + \lambda_n = -(-s_1) = s_1 = tr(A)$.

i.e., Sum of the eigen values = sum of the main diagonal elements of A. □

Property 1.3. Product of the eigen values of a square matrix A is equal to $|A|$.

Proof. For the square matrix A , the characteristic equation is $|A - \lambda I| = 0$.

It is of the form $\lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + \cdots + (-1)^n s_n = 0$, (1)

where $s_n = |A|$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of (1).

From the theory of equations, we have

Product of the roots $= (-1)^n \dfrac{\text{constant term}}{\text{coeff. of } \lambda^n}$.

i.e., $\lambda_1, \lambda_2, \ldots, \lambda_n = (-1)^n(-1)^n s_n = (-1)^{2n}|A|$.

i.e., Product of the eigen values = $|A|$. □

Worked Examples

Example 1.12. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$. [Dec 1999]

Solution. Sum of the eigen values = $tr(A) = -2 + 1 + 0 = -1$.

$$\text{Product of the eigen values} = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(-12) - 2(-6) - 3(-3) = 45.$$

Example 1.13. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. [March 1996]

Solution. Sum of the eigen values = Sum of the elements along the main diagonal

$$= -1 - 1 - 1 = -3.$$

$$\text{Product of the eigen values} = |A| = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1 - 1) - 1(-1 - 1) + 1(1 + 1)$$

$$= -1 \times 0 + 2 + 2 = 4.$$

Example 1.14. The product of the two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16, find the third eigen value. [Jan 2003,2012]

—

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values.

Given $\lambda_1\lambda_2 = 16$.

$$\text{But, product of eigen values} = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$i.e., \lambda_1\lambda_2\lambda_3 = 6(8) + 2(-4) + 2(-4)$$

$$16\lambda_3 = 48 - 8 - 8 = 32$$

$$\therefore \lambda_3 = 2.$$

Example 1.15. If 2 and 3 are the eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$, find the value of x.

Solution. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{pmatrix}$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

We know that, $\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 6$.

$$2 + 3 + \lambda_3 = 6$$

$$\lambda_3 = 1.$$

Also, product of the eigen values $= |A|$.

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 2 \end{vmatrix}$$

$$2.3.1 = 2(4) - 0(0) + 1(0 - 2x)$$

$$6 = 8 - 2x$$

$$2x = 2$$

$$x = 1.$$

—

Example 1.16. Find the values of a and b, such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 its eigen values. [May 2011]

Solution. Sum of the eigen values $= tr(A) = a + b$

$$3 - 2 = a + b$$

$$\Rightarrow a + b = 1 \qquad (1)$$

Product of the eigen values $= |A|$

$$\text{i.e., } 3 \times (-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix} = ab - 4$$

$$-6 = ab - 4$$

$$ab = -6 + 4 = -2$$

$$b = -\frac{2}{a}.$$

Substituting in (1) we get

$$a - \frac{2}{a} = 1$$

$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$(a - 2)(a + 1) = 0$$

$$a = 2, -1.$$

When $a = 2, b = -1$.

When $a = -1, b = 2$.

Example 1.17. If the sum of two eigen values and trace of a 3×3 matrix A are equal, find the value of $|A|$.

Solution. Let the eigen values be $\lambda_1, \lambda_2, \lambda_3$.

We know that $\lambda_1 + \lambda_2 + \lambda_3 = tr(A)$.

But, given that $\lambda_1 + \lambda_2 = tr(A)$.

$\therefore\ tr(A) + \lambda_3 = tr(A)$.

$\Longrightarrow \lambda_3 = 0$.

Now, $|A|$ = product of the eigen values = $\lambda_1 \times \lambda_2 \times 0 = 0$.

Property 1.4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the non zero eigen values of a square matrix A of order n, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

Proof. Let λ be a non zero eigen value of A. Then, there exists a non zero column matrix X such that

$$AX = \lambda X. \qquad (1)$$

Since all the eigen values are non zero, A is non-singular. Hence, A^{-1} exists.
Premultiplying both sides of (1) by A^{-1} we get,

$$A^{-1}(AX) = A^{-1}\lambda X$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X.$$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}.
This is true for all eigen values of A.
$\therefore \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}. □

Property 1.5. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then

(i) $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ are the eigen values of cA where $c \neq 0$.

(ii) $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$ are the eigen values of A^m where m is a positive integer.

Proof. Let λ be an eigen value of A.
Then, there exists a column matrix X such that

$$AX = \lambda X. \qquad (1)$$

—

(i). Let $c \neq 0$.

Multiply (1) by c we get, $c(AX) = c(\lambda X)$

$$(cA)X = (c\lambda)X.$$

$\Rightarrow c\lambda$ is an eigen value of cA.

This is true for all eigen values of A.

$\therefore c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ are the eigen values of cA.

(ii) We have

$$AX = \lambda X$$
$$\Longrightarrow A(AX) = A(\lambda X)$$
$$\Longrightarrow (AA)X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$$\Longrightarrow A^2X = \lambda^2 X.$$

$\Longrightarrow \lambda^2$ is an eigen value of A^2.

$$A(A^2X) = A(\lambda^2 X)$$
$$\Longrightarrow (AA^2)X = \lambda^2(AX)$$
$$\Longrightarrow A^3X = \lambda^2\lambda X = \lambda^3 X.$$

$\Longrightarrow \lambda^3$ is an eigen value of A^3. Proceeding like this, we arrive at $A^mX = \lambda^m X$.

$\Rightarrow \lambda^m$ is an eigen value of A^m.

This is true for all eigen values of A.

$\therefore \lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are the eigen values of A^m. □

—

Worked Examples

Example 1.18. Given $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$. Find the eigen values of A^2. [Jan 2010]

Solution. The given matrix is triangular.

Hence, the eigen values are the elements along the leading diagonal.

The Eigen values of A are $-1, -3, 2$.

$\therefore$ The Eigen values of A^2 are $(-1)^2, (-3)^2, 2^2$.

i.e., $1, 9, 4$.

Example 1.19. If 2 and 3 are the eigen values of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$, find the eigen values of A^{-1} and A^3. [Dec 2000]

Solution. Let λ be the third eigen value.

we know that, sum of the eigen values = $tr(A)$.

$$2 + 3 + \lambda = 3 - 3 + 7$$

$$5 + \lambda = 7 \Rightarrow \lambda = 2.$$

The Eigen values of A are $2, 2, 3$.

The Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$.

The Eigen values of A^3 are 2^3, 2^3, 3^3 (i.e) 8, 8, 27.

Example 1.20. Two of the eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to 1 each. Find the eigen values of A^{-1}. [Dec.2002]

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given $\lambda_1 = \lambda_2 = 1$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3$ = sum of the elements along the main diagonal

—

$$1 + 1 + \lambda_3 = 2 + 3 + 2$$
$$2 + \lambda_3 = 7$$
$$\lambda_3 = 5.$$

$\therefore$ The eigen values of A are 1,1,5

$\therefore$ Eigen values of A^{-1} are 1, 1, $\frac{1}{5}$.

Example 1.21. Two of the eigen values of the matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. Find the eigen values of A^{-1}. [May 2002]

Solution. Let λ_1, λ_2 and λ_3 be the eigen values of the given matrix A.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 =$ sum of the elements along the main diagonal

$$3 + 6 + \lambda_3 = 3 + 5 + 3$$
$$9 + \lambda_3 = 11$$
$$\lambda_3 = 2.$$

The eigen values of A are 3, 6, 2.

$\therefore$ The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$.

Example 1.22. Find the eigen values of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$. Hence, find the matrix whose eigen values are $\frac{1}{6}$ and -1. [Jan 2001]

Solution. Let $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.

The characteristic equation is $|A - \lambda I| = 0$.

This is of the form $\lambda^2 - s_1\lambda + s_2 = 0$.

$$s_1 = tr(A) = 1 + 4 = 5$$

$$s_2 = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6.$$

—

The characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 6$.

We know that, the matrix whose eigen values are $\frac{1}{-1}$ and $\frac{1}{6}$ is A^{-1}.

$$\therefore A^{-1} = \frac{1}{|A|} A_c^T = \frac{-1}{6} \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}.$$

Example 1.23. Find the eigen values of $adj(A)$ if $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Solution. Given $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Since A is triangular, the eigen values are the elements along the main diagonal.

$\therefore$ The eigen values are $3, 4, 1$.

We have $A^{-1} = \frac{1}{|A|} adj(A)$

$\Rightarrow adj(A) = |A|\, A^{-1}$

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

$\therefore$ The eigen values of $adj(A)$ are $|A| \times \frac{1}{3}, |A| \times \frac{1}{4}, |A| \times 1$

Now, $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \times 4 \times 1 = 12.$

$\therefore$ Eigen values of $adj(A)$ are $12 \times \frac{1}{3}, 12 \times \frac{1}{4}, 12 \times 1$

i.e., $4, 3, 12$.

Example 1.24. Find the eigen values of $adj(A)$ if $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Step 1. To find the eigen values of A

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

—

Now, s_1 = sum of the main diagonal elements

$= 2 + 2 + 2 = 6.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$= 4 - 0 + 4 - 1 + 4 - 0$

$= 4 + 3 + 4 = 11.$

$s_3 = |A|$

$$= \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 2(4-0) - 0 + 1(0-2) = 8 - 2 = 6.$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division, we have

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 2, 3.$$

The eigen values of A are $1, 2, 3$.

We know that $adj(A) = |A|\, A^{-1}$.

—

$\therefore$ The eigen values of $adj(A)$ are $|A| \times \frac{1}{1}, |A| \times \frac{1}{2}, |A| \times \frac{1}{3}$

$$\text{i.e.,} 6 \times 1, 6 \times \frac{1}{2}, 6 \times \frac{1}{3}$$

$$\text{i.e.,} 6, 3, 2$$

Property 1.6. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then

(i) $\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the eigen values of $A - kI$.

(ii) $a_0\lambda_1^2 + a_1\lambda_1 + a_2, a_0\lambda_2^2 + a_1\lambda_2 + a_2, \ldots, a_0\lambda_n^2 + a_1\lambda_n + a_2$ are the eigen values of $a_0A^2 + a_1A + a_2I$.

Proof. (i) let λ be an eigen value of A.

Then, there exists a column matrix X such that $AX = \lambda X$.

Since $X \neq 0$ is a column matrix, we have

$$AX - kX = \lambda X - kX$$

$$\Rightarrow (A - kI)X = (\lambda - k)X.$$

$\Rightarrow \lambda - k$ is an eigen value of $A - kI$.

This is true for all eigen values of A.

$\therefore\ \lambda_1 - k, \lambda_2 - k, \ldots, \lambda_n - k$ are the eigen values of $A - kI$.

(ii) We have

$$AX = \lambda X \Rightarrow A^2X = \lambda^2X \Rightarrow a_0(A^2X) = a_0(\lambda^2X)$$

$$\Longrightarrow (a_0A^2)X = (a_0\lambda^2)X.$$

$$a_1(AX) = a_1(\lambda X) \Rightarrow (a_1A)X = (a_1\lambda)X.$$

$$(a_0A^2)X + (a_1A)X = (a_0\lambda^2)X + (a_1\lambda)X.$$

Add a_2X both sides we get,

$$(a_0A^2 + a_1A + a_2I)X = (a_0\lambda^2 + a_1\lambda + a_2)X$$

$\Rightarrow a_0\lambda^2 + a_1\lambda + a_2$ is an eigen value of $a_0A^2 + a_1A + a_2I$.

This is true for all eigen values of A.

$\therefore a_0\lambda_1^2 + a_1\lambda_1 + a_2, a_0\lambda_2^2 + a_1\lambda_2 + a_2, \ldots$ are the eigen values of $a_0A^2 + a_1A + a_2I$. □

—

Worked Examples

Example 1.25. If $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$, find the eigen values of $A^2 - 2A + I$.

Solution. Since the given matrix is triangular, the eigen values are $3, 2, 5$.

Therefore, eigen values of $A^2 - 2A + I$ are $3^2 - 2(3) + 1, 2^2 - 2(2) + 1$ and $5^2 - 2(5) + 1$.

i.e., $4, 1, 16$.

Example 1.26. Form the matrix whose eigen values are $\alpha - 5, \beta - 5$ and $\gamma - 5$ where α, β and γ are the eigen values of $A = \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}$. [Jan 2001]

Solution. Using (i) of Property 1.6, we obtain the matrix whose eigen values are $\alpha - 5, \beta - 5$ and $\gamma - 5$ is $A - 5I$.

$\therefore$ The required matrix is

$$A - 5I = \begin{pmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{pmatrix}.$$

Example 1.27. What is the sum of the squares of the eigen values of $\begin{pmatrix} 1 & 7 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & 5 \end{pmatrix}$. [Jan 2001]

Solution. The given matrix is triangular.

$\therefore$ Eigen values are $1, 2, 5$.

Sum of squares of eigen values $= 1^2 + 2^2 + 5^2 = 30$.

Example 1.28. Find the eigen values of the matrix. $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$. Find also

—

the sum of the cubes of the eigen values.

Solution. Since A is triangular, the eigen values are the elements along the leading diagonal.

$\therefore$ The eigen values are $1, -1$ and 2.

$$\text{Sum of the cubes of the eigen values } = 1^3 + (-1)^3 + 2^3 = 1 - 1 + 8 = 8.$$

Example 1.29. Find the sum of the fourth powers of the eigen values of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}$.

Solution. Since the matrix A is triangular, the eigen values are the elements along the leading diagonal.

$\therefore$ The eigen values are $1, 2$ and -2.

$$\text{Sum of the fourth powers of the eigen values} = 1^4 + 2^4 + (-2)^4 = 1 + 16 + 16 = 33.$$

Example 1.30. If $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigen vector of the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, find the corresponding eigen value.

Solution. Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ and $X = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Let λ be the corresponding eigen value.

$$\text{Now, } (A - \lambda I)X = 0.$$

$$\text{i.e., } \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 0$$

$$4(5-\lambda) + 4 = 0 \Rightarrow 5 - \lambda = -1 \Rightarrow \lambda = 6.$$

$$\text{Also, } 4 + 2 - \lambda = 0 \Rightarrow \lambda = 6.$$

Hence, the eigen value corresponding to the given eigen vector is 6.

—

Example 1.31. If $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the eigen vectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$, find the corresponding eigen values.

Solution. Let λ_1 be the eigen value corresponding to the eigen vector $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$$\therefore (A - \lambda_1 I)X_1 = 0.$$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 3 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0.$$

$$i.e., \ 1 - \lambda_1 - 3 = 0 \Rightarrow \lambda_1 = -2.$$

Also the second equation is

$$3 - 3(-1 - \lambda_1) = 0$$
$$3 + 3 + 3\lambda_1 = 0$$
$$3\lambda_1 = -6$$
$$\lambda_1 = -2.$$

Let λ_2 be the eigen value corresponding to the eigen vector $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\therefore (A - \lambda_2 I)X_2 = 0.$$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 3 & -1-\lambda_2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow 1 - \lambda_2 + 1 = 0$$

$$\lambda_2 = 2.$$

Also $3 - 1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 2$

$\therefore$ The eigen values of A are -2 and 2.

—

Definition [Linearly dependent and independent vectors]

(i) The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly dependent if there exist real numbers $c_1, c_2, \ldots, c_n$ not all zero such that $c_1X_1 + c_2X_2 + \cdots + c_nX_n = 0$.

(ii) The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly independent if they are not linearly dependent.

Results

(i) If the vectors $X_1, X_2, \ldots, X_n$ are linearly independent and satisfying the relation $c_1X_1 + c_2X_2 + \cdots + c_nX_n = 0$, then $c_1 = c_2 = \cdots = c_n = 0$.

(ii) If the vectors $X_1, X_2, \ldots, X_n$ are linearly dependent, then any one vector can be written as the linear combination of all the other vectors.

Note

(i) The eigen values of the unit matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are $1, 1, 1$, and the corresponding eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors are linearly independent.

(ii) The eigen values of the triangular matrix $\begin{pmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix}$ are $\lambda_1, \lambda_2, \lambda_3$.

i.e., In a triangular matrix, the eigen values are the elements in the main diagonal.

(iii) The eigen values of $A, A^2, \ldots, A^m$ are $\lambda, \lambda^2, \ldots, \lambda^m$. which are all different. But they all have the same eigen vector X.

(iv) λ and $a_0\lambda^2 + a_1\lambda + a_2$ are the eigen values of A and $a_0A^2 + a_1A + a_2I$ respectively. But they have the same eigen vector X.

—

Exercise I(B)

1. Find the sum and product of the eigen values of the matrix $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{pmatrix}$.

2. If α and β are the eigen values of $\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$, form the matrix whose eigen values are α^3 and β^3.

3. Prove that $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $-3A^{-1}$ have the same eigen values.

4. Two of the eigen values of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are 3 and 15. Find the third eigen value.

5. Two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to unity each. Find the eigen values of A^{-1}.

6. If the product of two of the eigen values of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is 2, find the third eigen value.

7. Find the constants a and b such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and −2 as eigen values.

8. Two eigen values of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and are double the third.

—

Find the eigen values of A^2.

9. If the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ has an eigen vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, find the corresponding eigen value of A.

10. If the eigen values of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are $-2, 3, 6$, find the eigen values of A^T.

1.4 Cayley Hamilton Theorem

Statement. Every square matrix satisfies its own characteristic equation.

Example. If the characteristic equation of a square matrix A is $\lambda^3 - 6\lambda^2 + 5\lambda + 3 = 0$, then we have, $A^3 - 6A^2 + 5A + 3I = 0$.

Results. Cayley Hamilton theorem is useful for

(i) finding the inverse of a non-singular matrix,and

(ii) finding the higher powers of A.

The above results are demonstrated in the following examples.

Worked Examples

Example 1.32. If the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+i & -1 & 0 & 0 \\ -3 & 2i & i & 0 \\ 4 & -i & 1 & -i \end{pmatrix}$ where $i = \sqrt{-1}$, then using Cayley Hamilton theorem, prove that $A^4 = I$. [Jun 2011]

Solution. A is a triangular matrix.

The eigen values of A are $1, -1, i, -i$.

The characteristic equation is

—

$$(\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)=0$$
$$(\lambda^2-1)(\lambda^2-i^2)=0$$
$$(\lambda^2-1)(\lambda^2+1)=0$$
$$\lambda^4-1=0$$

By Cayley Hamilton theorem, $A^4-I=0 \quad \Rightarrow A^4=I$.

Example 1.33. If $A=\begin{pmatrix}1 & 0\\ 4 & 5\end{pmatrix}$, express A^3 interms of A and I, using Cayley Hamilton theorem. [Jan 2001]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2-s_1\lambda+s_2=0.$$

Now, s_1 = sum of the main diagonal elements $=1+5=6$.

$$s_2=|A|=\begin{vmatrix}1 & 0\\ 4 & 5\end{vmatrix}=5-0=5.$$

The characteristic equation is $\lambda^2-6\lambda+5=0$.

By Cayley Hamilton theorem we get

$$A^2-6A+5I=0. \quad (1)$$

Premultiplying by A, we get

$$A^3-6A^2+5A=0$$
$$\begin{aligned}A^3 &= 6A^2-5A\\ &= 6(6A-5I)-5A \qquad [\text{ by (1)}]\\ &= 36A-30I-5A\\ &= 31A-30I.\end{aligned}$$

Example 1.34. Find the value of A^4 if $A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$, using Cayley Hamilton theorem. [Jan 2003]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2-s_1\lambda+s_2=0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + (-1) = 0.$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5.$$

The characteristic equation is

$$\lambda^2 - 5 = 0.$$

By Cayley Hamilton theorem, $A^2 - 5I = 0$.

$$\text{i.e., } A^2 = 5I$$

$$\text{Now, } A^4 = A^2.A^2$$

$$= 5I.5I = 25I = 25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}.$$

Example 1.35. Using Cayley Hamilton theorem, find the inverse of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

[Jan 2003]

Solution. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements $= 1 + 3 = 4$.

$s_2 = |A| = 3 - 8 = -5.$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$.

By Cayley Hamilton theorem, we have

$$A^2 - 4A - 5I = 0.$$

Multiplying by A^{-1} we get,

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$5A^{-1} = A - 4I$$

$$A^{-1} = \frac{1}{5}[A - 4I]$$
$$= \frac{1}{5}\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}.$$

Example 1.36. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. [Jun 2009]

Step 1. To find the characteristic equation

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements $= 1 + 3 = 4$.

$s_2 = |A| = 3 - 8 = -5$.

The characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

Step 2. Verification of Cayley Hamilton theorem

By cayley Hamilton theorem, we have

$$A^2 - 4A - 5I = 0.$$

Now, $A^2 = A.A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$.

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

—

Step 3. To find A^{-1}

we have

$$A^2 - 4A - 5I = 0.$$

Multiplying by A^{-1} we get,

$$\begin{aligned} A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I &= 0 \\ A - 4I - 5A^{-1} &= 0 \\ 5A^{-1} &= A - 4I \\ A^{-1} = \frac{1}{5}[A - 4I] = \frac{1}{5}\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}. \end{aligned}$$

Step 4. To find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

Consider the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$.

Divide by $\lambda^2 - 4\lambda - 5$ we get,

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5.$

Replacing λ by A we get,

$$\begin{aligned} A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I &= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I \\ &= 0 + A + 5I \quad \text{[Using Cayley Hamilton theorem]} \\ &= A + 5I. \end{aligned}$$

Example 1.37. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Find A^{-1} and A^4. [Jan 2009]

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0. \tag{1}$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 2 + 2 = 6.$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$= 4 - 1 + 4 - 1 + 4 - 1$

$= 3 + 3 + 3$

$= 9.$

$$s_3 = |A| \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(3) + 1(-1) + 1(-1) = 6 - 1 - 1 = 4.$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

Step 2. Verification of Cayley Hamilton theorem.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^2 = A \cdot A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

—

$$= \begin{pmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 6A^2 + 9A - 4I = 0$.

Multiplying by A^{-1} we get,

$$A^2 - 6A + 9I - 4A^{-1} = 0.$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I.$$

$$\Rightarrow A^{-1} = \frac{1}{4}[A^2 - 6A + 9I].$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{4}\left(\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right).A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Step 4. To find A^4

Consider λ^4. Dividing λ^4 by $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ we get,

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$\lambda^4 = (\lambda+6)(\lambda^3 - 6\lambda^2 + 9\lambda - 4) + 27\lambda^2 - 50\lambda + 24.$

Replacing λ by A we get,

$$A^4 = (A+6I)(A^3 - 6A^2 + 9A - 4I) + 27A^2 - 50A + 24I.$$

$$= 27A^2 - 50A + 24I \quad \text{[By Cayley Hamilton theorem]}.$$

$$= \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}.$$

Example 1.38. If $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, verify Cayley Hamilton theorem and hence find A^{-1}. [Jan 2005]

Solution.

Step 1. To find the characteristic equation

Since A is 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$$= 1 + 3 + 1 = 5$$

s_2 = sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 3 - 0 + 1 - 0 + 3 + 2$$

$$= 3 + 1 + 5 = 9$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) - 2(2-0) = 3 + 2 - 4 = 1.$$

—

The characteristic equation is $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$.

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$.

Step2. Verification of Cayley Hamilton theorem

$$A^2 = A.A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix}$$

Now $A^3 - 5A^2 + 9A - I = \begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 5\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Hence, Cayley Hamilton theorem is verified.

Step 3. To find A^{-1}

By Cayley Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$.

Multiplying by A^{-1} we get

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore\ A^{-1} = A^2 - 5A + 9I = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Example 1.39. Find the matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, using Cayley Hamilton theorem. [Jun 2009]

—

Solution.

Step 1. To find the characteristic equation

Since A is a 3×3 matrix, the characteristic equation takes the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 1 + 2 = 5$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 + 3 + 2 = 7.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2-0) - 1(0-0) + 1(0-1) = 4 - 1 = 3.$$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

By Cayley Hamilton theorem we get $A^3 - 5A^2 + 7A - 3I = 0$.

Step 2. To find the value of given matrix expression

Consider the polynomial $\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$.

Divide this polynomial by $\lambda^3 - 5\lambda^2 + 7\lambda - 3$ we get

$$\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 = (\lambda^5 + 8\lambda + 35)(\lambda^3 - 5\lambda^2 + 7\lambda - 3) + 127\lambda^2 - 223\lambda + 106.$$

Replacing λ by A we get

$$\begin{aligned} A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I &= (A^5 + 8A + 35I)(A^3 - 5A^2 + 7A - 3I) + 127A^2 - 223A + 106I \\ &= 127A^2 - 223A + 106I \quad \text{[By Cayley Hamilton theorem].} \\ &= \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}. \end{aligned}$$

—

Example 1.40. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Hence find A^{50}. [Jan 2006]

Solution. Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 0 + 0 = 1$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$= 0 - 1 - 0 + 0 = -1.$

$$s_3 = |A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$= 1(0 - 1) = -1.$

$\therefore$ The characteristic equation is $\lambda^3 - \lambda^2 - \lambda - 1 = 0$.

By Cayley Hamilton theorem we have,

$$A^3 - A^2 - A + I = 0$$

$$A^3 - A^2 = A - I$$

$$A^4 - A^3 = A^2 - A$$

$$A^5 - A^4 = A^3 - A^2$$

$$\cdots\cdots\cdots\cdots$$

$$A^n - A^{n-1} = A^{n-2} - A^{n-3}$$

Adding all these equations we get, $A^n - A^2 = A^{n-2} - I$.

$$\therefore\ A^n = A^{n-2} + A^2 - I$$

—

$$A^{n-2} = A^{n-4} + A^2 - I.$$

$$\therefore\ A^n = A^{n-4} + 2(A^2 - I).$$

$$A^{n-4} = A^{n-8} + 2(A^2 - I).$$

$$\therefore\ A^n = A^{n-8} + 4(A^2 - I).$$

$$A^{n-8} = A^{n-16} + 4(A^2 - I).$$

$$A^n = A^{n-16} + 8(A^2 - I).$$

If n is even, $A^n = A^{n-(n-2)} + \frac{n-2}{2}(A^2 - I)$

$$= A^2 + \frac{n-2}{2}(A^2 - I)$$

$$= A^2 + \frac{n-2}{2}A^2 - \frac{n-2}{2}I$$

$$A^n = \frac{n}{2}A^2 - \frac{n-2}{2}I.$$

When $n = 50$, we have

$$A^{50} = 25A^2 - 24I.$$

Now, $A^2 = A \cdot A$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{50} = 25\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 24\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix}.$$

—

Example 1.41. If n is a positive integer, find A^n for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ using Cayley Hamilton theorem. [Jan 2005]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form

$$\lambda^2 - s_1\lambda + s_2 = 0.$$

Now, s_1 = sum of the main diagonal elements

$$= 1 + 3 = 4.$$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 3 - 8 = -5.$$

The characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

$$(\lambda - 5)(\lambda + 1) = 0.$$

The eigen values are $-1, 5$.

By Cayley Hamilton theorem we get, $A^2 - 4A - 5I = 0$.

Dividing λ^n by $\lambda^2 - 4\lambda - 5$, let $Q(\lambda)$ be the quotient and $R(\lambda)$ be the remainder.

Since we are dividing by $\lambda^2 - 4\lambda - 5$, $R(\lambda)$ is of degree atmost 1.

$\therefore$ Let $R(\lambda) = a\lambda + b$.

Hence, we can write λ^n as

$$\lambda^n = (\lambda^2 - 4\lambda - 5)Q(\lambda) + a\lambda + b. \quad \text{(A)}$$

When $\lambda = -1$, we get

$$-a + b = (-1)^n \quad (1)$$

When $\lambda = 5$, we get

$$5a + b = 5^n \quad (2)$$

$$(2) - (1) \Rightarrow 6a = 5^n - (-1)^n. \Rightarrow a = \frac{1}{6}[5^n - (-1)^n].$$

Substituting in (1) we get

$$b = a + (-1)^n = \frac{1}{6}[5^n - (-1)^n] + (-1)^n = \frac{1}{6}[5^n + 5(-1)^n].$$

—

Replacing λ by A in (A) we get

$$A^n = (A^2 - 4A - 5I)Q(A) + aA + bI$$
$$= aA + bI \text{ [by C.H Theorem]}$$
$$= \frac{1}{6}[5^n - (-1)^n]A + \frac{1}{6}[5^n + 5(-1)^n]I.$$

Example 1.42. If $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ find A^n in terms of A and I. [Jun 2003]

Solution. Since A is a 2×2 matrix, the characteristic equation is of the form $\lambda^2 - s_1\lambda + s_2 = 0$.

Now, s_1 = sum of the main diagonal elements

$= 1 - 1 = 0.$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$= -1 - 4 = -5.$

The characteristic equation is $\lambda^2 - 5 = 0$

$$\text{i.e., } \lambda^2 - 5 = 0$$

$$\Longrightarrow \lambda = \pm\sqrt{5}.$$

By Cayley Hamilton theorem we get $A^2 - 5I = 0$.

Dividing λ^n by $\lambda^2 - 5$, let $Q(\lambda)$ be the quotient and $R(\lambda)$ be the remainder.

Since we are dividing λ^n by $\lambda^2 - 5$, $R(\lambda)$ is of degree atmost 1.

$$\therefore R(\lambda) = a\lambda + b.$$

$$\therefore \lambda^n = (\lambda^2 - 5)Q(\lambda) + a\lambda + b. \quad (1)$$

$$\text{When } \lambda = \sqrt{5}, \text{ we get} (\sqrt{5})^n = a\sqrt{5} + b. \quad (2)$$

$$\text{When } \lambda = -\sqrt{5}, \text{ we get } (-\sqrt{5})^n = -\sqrt{5}a + b \quad (3)$$

—

$(2)-(3) \Rightarrow 2\sqrt{5}a = (\sqrt{5})^n - (-\sqrt{5})^n$

$= (\sqrt{5})^n[1-(-1)^n]$

$a = (\sqrt{5})^{n-1}\dfrac{[1-(-1)^n]}{2}.$

substituting in (2) we get

$$b = (\sqrt{5})^n - a(\sqrt{5})$$
$$= (\sqrt{5})^n - (\sqrt{5})^n\frac{[1-(-1)^n]}{2}$$
$$= (\sqrt{5})^n\frac{[2-1-+(-1)]^n}{2}$$
$$b = (\sqrt{5})^n\frac{[1+(-1)^n]}{2}.$$

$\therefore \lambda^n = (\lambda^2-5)\phi(\lambda) + \dfrac{\sqrt{5}^{n-1}}{2}[1-(-1)^n]\lambda + \dfrac{(\sqrt{5})^n + (-\sqrt{5})^n}{2}.$

Replacing λ by A we get

$$A^n = \frac{(\sqrt{5})^{n-1}}{2}[1-(-1)^n]A + (\sqrt{5})^n\frac{[1+(-1)^n]}{2}.$$

Exercise I(C)

1. Verify Cayley Hamilton theorem find the inverse of $A = \begin{pmatrix} 13 & -3 & 5 \\ 0 & 4 & 0 \\ -15 & 9 & -7 \end{pmatrix}$.

2. Verify Cayley Hamilton theorem for the matrix $\begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$. Hence find its inverse.

3. Using Cayley Hamilton theorem find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}$.

4. Using Cayley Hamilton theorem, find the inverse of the following matrices.

   (i) $\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 2 \end{pmatrix}$

—

5. Show that the matrix $\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ satisfies Cayley Hamilton theorem.

6. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{pmatrix}$.

7. Using Cayley Hamilton theorem find A^4 for the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

8. Find the characteristic equation of the matrix A and hence Compute $2A^8 - 3A^5 + A^4 - 4I$ where A is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

9. If $A = \begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$, find A^n interms of A and I using Cayley Hamilton theorem. Hence find the value of A^3.

10. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, find A^n interms of A and I.

11. If $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, prove that $A^n = 7^{n-1} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.

12. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$.

13. If $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$, using Cayley Hamilton theorem obtain the value of $A^6 - 25A^2 + 122A$.

—

1.5 Diagonalization of Matrices - Similarity Transformation

Similar matrices. Let A and B be square matrices of order n. A is similar to B if there exists a nonsingular matrix P such that $A = P^{-1}BP$.

The transformation which transforms B into A is called a similarity transformation. The matrix P is called a similarity matrix.

Results.

1. If A and B are similar then $|A| = |B|$.
2. If A and B are similar, then they have the same eigen values.

Diagonalisation of a square matrix

A square matrix A is said to be diagonalizable, if there exists a nonsingular matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. The matrix P is called the modal matrix of A. D is also called as the spectral matrix of A.

Working rule to diagonalize a matrix

Step 1. Find the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 2. Find the linearly independent eigen vectors $X_1, X_2, \ldots, X_n$, corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 3. Form the modal matrix $P = [X_1, X_2, \ldots, X_n]$.

Step 4. Find P^{-1}.

Step 5. Find $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$.

Note. P is possible only when A has n linearly independent eigen vectors.

—

Computation of powers of a square matrix using similarity transformation

we have $D = P^{-1}AP$

$$PDP^{-1} = P(P^{-1}AP)P^{-1}$$
$$= (PP^{-1})A(PP^{-1})$$
$$= IAI = A$$
$$\therefore\ A = PDP^{-1}$$
$$A^2 = A \cdot A$$
$$= (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1}$$
$$= PDIDP^{-1}$$
$$A^2 = PD^2P^{-1}$$

In the same way we get $A^3 = PD^3P^{-1}$.

In general $A^n = PD^nP^{-1}$, where $D = \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n^n \end{pmatrix}$

Properties of eigen values of similar matrices

Property 1.7. If A and B are similar matrices, they have the same characteristic equation.

Proof. Since A and B are similar, there exists a matrix P such that

$$B = P^{-1}AP$$
$$\therefore\ B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - P^{-1}\lambda IP$$
$$= P^{-1}(AP - \lambda IP)$$
$$= P^{-1}(A - \lambda I)P$$

—

Hence, $|B - \lambda I| = |P^{-1}(A - \lambda I)P|$

$= |P^{-1}||A - \lambda I||P|$

$= |P^{-1}||P||A - \lambda I|$

$= |P^{-1}P||A - \lambda I|$

$= |I||A - \lambda I|$

$= |A - \lambda I|.$

The characteristic equations of A and B are respectively $|A - \lambda I| = 0$ and $|B - \lambda I| = 0$. Hence, they are equal. □

Corollary. Two similar matrices have the same eigen values.

Property 1.8. If A and B are $n \times n$ matrices and B is a non singular matrix, then A and $B^{-1}AB$ have the same eigen values.

Proof. The characteristic polynomial of $B^{-1}AB$ is

$|B^{-1}AB - \lambda I| = |B^{-1}AB - B^{-1}(\lambda I)B|$

$= |B^{-1}(A - \lambda I)B|$

$= |B^{-1}||A - \lambda I||B|$

$= |B^{-1}||B||A - \lambda I|$

$= |B^{-1}B||A - \lambda I|$

$= |I||A - \lambda I|$

$= |A - \lambda I|.$

= Characteristic polynomial of A.

$\Rightarrow$ $B^{-1}AB$ and A have the same characteristic equation.

$\Rightarrow$ They have the same eigen values. □

Result

If we are asked to find $f(A)$, then $f(A)$ can be evaluated by $f(A) = P f(D) P^{-1}$

Worked Examples

Example 1.43. Reduce the matrix $A = \begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}$ to the diagonal form.

Solution. Let the matrix be $A = \begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}$.

Step 1. To find the eigen values

$$s_1 = tr(A) = -19 + 16 = -3.$$

$$s_2 = |A| = \begin{vmatrix} -19 & 7 \\ -42 & 16 \end{vmatrix}$$

$$= -304 + 294 = -10.$$

The characteristic equation is

$$\lambda^2 - s_1\lambda + s_2 = 0$$

$$i.e., \lambda^2 + 3\lambda - 10 = 0$$

$$(\lambda + 5)(\lambda - 2) = 0$$

$$\lambda = 2, -5.$$

The eigen values are $\lambda_1 = 2$, $\lambda_2 = -5$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{array}{l} (-19-\lambda)x_1 + 7x_2 = 0 \\ -42x_1 + (16-\lambda)x_2 = 0. \end{array}\right\} \quad (A)$$

—

case(i) When $\lambda = 2$, (A) become

$$-21x_1 + 7x_2 = 0$$

$$-42x_1 + 14x_2 = 0.$$

Both the equations are reduced to one single equation

$$3x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{3}.$$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is $X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

case(ii) When $\lambda = -5$, (A) reduces to

$$-14x_1 + 7x_2 = 0$$

$$-42x_1 + 21x_2 = 0.$$

Both the equations are reduced to one single equation

$$-2x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore$ The eigen vector corresponding to $\lambda = -5$ is $X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Step 3. To form the modal matrix P

The modal matrix P is obtained with X_1, X_2 as columns.

$$\therefore P = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}.$$

Step 4. To find P^{-1}

We know that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Hence $P^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$.

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -19 & 7 \\ -42 & 16 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -19+21 & -19+14 \\ -42+48 & -42+32 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 6 & -10 \end{pmatrix}$$
$$= \begin{pmatrix} -4+6 & 10-10 \\ 6-6 & -15+10 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

which is the required diagonal form of A.

Example 1.44. Reduce the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ to the diagonal form. Hence find A^4.

Solution.

Step 1. To find the eigen values

$s_1 = tr(A) =$ sum of the elements along the main diagonal $= 1 + 2 + 0 = 3$.

$s_2 =$ sum of the minors of the elements of the main diagonal

$$= \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0 + 1 + 0 - 2 + 2 - 2 = -1.$$

$$s_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{vmatrix} = 1(0+1) - 2(0+1) - 2(-1+2) = 1 - 2 - 2 = -3.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda + s_3 = 0$$

—

$$i.e., \lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -3 & -1 & 3 \\ & 0 & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$\therefore$ The characteristic equation becomes

$$(\lambda - 1)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1, 1, 3.$$

The eigen values are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ -x_1 - x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \quad \text{(A)}$$

case(i) When $\lambda = -1$, (A) become

$$2x_1 + 2x_2 - 2x_3 = 0. \quad (1)$$

—

$$x_1 + 3x_2 + x_3 = 0. \tag{2}$$

$$-x_1 - x_2 + x_3 = 0. \tag{3}$$

The above three equations are reduced to the following two equations.

$$x_1 + x_2 - x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0.$$

By the rule of cross multiplication we have,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & -1 & \times & 1 & \times & 1 \\ 3 & & 1 & & 1 & & 3 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{-1-1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}.$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

case(ii) <u>When $\lambda = 1$</u>, (A) become

$$2x_2 - 2x_3 = 0 \tag{4}$$

$$x_1 + x_2 + x_3 = 0 \tag{5}$$

$$-x_1 - x_2 - x_3 = 0. \tag{6}$$

The above three equations are reduced to

$$x_2 = x_3$$

$$\text{and } x_1 + x_2 + x_3 = 0.$$

$$\text{let } x_3 = 1 \Rightarrow x_2 = 1.$$

$$\text{Hence } \; x_1 = -2$$

$$\therefore X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

case(iii) When $\lambda = 3$, (A) become

$$-2x_1 + 2x_2 - 2x_3 = 0 \tag{7}$$

$$x_1 - x_2 + x_3 = 0 \tag{8}$$

$$-x_1 - x_2 - 3x_3 = 0. \tag{9}$$

The above three equations are reduced to the following two equations.

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 + 3x_3 = 0.$$

By the rule of cross multiplication we have

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & \times & 1 & \times & 1 & \times & -1 \\ 1 & & 3 & & 1 & & 1 \end{array}$$

$$\frac{x_1}{-3-1} = \frac{x_2}{1-3} = \frac{x_3}{1+1}$$
$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{2}$$
$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-1}.$$
$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

—

Step 3. To form the modal matrix P

The modal matrix P is formed with the eigen vectors X_1, X_2, X_3 as columns.

$$i.e, \quad P = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 2(-1-1) + 2(1-1) + 2(-1-1)$$

$$= 2(-2) + 2(0) + 2(-2) = -4 + 0 - 4 = -8.$$

$$\text{cofactor of } 2 = +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2.$$

$$\text{cofactor of } -2 = -\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = -(1 - 1) = 0.$$

$$\text{cofactor of } 2 = +\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2.$$

$$\text{cofactor of } -1 = -\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = -(2 - 2) = 0.$$

$$\text{cofactor of } 1 = +\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = -2 - 2 = -4.$$

$$\text{cofactor of } 1 = -\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -(2 + 2) = -4.$$

$$\text{cofactor of } 1 = +\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -2 - 2 = -4.$$

$$\text{cofactor of } 1 = -\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} = -(2 + 2) = -4.$$

—

cofactor of $-1 = +\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 2 - 2 = 0.$

$$\therefore P_c = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -4 & -4 \\ -4 & -4 & 0 \end{pmatrix}.$$

$$P^{-1} = \frac{1}{|P|} P_c^T = \frac{-1}{8} \begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{-1}{8} \begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8} \begin{pmatrix} -2+0+4 & -4+0+4 & 4+0+0 \\ 0-4+4 & 0-8+4 & 0-4+0 \\ -2-4+0 & -4-8+0 & 4-4+0 \end{pmatrix} \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8} \begin{pmatrix} 2 & 0 & 4 \\ 0 & -4 & -4 \\ -6 & -12 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{8} \begin{pmatrix} 4+0+4 & -4+0+4 & 4+0-4 \\ 0+4-4 & 0-4-4 & 0-4+4 \\ -12+12+0 & 12-12+0 & -12-12+0 \end{pmatrix}$$

$$= \frac{-1}{8} \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -24 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

which is the required diagonal form.

—

Step 6. To find A^4

By similarity transformation, $D = P^{-1}AP$.

$$\therefore D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence

$$D^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

Now $A^4 = PD^4P^{-1}$

$$= \frac{-1}{8}\begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 2+0+0 & 0-2+0 & 0+0+162 \\ -1+0+0 & 0+1+0 & 0+0+81 \\ 1+0+0 & 0+1+0 & 0+0-81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} 2 & -2 & 162 \\ -1 & 1 & 81 \\ 1 & 1 & -81 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -4+0-324 & 0+8-648 & -8+8+0 \\ 2+0-162 & 0-4-324 & 4-4+0 \\ -2+0+162 & 0-4+324 & -4-4+0 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -328 & -640 & 0 \\ -160 & -328 & 0 \\ 160 & 320 & -8 \end{pmatrix} = \begin{pmatrix} 41 & 80 & 0 \\ 20 & 41 & 0 \\ -20 & -40 & 1 \end{pmatrix}$$

Result. A simple way to find the inverse of a 3×3 matrix.

—

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Step1. Rearrange the elements in the following way

$$\begin{matrix} b_2 & b_3 & b_1 \\ c_2 & c_3 & c_1 \\ a_2 & a_3 & a_1 \end{matrix}$$

Step2. Attach the first column as the 4^{th} column

$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \end{matrix}$$

Step3. Attach the first row as the 4^{th} row

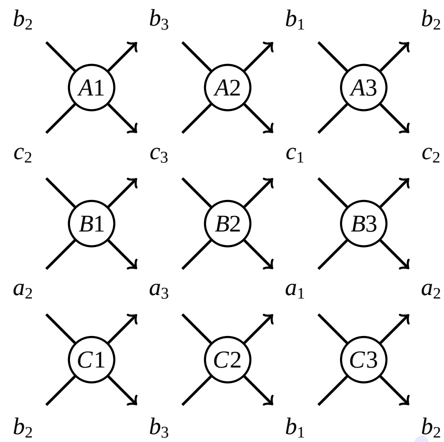
$$\begin{matrix} b_2 & b_3 & b_1 & b_2 \\ c_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & a_2 \\ b_2 & b_3 & b_1 & b_2 \end{matrix}$$

Step4. The matrix A_c formed by the Cofactors of the elements of A is given by

$$A_c = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}.$$

A_i^s, B_i^s and C_i^s are calculated as follows, similar to the rule of Cross multiplication.

—



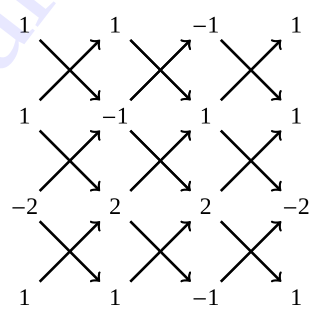
$$A_c = \begin{pmatrix} b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & c_3a_1 - c_1a_3 & c_1a_2 - c_2a_1 \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \end{pmatrix}$$

Now $A^{-1} = \frac{1}{|A|} A_c^T$.

Example. Consider the previous example.

To find P^{-1} we have

$$P = \begin{pmatrix} 2 & -2 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$



$$P_c = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -4 & -4 \\ -4 & -4 & 0 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T.$$

$$= -\frac{1}{8}\begin{pmatrix} 2 & 0 & -4 \\ 0 & -4 & -4 \\ -2 & -4 & 0 \end{pmatrix}$$

Note. We shall apply this procedure to find the inverse of any 3×3 matrix here after.

Example 1.45. Diagonalize the matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Solution. Given $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 1 - 1 = 2.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$

$= -4 - 5 + 4 = -5.$

$$s_3 = |A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$= 2(-1-3) + 2(-1-1) + 3(3-1)$

$= -8 - 4 + 6$

$= -6.$

—

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0.$$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = -2, 3.$$

The eigen values are $1, -2, 3$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$x_1 - 2x_2 + 3x_3 = 0 \tag{2}$$

—

$$x_1 + x_3 = 0 \tag{3}$$

$$x_1 + 3x_2 - 2x_3 = 0. \tag{4}$$

$(3) \Rightarrow x_1 = -x_3$

$\frac{x_1}{-1} = \frac{x_3}{1}$

$\Rightarrow x_1 = -1, x_3 = 1.$

Substituting in (2) we get

$-1 - 2x_2 + 3 = 0$

$2 = 2x_2$

$x_2 = 1.$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$

When $\lambda = -2$ (1) becomes

$$\begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$4x_1 - 2x_2 + 3x_3 = 0$

$x_1 + 3x_2 + x_3 = 0.$

By the rule of cross multiplication, we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & \times & 3 & \times & 4 & \times & -2 \\ 3 & & 1 & & 1 & & 3 \end{array}$$

$$\frac{x_1}{-2-9} = \frac{x_2}{3-4} = \frac{x_3}{12+2}$$

$$\frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$$\therefore x_1 = 11, x_2 = 1, x_3 = -14$$

—

$$\therefore X_2 = \begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$$-x_1 - 2x_2 + 3x_3 = 0 \qquad (5)$$

$$x_1 - 2x_2 + x_3 = 0 \qquad (6)$$

$$x_1 + 3x_2 - 4x_3 = 0. \qquad (7)$$

All the three equations are different.

Taking (5) & (6) and by the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & \times & 3 & \times & -1 & \times & -2 \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{-2+6} = \frac{x_2}{3+1} = \frac{x_3}{2+2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\therefore x_1 = x_2 = x_3 = 1$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

—

Step 3. To form the modal matrix P

The modal matrix P is formed with the eigen vectors X_1, X_2, X_3 as columns.

$$\text{i.e., } P = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{vmatrix}$$

$$= -1(1+14) - 11(1-1) + 1(-14-1)$$

$$= -15 + 0 - 15$$

$$= -30.$$

To find P_c

$$\begin{matrix} 1 & 1 & 1 & 1 \\ -14 & 1 & 1 & -14 \\ 11 & 1 & -1 & 11 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$$P_c = \begin{pmatrix} 15 & 0 & -15 \\ -25 & -2 & -3 \\ 10 & 2 & -12 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T$$

$$= -\frac{1}{30} \begin{pmatrix} 15 & -25 & 10 \\ 0 & -2 & 2 \\ -15 & -3 & -12 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{pmatrix}$$

—

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{1}{30}\begin{pmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{pmatrix}\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix} -15 & 25 & -10 \\ 0 & -4 & 4 \\ 45 & 9 & 36 \end{pmatrix}\begin{pmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{pmatrix}$$

$$= \frac{1}{30}\begin{pmatrix} 30 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & 90 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

which is the required diagonal form.

Example 1.46. Diagonalize the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Solution. Let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Step 1. To find the eigen values

$s_1 = tr(A) = 6 + 3 + 3 = 12.$

s_2 = sum of the minors of the main diogonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$= 9 - 1 + 18 - 4 + 18 - 4$

$= 8 + 14 + 14 = 36.$

—

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32.$$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$i.e., \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$$

$$\lambda = 2 \text{ is a root.}$$

By synthetic division we have

2	1	−12	36	−32
	0	2	−20	32
	1	−10	16	0

Hence, the characteristic equation becomes

$$(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

$$\lambda = 2, 2, 8.$$

$\therefore$ The eigen values are $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 8$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0.$$

—

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\}. \qquad \text{(A)}$$

case(i) when $\lambda = 2$, (A) reduces to

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0.$$

The above three equations are reduced to the single equation

$$2x_1 - x_2 + x_3 = 0.$$

Since $\lambda = 2$ is a repeated root, we have to find two eigen vectors by assigning a particular value to two variables, one at a time.

Assigning $x_3 = 0$, we obtain

$$2x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Assigning $x_2 = 0$, we obtain

$$2x_1 = -x_3$$

$$\frac{x_1}{-1} = \frac{x_3}{2}.$$

—

$$\therefore X_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

case(ii) when $\lambda = 8$, (A) reduces to

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad (2)$$

$$2x_1 - x_2 - 5x_3 = 0. \quad (3)$$

Since all the three equations are different, we can consider the the first two equations and solve for $x_1, x_2, \&, x_3$ by the method of cross multiplication

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -2 & \times & 2 & \times & -2 & \times & -2 \\ -5 & & -1 & & -2 & & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$
$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$
$$\therefore x_1 = 2, x_2 = -1, x_3 = 1.$$

x_1, x_2, x_3 satisfy (3).

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Step 3. To form the modal matrix P

P can be obtained by considering X_1, X_2, X_3 as columns.

$$\therefore P = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

—

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1(0 + 2) + 1(2 - 0) + 2(4 - 0) = 2 + 2 + 8 = 12.$$

To find P_c

$$\begin{matrix} 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -1 \\ 0 & -1 & 2 & 0 \end{matrix}$$

$$P_c = \begin{pmatrix} 2 & -2 & 4 \\ 5 & 1 & -2 \\ 1 & 5 & 2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} P_c^T.$$

$$= \frac{1}{12}\begin{pmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$P^{-1}AP = \frac{1}{12}\begin{pmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{pmatrix}\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 12 - 10 + 2 & -4 + 15 - 1 & 4 - 5 + 3 \\ -12 - 2 + 10 & 4 + 3 - 5 & -4 - 1 + 15 \\ 24 + 4 + 4 & -8 - 6 - 2 & 8 + 2 + 6 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

—

$$= \frac{1}{12}\begin{pmatrix} 4 & 10 & 2 \\ -4 & 2 & 10 \\ 32 & -16 & 16 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 4+20+0 & -4+0+4 & 8-10+2 \\ -4+4+0 & 4+0+20 & -8-2+10 \\ 32-32+0 & -32+0+32 & 64+16+16 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 96 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

which is the required diagonal form.

Example 1.47. Find e^A and 4^A if $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Solution.

Step 1. To find the eigen values

$$s_1 = \text{sum of the main diagonal elements.}$$
$$= 3 + 3 = 6.$$
$$s_2 = |A| = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8.$$

The characteristic equation is

$$\lambda^2 - s_1\lambda + s_2 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda - 2)(\lambda - 4) = 0$$
$$\lambda = 2, 4.$$

The eigen values are $\lambda_1 = 2$, $\lambda_2 = 4$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$i.e. \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\left.\begin{aligned} (3-\lambda)x_1 + x_2 &= 0 \\ x_1 + (3-\lambda)x_2 &= 0. \end{aligned}\right\} \qquad \text{(A)}$$

case(i) when $\lambda = 2$, (A) reduces to one single equation

$$x_1 + x_2 = 0$$

$$i.e. x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}.$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

case(ii) when $\lambda = 4$, (A) reduces to

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

Both the equations are reduced to one single equation

$$x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}.$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

—

Step 3. To form the modal matrix P

P is obtained by taking X_1, X_2 as columns.

$$\therefore P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Step 4. To find P^{-1}

$$|P| = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2.$$

$$P^{-1} = \frac{1}{|P|} P_c^T = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Step 5. To find $P^{-1}AP$

$$\begin{aligned} P^{-1}AP &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -3+1 & -1+3 \\ 3+1 & 1+3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2+2 & -2+2 \\ -4+4 & 4+4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}. \end{aligned}$$

Step 6. To find e^A and 4^A

By similarity transformation

$$D = P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

—

Let $f(A) = e^A$, then $f(D) = e^D = \begin{pmatrix} e^2 & 0 \\ 0 & e^4 \end{pmatrix}$.

$$\text{Now, } e^A = Pf(D)P^{-1}$$
$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^4 \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} -e^2 & e^4 \\ e^2 & e^4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$e^A = \frac{1}{2}\begin{pmatrix} e^2 + e^4 & -e^2 + e^4 \\ -e^2 + e^4 & e^2 + e^4 \end{pmatrix}.$$

Replacing e by 4, we get

$$4^A = \frac{1}{2}\begin{pmatrix} 4^2 + 4^4 & -4^2 + 4^4 \\ -4^2 + 4^4 & 4^2 + 4^4 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 16 + 256 & -16 + 256 \\ -16 + 256 & 16 + 256 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 272 & 240 \\ 240 & 272 \end{pmatrix} = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}.$$

Exercise I(D)

1. Find a nonsingular matrix P such that $P^{-1}AP$ is in a diagonal form where A is given by

i. $\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$ ii. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ iii. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

iv. $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ v. $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ vi. $\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -2 & -4 & -3 \end{pmatrix}$

—

2. Reduce the following matrices to the diagonal form using similarity transformation.

i. $\begin{pmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ ii. $\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ iii. $\begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$

iv. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ v. $\begin{pmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{pmatrix}$ vi. $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$ vii. $\begin{pmatrix} 2 & -1 & 0 \\ 9 & 4 & 6 \\ -8 & 0 & -3 \end{pmatrix}$.

3. Reduce the matrix $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$ to the diagonal form and hence evaluate A^n and A^4.

4. Find A^4 if $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

5. Reduce the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ to the diagonal form by similarity transformation. Hence find A^3.

1.6 Diagonlization by orthogonal transformation

A real square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

Result. A real square matrix A is orthogonal if $A^T = A^{-1}$.

Normalized eigen vector

Let $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be an eigen vector corresponding to an eigen value λ. Then the normalized eigen vector of X is defined as $\dfrac{X}{\sqrt{a^2+b^2+c^2}}$.

Example

If $X = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is an eigen vector, then its normalised eigen vector is $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$.

Diagonalization by orthogonal transformation

A real square matrix A is said to be orthogonally diagonalizable, if there exists an orthogonal matrix N such that $N^{-1}AN = D \Rightarrow N^TAN = D$. This transformation which transforms A into D is called an orthogonal transformation or orthogonal reduction.

Symmetric Matrices

A real square matrix A is said to be symmetric if $A^T = A$.

Working rule for orthogonal reduction

Step 1. Find the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 2. Find eigen vectors $X_1, X_2, \ldots, X_n$ which are pairwise orthogonal.

Step 3. Form the normalised modal matrix N with the normalised eigen vectors as columns.

Step 4. Find N^T and $N^TAN = D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$.

Definition 1.1. If $X_i, X_j, i, j = 1, 2, \ldots, n$ are orthogonal, then $X_i^T X_j = 0, i \neq j$.

Properties of eigen values of orthogonal matrices

Property 1.9. The eigen values of a real symmetric matrix are real.

—

Proof. Let λ (real or complex) be an eigen value of A.

Let X be the eigen vector corresponding to the eigen value λ.

$$\therefore AX = \lambda X. \tag{1}$$

Let $\bar{X}$ be the Complex conjugate of X.

Premultiply (1) by $\bar{X}^T$ we get

$$\bar{X}^T AX = \lambda \bar{X}^T X$$

$$\overline{\bar{X}^T AX} = \overline{\lambda \bar{X}^T X}$$

i.e., $X^T \bar{A}\bar{X} = \bar{\lambda} X^T \bar{X}$.

Since A is a real symmetric matrix, $\bar{A} = A$ & $A^T = A$.

$$\therefore \; X^T A\bar{X} = \bar{\lambda} X^T \bar{X}.$$

Taking transpose on both sides we get

$$\left(X^T A\bar{X}\right)^T = \left(\bar{\lambda} X^T \bar{X}\right)^T$$

$$\bar{X}^T A^T X = \bar{X}^T X \bar{\lambda}^T = \bar{\lambda}\bar{X}^T X$$

$$\bar{X}^T AX = \bar{\lambda}\bar{X}^T X$$

$$\bar{X}^T \lambda X = \bar{\lambda}\bar{X}^T X$$

$$\lambda \bar{X}^T X = \bar{\lambda}\bar{X}^T X.$$

Now X is an $n \times 1$ matrix and $\bar{X}^T$ is a $1 \times n$ matrix.

Hence, $\bar{X}^T X$ is a 1×1 matrix, which is a positive real.

$$\Rightarrow \; \lambda = \bar{\lambda}.$$

$\Rightarrow$ λ is a real number. □

Property 1.10. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal. [Dec.2002]

Proof. By property 1.9, we know that, if A is a real symmetric matrix, then its eigen values are real.

Let λ_1 and λ_2 be two real eigen values such that $\lambda_1 \neq \lambda_2$.

—

Let X_1 and X_2 be the eigen vectors corresponding to the eigen values λ_1 and λ_2 respectively.

$$\therefore \; AX_1 = \lambda_1 X_1 \tag{1}$$

$$AX_2 = \lambda_2 X_2. \tag{2}$$

Premultiplying (1) by X_2^T we get

$$X_2^T AX_1 = X_2^T \lambda_1 X_1 = \lambda_1 X_2^T X_1.$$

Taking transpose both sides we get

$$\left(X_2^T AX_1\right)^T = \left(\lambda_1 X_2^T X_1\right)^T$$

$$\Rightarrow \; X_1^T A^T X_2 = \lambda_1 X_1^T X_2$$

$$\text{i.e., } X_1^T AX_2 = \lambda_1 X_1^T X_2 \quad [\text{since, } A \text{ is symmetric, } A^T = A]. \tag{3}$$

Premultiplying (2) by X_1^T we get

$$X_1^T AX_2 = X_1^T \lambda_2 X_2 = \lambda_2 X_1^T X_2. \tag{4}$$

From (3) and (4) we get

$$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$$

$$\text{i.e., } \lambda_1 X_1^T X_2 - \lambda_2 X_1^T X_2 = 0$$

$$(\lambda_1 - \lambda_2) X_1^T X_2 = 0.$$

$$\text{Since } \lambda_1 \neq \lambda_2, \; \lambda_1 - \lambda_2 \neq 0.$$

$$\therefore \; X_1^T X_2 = 0$$

$\Rightarrow X_1$ and X_2 are orthogonal.

□

Property 1.11. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

—

Proof. Let A be an orthogonal matrix.

$$\Rightarrow \; A^T = A^{-1}. \qquad (1)$$

We know that A and A^T have the same eigen values.

Also we know that if λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

From (1), we get $\frac{1}{\lambda}$ is an eigen value of A^T.

$\Rightarrow$ $\frac{1}{\lambda}$ is an eigen value of A. □

Results

1. Diagonalization of orthogonal transformation is possible only for a real symmetric martix.
2. If A is a real symmetric matrix, then the eigen vectors of A will be not only linearly independent but also pairwise orthogonal.
3. If A is orthogonal, then A^T is orthogonal.
4. If A is an orthogonal matrix, then $|A| = \pm 1$.
5. The eigen values of an orthogonal matrix are of magnitude 1.

Worked Examples

Example 1.48. Prove that $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal. [Jan 2009]

Solution. $|A| = \cos^2\theta + \sin^2\theta = 1$, which is nonsingular.

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore$ A is orthogonal.

—

Example 1.49. Show that $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$ is orthogonal.

Solution. Given $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$

$$A^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$AA^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\therefore$ A is orthogonal.

Example 1.50. If A is an orthogonal matrix, prove that A^{-1} is also orthogonal.

Solution. Since A is orthogonal, $A^T = A^{-1}$.

Let $A^T = A^{-1} = B$.

Now $B^T = (A^{-1})^T = (A^T)^{-1} = B^{-1}$.

$\therefore$ B is orthogonal.

i.e., A^{-1} is orthogonal.

Example 1.51. Prove that if A and B are orthogonal matrices, then AB is orthogonal.

Solution. Given A & B are orthogonal.

$\therefore\ A^T = A^{-1}$ and $B^T = B^{-1}$.

Now, $(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$ $\quad \Rightarrow AB$ is orthogonal.

—

Example 1.52. Diagonalise the symmetric matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ by an orthogonal transformation. [Jan 2004]

Solution. A is a real symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

Now s_1 = sum of the main diagonal elements

$$= 2 + 1 + 1 = 4.$$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1 - 4 + 2 - 1 + 2 - 1$$

$$= -3 + 1 + 1 = -1.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 2(-3) - 1(-1) - 1(-1)$$

$$= -6 + 1 + 1 = -4.$$

The characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$.

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we get,

1	1	−4	−1	4
	0	1	−3	−4
	1	−3	−4	0

$$\lambda^2 - 3\lambda - 4 = 0.$$

$$(\lambda + 1)(\lambda - 4) = 0.$$

$\therefore$ The characteristic equation becomes $(\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$.

$\therefore$ The eigen values are $-1, 1, 4$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to λ. Then,

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (1)$$

When $\lambda = -1$, (1) becomes

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are reduced to two equations

$$3x_1 + x_2 - x_3 = 0.$$

$$x_1 + 2x_2 - 2x_3 = 0.$$

By the rule of cross multiplication we get,

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & -1 & \times & 3 & \times & 1 \\ 2 & & -2 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5} \Longrightarrow x_1 = 0, x_2 = 1, x_3 = 1.$$

$$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

—

When $\lambda = 1$ (1) become

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$\text{i.e.,} \;\; x_1 + x_2 - x_3 = 0.$$

$$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \Rightarrow \frac{x_1}{2} = \frac{x_3}{1} \Longrightarrow x_1 = 2, x_3 = 1.$$

$$-x_1 - 2x_2 = 0 \Rightarrow x_1 = -2x_2 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} \Longrightarrow x_1 = 2, x_2 = -1.$$

$$\therefore \;\; x_1 = 2, x_2 = -1, x_3 = 1$$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 4$, (1) become

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

i.e., $-2x_1 + x_2 - x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0.$

$x_1 - 3x_2 - 2x_3 = 0.$

$-x_1 - 2x_2 - 3x_3 = 0 \Rightarrow x_1 + 2x_2 + 3x_3 = 0.$

All the three equations are different.

Taking the first two equations and solving by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & \times & 1 & \times & 2 & \times & -1 \\ -3 & & -2 & & 1 & & -3 \end{array}$$

$$\text{i.e.,} \; \frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}.$$

$$\Longrightarrow x_1 = 1, x_2 = 1, x_3 = -1.$$

—

$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Since A is symmetric and all the eigen values are different X_1, X_2, X_3 are pairwise orthogonal.

Step 3. To form the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$.

The modal matrix N is formed with the normalized eigen vectors as columns.

i.e., $N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$.

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D.$$

Example 1.53. Diagonalise the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ by an orthogonal reduction.

Solution. Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

A is a real symmetric matrix.

—

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements of A.

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= 5 - 1 + 1 - 9 + 5 - 1$

$= 4 - 8 + 4 = 0.$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$= 1(5 - 1) - 1(1 - 3) + 3(1 - 15)$

$= 4 + 2 - 42$

$= -36.$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\lambda = -2$ is a root.

By synthetic division we have

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0$

$\lambda = 3, \lambda = 6.$

The eigen values are $-2, 3, 6$.

—

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to

$$3x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 7x_2 + x_3 = 0.$$

By the rule of cross multiplication we obtain

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & 3 & \times & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore \ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -2 & \times & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$
$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$
$$\therefore \; x_1 = 1, x_2 = -1, x_3 = 1.$$
$$\therefore \; X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0.$$

Taking the first two equations and applying rule of cross multiplication we get

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -5 & \times & 1 \\ -1 & & 1 & & 1 & & -1 \end{matrix}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\therefore \; x_1 = 1, x_2 = 2, x_3 = 1.$$

$$\therefore \; X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Since A is symmetric and all the eigen values are different x_1, x_2, x_3 are pairwise orthogonal.

Step 3. To form the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore \; N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Step 4. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

—

Example 1.54. Diagonalise the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by means of an orthogonal transformation. [Jan 2004]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now s_1 = sum of the main diagonal elements

$= 3 + 3 + 3 = 9.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 9 - 1 + 9 - 1 + 9 - 1 = 24.$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3(9 - 1) - 1(3 + 1) + 1(-1 - 3)$

$= 24 - 4 - 4 = 16.$

$\therefore$ The characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$

$$\lambda = 1 \text{ is a root.}$$

By synthetic division we have

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$$\Longrightarrow \lambda^2 - 8\lambda + 16 = 0.$$

—

$$(\lambda - 4)(\lambda - 4) = 0.$$

The characteristic equation becomes $(\lambda - 1)(\lambda - 4)(\lambda - 4) = 0$.

$\therefore$ The eigen values are $1, 4, 4$.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to λ. Therefore,

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$2x_1 + x_2 + x_3 = 0.$$

$$x_1 + 2x_2 - x_3 = 0.$$

$$x_1 - x_2 + 2x_3 = 0.$$

Taking the first two equations and solving for x_1, x_2, x_3 by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 1 & \times & 2 & \times & 1 \\ 2 & & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\Longrightarrow \frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}.$$

$$\Longrightarrow x_1 = -1, x_2 = 1, x_3 = 1.$$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$

When $\lambda = 4$, (1) becomes

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$x_1 - x_2 - x_3 = 0. \tag{1}$$

Put $x_3 = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \Longrightarrow x_1 = 1, x_2 = 1.$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

By the condition of orthogonality, $X_3^T X_2 = 0$.

$$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\Longrightarrow a + b = 0 \Rightarrow a = -b.$$

Also X_3 should satisfy (1).

$$\therefore a - b - c = 0 \Rightarrow -2b - c = 0 \Rightarrow 2b = -c$$

$$\Rightarrow \frac{b}{-1} = \frac{c}{2} \Rightarrow b = -1, c = 2, a = 1.$$

—

$$\therefore X_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Step 3. To find the modal matrix N

The normalised eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}.$$

The normalised modal matrix is formed by the normalised eigen vectors as columns.

$$\therefore N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}, N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Step 4. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{-4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{8}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

which is the required diagonal form.

Computation of powers of a square matrix

Let A be the given square matrix. Diagonalising A, we obtain,

$$D = N^TAN = N^{-1}AN.$$

$$D^2 = D.D = (N^{-1}AN)(N^{-1}AN) = N^{-1}A(NN^{-1})AN$$

—

$$= N^{-1}AAN = N^{-1}A^2N$$

$$D^3 = N^{-1}A^3N$$

$$\text{In general, } D^r = N^{-1}A^rN$$

Where $D^r = \begin{pmatrix} \lambda_1^r & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n^r \end{pmatrix}$.

Premultiplying by N and postmultiplying by N^{-1} we get,

$$A^r = ND^rN^{-1}.$$

Example 1.55. Diagonalise the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Hence find A^3. [Jan 2006]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now s_1 = sum of the main diagonal elements

$= 2 + 3 + 2 = 7.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$= 6 - 0 + 4 - 1 + 6 - 0$

$= 6 + 3 + 6 = 15.$

$$s_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

—

$$= 2(6 - 0) + 1(0 - 3)$$
$$= 12 - 3 = 9.$$

The characteristic equation is $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$.

$$\lambda = 1 \text{ is a root.}$$

By synthetic division, we get

$$\begin{array}{c|cccc} 1 & 1 & -7 & 15 & -9 \\ & 0 & 1 & -6 & 9 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

$$\lambda^2 - 6\lambda + 9 = 0.$$
$$(\lambda - 3)(\lambda - 3) = 0$$
$$\Rightarrow \lambda = 3, 3.$$

The eigen values are 1, 3, 3.

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore (A - \lambda I)X = 0$$

$$\text{i.e., } \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

—

The above equations become

$$x_1 + x_3 = 0$$

$$2x_2 = 0.$$

From the last equation we get $x_2 = 0$.

Also $x_1 = -x_3$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

$$\therefore\ x_1 = 1, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$x_1 - x_3 = 0 \qquad (2)$$

$$\Rightarrow\ x_1 = x_3$$

i.e., $x_1 = 1, x_3 = 1$ and $x_2 =$ any value $= 0$ (say)

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

$$\therefore\ X_3^T X_2 = 0.$$

$$\text{i.e., } \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad \Longrightarrow a + c = 0. \qquad (3)$$

—

i.e., Also X_3 will satisfy by (2).

Solving (2) & (3) we get $a = 0, c = 0$. Choose $b = 1$.

We get the corresponding eigen vector $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Step 3. To find the modal matrix N

The normalised eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The modal matrix N is formed with the normalised eigen vectors as columns.

$$\therefore \; N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Step 4. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$

Step 5. To find A^3

$$D^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$\text{Now, } A^3 = ND^3N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 13 \\ 0 & 27 & 0 \\ 13 & 0 & 14 \end{pmatrix}.$$

—

Example 1.56. Diagonalise the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. Hence find the value of A^4. [Jan 2013]

Solution. A is a symmetric matrix.

Step 1. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements.

$= 8 + 7 + 3 = 18.$

s_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$$= 21 - 16 + 24 - 4 + 56 - 36$$

$$= 5 + 20 + 20 = 45.$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$

$$= 40 - 60 + 20 = 0$$

$\therefore$ The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

The eigen values are $0, 3, 15$

—

Step 2. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to an eigen value λ.

$$\therefore\ (A - \lambda I)X = 0$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \quad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & 8 & \times & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$x_1 = 1, x_2 = 2, x_3 = 2.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$, (1) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 = 0$$

i.e., $x_1 = 2x_2$

$$\frac{x_1}{2} = \frac{x_2}{1}$$

$\therefore\ x_1 = 2, x_2 = 1.$

Substituting in the first equation we get

$$10 - 6 + 2x_3 = 0$$
$$4 + 2x_3 = 0$$
$$2x_3 = -4$$
$$x_3 = -2.$$

$$\therefore\ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

When $\lambda = 15$, (1) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 - 8x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & -7 & \times & -6 \\ 8 & & 4 & & 6 & & 8 \end{array}$$

$$\frac{x_1}{-24-16} = \frac{x_2}{12+28} = \frac{x_3}{-56+36}$$

$$\frac{x_1}{-40} = \frac{x_2}{40} = \frac{x_3}{-20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$\therefore\ x_1 = 2, x_2 = -2, x_3 = 1.$$

$$\therefore\ X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Since A is a symmetric matrix and all the eigen values are different, X_1, X_2 & X_3 are pairwise orthogonal.

Step 3. To find the modal matrix N

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore\ N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step 4. To find N^TAN

$$N^TAN = \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}\begin{pmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 0 & 0\\ 6 & 3 & -6\\ 30 & -30 & 15\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 0 & 0\\ 0 & 27 & 0\\ 0 & 0 & 135\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix} = D$$

Step 5. To find A^4

$$D = \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 15\end{pmatrix}.$$

$$D^4 = \begin{pmatrix}0 & 0 & 0\\ 0 & 3^4 & 0\\ 0 & 0 & 15^4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 50625\end{pmatrix}.$$

Now $A^4 = ND^4N^T$

$$= \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 50625\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}0 & 162 & 101250\\ 0 & 81 & -101250\\ 0 & -162 & 50625\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}202824 & -202338 & 100926\\ -202338 & 202581 & -101412\\ 100926 & -101412 & 50949\end{pmatrix} = \begin{pmatrix}22536 & -22482 & 11214\\ -22482 & 22509 & -11268\\ 11214 & -11268 & 5661\end{pmatrix}.$$

—

Example 1.57. The eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values $2, 3, 6$ are $[1\ 0\ -1]^T, [1\ 1\ 1]^T, [-1\ 2\ -1]^T$ respectively. Find A.

[Jan 2012]

Solution. Given, A is symmetric and the eigen values are different.

$\therefore$ The eigen vectors are pairwise disjoint.

$$\text{Now, } N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}.$$

$\therefore$ By the orthogonal transformation, we obtain

$$N^T AN = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

$$\text{Now, } A = NDN^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Exercise I(E)

1. Prove that the matrix $B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal.

2. Show that the matrix $\frac{1}{7}\begin{pmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{pmatrix}$ is orthogonal.

—

3. Prove that the following matrices are orthogonal.

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\frac{1}{9}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$.

4. Diagonalise the matrix $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

5. Reduce $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ to the diagonal form by an orthogonal reduction.

6. Reduce the matrix $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ to the diagonal form.

7. By an orthogonal transformation, diagonalise the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

8. Reduce the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ to diagonal form by an orthogonal reduction. Hence find A^3.

9. The eigen vectors corresponding to the eigen values $1, 2, 3$ of the symmetric matrix A are $[1\ -1\ 0]^T, [0\ 0\ 1]^T$ and $[1\ 1\ 0]^T$ respectively. Find A.

10. $[1\quad -1]^T$ and $[1\ \ 1]^T$ are the eigen vectors of the symmetric matrix A corresponding to the eigen values 0 and 2 respectively. Find A.

—

1.7 Quadratic Form

Definition. A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Example.

(1) $x^2 + 2xy + 2y^2$ is a quadratic form in 2 variables.

(2) $ax^2 + 2hxy + by^2 + cz^2 + 2gyz + 2fzx$ is a quadratic form in 3 variables.

(3) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2\ell xw + 2myw + 2nzw$ is a quadratic form in 4 variables.

Definition. The general quadratic form in n variables $x_1, x_2, \dots, x_n$ is $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ where a_{ij}'s are real numbers such that $a_{ij} = a_{ji}$ for all $i, j = 1, 2, 3, \dots, n$.

Notation. Usually the quadratic form is denoted by Q.

$$\therefore\ Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j.$$

Matrix form of Q

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ where $a_{ij} = a_{ji}$. Then, A is a symmetric matrix.

Now the quadratic form $Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ can be written as $Q = X^TAX$. A is called the matrix of the quadratic form.

Example. Consider the quadratic form $x^2 + 4xy + y^2$.

i.e., $x^2 + 2xy + 2yx + y^2$.

The quadratic form is $[x\ y]\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Hence $X = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

—

1.8 Canonical form of Q

A quadratic form of Q which contains only the square terms of the variables is said to be in canonical form.

Example. $x^2 + 2y^2, x^2 - y^2, x^2 + y^2 + z^2, x_1^2 + x_2^2 + x_3^2 + 4x_4^2$ are in canonical forms.

Reduction of Q to canonical form by orthogonal transformation

Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$ and $A = [a_{ij}]$ be the symmetric matrix of order n of the quadratic form. We shall reduce A to the diagonal form by an orthogonal transformation. Let $X = NY$ where N is the normalized modal matrix of A so that $N^T AN = D$, where $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A.

$$\text{We have } Q = X^T AX = (NY)^T A(NY) = (y^T N^Y)A(NY)$$
$$= Y^T(N^T AN)Y = Y^T DY.$$

If $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ then $Q = [y_1\ y_2\ \ldots\ y_n] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

which is the required canonical form.

Note

(i) In the above canonical form, some λ_i's may be positive or negative or zero.

(ii) In the canonical form, the coefficients are the eigen values of the matrix A.

(iii) A quadratic form is said to be real if the elements of the symmetric matrix are real.

—

Definition

If A is the matrix of the quadratic form Q in the variables $x_1, x_2, \ldots, x_n$, then the rank of Q is equal to the rank of A. The rank of A is denoted by $\rho(A)$. If rank of $A < n$, where n is the number of variables or order of A, then $|A| = 0$ and Q is called a singular form.

Definition

Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$, where $X = [x_1\ x_2\ \ldots\ x_n]^T$ and A is the matrix of the quadratic form.

(i) The number of positive and negative eigen values of A is called the *rank* of the quadratic form. It is denoted by r.

(ii) The number of positive eigen values of A is defined as the *index* of the quadratic form. It is also denoted by p. It is equal to the number of positive terms in the canonical form.

(iii) The difference between the number of positive and negative eigen values of A is called the *signature* of the quadratic form. It is denoted by s. It is also equal to the difference between the number of positive and negative terms in the canonical form. i.e., $s = p - (r - p) = 2p - r$ where $\rho(A) = r$.

(iv) Q is said to be *positive definite* if all the n eigen values of A are positive. i.e., $r = n, p = r$.

Ex. $y_1^2 + y_2^2 + \cdots + y_n^2$ is positive definite.

(v) Q is said to be *negative definite* if all the n eigen values of A are negative. i.e.,$r = n, p = 0$.

Ex. $-y_1^2 - y_2^2 - \cdots - y_n^2$ is negative definite.

(vi) Q is said to be *positive semi definite* if all the n eigen values are ≥ 0 with atleast one eigen value $= 0$.

i.e., if $r < n$ and $p = r$.

Ex. $y_1^2 + y_2^2 + y_4^2 + \cdots + y_r^2$ is positive semi definite.

—

(vii) Q is said to be *negative semi definite* if all the n eigen values of A are ≤ 0 with atleast one value $= 0$.

i.e., if $r < n$ and $p = 0$.

Ex. $-y_1^2 - y_2^2 - y_5^2 - \cdots - y_r^2, (r < n)$ is negative semi definite.

(viii) Q is said to be *indefinite* if A has positive and negative eigen values.

Ex. $y_1^2 - y_2^2 + y_3^2 - y_4^2 - y_5^2 + \cdots + y_n^2$ is indefinite.

Result. The nature of the quadratic form can also be found without finding the eigen values of A or without reducing to canonical form but by using the principal minors of A.

Definition. Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$ and let the matrix of the form A be

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Let $D_1 = |a_{11}|, D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ etc.

Finally $D_n = |A|$. The determinants $D_1, D_2, \ldots, D_n$ are called the principal minors of A. The quadratic form Q is said to be

(i) Positive definite if $D_i > 0$ for all $i = 1, 2, \ldots, n$.

(ii) Negative definite if $(-1)^i D_i > 0$ for all $i = 1, 2, \ldots, n$. i.e.,$D_1, D_3, D_5, \ldots$ are negative and $D_2, D_4, D_6, \ldots$ are positive.

(iii) Positive semidefinite if $D_i \geq 0$ for all $i = 1, 2, 3, \ldots, n$ and atleast one $D_i = 0$.

(iv) Negative semidefinite if $(-1)^i D_i \geq 0$ for all $i = 1, 2, 3, \ldots, n$ and atleast one $D_i = 0$.

(v) Indefinite in all other cases.

—

Law of inertia of a quadratic form

The index of a real quadratic form is invariant under a real non singular transformation. This property is called the law of inertia of the quadratic form.

Worked Examples

Example 1.58. Write down the matrix of the quadratic form $2x^2 + 3y^2 + 6xy$.

[Jan 2001]

Solution. The given quadratic form is in two variables.

Hence, the matrix of the quadratic form is a 2×2 symmetric matrix. Let A be the required matrix. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

a_{11} = coefficient of $x^2 = 2$.

$a_{12} = a_{21} = \frac{1}{2}$ coefficient of $xy = \frac{1}{2} \times 6 = 3$.

a_{22} = coefficient of $y^2 = 3$.

$\therefore\ A = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$

Example 1.59. Write down the matrix of the quadratic form

$2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$. [Jan 2002]

Solution. The given quadratic form contains 3 variables. Hence, the matrix A of the quadratic form is a 3×3 symmetric matrix. The matrix of the quadratic form is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

a_{11} = Coeff. of $x_1^2 = 2$.

a_{22} = Coeff. of $x_2^2 = -2$.

a_{33} = Coeff. of $x_3^2 = 4$.

$a_{12} = a_{21} = \frac{1}{2}$ Coeff. of $x_1x_2 = \frac{1}{2}2 = 1$.

$a_{13} = a_{31} = \frac{1}{2}$ Coeff. of $x_1x_3 = \frac{1}{2}(-6) = -3$.

—

$a_{23} = a_{32} = \frac{1}{2}$ Coeff. of $x_2x_3 = \frac{1}{2}6 = 3.$

$$\therefore A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix}.$$

Example 1.60. Write down the matrix of the quadratic form $2x^2+8z^2+4xy+10xz-2yz$ [Jun 2013, Dec 2001].

Solution. The given quadratic form contain 3 variables. Hence the matrix A of the quadratic form is a 3×3 matrix.

$$\text{Hence } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned}
\text{Now, } a_{11} &= \text{Coeff.of } x^2 = 2 \\
a_{12} = a_{21} &= \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 4 = 2. \\
a_{13} = a_{31} &= \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 10 = 5. \\
a_{22} &= \text{Coeff.of } y^2 = 0 \\
a_{23} = a_{32} &= \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1. \\
a_{33} &= \text{Coeff.of } z^2 = 8.
\end{aligned}$$

$$\therefore A = \begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}.$$

Example 1.61. Write down the quadratic form of the matrix $\begin{pmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{pmatrix}$.[Jan 2003]

Solution. The given matrix is a 3×3 symmetric matrix. Hence, the quadratic form is in 3 variables. The required quadratic form must be of the form

$$Q = a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_3z^2.$$

—

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we get,

$a_{11} = 2, a_{12} = 4, a_{13} = 5, a_{22} = 3, a_{23} = 1, a_{33} = 1.$

$\therefore Q = 2x^2 + 8xy + 10xz + 3y^2 + 2yz + z^2.$

$= 2x^2 + 3y^2 + z^2 + 8xy + 10xz + 2yz.$

Example 1.62. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}$.

Solution. Since the given matrix is a 3×3 symmetric matrix, the quadratic form must be in 3 variables x, y, z. Let Q be of the required quadratic form

$\therefore\ Q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz.$

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we have

$a_{11} = 2, a_{22} = 0, a_{33} = 8, a_{12} = 2, a_{13} = 5, a_{23} = -1.$

$\therefore\ Q = 2x^2 + 8z^2 + 4xy - 2yz + 10xz.$

Example 1.63. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$. [Jan 2013]

Solution. Since the given matrix is a 3×3 symmetric matrix, the quadratic form must be in 3 variables x, y, z. Let Q be the required quadratic form.

$\therefore\ Q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz.$

Comparing the given matrix with $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we obtain

$$a_{11} = 2, a_{12} = 0, a_{13} = -2, a_{22} = 2, a_{23} = 1, a_{33} = -2.$$

$$\therefore\ Q = 2x^2 + 2y^2 - 2z^2 - 4xz + 2yz.$$

—

Example 1.64. Determine the nature of the quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$.

[Jan 2003]

Solution. Since it contains only square terms it is of the canonical form. The canonical form contains 3 variables but the RHS expression contains only two terms. The coefficients of the canonical form are the eigen values of the matrix of the quadratic form. Hence, $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 0$. Hence, the given quadratic form is positive semi-definite.

Example 1.65. Find the nature of the quadratic form $2x^2 + 2xy + 3y^2$.

Solution. Let us form the matrix A of the given quadratic form. Since the Q.F contains only two variables, A is a 2×2 matrix.

$\therefore\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. From the given Q.E, we have

$$a_{11} = \text{Coeff.of } x^2 = 2.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2 = 1$$

$$a_{22} = \text{Coeff.of } y^2 = 3.$$

$$\therefore\ A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Let us analyse the nature of the Q.F with the help of the principal minors.

$$D_1 = |2| = 2.$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5.$$

Since all $D_i > 0$, the quadratic form is positive definite.

Example 1.66. Find the nature of the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$.

[Jan 2010]

Solution. Since the given Q.F is in 3 variables, the matrix A of the Q.E is a 3×3 matrix.

—

$$\therefore \ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

From the given Q.F we have

$$a_{11} = \text{Coeff.of } x^2 = 6.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-4) = -2$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 3.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1.$$

$$a_{33} = \text{Coeff.of } z^2 = 3$$

$$\therefore \ A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

The principal minors are $D_1 = 6 > 0$.

$$D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 18 - 4 = 14 > 0.$$

$D_3 = |A| = 6(8) + 2(-4) + 2(-4) = 48 - 8 - 8 = 32 > 0.$

Since D_1, D_2, D_3 are positive, the Q.F is positive definite.

Example 1.67. Determine λ so that the quadratic form $\lambda(x^2+y^2+z^2)+2xy-2yz+2zx$ is positive definite.

Solution. If A is the matrix of the given quadratic form, then $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Now, $a_{11} = \text{Coeff.of } x^2 = \lambda$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times 2 = 1$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 2 = 1$$

$$a_{22} = \text{Coeff.of } y^2 = \lambda$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1$$

$$a_{33} = \text{Coeff.of } z^2 = \lambda$$

$$\therefore\ A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{pmatrix}.$$

Let us find out the principal minors D_1, D_2 and D_3.

$$D_1 = \lambda.$$

$$D_2 = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1).$$

$$D_3 = \lambda(\lambda^2 - 1) - 1(\lambda + 1) + 1(-1 - \lambda)$$

$$= \lambda(\lambda - 1)(\lambda + 1) - (\lambda + 1) - (\lambda + 1) = (\lambda + 1)(\lambda^2 - \lambda - 1 - 1)$$

$$= (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)(\lambda - 2)(\lambda + 1) = (\lambda + 1)^2(\lambda - 2).$$

Since the Q.F is positive definite, we have $D_1 > 0, D_2 > 0$ and $D_3 > 0$.

$$D_1 > 0 \Longrightarrow \lambda > 0$$

$$D_2 > 0 \Longrightarrow (\lambda + 1)(\lambda - 1) > 0.$$

$$\Longrightarrow \lambda + 1 > 0 \text{ and } \lambda - 1 > 0.$$

$$\Longrightarrow \lambda > -1 \text{ and } \lambda > 1.$$

OR

$$\lambda + 1 < 0 \text{ and } \lambda - 1 < 0$$

$$\Longrightarrow \lambda < -1 \text{ and } \lambda < 1.$$

$$D_3 > 0 \Longrightarrow (\lambda + 1)^2(\lambda - 2) > 0$$

$$\Longrightarrow \lambda > 2 \text{ [since } (\lambda + 1)^2 \text{ is always } > 0].$$

—

We have to consider the following cases.

case(i). $\lambda > 0, \lambda > -1, \lambda > 1, \lambda > 2.$

or

case(ii). $\lambda > 0, \lambda < -1, \lambda < 1, \lambda > 2.$ In the above cases the value of λ that satisfies the above conditions is $\lambda > 2$.

Example 1.68. Show that the Q.F $ax_1^2 - 2bx_1x_2 + cx_2^2$ is positive definite if $a > 0$ and $ac - b^2 > 0$.

Solution. The given Q.F. is in 2 variables. Hence, the matrix A of the Q.F is a 2×2 matrix.

Let A be $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$$\text{Now, } a_{11} = \text{Coeff.of } x_1^2 = a.$$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times (-2b) = -b.$$

$$a_{22} = \text{Coeff.of } x_2^2 = c.$$

$$\therefore A = \begin{pmatrix} a & -b \\ -b & c \end{pmatrix}.$$

We have $D_1 = a, D_2 = ac - b^2$.

Given $a > 0$ and $ac - b^2 > 0$.

$\implies D_1 > 0$ and $D_2 > 0$.

$\therefore$ The Q.F is positive definite.

Example 1.69. Find the index, signature, rank and nature of the quadratic form in 3 variables $x^2 + 2y^2 - 3z^2$. [May 2011]

Solution. Since the given Q.F contains only square terms, it is in the canonical form in 3 variables.

The eigen values are the coefficients of x^2, y^2 and z^2.

$$\therefore \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -3$$

—

No. of positive eigen values = 2.

$\therefore$ Index = 2.

No. of negative eigen values = 1.

Signature = Difference of the number of positive and negative eigen values

= 2 − 1 = 1.

Rank = 3 = Number of positive and negative eigen values.

Nature of the Q.F is indefinite.

Example 1.70. If the quadratic form $ax^2+2bxy+cy^2$ is positive definite (or negative definite), then prove that the quadratic equation $ax^2 + 2bx + c = 0$ has imaginary roots.

Solution. Since the given Q.F contains two variables, the matrix of the Q.F. A is a 2×2 matrix.

Let A be $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Now, a_{11} = Coeff.of $x^2 = a$.

$a_{12} = a_{21} = \frac{1}{2} \times$ Coeff.of $xy = \frac{1}{2} \times 2b = b$.

a_{22} = Coeff.of $y^2 = c$.

$\therefore\ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

The principal minors are $D_1 = a, D_2 = ac - b^2$.

If the quadratic form is positive definite then $D_1 > 0$ and $D_2 > 0$.

$$\implies a > 0 \text{ and } ac - b^2 > 0.$$

$$\implies a > 0 \text{ and } b^2 - ac < 0. \quad (1)$$

If the quadratic form is negative definite, then $(-1)^i D_i > 0$ for all i.

$$\implies -D_1 > 0 \text{ and } D_2 > 0$$

$$\implies -a > 0 \text{ and } ac - b^2 > 0.$$

—

$$\Longrightarrow a < 0 \text{ and } b^2 - ac < 0. \tag{2}$$

Consider the quadratic equation

$$ax^2 + 2bx + c = 0. \tag{3}$$

The discriminant of the quadratic equation is

$$\Delta = 4b^2 - 4ac = 4(b^2 - ac) < 0.$$

$\therefore$ The roots of (3) are imaginary.

Example 1.71. Reduce the quadratic form $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$ to canonical form through an orthogonal transformation. [Jan 2006]

Solution.

Step 1. To form the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F is in 3 variables, A must be a 3×3 matrix.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\begin{aligned}
\text{Now, } a_{11} &= \text{Coeff.of } x_1^2 = 1 \\
a_{12} = a_{21} &= \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times 2 = 1 \\
a_{13} = a_{31} &= \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 6 = 3 \\
a_{22} &= \text{Coeff.of } x_2^2 = 5. \\
a_{23} = a_{32} &= \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1. \\
a_{33} &= \text{Coeff.of } x_3^2 = 1.
\end{aligned}$$

$\therefore\ A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 5 + 1 = 7.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= 5 - 1 + 1 - 9 + 5 - 1$

$= 4 - 8 + 4 = 0.$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$= 1(5 - 1) - 1(1 - 3) + 3(1 - 15)$

$= 4 + 2 - 42$

$= -36.$

$\therefore$ The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0.$

$\lambda = -2$ is a root.

By synthetic division, we get

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0.$$

$$(\lambda - 3)(\lambda - 6) = 0.$$

The characteristic equation becomes $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$

The eigen values are $-2, 3, 6.$

—

Step 3.To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore\ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above two equations are reduced to two equations

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0.$$

Solving the two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & 3 & \times & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\Longrightarrow\ x_1 = 1, x_2 = 0, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

—

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-2x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$3x_1 + x_2 - 2x_3 = 0.$$

From the first two equations by the rule of cross multiplication we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -2 & \times & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Longrightarrow x_1 = 1, x_2 = -1, x_3 = 1.$$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes $\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations are reduced to

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0.$$

By the rule of cross multiplication, from the first two equations we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & \times & 3 & \times & -5 & \times & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Longrightarrow x_1 = 1, x_2 = 2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since the eigen values are different, the eigen vectors are mutually orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with columns as the normalized eigen vectors.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

—

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{6} & 2\sqrt{6} & \sqrt{6} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^T AX$.

$$= (NY)^T A(NY)$$

$$= Y^T N^T ANY$$

$$= Y^T DY$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -2y_1^2 + 3y_2^2 + 6y_3^2$$

which is canonical.

Example 1.72. Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ to the canonical form by an orthogonal transformation. Find one set of values of x, y, z (not all zero) which will make the Quadratic form zero. [Jan 2012]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

—

$$\therefore\ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Now, $a_{11} = \text{Coeff.of } x_1^2 = 8$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-12) = -6$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 7.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-8) = -4.$$

$$a_{33} = \text{Coeff.of } z^2 = 3.$$

$$\therefore\ A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 8 + 7 + 3 = 18.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= 21 - 16 + 24 - 4 + 56 - 36$

$= 5 + 20 + 20 = 45.$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$

—

$$= 40 - 60 + 20 = 0.$$

The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations become

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we obtain

—

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & 8 & \times & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\implies x_1 = 1, x_2 = 2, x_3 = 2.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations become

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 = 0.$$

From the last equation we get

$$2x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{2}$$

$\Rightarrow x_1 = 2, x_2 = 1.$

Substituting in the first equation we get

$$10 - 6 + 2x_3 = 0$$

$$4 + 2x_3 = 0$$

$$2x_3 = -4$$

$$x_3 = -2.$$

$\therefore \ X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$

When $\lambda = 15$, (1) becomes $\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The above equations become

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0.$$

Taking the first two equations and applying the rule of cross multiplication, we get

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & \times & 2 & \times & -7 & \times & -6 \\ -8 & & -4 & & -6 & & -8 \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \Longrightarrow x_1 = 2, x_2 = -2, x_3 = 1.$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Step 4. To form the modal matrix

Since all the eigen values are different, X_1, X_2, X_3 are pairwise orthogonal.

The modal matrix N is formed with the normalized eigen vectors are columns.

—

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$.

$$\therefore\ N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Step 5. To find $N^T AN$

$$N^T AN = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 30 & -30 & 15 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

$$\text{The given Q.F is } Q = X^T AX.$$

$$= (NY)^T A(NY)$$

$$= Y^T N^T ANY$$

$$= Y^T DY$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

—

$$= 3y_2^2 + 15y_3^2.$$

which is canonical.

we have $X = NY$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x = \frac{1}{3}(y_1 + 2y_2 + 2y_3)$$

$$y = \frac{1}{3}(2y_1 + y_2 - 2y_3)$$

$$z = \frac{1}{3}(2y_1 - 2y_2 + y_3).$$

Equating the quadratic form to zero we get

$$3y_2^2 + 15y_3^2 = 0.$$

$$\Rightarrow \; y_2 = 0, y_3 = 0, \text{ and assume } y_1 = 3.$$

Substituting these values in x, y, z we obtain $x = 1, y = 2, z = 2$.

These set of values will make the quadratic form zero.

Example 1.73. Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Also give a nonzero set of values (x_1, x_2, x_3) which makes the quadratic form to zero. [Jan 2009]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

—

Now, $a_{11} = \text{Coeff.of } x_1^2 = 1$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = \frac{1}{2} \times (-2) = -1$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 0 = 0$$

$$a_{22} = \text{Coeff.of } x_2^2 = 2.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 2 = 1$$

$$a_{33} = \text{Coeff.of } x_3^2 = 1.$$

$$\therefore \; A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 1 + 2 + 1 = 4.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$= 2 - 1 + 1 - 0 + 2 - 1$

$= 3.$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$= 1(2 - 1) + 1(-1 - 0) + 0$

$= 1 - 1 = 0.$

—

$\therefore$ The characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 0, 1, 3.$$

The eigen values are $0, 1, 3$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \qquad (1)$$

When $\lambda = 0$, (1) becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$x_1 - x_2 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$x_2 + x_3 = 0.$$

From the first equation we get

$$x_1 = x_2 \implies x_1 = 1, \ x_2 = 1.$$

From the last equation we get

$$x_2 = -x_3$$

$$\Rightarrow \; x_3 = -1.$$

$$\therefore \; X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

When $\lambda = 1$, (1) becomes

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are

$$x_2 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

Substituting $x_2 = 0$, the second equation becomes

$$x_1 = x_3 \;\Rightarrow\; x_1 = 1, \; x_3 = 1.$$

$$\therefore \; X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 3$, (1) become $\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

The equations are

$$2x_1 + x_2 = 0$$

$$x_1 + x_2 - x_3 = 0$$

$$x_2 - 2x_3 = 0.$$

From the first equation we have

$$2x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{2}$$

—

$\Rightarrow x_1 = -1, x_2 = 2.$

From the last equation we have

$$x_2 = 2x_3$$
$$\frac{x_2}{2} = \frac{x_3}{1}$$
$$\Longrightarrow x_2 = 2, x_3 = 1.$$

$$\therefore\ X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Step 4. To find the modal matrix

Since all the eigen values are different, X_1, X_2, X_3 are pairwise orthogonal.

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

The modal matrix N is formed with the normalized eigen vectors as columns.

$$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-3}{\sqrt{6}} & \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$

—

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^T AX$.

$$\begin{aligned}
&= (NY)^T A(NY) \\
&= Y^T N^T ANY \\
&= Y^T DY \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= y_2^2 + 3y_3^2.
\end{aligned}$$

which is canonical.

Step 7. To find the nature of the Q.F

Since two eigen values are positive and one eigen value is 0, the given quadratic form is positive semidefinite.

Step 8. To find the value of x_1, x_2, x_3 for which the $Q.F = 0$

Taking $y_2^2 + 3y_3^2 = 0 \Longrightarrow y_2 = 0, y_3 = 0$.

Now $X = NY$ is reduced to the equations.

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3.$$

Taking $y_1 = \sqrt{3}$, we get $x_1 = 1, x_2 = 1, x_3 = -1$.

$\therefore x_1 = 1, x_2 = 1, x_3 = -1$ make the given quadratic form zero.

—

Example 1.74. Reduce the Q.F $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form. State the type of the canonical form. [Jan 2008]

Solution.

Step 1. To find the matrix of the Q.F

Let A be the matrix of the Q.F.

Since the given Q.F contains 3 variables, A must be a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, $a_{11} = \text{Coeff.of } x_1^2 = 2.$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } x_1x_2 = 0.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } x_1x_3 = \frac{1}{2} \times 8 = 4$$

$a_{22} = \text{Coeff.of } x_2^2 = 6.$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } x_2x_3 = \frac{1}{2} \times 0 = 0.$$

$a_{33} = \text{Coeff.of } x_3^2 = 2.$

$$\therefore\ A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 2 + 6 + 2 = 10.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$$

—

$$= 12 - 0 + 4 - 16 + 12 - 0$$

$$= 12 - 12 + 12 = 12.$$

$$s_3 = |A| = \begin{vmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{vmatrix}$$

$$= 2(12 - 0) - 0 + 4(0 - 24)$$

$$= 24 - 96$$

$$= -72.$$

$\therefore$ The characteristic equation is $\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

$$\lambda = -2 \text{ is a root.}$$

By synthetic division we get

−2	1	−10	12	72
	0	−2	24	−72
	1	−12	36	0

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = 6, 6.$$

The eigen values are $-2, 6, 6$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ.

$$\therefore \ (A - \lambda I)X = 0.$$

—

$$\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{1}$$

When $\lambda = -2$, (1) becomes

$$\begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The equations are reduced to

$$4x_1 + 4x_3 = 0$$

$$8x_2 = 0$$

$$\therefore\ x_2 = 0.$$

$$\text{and } 4x_1 = -4x_3$$

$$\frac{x_1}{4} = \frac{x_3}{-4}$$

$$\Rightarrow\ x_1 = 1, x_3 = -1.$$

$$\therefore\ X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

When $\lambda = 6$, (1) becomes

$$\begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

The above equations are reduced to one single equation

$$-4x_1 + 4x_3 = 0$$

$$\text{i.e., } 4x_1 = 4x_3$$

$$x_1 = x_3.$$

—

$$\Rightarrow\ x_1 = 1, x_3 = 1, x_2 = \text{any value. Take } x_2 = 0.$$

$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ orthogonal to X_2.

$\therefore\ X_3^T X_2 = 0$

$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

$a + c = 0$ (2)

X_3 also satisfy

$-4a + 4c = 0$

i.e., $a - c = 0$ (3)

Solving (2) and (3) we get

$a = 0, c = 0$ and b =any value.

Take $b = 1$.

$\therefore\ X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Step 4. To find the modal matrix

The normalized eigen vectors are $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The modal matrix is formed with the normalised eigen vectors as columns.

$\therefore\ N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$.

—

Step 5. To find N^TAN

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 3\sqrt{2} & 0 & 3\sqrt{2} \\ 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D.$$

Step 6. Reduction to Canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

The given Q.F is $Q = X^TAX$.

$$= (NY)^TA(NY)$$

$$= Y^TN^TANY$$

$$= Y^TDY$$

$$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -2y_1^2 + 6y_2^2 + 6y_3^2$$

which is canonical.

Step 7. To find the nature

Since two of the eigen values are positive and one eigen value is negative, the given Q.F is indefinite.

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Example 1.75. Reduce $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal reduction and find the rank, signature, index and nature of the quadratic form. [Jan 2015, Jan 2005]

Solution.

Step 1. To find the matrix of the Q.F

Since the given Q.F contains 3 variables, the matrix of the Q.F A is a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, $a_{11} = \text{Coeff.of } x^2 = 6$

$$a_{12} = a_{21} = \frac{1}{2} \times \text{Coeff.of } xy = \frac{1}{2} \times (-4) = -2.$$

$$a_{13} = a_{31} = \frac{1}{2} \times \text{Coeff.of } xz = \frac{1}{2} \times 4 = 2$$

$$a_{22} = \text{Coeff.of } y^2 = 3.$$

$$a_{23} = a_{32} = \frac{1}{2} \times \text{Coeff.of } yz = \frac{1}{2} \times (-2) = -1.$$

$a_{33} = \text{Coeff.of } z^2 = 3.$

$$\therefore\ A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

Step 2. To find the eigen values

Since A is a 3×3 matrix, the characteristic equation is of the form

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0.$$

Now, s_1 = sum of the main diagonal elements

$= 6 + 3 + 3 = 12.$

s_2 = sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

—

$$= 9 - 1 + 18 - 4 + 18 - 4$$

$$= 8 + 14 + 14 = 36.$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$$

$$= 48 - 8 - 8$$

$$= 32.$$

$\therefore$ The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$\lambda = 2$ is a root. By synthetic division we get

2	1	−12	36	−32
	0	2	−20	32
	1	−10	16	0

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0.$$

The characteristic equation becomes $(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$.

The eigen values are $2, 2, 8$.

Step 3. To find the eigen vectors

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to the eigen value λ.

$$\therefore \; (A - \lambda I)X = 0.$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

—

$$\left.\begin{aligned}(6-\lambda)x_1 - 2x_2 + 2x_3 &= 0\\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0\\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0.\end{aligned}\right\} \quad (1)$$

When $\lambda = 8$ (1) $\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0.$

$$-2x_1 - 5x_2 - x_3 = 0 \Rightarrow 2x_1 + 5x_2 + x_3 = 0.$$

$$2x_1 - x_2 - 5x_3 = 0 \Rightarrow 2x_1 - x_2 - 5x_3 = 0.$$

From the first two equations using the rule of cross multiplication we get

$$\frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow x_1 = 2, x_2 = -1, x_3 = 1.$$

$$\therefore\ X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

When $\lambda = 2$, (1) is reduced to a single equation, $2x_1 - x_2 + x_3 = 0$.
Choose $x_3 = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2 \Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{2} \Rightarrow x_1 = 1, x_2 = 2.$

$$\therefore\ X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be orthogonal to X_2.

$$\Longrightarrow\ X_2^T X_3 = 0 \Rightarrow a + 2b = 0. \quad (2)$$

$$X_3 \text{ also satisfy } 2a - b + c = 0. \quad (3)$$

(2) $\Rightarrow a = -2b \Longrightarrow \dfrac{a}{-2} = \dfrac{b}{1}.$

(3) $\Rightarrow -4b - b + c = 0 \Rightarrow 5b = c \Rightarrow \dfrac{b}{1} = \dfrac{c}{5}.$

—

Combining the above two equations we obtain $\frac{a}{-2} = \frac{b}{1} = \frac{c}{5}$.

$$\text{Hence, } b = 1, c = 5, a = -2.$$

$$\therefore \; X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}.$$

Step 4. To find the modal matrix

The normalised eigen vectors are

$\left[\frac{2}{\sqrt{6}} \; \frac{-1}{\sqrt{6}} \; \frac{1}{\sqrt{6}}\right]^T, \left[\frac{1}{\sqrt{5}} \; \frac{2}{\sqrt{5}} \; 0\right]^T, \left[\frac{-2}{\sqrt{30}} \; \frac{1}{\sqrt{30}} \; \frac{5}{\sqrt{30}}\right]^T$

$$\therefore \; N = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}.$$

Step5. To find $N^T AN$

$$N^T AN = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{16}{\sqrt{6}} & \frac{-8}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-8}{\sqrt{5}} & 0 \\ \frac{-4}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{10}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

Step 6. Reduction to canonical form

Let $X = NY$, where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

—

The given Q.F. is $Q = X^T AX$

$$= (NY)^T ANY$$
$$= Y^T N^T ANY$$
$$= Y^T DY$$

$$Y^T DY = [y_1\ y_2\ y_3]\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= [y_1\ y_2\ y_3]\begin{pmatrix} 8y_1 \\ 2y_2 \\ 2y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

which is the canonical form.

Rank of the Q.F = 3.

Index = 3.

Signature = 3.

Since all the eigen values are positive, the Q.F is positive definite.

Example 1.76. Reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ in to canonical form. Discuss also its nature. [Jan 2013].

Solution.

Step 1. To find the matrix of the quadratic form

Since given Q.F. contains 3 variables, the matrix of the Q.F. A is a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$